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Supersymmetric Path Integrals

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Abstract. The supersymmetric path integral is constructed for quantum mechanical models on flat space as a supersymmetric extension of the Wiener integral. It is then pushed forward to a compact Riemannian manifold by means of a Malliavin-type construction. The relation to index theory is discussed.

Introduction

An interesting new branch of mathematical physics is supersymmetry. With the advent of the theory of superstrings [1], it has become important to analyze the quantum field theory of supersymmetric maps from R^2 to a manifold. This should probably be done in a supersymmetric way, that is, based on the theory of supermanifolds, and in a space-time covariant way as opposed to the Hamiltonian approach. Accordingly, one wishes to make sense of supersymmetric path integrals. As a first step we study a simpler case, that of supersymmetric maps from R^1 to a manifold, which gives supersymmetric quantum mechanics. As Witten has shown, supersymmetric quantum mechanics is related to the index theory of differential operators [2]. In this particular case of a supersymmetric field theory, the Witten index, which gives a criterion for dynamical supersymmetry breaking, is the ordinary index of a differential operator. If one adds the adjoint to the operator and takes the square, one obtains the Hamiltonian of the quantum mechanical theory. These indices can be formally computed by supersymmetric path integrals. For example, the Euler characteristic of a manifold M is supposed to be given by integrating e^{-L}, with

$$L=\frac{1}{2}g_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j-\frac{1}{2}g_{ij}(\phi)\psi^{i\dagger}\frac{D}{dt}\psi^j-\frac{1}{8}R_{ijkl}(\phi)\psi^{i\dagger}\psi^j\psi^{k\dagger}\psi^l,$$

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over periodic ϕ's and ψ's, ϕ being a map from S^1 to M and ψ being its fermionic counterpart [3]. These formal considerations have given rise to a rigorous method of computing index densities by means of a quadratic approximation to the operator, which is in fact independent of any considerations of supersymmetry [4, 5].

There is an intimate relation between supersymmetric quantum mechanics and the geometry of loop spaces, as was noted by Atiyah and Witten in [11, 15]. (The reader may wish to look at [11] to understand some of the constructions in the present paper.) They remarked that the generator of the supersymmetry transformation (in the Lagrangian approach) can be formally represented by $d+i_{\dot{\gamma}}$ acting on differential forms on the loop space ΩM of M. The super Lagrangian (for $N=1/2$ supersymmetry) was identified as $E+\omega$, where E is the energy of a loop and ω is the natural presymplectic form on ΩM. The formal application of the Duistermaat-Heckman integration formula gave the identification of the Feynman-Kac expression for the index of the Dirac operator with the index theorem expression (as an integral over M). This shows a connection between the cohomology of loop spaces and the Wiener measure. We do not explore this question, but instead study the supersymmetric path integral as an object in its own right.

We wish to show that the supersymmetric path integral can be rigorously defined. This is done by means of a Malliavin-type construction, after the flat space supermeasure is constructed by hand. The organization of this paper is as follows:

Section I consists of a construction of the fermionic (Berezin) path integral.

Section II uses this to construct the $N=1/2$ supermeasure for supermaps of R^1 to a flat space.

Section III does the same for $N=1$ supersymmetry with superpotential added and shows the superinvariance of the supermeasure.

Section IV proves an index theorem for the operator corresponding to the supercharge of the previous section, namely $e^V de^{-V}+e^{-V}d^*e^V$. This is done by first performing the fermionic integral explicitly. The answer obtained is the same as from the corresponding zeta function determinant, but with the relative sign fixed. Then a semiclassical approximation is done, which in this case is equivalent to the scaling of V used in [15]. We show that the quadratic approximation then gives the exact formula for the index.

Section V extends the $N=1/2$ supermeasure to the case of an arbitrary compact spin manifold M. First, the supermeasure is considered as a linear functional on the superfunctions on the supermanifold of maps from $S^{1,1}$ to M, which is formally shown to be the cross-sections of the Grassmannian of the tangent bundle of ΩM. The algebra of observables and its supermeasure are constructed using the Cartan development. Superinvariance is shown and the corresponding Hamiltonian operator is shown to be the square of the Dirac operator. In terms of forms on ΩM, the algebra of observables is generated by the pullback of Λ^*M under $\gamma\in\Omega M\to\gamma(t)\in M$, when smoothed out in t. The supertransformation is the aforementioned $d+i_{\dot{\gamma}}$.

Section VI covers the case of an added external connection which lies on a vector bundle over M.

Notation. For a vector space V, let $\mathrm{Cl}(V)$ denote the Clifford algebra on V generated by $\{\gamma(v),\gamma(v')\}=2\langle v,v'\rangle$. For a vector bundle E, let Λ^*E denote the Grassmannian of E and let $\Gamma^k(\Lambda^*E)$ denote its C^k sections. Let $[M,N]^k$ denote the C^k maps between two manifolds M and N and if N is linear, let $[M,N]_0^k$ denote those of compact support. Define $h^i_{[a,b]}\in[\mathbb{R},\mathbb{R}^{2n}]^\infty$ to be $\phi(x)e^i$ for some $\phi\in C_0^\infty(\mathbb{R})$, with $\phi\geqq 0$, $\operatorname{supp}\phi\subset[a,b]$ and $\int\phi=1$. The Einstein summation convention is used freely.

I. Fermionic Integrals

The fermionic integral given here is based on the work of [6], with some modifications. Let V be a real $2n$-dimensional inner product space and let M be an invertible skew-adjoint operator on V. Consider M also as an element of $\Lambda^2(V^*)$ by $M(V_1,V_2)\equiv\langle V_1,MV_2\rangle$. Define a linear functional on $\Lambda^*(V)$, the Berezin integral, by $\eta\in\Lambda^*(V)\to\int\eta\equiv$(the coefficient of the $\Lambda^{2n}(V)$ term of $e^{\frac{1}{2}M}\eta$).

Proposition 1. *For* $\{v_i\}_{i=1}^k\in V$,

$$\int\bigwedge_{i=1}^k v_i=(-)^{k/2}\,\mathrm{Pf}(M)\times\sum_{\substack{\text{distinct pairings}\\(a_1,a_2)\dots(a_{k-1},a_k)\text{ of }(1,\dots,k)}}(-)^{\sigma(a_1,\dots,a_k)}(v_{a_1},M^{-1}v_{a_2})\dots(v_{a_{k-1}},M^{-1}v_{a_k})\,.$$

Proof. See [7]. □

We wish to generalize this integral to the case of an infinite-dimensional Hilbert space. Clearly, it no longer makes sense to pick out the highest term in $\Lambda^*(V)$. However, it is possible to rewrite the finite-dimensional integral in a way that will extend to infinite dimensions.

Let $d:V\to V^*$ be the map induced by the inner product on V. Construct the Clifford algebra $A_F(V\oplus V^*)$ with the generating relationship

$$\{v_1\oplus w_1,v_2\oplus w_2\}=w_1(|M|^{-1}v_2)+w_2(|M|^{-1}v_1)\,.$$

Denote the image of $v_1\oplus d(v_2)$ in A_F by $a(v_1)\oplus a^*(v_2)$ and define a duality on A_F generated by $(a(v_1)\oplus a^*(v_2))^*=a(v_2)\oplus a^*(v_1)$. Put $\psi(v)=a^*(v)+a\left(\frac{M}{|M|}v\right)$. Then

$$\begin{aligned}\{\psi(v_1),\psi(v_2)\}&=\left\{a^*(v_1)+a\left(\frac{M}{|M|}v_1\right),\,a^*(v_2)+a\left(\frac{M}{|M|}v_2\right)\right\}\\&=\left(v_1,\frac{M}{|M|}v_2\right)+\left(v_2,\frac{M}{|M|}v_1\right)=0\,,\end{aligned}$$

and so ψ generates a monomorphism $\psi:\Lambda^*(V)\to A_F$. There is a unique pure state $\langle\ \rangle$, the Fock state, on A_F which satisfies $\langle xa(v)\rangle=\langle a^*(v)x\rangle=0$ for all $x\in A_F$ and $v\in V$.

Proposition 2. *For all* $\eta\in\Lambda^*(V)$, $\langle\psi(\eta)\rangle=\int\eta/\int 1$.

Proof. We have $\{\psi(v), a^*(v')\} = \left\{a^*(v) + a\left(\frac{M}{|M|}v\right),\ a^*(v')\right\} = (v', M^{-1}v)$. To prove the desired formula, it suffices to compute $\left\langle \prod_{i=1}^{k} \psi(v_i)\right\rangle$. For $k=0$ or 1, the truth of the formula is clear. For $k>1$,

$$\begin{aligned}\left\langle \prod_{i=1}^{k} \psi(v_i)\right\rangle &= \left\langle \prod_{i=1}^{k-1} \psi(v_i) a^*(v_k)\right\rangle \\ &= \sum_{i=1}^{k-1} (-)^{i+1} \langle v_k, M^{-1} v_{k-i}\rangle \langle \psi(V_1) \dots \widehat{\psi(v_{k-i})} \dots \psi(v_{k-1})\rangle .\end{aligned}$$

Assuming the truth for (degree $\eta) \leqq k-1$, we have

$$\begin{aligned}\left\langle \prod_{i=1}^{k} \psi(v_i)\right\rangle &= \sum_{i=1}^{k-1} (-)^{i+1} (v_k, M^{-1} v_{k-i}) \\ &\times \sum_{\substack{\text{pairings } (a_1,a_2),\dots,(a_{k-3},a_{k-2}) \\ \text{of } \{1,2,\dots,\widehat{k-i},\dots,k-1\}}} (-)^{\sigma(a_1,\dots,a_{k-2})} \\ &\times (-)^{i+1} (v_{a_1}, M^{-1} v_{a_2}) \dots (v_{a_{k-3}}, M^{-1} v_{a_{k-2}}) \\ &= (-)^{k/2} \sum_{i=1}^{k-1} (-)^{i+1} (v_{k-i}, M^{-1} v_k) \\ &\times \sum_{\substack{\text{pairings } (a_1,a_2),\dots,(a_{k-1},a_k) \\ \text{of } \{1,\dots,k\} \text{ s.t. } a_{k-1}=k-i,\ a_k=k}} (-)^{\sigma(a_1,\dots,a_k)} \\ &\times (-)^{i+1} (v_{a_1}, M^{-1} v_{a_2}) \dots (v_{a_{k-3}}, M^{-1} v_{a_{k-2}}) \\ &= (-)^{k/2} \sum_{\substack{\text{pairings } (a_1,a_2),\dots,(a_{k-1},a_k) \\ \text{of } \{1,\dots,k\}}} (-)^{\sigma(a_1,\dots,a_k)} \\ &\times (v_{a_1}, M^{-1} v_{a_2}) \dots (v_{a_{k-1}}, M^{-1} v_{a_k}) \\ &= \int \eta / \int 1 .\end{aligned}$$

The proposition follows by induction. □

Note that the measurables are in $\Lambda^*(V)$; the value of the state on the rest of A_F is immaterial.

Given a real Hilbert space $\mathscr{H}_F$ and a bounded invertible real skew-adjoint operator M on $\mathscr{H}_F$, let $\langle , \rangle$ be the inner product on $\mathscr{H}_F$ defined by $\langle v_1, v_2\rangle = (v_1, |M|^{-1} v_2)$. Form the CAR algebra A_F based on $\mathscr{H}_F$ with generating relationship $\{a^*(v_1), a(v_2)\} = \langle v_1, v_2\rangle$. Then there is a unique Fock state $\langle\ \rangle_F$ on A_F. Put $\psi(v) = a^*(v) + a\left(\frac{M}{|M|}v\right)$, and let $\mathscr{A}_F$ be the Banach subalgebra of A_F generated by $\{\psi(v)\}$. Define the normalized Berezin integral on $\mathscr{A}_F$ by $\int \eta = \langle \eta\rangle_F$. (The use of a CAR algebra here has nothing to do with the use of CAR algebras in Hamiltonian formulations of fermion theories.)

When one wishes to quantize Majorana fermions, the above applies when the Euclidean Dirac operator is real and skew-adjoint, that is, in spacetime dimensions $\equiv 0, 1, 2 \pmod 8$, and one avoids the fermion doubling problem of [6].

II. The Free $N=1/2$ Supersymmetric Field

The Lagrangian for $N=1/2$ supersymmetry is $L=\frac{1}{2}\int_{-\infty}^{\infty}\left(\left\langle\frac{dA}{dT},\frac{dA}{dT}\right\rangle-\left\langle\psi,\frac{d\psi}{dT}\right\rangle\right)dT$. Here $A,\omega\in[\mathbb{R},\mathbb{R}^{2n}]_0^\infty$ and ψ is formally of odd degree (i.e., anticommuting). (For a more meaningful description, see Sect. V.) If $\varepsilon\in[\mathbb{R},\mathbb{R}]^\infty$ is a real constant of odd degree then L is invariant under the infinitesimal variation $\delta A=\varepsilon\psi$, $\delta\psi=\varepsilon\frac{dA}{dT}$. In order to quantize this Lagrangian we wish to make sense of $\int e^{-L}\mathcal{O}(A,\psi)\mathcal{D}A\mathcal{D}\psi$ with $\mathcal{O}$ being some functional of A and ψ. For the A field this formal integral has a precise meaning using the Wiener measure $d\mu$ on $[\mathbb{R},\mathbb{R}^{2n}]^0$, which can also be thought of as giving a state on the commutative algebra $L^\infty(d\mu)$. The supersymmetric Wiener integral should then be a linear functional on the noncommutative algebra of measurables.

Definition. Put $H^s=\{f\in\mathcal{S}'[\mathbb{R},\mathbb{R}^{2n}]$: the Fourier transform $F(f)$ of f has $\int|k|^{2s}|F(f)(k)|^2dk<\infty\}$. Let A_B be the Weyl algebra based on H^{-1} with the relation

$$U(v_1,w_1)U(v_2,w_2)=e^{i((v_2,w_1)-(v_1,w_2))}U(v_2,w_2)U(v_1,w_1)$$

for $v_1,v_2,w_1,w_2\in H^{-1}$. Let $\mathcal{A}_B$ be the commutative Banach subalgebra generated by $\{U(v,0)\}$. Let M be the Hilbert transform $\frac{d}{dT}\Big/\left|\frac{d}{dT}\right|$ acting on $H^{-1/2}$. Form the algebras A_F and $\mathcal{A}_F$ of the previous section. The algebra of measurables is $\mathcal{A}=\mathcal{A}_B\otimes\mathcal{A}_F$ with the linear functional $\langle\ \rangle=\langle\ \rangle_B\otimes\langle\ \rangle_F$ induced from the Fock states on A_B and A_F.

As $\langle\ \rangle_B$ is a faithful state, it gives a positive probability measure $d\mu$ on the maximal ideal space Δ of $\mathcal{A}_B$. If $A(f)=-i\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}U(\varepsilon f,0)$ then $A(f)=-i\ln U(f,0)(\mathrm{mod}\,2\pi)$, and so $A(f)$ is Borel measurable on Δ. Given a sequence $\{f_i\}_{i=1}^m$ in H^{-1}, we have $\int d\mu\left(\prod_{i=1}^m A(f_i)\right)^2=\left\langle\left(\prod_{i=1}^m A(f_i)\right)^2\right\rangle_B$ which is finite by Wick's theorem and the fact that $\langle A(f)A(f')\rangle=(f,f')_{-1}$. Thus $\prod_{i=1}^m A(f_i)\in L^2(d\mu)\subset L^1(d\mu)$. Let $\langle\ \rangle_B$ also denote integration on $L^1(d)$.

Definition. The supersymmetry transformation S is a densely defined (graded) derivation on $L^1(d\mu)\otimes\mathcal{A}_F$ such that if $\{f_i\}_{i=1}^m$ and $\{g_j\}_{j=1}^{m'}$ are in the Fourier transform of $[\mathbb{R},\mathbb{R}^{2n}]_0^\infty$ with $F(f_i)(0)=F(g_j)(0)=0$, then

$$S\left(\prod_{i=1}^m A(f_i)\prod_{j=1}^{m'}\psi(g_j)\right)=\sum_{i=1}^m\left(\prod_{\substack{i'=1\\ i'\neq i}}^m A(f_{i'})\right)\psi(f_i)\prod_{j=1}^{m'}\psi(g_j)+\sum_{j=1}^{m'}(-)^j\left(\prod_{i=1}^m A(f_i)\right)A\left(\frac{d}{dT}g_j\right)\prod_{\substack{j'=1\\ j'\neq j}}^{m'}\psi(g_{j'}).$$

Proposition 3. *For all* $\mathcal{O}\in\mathrm{Dom}(S)$, $\langle S\mathcal{O}\rangle=0$.

Proof. Take $\mathcal{O}=\prod_{i=1}^{m} A(f_i)\prod_{j=1}^{m'}\psi(g_j)$. WLOG, assume that m and n are odd. Now

$$\left\langle\left(\prod_{i=1}^{m} A(f_i)\right)A\left(\frac{d}{dT}g_j\right)\right\rangle_B=\sum_{i=1}^{m}\left\langle A(f_i)A\left(\frac{d}{dT}g_j\right)\right\rangle_B\left\langle\prod_{\substack{i'=1\\ i'\neq 1}}^{m} A(f_{i'})\right\rangle_B,$$

and

$$\left\langle\psi(f_i)\prod_{j=1}^{m'}\psi(g_j)\right\rangle_F=\sum_{j=1}^{m'}(-)^{j+1}\langle\psi(f_i)\psi(g_j)\rangle_F\left\langle\prod_{\substack{j'=1\\ j'\neq j}}^{m'}\psi(g_{j'})\right\rangle.$$

The proposition follows because

$$\left\langle A(f)A\left(\frac{d}{dT}g\right)\right\rangle_B=\left(f,\frac{d}{dT}g\right)_{-1}=\left(\mathrm{f},\left|\frac{d}{dT}\right|\frac{d}{dT}\Big/\left|\frac{d}{dT}\right|g\right)_{-1}$$
$$=\left(f,\frac{d}{dT}\Big/\left|\frac{d}{dT}\right|g\right)_{-1/2}=\langle\psi(f)\psi(g)\rangle_F.\quad\square$$

This shows the supersymmetry of the vacuum state of the free theory. We will also need the supersymmetric state given by making time periodic of period β. This requires considering the conditional Wiener measure on paths from a point to itself, and then integrating over $\mathbb{R}^{2n}$.

In the preceding, because of the masslessness of the fields, it was natural to restrict to fermion fields of the form $\psi(f)(0)=0$. This restriction can be evaded by using the fact that only $\mathscr{A}_F$ expectations are taken and the rest of A_F does not matter. Thus the Hilbert space used to define A_F can be varied provided that the ψ fields are changed accordingly.

Definition. Given $-\infty<a<b<\infty$, put $H'=\{f\in[[a,b],\mathbb{R}^{2n}]: f\in L^2([a,b])\}$ and form the CAR algebra $A_{F'}$ based on H'. Define $T'\in B(H')$ by $(T'f)(x)=\frac{1}{2}\int_a^b \operatorname{sign}(x-y)f(y)dy$. Put $\psi'(f)=a^*(f)+a(T'f)\in A_{F'}$ and let these generate the Grassmann algebra $\mathscr{A}_{F'}$. Let $\langle\ \rangle_{F'}$ denote the linear functional on $\mathscr{A}_{F'}$ induced from the Fock state on $A_{F'}$.

Lemma 1. *For* $\{g_j\}_{j=1}^{m'}$ *as in Proposition 3,* $\left\langle\prod_{j=1}^{m'}\psi(g_j)\right\rangle_F=\left\langle\prod_{j=1}^{m'}\psi'(g_j)\right\rangle_{F'}$.

Proof. By Wick's theorem, it suffices to show $\langle\psi(g_1)\psi(g_2)\rangle_F=\langle\psi'(g_1)\psi'(g_2)\rangle_{F'}$. Now

$$\langle\psi(g_1)\psi(g_2)\rangle_F=\langle(a^*(g_1)+a(Mg_1))(a^*(g_2)+a(Mg_2))\rangle_F=\langle a(Mg_1)a^*(g_2)\rangle_F$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g_2(x)g_1(y)\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{il}e^{il(x-y)}dl\,dx\,dy$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g_2(x)g_1(y)\frac{1}{2}\operatorname{sign}(x-y)dx\,dy,$$

and

$$\langle\psi'(g_1)\psi'(g_2)\rangle_{F'}=\langle(a^*(g_1)+a(T'g_1))(a^*(g_2)+a(T'g_2))\rangle_{F'}=\langle a(T'g_1)a^*(g_2)\rangle_{F'}$$
$$=(g_2,T'g_1)_{H'}$$
$$=\int_{-\infty}^{\infty} g_2(x)\int_{-\infty}^{\infty}\frac{1}{2}\operatorname{sign}(x-y)g_1(y)dydx\,. \quad \square$$

It follows that $L^1(d\mu|_{[a,b]})\otimes\mathscr{A}_{F'}$ has the supersymmetric linear functional $\langle\ \rangle_B\otimes\langle\ \rangle_{F'}$. The point of using $\mathscr{A}_{F'}$ is that one can consider $\psi(g)$ with $\int g\neq 0$. Henceforth $\mathscr{A}_{F'}$ and $\langle\ \rangle_{F'}$ will be used exclusively and the primes dropped.

We can now give the Hamiltonian version of the fermion path integral. In one spacetime dimension the fermion Hamiltonian vanishes and all that matters is the factor ordering.

Definition. Let $\{e_i\}_i^{2n}=1$ be an orthonormal basis for $\mathbb{R}^{2n}$ and put

$$\gamma_{2n+1}=i^{n(2n-1)}\prod_{j=1}^{2n}\gamma(e_j)\in \mathrm{Cl}(\mathbb{R}^{2n}),$$

so that $\gamma_{2n+1}^2=1$.

Proposition 4. *Take* $\{g_j\}_{j=1}^{m'}$ *to be a sequence in* $[(a,b),\mathbb{R}^{2n}]_0^\infty$ *with* $\operatorname{supp} g_1 \leqq \ldots \leqq \operatorname{supp} g_{m'}$. *Then*

$$\left\langle\prod_{j=1}^{m'}\psi(g_j)\right\rangle_F=2^{-n}\int \operatorname{Tr}\prod_{j=1}^{m'}\frac{1}{\sqrt{2}}\gamma(g_j(T_j))dT_1\ldots dT_{m'}\,.$$

Proof. Because the dimension of the spinor space is 2^n, the proposition is true for $m'=0,1$. By induction,

$$\left\langle\prod_{j=1}^{m'}\psi(g_j)\right\rangle_F=\sum_{j=2}^{m'}(-)^j\langle\psi(g_1)\psi(g_j)\rangle_F\left\langle\prod_{\substack{j'=2\\ j'\neq j}}^{m'}\psi(g_{j'})\right\rangle_F$$
$$=\sum_{j=2}^{m'}(-)^j\int\frac{1}{2}\langle g_1(T_1),g_j(T_j)\rangle dT_1dT_j2^{-n}$$
$$\times\int dT_2\ldots\widehat{dT_j}\ldots dT_{m'}\operatorname{Tr}\prod_{\substack{j'=2\\ j'\neq j}}^{m'}\frac{1}{\sqrt{2}}\gamma(g_{j'}(T_{j'}))\,.$$

On the other hand, by anticommuting $\gamma(g_1(T_1))$ to the right,

$$\operatorname{Tr}\prod_{j=1}^{m'}\frac{1}{\sqrt{2}}\gamma(g_j(T_j))=\sum_{j=2}^{m'}(-)^j\frac{1}{2}\langle g_j(T_j),g_1(T_1)\rangle\operatorname{Tr}\prod_{\substack{j'=2\\ j'\neq j}}^{m'}\frac{1}{\sqrt{2}}\gamma(g_{j'}(T_{j'}))\,,$$

and so

$$\left\langle\prod_{j=1}^{m'}\psi(g_j)\right\rangle_F=2^{-n}\int\operatorname{Tr}\prod_{j=1}^{m'}\frac{1}{\sqrt{2}}\gamma(g_j(T_j))dT_1\ldots dT_{m'}\,. \quad\square$$

Let $d\mu_{x,y,\beta}$ be the conditional Wiener measure on $\{A\in[(0,\beta),\mathbb{R}^{2n}]^0$ with $\gamma(0)=x,\ \gamma(\beta)=y\}$. Then integration gives a linear functional on $L^1(d\mu_{x,y,\beta})$. For $G\in C_0^\infty(\mathbb{R}^{2n})$ and $f\in[(0,\beta),\mathbb{R}^{2n}]_0^\infty$, $A\to\int_0^\beta f(T)G(A(T))dT$ is in $L^\infty(d\mu_{x,y,\beta})$.

Definition. Let $\mathscr{A}_{F,\beta}$ be the Grassmann algebra generated by $L^2([a,b])$ with $a \ll 0 < \beta \ll b$. The linear functional $\langle\ \rangle_{x,y,\beta}$ on $L^1(d\mu_{x,y,\beta}) \otimes \mathscr{A}_{F,\beta}$ is defined by

$$\langle \mathcal{O}_B \otimes \mathcal{O}_F \rangle_{x,y,\beta} = i^{n(2n-1)} 2^{2n} \langle \mathcal{O}_B \rangle_{x,y,\beta} \left\langle \left(\prod_{k=1}^{2n} \psi(h^k_{[-2n+k-1,\,-2n+k]}) \right) \mathcal{O}_F \right\rangle_F .$$

We now give the Feynman-Kac formula relating the above expectation to the heat kernel of an operator. Let S be the spinor bundle over $\mathbb{R}^{2n}$ and let $\not{D}$ denote the Dirac operator, essentially s.a. on a dense subspace of $L^2(S)$. Let $\tilde{A}$ denote the position operator on $L^2(S)$: $(\tilde{A}S)(x) = xS(x)$, and for $v \in \mathbb{R}^{2n}$, let $\gamma(v)$ denote Clifford multiplication on $L^2(S)$: $(\gamma(v)S)(x) = \gamma(v)S(x)$.

Corollary 1. *Let $\{f_i\}_{i=1}^m$ and $\{g_j\}_{j=1}^m$ be sequences in $[(0,\beta), \mathbb{R}^{2n}]_0^\infty$ with $\operatorname{supp} f_1 \leqq \operatorname{supp} g_1 \leqq \operatorname{supp} f_2 \leqq \ldots \leqq \operatorname{supp} g_m$. (Some elements can be considered missing in the sequence.) Let $\{G_i\}_{i=1}^m$ be a sequence in $C_0^\infty(\mathbb{R}^{2n})$. Put $H = \frac{1}{2}\not{D}^2$. Then*

$$\left\langle \prod_{i=1}^m \left(\int f_i(T_i) G_i(A(T_i)) dT_i\right) \psi(g_i) \right\rangle_{x,y,\beta}$$
$$= \operatorname{Tr} \gamma_{2n+1} e^{-\beta H} \prod_{i=1}^m \int f_i(T_i) e^{-T_i H} G_i(\tilde{A}) e^{T_i H} dT_i$$
$$\times \left(\int e^{-T_i' H} \frac{1}{\sqrt{2}} \gamma(g_i(T_i')) e^{T_i' H} dT_i' \right)(y,x).$$

(The trace is on the Clifford algebra component.)

Proof. This follows from Proposition 4 and the Feynman-Kac formula for the Laplacian, as on $\mathbb{R}^{2n}$, $\not{D}^2$ acts as $\nabla^\dagger \nabla$ and commutes with Clifford multiplication. □

Note. The appearance of the γ_{2n+1} in the Corollary is to ensure that the fermionic integration is over formally periodic fields on $[0,\beta]$. If all the fields are periodic then the Lagrangian is formally superinvariant, and one might expect that $\langle\ \rangle_{x,x,\beta}$ is superinvariant. However, this is not the case. For example, with $n=1$,

$$\left\langle S\left(A\left(h^1\left[0, \tfrac{\beta}{2}\right]\right) \psi\left(h^2\left[\tfrac{\beta}{2}, \beta\right]\right)\right)\right\rangle_{x,x,\beta}$$
$$= \left\langle \psi\left(h^1\left[0, \tfrac{\beta}{2}\right]\right) \psi\left(h^2\left[\tfrac{\beta}{2}, \beta\right]\right) - A\left(h^1\left[0, \tfrac{\beta}{2}\right]\right) A\left(\frac{d}{dT} h^2\left[\tfrac{\beta}{2}, \beta\right]\right)\right\rangle_{x,x,\beta} \sim \operatorname{Tr} \gamma_3 \gamma_1 \gamma_2 \neq 0 .$$

The superinvariance is only recovered when one can integrate over x.

III. The $N=1$ Supersymmetric Field

The Lagrangian for $N=1$ supersymmetry is

$$L = \frac{1}{2} \int_{-\infty}^{\infty} \left[\left\langle \frac{dA}{dT}, \frac{dA}{dT} \right\rangle - \left\langle \psi_1, \frac{d}{dT} \psi_1 \right\rangle - \left\langle \psi_2, \frac{d}{dT} \psi_2 \right\rangle + \langle F, F \rangle \right] dT .$$

Here A, ψ_1, ψ_2, $F \in [\mathbb{R}, \mathbb{R}^{2n}]_0^\infty$ and ψ_1 and ψ_2 are of odd degree. L is formally invariant under $\delta A = \varepsilon_1 \psi_1 + \varepsilon_2 \psi_2$, $\delta \psi_1 = \dot{A}\varepsilon_1 - F\varepsilon_2$, $\delta \psi_2 = \dot{A}\varepsilon_2 + F\varepsilon_1$, $\delta F = \varepsilon_1 \dot{\psi}_2 - \varepsilon_2 \dot{\psi}_1$, with $\varepsilon_1, \varepsilon_2 \in [\mathbb{R}, \mathbb{R}]^\infty$ being odd degree constants. Just as before, we can compute vacuum expectations of sums of products of the form $A(f)\psi_1(g)\psi_2(g')F(h)$ with $f \in H^{-1}$, $g, g' \in H^{-1/2}$ and $h \in H^0$, and show supersymmetry of the vacuum state.

For the case when time is periodic we will not measure the F field and so integrate it out immediately. By writing $\psi_1(g) + \psi_2(g')$ as $\psi(g \oplus g')$, construct the algebra $\mathscr{A}_F$ generated by $\{\psi_1(g)\}$ and $\{\psi_2(g)\}$ for $g \in L^2([a,b])$, with the linear functional $\langle\ \rangle_F$. The algebra of measurables is $L^\infty(d\mu_{x,y,\beta}) \otimes \mathscr{A}_F$ with the state $\langle\ \rangle_{x,y,\beta}$ given by

$$\langle \mathcal{O}_B \otimes \mathcal{O}_F \rangle_{x,y,\beta} = 2^{4n} \langle \mathcal{O}_B \rangle_{x,y,\beta} \left\langle \prod_{k=1}^{2n} \psi_1(h^k_{[-2n+k-1,\,-2n+k]}) \psi_2(h^k_{[-2n+k-1,\,-2n+k]}) \mathcal{O}_F \right\rangle_F .$$

Proposition 5 (Free Feynman-Kac Formula). *For $v \in \mathbb{R}^{2n}$, let $E(v)$ denote exterior multiplication by v on $L^2(\Omega^* \mathbb{R}^{2n})$ and let $I(v)$ denote interior multiplication by v on $L^2(\Omega^* \mathbb{R}^{2n})$. Let $(-)^F$ be the operator on $L^2(\Omega^* \mathbb{R}^{2n})$ which is $(-)^p$ on $\Omega^p \mathbb{R}^{2n}$ and let $H = \frac{1}{2}\Delta$ be the Laplacian, ess.s.a. on a dense domain in $L^2(\Omega^* \mathbb{R}^{2n})$. Let $\{f_i\}_{i=1}^m$, $\{g_i\}_{i=1}^m$, and $\{g_i'\}_{i=1}^m$ be sequences in $[(0,\beta), \mathbb{R}^{2n}]_0^\infty$ with $\operatorname{supp} f_1 \leqq \operatorname{supp} g_1 \leqq \ldots \leqq \operatorname{supp} g_m'$ and let $\{G_i\}_{i=1}^m$ be a sequence in $C_0^\infty(\mathbb{R}^{2n})$. Then*

$$\begin{aligned}
&\left\langle \prod_{i=1}^m \left(\int f_i(T_i) G_i(A(T_i)) dT_i \psi_1(g_i) \psi_2(g_i')\right) \right\rangle_{x,y,\beta} \\
&= \left[\operatorname{Tr}(-)^F e^{-\beta H} \prod_{i=1}^m \left(\int f_i(T_i) e^{-T_i H} G_i(\tilde{A}) e^{T_i H} dT_i \right.\right. \\
&\quad \times \int e^{-T_i' H} \frac{1}{\sqrt{2}} (E+I)(g_i(T_i')) e^{T_i' H} dT_i' \\
&\quad \left.\left. \times \int e^{-T_i'' H} \frac{1}{\sqrt{2}} i(E-I)(g_i'(T_i'')) e^{T_i'' H} dT_i'' \right) \right] (y,x) .
\end{aligned}$$

(The local trace is over $\Omega^*(\mathbb{R}^{2n})$.)

Proof. The same as for Corollary 1. □

With $N=1$ supersymmetry one can add supersymmetric interactions. For $V(A) \in C^\infty(\mathbb{R}^{2n})$, the term $L_{\text{int}} = \int_0^\beta [-F_j \partial_j V(A) - i\psi_{1i}\psi_{2j}\partial_i\partial_j V(A)] dT$ is formally superinvariant provided that the fields are periodic. Integrating out the F field gives

$$L_{\text{int}} \to \int_0^\beta \left[\frac{1}{2} |\nabla V|^2(A) - i\psi_{1i}\psi_{2j}\partial_i\partial_j V(A) \right] dT .$$

We wish to define $\langle e^{-L_{\text{int}}} \mathcal{O} \rangle_{x,y,\beta}$ for $\mathcal{O} \in \mathscr{A}$; however, in general L_{int} has no hermiticity properties and $e^{-L_{\text{int}}}$ need not be in $\mathscr{A}$. To circumvent this, one can use the fact that $\langle\ \rangle_F$ comes from the Fock state on A_F, and is given by the vacuum state $|0\rangle_F$ in the Fock space $H_F = \bigoplus_k \Omega^k(L^2([a,b]))$. One can show [6] that for fixed A, $\exp i \int_0^\beta \psi_{1i}\psi_{2j}(\partial_i\partial_j V)(A) dT$ is an operator on H_F densely defined on the finite particle subspace of H_F, and that on this subspace it is the strong limit of $\sum_{n=0}^{\infty}\frac{1}{n!}\left(i\int_0^\beta \psi_{1i}\psi_{2j}(\partial_i\partial_j V)(A)dT\right)^n$. Furthermore, $\exp i\int_0^\beta \psi_{1i}\psi_{2j}(\partial_i\partial_j V)(A)dT$ formally commutes with $\mathscr{A}_F$.

Definition. For $\mathcal{O}\in L^1(d\mu_{x,y,\beta})\otimes\mathscr{A}_F$, define $\langle e^{-L_{\text{int}}}\mathcal{O}\rangle_{x,y,\beta}$ to be

$$
\begin{aligned}
&2^{4n}\int d\mu_{x,y,\beta}e^{-1/2\int_0^\beta |\nabla V|^2(A)dT}\\
&\quad\times\langle 0_F|\prod_{k=1}^{2n}\psi_1(h^k_{[-2n+k-1,-2n+k]})\psi_2(h^k_{[-2n+k-1,-2n+k]})\mathcal{O}\\
&\quad\left(\exp i\int_0^\beta \psi_{1i}\psi_{2j}(\partial_i\partial_j V)(A)dT|0_F\rangle\right).
\end{aligned}
$$

Proposition 6 (Feynman-Kac Formula). *With the sequences of Proposition 5,*

$$H=\tfrac{1}{2}(e^v de^{-v}+e^{-v}d^*e^v)^2$$

and

$$\mathcal{O}=\prod_{i=1}^m\left(\int f_i(T_i)G_i(A(T_i))dT_i\psi_1(g_i)\psi_2(g_i')\right),$$

one has

$$
\begin{aligned}
\langle e^{-L_{\text{int}}}\mathcal{O}\rangle_{x,y,\beta}=\Bigg[&\operatorname{Tr}(-)^F e^{-\beta H}\prod_{i=1}^m\Bigg(\int f_i(T_i)e^{-T_iH}G_i(\tilde{A})e^{T_iH}dT_i\\
&\times\int e^{-T_i'H}\frac{1}{\sqrt{2}}(E+I)(g_i(T_i'))e^{T_i'H}dT_i'\\
&\times\int e^{-T_i''H}\frac{1}{\sqrt{2}}i(E-I)(g_i'(T_i''))e^{T_i''H}dT_i''\Bigg)\Bigg](y,x).
\end{aligned}
$$

Proof. Put $H_0=\frac{1}{2}(d+d^*)^2$. Because $\langle e^{-L_{\text{int}}}\mathcal{O}\rangle_{x,y,\beta}$ is continuous in $\{g_i\}$ and $\{g_i'\}$, there is a Schwartz kernel which is given by

$$
\begin{aligned}
\langle e^{-L_{\text{int}}}\mathcal{O}\rangle_{x,y,\beta}&=\int\left(\prod_{i=1}^m dT_i dT_i' dT_i'' f_i(T_i)g_i(T_i')g_i'(T_i'')\right)\\
&\quad\times\left\langle e^{-L_{\text{int}}}\prod_{i=1}^m G_i(A(T_i))\psi_1(T_i')\psi_2(T_i'')\right\rangle_{x,y,\beta}\\
&=2^{4n}\int\left(\prod_{i=1}^m dT_i dT_i' dT_i'' f_i(T_i)g_i(T_i')g_i'(T_i'')\right)\\
&\quad\times\int d\mu_{x,y,\beta}e^{-\frac{1}{2}\int_0^\beta |\nabla V|^2(A)dT}\left(\prod_{i=1}^m G_i(A(T_i))\right)\\
&\quad\times\langle 0_F|\prod_{k=1}^{2n}\psi_1(h^k_{[-2n+k-1,-2n+k]})\psi_2(h^k_{[-2n+k-1,-2n+k]})\\
&\quad\times\prod_{i=1}^m\psi_1(T_i')\psi_2(T_i'')\left(\exp i\int_0^\beta \psi_{1i}\psi_{2j}(\partial_i\partial_j V)(A)dT|0_F\rangle\right).
\end{aligned}
$$

If $\{\eta_l\}$ is an orthonormal basis of H_F consisting of finite particle vectors then the last factor is

$$\sum_{l,l'} \langle 0_F| \prod_{k=1}^{2n} \psi_1(h^k_{[-2n+k-1,-2n+k]})\psi_2(h^k_{[-2n+k-1,-2n+k]})|\eta_{l_1}\rangle$$
$$\times \left(\prod_{i=1}^{m} \langle \eta_{l_i}|\psi_1(T_i')|\eta_{l_i'}\rangle \langle \eta_{l_i'}|\psi_2(T_i'')|\eta_{l_{i+1}}\rangle \right)$$
$$\times \langle \eta_{l_{m+1}}| \exp i \int_0^\beta \psi_{1i}\psi_{2j}(\partial_i\partial_j V)(A)dT|0_F\rangle .$$

Expanding the exponential as a strong limit and commuting the various terms to the left, one obtains

$$\sum_{l,l'} \langle 0_F| \prod_{k=1}^{2n} \psi_1(h^k_{[-2n+k-1,-2n+k]})\psi_2(h^k_{[-2n+k-1,-2n+k]})$$
$$\times \exp i \int_0^{T_1'} \psi_{1i}\psi_{2j}(\partial_i\partial_j V)(A)dT|\eta_{l_1}\rangle$$
$$\times \prod_{i=1}^{m} \left(\langle \eta_{l_i}|\psi(T_i')\exp i \int_{T_i'}^{T_i''} \psi_{1i}\psi_{2j}(\partial_i\partial_j V)(A)dT|\eta_{l_i'}\rangle \right.$$
$$\left. \times \langle \eta_{l_i'}|\psi(T_i'')\exp i \int_{T_i''}^{T_{i+1}'} \psi_{1i}\psi_{2j}(\partial_i\partial_j V)(A)dT|\eta_{l_{i+1}}\rangle \right)$$

with $T_{m+1}'=\beta$ and $\eta_{l_{m+1}}=0_F$. Then, by Proposition 5,

$$\langle e^{-L_{\text{int}}}\mathcal{O}\rangle_{\beta,x,y} = \int d\mu_{x,y,\beta}(A)e^{-\frac{1}{2}\int |\nabla V|^2(A)dT}$$
$$\times \left(\prod_{i=1}^{m} dT_i dT_i' dT_i'' f(T_i)G_i(A(T_i)) \right) \mathrm{Tr}(-)^F$$
$$\times \left[\exp i \int_0^{T_1'} \frac{1}{\sqrt{2}}(E+I)(e_i)\frac{1}{\sqrt{2}} i(E-I)(e_j)(\partial_i\partial_j V)(A)dT \right.$$
$$\times \prod_{i=1}^{m} \frac{1}{\sqrt{2}}(E+I)(g_i(T_i)) \left(\exp i \int_{T_i'}^{T_i''} \frac{1}{\sqrt{2}}(E+I)(e_i) \right.$$
$$\left. \times \frac{1}{\sqrt{2}} i(E-I)(e_j)(\partial_i\partial_j V)(A)dT \right)$$
$$\times \frac{1}{\sqrt{2}} i(E-I)(g_i'(T_i')) \left(\exp i \int_{T_i''}^{T_{i+1}'} \frac{1}{\sqrt{2}}(E+I)(e_i) \right.$$
$$\left. \left. \times \frac{1}{\sqrt{2}} i(E-I)(e_j)(\partial_i\partial_j V)(A)dT \right) \right] (y,x)$$

with $T_{m+1}'=\beta$. By the Feynman-Kac formula for tensor fields [8], this equals the RHS of the desired formula when

$$H = H_0 + \tfrac{1}{2}(E+I)(e_i)(E-I)(e_j)(\partial_i\partial_j V)(\tilde{A}) + \tfrac{1}{2}|\nabla V|^2(\tilde{A})$$
$$= -\tfrac{1}{2}\partial^2 + \tfrac{1}{2}(I(e_i)E(e_j) - E(e_i)I(e_j))(\partial_i\partial_j V)(\tilde{A}) + \tfrac{1}{2}|\nabla V|^2(\tilde{A}) .$$

On the other hand,

$$(e^V de^{-V}+e^{-V}d^*e^V)^2=(E(e_j)(\partial_j-\partial_j V)-I(e_j)(\partial_j+\partial_j V))^2$$
$$=(d^*d+dd^*)+(I(e_i)E(e_j)-E(e_i)I(e_j))\partial_i\partial_j V+|\nabla V|^2.$$

Thus $H=\frac{1}{2}(e^V de^{-V}+e^{-V}d^*e^V)^2$. □

Proposition 7. *Suppose that* $e^{-\frac{1}{2}\beta(|\nabla V|^2-\|\nabla\nabla V\|)}\in L^1(\mathbb{R}^{2n})$. *Then* $\langle\mathcal{O}\rangle_\beta \equiv \int dx\langle e^{-L_{\mathrm{int}}}\mathcal{O}\rangle_{x,x,\beta}$ *defines a superinvariant linear functional. That is, if* f, g, *and* g' *are in* $([0,\beta],\mathbb{R}^{2n})_0^\infty$ *and* $G\in C_0^\infty(\mathbb{R}^{2n})$, *define the graded derivations* S_1 *and* S_2 *by*

$$S_1\int_0^\beta f(T)G(A(T))dT=\psi_1(f(T)\nabla G(A(T))),$$

$$S_2\int_0^\beta f(T)G(A(T))dT=\psi_2(f(T)\nabla G(A(T))),$$

$$S_1\psi_1(g)=-\int_0^\beta\left\langle\frac{dg}{dT},A(T)\right\rangle dT,$$

$$S_2\psi_1(g)=-i\int_0^\beta\langle g(T),\nabla V(A(T))\rangle dT,$$

$$S_1\psi_2(g')=i\int_0^\beta\langle g'(T),\nabla V(A(T))\rangle dT,$$

$$S_2\psi_2(g')=-\int_0^\beta\left\langle\frac{dg'}{dT},A(T)\right\rangle dT.$$

Then $\left\langle S_k\prod_{i=1}^m\left(\int f_i(T_i)G_i(A(T_i))dT_i\right)\psi_1(g_i)\psi_2(g_i')\right\rangle_\beta=0$ *for* $k=1,2$.

Proof. With the assumption on V, by Symanzik's inequality [9], $e^{-\beta H}$ is trace class on $L^2(\Omega^*\mathbb{R}^{2n})$. Put $Q_1=\frac{1}{\sqrt{2}}[e^V de^{-V}-e^{-V}d^*e^V]$ and $Q_2=\frac{1}{\sqrt{2}}i[e^V de^{-V}+e^{-V}d^*e^V]$. Then $Q_1^2=Q_2^2=-H$, $\{Q_1,Q_2\}=0$, and $\{Q_1,(-)^F\}=\{Q_2,(-)^F\}=0$. Thus $\mathrm{Tr}(-)^F e^{-\beta H}\{Q_k,\tilde{\mathcal{O}}\}=0$ for any $\tilde{\mathcal{O}}\in B(L^2(\Omega^*\mathbb{R}^{2n}))$ with $k=1,2$. Now Q_1 acts by commutation as a graded derivation on bounded operators and

$$\left[Q_1,\int_0^\beta f(T)e^{-TH}G(\tilde{A})e^{TH}dT\right]=\int_0^\beta f(T)e^{-TH}\left\langle e_iG(\tilde{A}),\frac{1}{\sqrt{2}}(E+I)(e_i)\right\rangle e^{TH}dT,$$

$$\left\{Q_1,\int_0^\beta g(T)e^{-TH}\frac{1}{\sqrt{2}}(E+I)(e_i)e^{TH}dT\right\}=-\int_0^\beta g(T)e^{-TH}[H,\tilde{A}_i]e^{TH}dT,$$

and

$$\left\{Q_1,\int_0^\beta g'(T)e^{-TH}\frac{1}{\sqrt{2}}i(E-I)(e_i)e^{TH}dT\right\}=\int_0^\beta g'(T)ie^{-TH}(e_iV)(\tilde{A})e^{TH}dT.$$

If $\mathcal{O}$ is a measurable in $\mathrm{Dom}(S_1)$ and $\tilde{\mathcal{O}}$ is its translation into an operator via Proposition 6, then

$$\langle e^{-L_{\mathrm{int}}} S_1 \mathcal{O}\rangle_\beta = \mathrm{Tr}(-)^F e^{-\beta H} S_1 \tilde{\mathcal{O}} = \mathrm{Tr}(-)^F e^{-\beta H}\{Q_1, \tilde{\mathcal{O}}\} = 0 .$$

One can proceed similarly for S_2. □

IV. An Index Theorem

As a simple example of how supersymmetry is related to index theory, one can prove a Morse-type theorem on $\mathbb{R}^{2n}$.

To do a semiclassical analysis, one must add an explicit factor of $\hbar$ to the path integral by changing L to $\frac{1}{\hbar}L$. The only effect is to multiply free vacuum expectations by appropriate powers of $\hbar$ and to replace L_{int} by $\frac{1}{\hbar}L_{\mathrm{int}}$. As $\hbar \to 0$, one expects that the supermeasure becomes concentrated around the minima of the bosonic part of L. Let $H_\hbar$ denote the Hamiltonian corresponding to $\frac{1}{\hbar}L$.

Consider the operator $e^V d e^{-V} + e^{-V} d^* e^V$ of Proposition 6 mapping $\Lambda^{\mathrm{even}}(\mathbb{R}^{2n}) \to \Lambda^{\mathrm{odd}}(\mathbb{R}^{2n})$. The index is $\mathrm{Tr}(-)^F e^{-\beta H}$. By homotopy invariance of the index, this equals $\mathrm{Tr}(-)^F e^{-\frac{\beta}{\hbar}H_\hbar} = \left\langle e^{-\frac{1}{\hbar}L_{int}} \right\rangle_{\beta,\hbar}$, where we have noted an $\hbar$ dependence in the linear functional $\langle\ \rangle_{\beta,\hbar}$. [The measure $d\mu_{x,x,\beta,\hbar}$ is normalized to have total mass $(2\pi\beta\hbar)^{-n}$.]

The derivation of the index formula is done by first integrating out the fermions. This leaves a standard Feynman-Kac expression for the index with an $\hbar$ dependence (and no explicit supersymmetry). Then the $\hbar \to 0$ limit is taken.

Proposition 8. *Suppose that $V \in C^\infty(\mathbb{R}^{2n})$ is such that its critical points are finite and nondegenerate, $|\nabla V|^2$ goes to ∞ at ∞, and $e^{-a|\nabla V|^2 + b\|\nabla\nabla V\|} \in L^1(\mathbb{R}^{2n})$ for all $a, b > 0$. Then* $\mathrm{Index}(e^V d e^{-V} + e^{-V} d^* e^V) = \sum_{c_i} (-)^{\mathrm{index}(\mathrm{Hess}\, V)(c_i)}$, *the sum being over the critical points $\{c_i\}$.*

Proof. We have

$$\begin{aligned}\mathrm{Index}(e^V d e^{-V} + e^{-V} d^* e^V) = 2^{4n}\hbar^{-2n} \int dx \int d\mu_{x,x,\beta,\hbar} \\ \times \left\langle \prod_{k=1}^{2n} \psi_1(h^k_{(-2n+k-1,\,-2n+k]})\psi_2(h^k_{(-2n+k-1,\,-2n+k]}) \right. \\ \left. \times \exp \frac{1}{\hbar} i \int_0^\beta \psi_{1i}\psi_{2j}(\partial_i\partial_j V)(A)\,dT \right\rangle_{F,\hbar} .\end{aligned}$$

Because the fermion fields are quadratic in the exponential, the fermion integral can be evaluated explicitly.

Lemma 2. *For a fixed A field,*

$$2^{4n}\hbar^{-2n}\left\langle \prod_{k=1}^{2n} \psi_1(h^k_{(-2n+k-1,\,-2n+k]})\psi_2(h^k_{(-2n+k-1,\,-2n+k]}) \times \exp\frac{1}{\hbar} i \int_0^\beta \psi_{1i}\psi_{2j}(\partial_i\partial_j V)(A)dT \right\rangle_{F,\hbar}$$

$$= \mathrm{Tr}(-)^F P \exp - \int_0^\beta \frac{1}{2}[I(e_i), E(e_j)](\partial_i\partial_j V)(A(T))dT$$

(where P denotes path ordering).

Proof. The expectation equals

$$2^{4n}\hbar^{-2n}\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{i}{\hbar}\right)^m \langle 0_F|\left(\prod_{k=1}^{2n}\psi_1(h^k_{[-2n+k-1,\,-2n+k]}) \times \psi_2(h^k_{[-2n+k-1,\,-2n+k]})\right)\left(\int_0^\beta \psi_{1i}(T)\psi_{2j}(T)(\partial_i\partial_j V)(A)dT\right)^m |0_F\rangle$$

$$= 2^{4n}\hbar^{-2n}\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{i}{\hbar}\right)^m \mathrm{Tr}\left[\left(\prod_{k=1}^{2n}\sqrt{\frac{\hbar}{2}}(E+I)(e_k)\sqrt{\frac{\hbar}{2}}i(E-I)(e_k)\right) \times P\prod_{l=1}^{m}\left[\int_0^\beta dT_l \sqrt{\frac{\hbar}{2}}(E+I)(e_i)\sqrt{\frac{\hbar}{2}}i(E-I)(e_j)(\partial_i\partial_j V)(A(T))\right]\right]$$

$$= \mathrm{Tr}(-)^F P \exp - \int_0^\beta \frac{1}{2}[I(e_i), E(e_j)](\partial_i\partial_j V)(A(T))dT. \quad \square$$

Thus

$$\mathrm{Index} = \int dx \int d\mu_{x,x,\beta,\hbar}(A) e^{-\frac{1}{2}\frac{1}{\hbar}\int_0^\beta |\nabla V|^2(A)dT} \times \mathrm{Tr}(-)^F P \exp - \int_0^\beta \frac{1}{2}[I(e_i), E(e_j)](\partial_i\partial_j V)(A(T))dT.$$

By homotopy invariance of the index, we can perform a relatively compact perturbation of the operator to make V exactly quadratic in a neighborhood of each of the critical points without changing the Hessian of V at the critical points, while leaving the index invariant. Let $\{B(C_k, 2\varepsilon)\}$ be disjoint open balls in this neighborhood and let C denote $\mathbb{R}^{2n}\setminus\bigcup B(C_k, 2\varepsilon)$. Put $\delta = \inf_{x\in\mathbb{R}^{2n}\setminus\bigcup_k B(C_k,\varepsilon)} |\nabla V|^2(X) > 0$.

Lemma 3.

$$\lim_{\hbar\to 0}\int_C dx \int d\mu_{x,x,\beta,\hbar}(A) e^{-\frac{1}{2}\frac{1}{\hbar}\int_0^\beta |\nabla V|^2(A)dT} \times \mathrm{Tr}(-)^F P \exp - \int_0^\beta \frac{1}{2}[I(e_i), E(e_j)](\partial_i\partial_j V)(A(T))dT = 0.$$

Proof. Let Z denote the preceding integrand. By Jensen's inequality,

$$Z \leqq \int_C dx \int d\mu_{x,x,\beta,\hbar}(A) 2^{2n} \int_0^\beta \frac{dT}{\beta} \times \exp - \left[\frac{1}{2}\frac{1}{\hbar} |\nabla V|^2 (A(T)) - \frac{1}{2} \|\nabla\nabla V\| (A(T)) \right]$$

$$= \int_C dx \int d\mu_{x,x,\beta,\hbar}(A) 2^{2n} \int_0^\beta \frac{dT}{\beta} \int dy \delta(y - A(T)) \times \exp - \left[\frac{1}{2}\frac{1}{\hbar} |\nabla V|^2 (y) - \frac{1}{2} \|\nabla\nabla V\| (y) \right].$$

Let W denote $\frac{1}{\hbar}|\nabla V|^2 - \|\nabla\nabla V\|$. Then

$$Z \leqq 2^{2n} \int dy e^{-\frac{1}{2}W(y)} \int_C dx \int d\mu_{x,x,\beta,\hbar}(A) \int_0^\beta \frac{dT}{\beta} \delta(y - A(T))$$

$$\leqq 2^{2n} \int dy e^{-\frac{1}{2}W(y)} \int_C dx (1/(\pi\hbar\beta))^{2n} e^{-\frac{2(x-y)^2}{\hbar\beta}}$$

$$\leqq 2^{2n} \int dy e^{-\frac{1}{2}W(y)} \times \min\left((2\pi\hbar\beta)^{-n}, \int_{|x-y| \geqq d(y,C)} dx \left(\frac{1}{\pi\hbar\beta}\right)^{2n} e^{-\frac{2(x-y)^2}{\hbar\beta}} \right)$$

$$\leqq 2^{2n} \int dy e^{-\frac{1}{2}W(y)} \times \min\left((2\pi\hbar\beta)^{-n}, e^{-\frac{2d^2(y,C)}{\hbar\beta}} \left(a_1 d^{-2n}(y,C) + a_2 \left(\frac{d(y,C)}{\pi\hbar\beta} \right)^{2n} \right) \right)$$

for constants a_1 and a_2.

The coefficient of $\hbar$ in the exponent is $\frac{1}{2}|\nabla V|^2(y) + 2\frac{d^2(y,C)}{\beta}$. For $y \in \bigcup B(c_k, \varepsilon)$, this is $\geqq \frac{2\varepsilon^2}{\beta}$. For $y \notin \bigcup B(c_k, \varepsilon)$, it is $\geqq \frac{1}{2}\delta$. By dominated convergence, $\lim_{\hbar \to 0} Z = 0$. □

Over any fixed ball $B(c_k, 2\varepsilon)$, V is a nondegenerate quadratic. Let Q_k be the extension of this quadratic to $\mathbb{R}^{2n}$. By the same argument as in Lemma 3,

$$\int_{B(c_k, 2\varepsilon)} dx \int d\mu_{x,x,\beta,\hbar}(A) e^{-\frac{1}{2}\frac{1}{\hbar}\int_0^\beta |\nabla V|^2(A) dT}$$

$$\mathrm{Tr}(-)^F P \exp - \int_0^\beta \frac{1}{2} [I(e_i), E(e_j)] (\partial_i \partial_j V)(A(T)) dT$$

differs from the same expression, but with V replaced by Q_k and the integration done over $\mathbb{R}^{2n}$, by something which decreases exponentially in $\frac{1}{\hbar}$. Thus

$$\mathrm{Index} = \lim_{\hbar \to 0} \sum_k \int dx \int d\mu_{x,x,\beta,\hbar}(A) e^{-\frac{1}{2}\frac{1}{\hbar}\int_0^\beta |\nabla Q_k|^2(A) dT} \times \mathrm{Tr}(-)^F P \exp - \int_0^\beta \frac{1}{2} [I(e_i), E(e_j)] (\partial_i \partial_j Q_k)(A(T)) dT.$$

Lemma 4.

$$\int dx \int d\mu_{x,x,\beta,\hbar}(A) e^{-\frac{1}{2}\frac{1}{\hbar}\int_0^\beta \sum_k \lambda_k A_k^2} \prod_{k=1}^{2n} 2\mathrm{SINH}\frac{1}{2}\beta\lambda_k = (-)^{(\#\text{ of }\lambda_k<0)}.$$

Proof. By the Feynman-Kac formula,

$$\int dx \int d\mu_{x,x,\beta,\hbar}(A) e^{-\frac{1}{2}\frac{1}{\hbar}\int_0^\beta \sum_k \lambda_k A_k^2} = \mathrm{Tr}\, e^{-\frac{\beta}{\hbar}H_\hbar}$$

with $H_\hbar = \frac{\hbar^2}{2}\Delta + \frac{1}{2}\sum \lambda_k^2 \tilde{A}_k^2$. By separation of variables, this equals

$$\prod_{k=1}^{2n} \mathrm{Tr}\, e^{-\frac{\beta}{\hbar}H_k} \quad \text{with} \quad H_k = -\frac{\hbar^2}{2}\partial_x^2 + \frac{1}{2}\lambda_k^2 X^2.$$

The eigenvalues of H_k are $\left\{\frac{\hbar}{2}(2n+1)|\lambda_k| : n \in Z,\ n \geqq 0\right\}$, and so

$$\mathrm{Tr}\exp - \frac{\beta}{\hbar}H_k = \sum_{n=0}^{\infty} \exp - \frac{1}{2}\beta(2n+1)|\lambda_k| = \frac{1}{2\mathrm{SINH}\beta|\lambda_k|/2}.$$

Thus the desired integral is

$$\prod_{k=1}^{2n}\left(2\mathrm{SINH}\frac{1}{2}\beta\lambda_k / 2\mathrm{SINH}\frac{1}{2}\beta|\lambda_k|\right) = (-)^{(\#\text{ of }\lambda_k<0)}. \quad \square$$

By diagonalizing each Q_k and applying Lemma 4, one obtains

$$\mathrm{Index}(e^V d e^{-V} + e^{-V} d^* e^V) = \lim_{\hbar\to 0} \sum_{c_i} (-)^{\mathrm{index}\, Q(c_i)} = \sum_{c_i} (-)^{\mathrm{index}\,(\mathrm{Hess}\, V)(c_i)}. \quad \square$$

V. Compact Manifold

Let M be a compact $2n$-dimensional spin manifold with spinor bundle S. The standard Brownian motion is a measure on $M = [S^1, M]^0$. To form the super analogue it is necessary to look at certain supermanifolds. We recall from [10] that $R^{p,q}$ is the superspace over R^p with q Grassmannian generators; that is, the ring of superfunctions over $R^{p,q}$ is $C^\infty(R^{p,q}) \equiv C^\infty(R^p) \otimes \Lambda^*(R^q)$. $S^{1,q}$ is the analogous thing over S^1. We will want to consider a supermanifold of maps from $S^{1,1}$ to M. Let $[A, B]_{\mathrm{reg}}$ denote the space of maps between supermanifolds A and B as defined in [10], that is, homomorphisms from the superfunction sheaf over B to the superfunction sheaf over A. As this is not a supermanifold, following folklore we define $[A, B]_{\mathrm{sup}}$ to be the supermanifold such that $[\mathbb{R}^{p,q}, [A, B]_{\mathrm{sup}}]_{\mathrm{reg}} = [\mathbb{R}^{p,q} \times A, B]_{\mathrm{reg}}$ for all $p, q \geqq 0$.

Let Y denote the supermanifold given by $C^\infty(Y) = \Gamma^\infty(\Lambda^* T^* M)$; that is, the superfunctions over Y are cross-sections of the Grassmannian over M.

Claim. Formally, $[S^{1,1}, M]_{\mathrm{sup}} = X$, the supermanifold with $C^\infty(X) = \Gamma(\Lambda^*[S^1, T^*M])$ (where $[S^1, T^*M]$ is a vector bundle over $[S^1, M]$).

Corollary. *Formally,* $[S^1, Y]_{\mathrm{sup}} = X$.

Proof of Corollary. $[S^1, Y]_{\sup} = [S^1, [\mathbb{R}^{0,1}, M]_{\sup}]_{\sup} = [S^{1,1}, M]_{\sup} = X$.

Proof of Claim. Taking $p=q=0$, the base space of $[S^{1,1}, M]_{\sup}$ is $[S^1, M]$. One must show that $\forall p, q$, we have

$$\mathrm{Hom}(C^\infty(M), C^\infty(S^{1,1}\times \mathbb{R}^{p,q})) = \mathrm{Hom}(\Gamma^\infty(\Lambda^*[S^1, T^*M]), C^\infty(\mathbb{R}^{p,q})).$$

For

$$\eta \in \mathrm{Hom}(C^\infty(M), C^\infty(S^{1,1}\times \mathbb{R}^{p,q})) = \mathrm{Hom}(C^\infty(M), C^\infty(S^1 \times \mathbb{R}^{p,q+1})),$$

η covers a map $\phi: S^1 \times \mathbb{R}^p \to M$. For $f \in C^\infty(M)$, write $\eta(f) = \sum \eta_I(f)\theta^I$, where I is an even length increasing multi-index composed of $\{1, \ldots, q+1\}$, and $\eta_I \in C^\infty(S^1 \times \mathbb{R}^p)$. We have $\sum \eta_I(ff')\theta^I = \eta(ff') = \eta(f)\eta(f') = \sum \eta_J(f)\eta_K(f')\theta^J\theta^K$. In particular, $\eta_\phi(ff') = \eta_\phi(f)\eta_\phi(f')$, and so $\eta_\phi(f) = f \circ \phi$. At a fixed level I,

$$\begin{aligned}\eta_I(ff') &= \sum_{J,K,\theta^J\theta^K=\theta^I} \eta_J(f)\eta_K(f') \\ &= \eta_I(f)\eta_\phi(f') + \eta_\phi(f)\eta_I(f') + \sum_{J,K\neq\emptyset,\ \theta^J\theta^K=\theta^I} \eta_J(f)\eta_K(f').\end{aligned}$$

If $\tilde{\eta}_I(f)$ also satisfies this equation then $(\eta-\tilde{\eta})_I(ff') = (\eta-\tilde{\eta})_I(f)(f'\circ\phi) + (\eta-\tilde{\eta})_I(f')(f\circ\phi)$, the most general solution of which is $(\eta-\tilde{\eta})(f) = hf$ for some $h \in [S^1 \times \mathbb{R}^p, TM]$ covering ϕ. Thus at level I, the possible choices for η_I, given ϕ and $\{\eta_J\}_{\deg J < \deg I}$ form either nothing or an affine space with tangent space $T_\phi \equiv \{h \in [S^1 \times \mathbb{R}^p, TM]\colon h \text{ covers } \phi\}$.

Lemma 5. *Let* $\{V_I\}_{\substack{I\neq\emptyset\\ I \text{ of even degree}}}$ *be a sequence in* $[S^1 \times \mathbb{R}^p, \mathrm{Vect}(M)]$. *Define* $\eta: C^\infty(M) \to C^\infty(S^1 \times \mathbb{R}^{p,q+1})$ *by* $\eta(f)(z) = ((\exp V_I(z)\theta^I)f)(\phi(z))$ *for* $z \in S^1 \times \mathbb{R}^p$. *Then* η *is a homomorphism.*

Proof. It suffices to show that $\exp V_I(z)\theta^I$ is a homomorphism on $C^\infty(M)$. Each $V_I(z)\theta^I$ is in $\mathrm{Der}(C^\infty(M)\otimes \Lambda^{\mathrm{even}}(\mathbb{R}^{q+1}))$ and acts on $C^\infty(M)\otimes\Lambda^*(\mathbb{R}^{q+1})$. As

$$(V_I\theta^I)(f\theta^J g\theta^K) = V_I(fg)\theta^I\theta^J\theta^K = f\theta^J(V_I g\theta^I\theta^K) + (V_I f\theta^I\theta^J)g\theta^K,$$

$V_I(z)\theta^I$ acts as a derivation. Then $\exp V_I(z)\theta^I$ is a finite power series which is a homomorphism. □

Thus as a set, $\mathrm{Hom}((C^\infty(M), C^\infty(S^{1,1}\times\mathbb{R}^{p,q}))$ is

$$\bigcup_{\phi\in[S^1\times\mathbb{R}^p, M]} \prod_{\substack{I \text{ even}\\ I\neq\emptyset}} T_\phi = \bigcup_{\phi\in[S^1\times\mathbb{R}^p, M]} T_\phi^{2^q-1}.$$

On the other hand, for

$$\eta' \in \mathrm{Hom}(\Gamma(\Lambda^*[S^1, T^*M]), C^\infty(\mathbb{R}^{p,q})) = \mathrm{Hom}(\Gamma(\Lambda^* T^*\Omega M), C^\infty(\mathbb{R}^{p,q})),$$

η' covers a map $\phi' \in [\mathbb{R}^p, \Omega M] = [S^1 \times \mathbb{R}^p, M]$. For $f' \in C^\infty(\Omega M)$, write $\eta'(f') = \sum \eta'_I(f')\theta^I$. (The multi-index is now composed of $\{1, \ldots, q\}$.) As before, each η'_I forms an affine space with tangent space being the subspace of $[\mathbb{R}^p, T\Omega M] = [S^1 \times \mathbb{R}^p, TM]$ covering ϕ'. For $\omega' \in \Gamma(T^*\Omega M)$, write $\eta'(\omega') = \sum_{J \text{ odd}} \eta'_J(\omega')\theta^J$. The restriction on η' to be a homomorphism gives

$$\eta'(f'\omega') = \sum_{I \text{ odd}} \eta'_I(f'\omega')\theta^I = \eta'(f')\eta'(\omega') = \sum_{J,K} \eta'_J(f')\eta'_K(\omega')\theta^J\theta^K,$$

or

$$\eta'_I(f'\omega') = \sum_{JK=I} \eta'_J(f')\eta'_K(\omega') = (f'\circ\phi')\eta'_I(\omega') + \sum_{\substack{JK=I\\ J\neq\emptyset}} \eta'_J(f')\eta'_K(\omega').$$

If $\tilde{\eta}'_I$ also satisfies this equation then $(\eta'_I - \tilde{\eta}'_I)(f'\omega') = (f'\circ\phi')(\eta'_I - \tilde{\eta}'_I)(\omega')$. Thus at level I the possible choices for η'_I, given ϕ' and $\{\eta'_J\}_{\deg J < \deg I}$, form either nothing or an affine space with tangent space $\{h' \in [\mathbb{R}^p, T\Omega M]: h' \text{ covers } \phi'\} = T_{\phi'}$.

Lemma 6. *Let* $\{V'_I\}_{\substack{I\neq\emptyset\\ I \text{ of even degree}}}$ *and* $\{W'_J\}_{J \text{ of odd degree}}$ *be sequences in* $[\mathbb{R}^p, \mathrm{Vect}\,\Omega M]$. *Define* $\eta' : C^\infty(\Omega M) \to C^\infty(\mathbb{R}^{p,q})$ *by*

$$\eta'(f')(z') = ((\exp V'_I(z')\theta^I)f')(\phi'(z'))$$

for $z' \in \mathbb{R}^p$ *and* $\eta' : \Gamma(T^*\Omega M) \to C^\infty(\mathbb{R}^{p,q})$ *by*

$$\eta'(\omega')(z') = (\exp V'_I(z')\theta^I)(\langle W'_J(z), \omega'\rangle\theta^J)(\phi'(z')).$$

Then $\eta'(f'\omega') = \eta'(f')\eta'(\omega')$.

Proof. The same as for Lemma 5. □

Thus as a set, $\mathrm{Hom}(\Gamma(\Lambda^*[S^1, T^*M]), C^\infty(\mathbb{R}^{p,q}))$ is

$$\bigcup_{\phi'\in[S^1\times\mathbb{R}^p, M]} \left(\prod_{\substack{I \text{ even}\\ I\neq\emptyset}} T_{\phi'} \times \prod_{I \text{ odd}} T_{\phi'} \right) = \bigcup_{\phi'\in[S^1\times\mathbb{R}^p, M]} T_{\phi'}^{2^q-1}. \quad \square$$

As a consequence of the claim, the space of measurable S is formally $\Gamma(\Lambda^*[S^1, T^*M]$.

Definition. Define $E \in C([S^1, M]^\infty)$ by $E(\gamma) = \int_\gamma \langle\dot{\gamma}, \dot{\gamma}\rangle$, define $\theta \in \Gamma([S^1, T^*M]^\infty)$ such that $\forall V \in \Gamma([S^1, TM]^\infty)$, $\theta(V)(\gamma) = \int_\gamma \langle\dot{\gamma}, V_\gamma\rangle$, and define $\omega \in \Gamma(\Lambda^2[S^1, T^*M]^\infty)$ such that $\forall V, W \in \Gamma([S^1, TM]^\infty)$, $\omega(V, W)(\gamma) = -\int_\gamma \langle V_\gamma, \nabla_{\dot{\gamma}} W_\gamma\rangle$.

Lemma 7 [11]. $(d + i_{\dot{\gamma}})\theta = E + \omega$ *and* $(d + i_{\dot{\gamma}})(E + \omega) = 0$.

Proof. See [11]. □

We take the supersymmetric Lagrangian to be $L = \frac{1}{2}(E + \omega)$. In local coordinates,

$$L = \frac{1}{2}\int_0^\beta g_{\mu\nu}(\gamma)(\dot{\gamma}^\mu\dot{\gamma}^\nu - \psi^\mu(\nabla_{\dot{\gamma}}\psi)^\nu)dT$$

and the supersymmetric variation $d + i_{\dot{\gamma}}$ acts as $(d + i_{\dot{\gamma}})\gamma^\mu = \psi^\mu$, $(d + i_{\dot{\gamma}})\psi^\mu = \dot{\gamma}^\mu$. We wish to define the formal object $\int e^{-L}\eta$ for $\eta \in \Gamma(\Lambda^*[S', T^*M])$ such that $\int e^{-L}(d + i_{\dot{\gamma}})\eta = 0$. For $M = \mathbb{R}^{2n}$, this was done in the previous sections.

To establish notation, the Malliavin construction of the ordinary Wiener measure $d\mu_{m,m,\beta}(\gamma)$ {formally $e^{-\frac{1}{2}E}(\mathscr{D}\gamma)$ on $\Omega_m M = \{\gamma \in [S^1, M]^0 : \gamma(0) = m\}$ is given as follows [12]: Let $\{A_1, \ldots, A_{2n}\}$ be the canonical horizontal vector fields on the principal bundle $\mathrm{Spin}(2n) \to P \to M$. Solve the stochastic differential equation $dr_\omega = \sum_k A_k d_s b^k_\omega$ on P with the standard Brownian motions $\{b^k_\omega\}_{k=1}^{2n}$, subject to $\pi r_\omega(0) = m$. It can be shown that this has a continuous solution for almost all ω. If B

denotes the Wiener measure on $\{\omega:\mathbb{R}^+\to\mathbb{R}^{2n}: \omega\{0\}=0\}$ then the Wiener measure on $\Omega_m M$ is $E_{m,m,\beta}\pi_* r_* B$, with $E_{m,m,\beta}$ being the conditional expectation on paths with $\gamma(\beta)=m$.

Definition. Let $\mathscr{B}$ be the * algebra of finite linear sums of products of $\int_0^\beta f(T)F(\gamma(T))dT$, $\int_0^\beta g(T)dG(\gamma(T))dT$, and $\int_0^\beta h(T)dH^*(\gamma(T))dT$ with the relationship

$$\left\{\int_0^\beta g(T)dG(\gamma(T))dT, \int_0^\beta h(T)dH^*(\gamma(T))dT\right\} = \int_0^\beta g(T)h(T)\langle dG, dH\rangle(\gamma(T))dT.$$

Here f, g, and h are in $C^\infty_{0\mathbb{C}}([0,\beta])$ and F, G, and H are in $C^\infty_{\mathbb{C}}(M)$.

Definition. For a given $\gamma\in\Omega_m M$, let $r_\omega(T)$ be its horizontal lift in P starting from some $r_\omega(0)$ and let $\{e_i(T)\}_{i=1}^{2n}$ be the frame obtained by projecting $r_\omega(T)$ to the orthonormal frame bundle. Define a homomorphism $s_m:\mathscr{B}\to L^1(d\mu_{m,m,\beta})\otimes A_F$, the scalarization, by

$$s_m\left(\int_0^\beta f(T)F(\gamma(T))dT\right) = \int_0^\beta f(T)F(\gamma(T))dT,$$

$$s_m\left(\int_0^\beta g(T)dG(\gamma(T))dT\right) = \int_0^\beta g(T)(e_iG)(\gamma(T))a^i(T)dT,$$

and

$$s_m\left(\int_0^\beta h(T)dH^*(\gamma(T))dT\right) = \int_0^\beta h(T)(e_iH)(\gamma(T))a^{i*}(T)dT.$$

Define ϕ, a linear functional on B, by $\phi(b)=\int dm\int d\mu_{m,m,\beta}\langle s_m(b)\rangle_F$. [It follows from Wick's theorem that $\langle s_m(b)\rangle_F$ is measurable on $\Omega_m M$.]

Lemma 8. *$\forall b\in\mathscr{B}$, $\phi(b^*b)\geqq 0$ and $\phi(b^*a^*ab)\leqq \mathrm{const}(a)\phi(b^*b)$.*

Proof. $\phi(b^*b)=\int dm\int d\mu_{m,m,\beta}\langle s_m(b)^*s_m(b)\rangle_F\geqq 0$.

$$\begin{aligned}\phi(b^*a^*ab) &= \int dm d\mu_{m,m,\beta}\langle s_m(b)^*s_m(a)s_m(a)s_m(b)\rangle_F\\ &\leqq \int dm d\mu_{m,m,\beta}\|s_m(a)\|_F^2\langle s_m(b)^*s_m(b)\rangle\\ &\leqq \left(\sup_{m,\Omega_m M}\|s_m(a)\|_F^2\right)\phi(b^*b).\end{aligned}$$

Because all F, G, and H's are in $C^\infty(M)$, $\sup_{m,\Omega_m M}\|s_m(a)\|_F^2<\infty$. □

By the GNS construction, $\mathscr{B}$ is represented on a Hilbert space $\mathscr{H}$. Let G be the closure of $\mathscr{B}$ in $B(\mathscr{H})$ and let $\mathscr{A}$ be the subalgebra of G generated by $\int_0^\beta f(T)F(\gamma(T))dT$ and

$$\int_0^\beta g(T)\langle dG(\gamma(T)),\psi(T)\rangle dT \equiv \int_0^\beta g(T)\left(dG^*(\gamma(T))+\frac{1}{2}\int_0^\beta \mathrm{sign}(T-S)dG(\gamma(S))dS\right)dT.$$

In general, if one wishes to define an algebra of measurables which is formally $\Gamma(\Lambda^*[S^1, T^*M])$, then it must contain the continuous functions on $[S^1, M]^0$ in order for the bosonic part to carry the Wiener measure. One can treat ΩM as a C^{υ} Banach manifold and consider its C^{∞} differential forms [13]. These will look like the exterior products of vector-valued measures over each curve $\gamma \in \Omega M$. We expect that the algebra $\mathscr{A}$ will contain all such forms which are exterior products of vector-valued L^2 functions along each γ.

Definition. For a curve γ in $\Omega_m M$, let $T_\gamma \in \mathrm{Spin}(2n)$ denote the holonomy around γ from $r_\omega(0)$. Write T_γ in terms of the basis of $\mathrm{Cl}(\mathbb{R}^{2n})$ as $T_\gamma = \sum_{k=1}^{2n} T_{\boldsymbol{\mu}} \prod_{i=1}^{k} \gamma_{\mu_i}$. Define a linear functional $\langle\ \rangle_\beta$ on $\mathscr{B} \subset \mathscr{A}$ by

$$\langle b\rangle_\beta = i^{n(2n-1)} 2^{2n} \int dm \int d\mu_{m,m,\beta} e^{-\frac{1}{8}\int_\gamma R}$$
$$\times \left\langle \prod_{k=1}^{2n} \eta(h^k_{[-2n+k-1,\,-2n+k]}) s_m(b) \sum_{k=0}^{2n} \sum_{\boldsymbol{\mu}} 2^{k/2} T_{\boldsymbol{\mu}} \prod_{i=1}^{k} \eta(h^{\mu_i}_{[\beta+i,\,\beta+i+1]}) \right\rangle_F .$$

Extend $\langle\ \rangle_\beta$ to $\mathscr{A}$ by continuity.

Note. That the RHS of the expression for $\langle b\rangle_\beta$ is measurable on $\Omega_m M$ follows from the next proposition. The various terms of the expression have the following meaning: The $s(b)$ term is the translation of b to a flat space measurable using the Cartan development. The factor $e^{-\frac{1}{8}\int_\gamma R}$ comes from quantum effects. In the Hamiltonian approach there is a question of factor ordering and the $\frac{1}{8}R$ is the same as in the equation $\frac{1}{2}\not{D}^2 = \frac{1}{2}\nabla^+\nabla + \frac{1}{8}R$. The term involving T_γ is to ensure that in the integration is formally done over periodic fermion fields along γ.

Proposition 9. *Let M_F denote multiplication on $L^2(S)$ by F, let $\mathrm{Cl}(dG)$ denote Clifford multiplication on $L^2(S)$ by dG, and let H equal $\frac{1}{2}\not{D}^2$. Then for $b \in \mathscr{B}$ of the form*

$$b = \prod_{i=1}^{r} \int_0^\beta f_i(T_i) F_i(\gamma(T_i)) dT_i \int_0^\beta g_i(T_i') \langle dG(\gamma(T_i')), \psi(T_i')\rangle dT_i'$$

with $\mathrm{supp} f_1 \leqq \mathrm{supp} g_1 \leqq \ldots \leqq \mathrm{supp} g_r$,

$$\langle b\rangle_\beta = \mathrm{Tr}\gamma_{2n+1} e^{-\beta H} \prod_{i=1}^{r} \left(\int_0^\beta f_i(T_i) e^{-T_i H} M_{F_i} e^{T_i H} dT_i \right)$$
$$\times \left(\int_0^\beta g_i(T_i') e^{-T_i' H}\, \mathrm{Cl}(dG_i) e^{T_i' H} dT_i' \right).$$

Proof. By Proposition 4,

$$i^{n(2n-1)} 2^{2n} \left\langle \prod_{k=1}^{2n} \eta(h^k_{[-2n+k-1,\,-2n+k]}) \right.$$
$$\left. \times \left(\prod_{i=1}^{r} \left(\int g_i(T_i') \langle dG(\gamma(T_i')), \psi(T_i')\rangle dT_i' \right) \right) \right\rangle$$
$$= \mathrm{Tr}\gamma_{2n+1} \prod_{i=1}^{r} \left(\int g_i(T_i') \sum_j (e_j G_i)(\gamma(T_i')) \frac{1}{\sqrt{2}} \gamma(e_j) dT_i' \right).$$

Thus

$$\langle b\rangle_\beta = \int dm \int_{\Omega_m M} d\mu_{m,m,\beta}(\gamma)\,\mathrm{Tr}\,\gamma_{2n+1}$$
$$\times\left[\prod_{i=1}^{r} \int f_i(T_i) F_i(\gamma(T_i))\,dT_i \int g_i(T_i')\right.$$
$$\left.\times \sum_j (e_j G_i)(\gamma(T_i'))\frac{1}{\sqrt{2}}\gamma(e_j)\,dT_i'\right] T_\gamma e^{-\frac{1}{8}\int_\gamma R}.$$

On the other hand,

$$\mathrm{Tr}\,\gamma_{2n+1} e^{-\beta H} \prod_{i=1}^{r} \int f_i(T_i) e^{-T_i H} M_{F_i} e^{T_i H} dT_i \int g_i(T_i') e^{-T_i' H}\,\mathrm{Cl}(dG_i) e^{T_i' H} dT_i'$$
$$= \mathrm{Tr}\,\gamma_{2n+1} \int d^r T d^r T' \prod_{i=1}^{r} f_i(T_i) g_i(T_i') M_{F_i}$$
$$\times e^{-(T_i'-T_i)H}\,\mathrm{Cl}(dG_i) e^{-(T_{i+1}-T_i')H} \quad (\text{with } T_{r+1}=\beta+T_1)$$
$$= \int d^r T d^r T'\left[\prod_{i=1}^{r} f_i(T_i) g_i(T_i')\right]\int d^r m\, d^r n\,\mathrm{Tr}\,\gamma_{2n+1}$$
$$\times \prod_{i=1}^{r} F_i(m_i) e^{-(T_i'-T_i)H}(m_i,n_i)\,\mathrm{Cl}(dG_i)(n_i)$$
$$\times e^{-(T_{i+1}-T_i')H}(n_i,m_i) \quad (\text{with } m_{r+1}=m_1).$$

Let ψ_i be an orthonormal basis of spinors at m_i. Then the above equals

$$\int d^r T d^r T'\left[\prod_{i=1}^{r} f_i(T_i) g_i(T_i')\right]\int d^r m\, d^r n \sum_{\psi} \mathrm{Tr}\,\gamma_{2n+1}|\psi_1\rangle$$
$$\otimes\langle\psi_1|\prod_{i=1}^{r} F_i(m_i) e^{-(T_i'-T_i)H}(m_i,n_i)\,\mathrm{Cl}(dG_i)(n_i) e^{-(T_{i+1}-T_i')H}(n_i,m_{i+1})|\psi_{i+1}\rangle$$
$$\otimes\langle\psi_{i+1}| \quad (\text{with } \psi_{r+1}=\psi_1).$$

Let $\gamma\in\Omega_m M$ pass through m_i at time T_i and n_i at time T_i'. Let $\mathrm{Sc}(\mathrm{Cl}(dG_i))(n_i)$ be the scalarization of $\mathrm{Cl}(dG_i)(n_i)$ and $\mathrm{Sc}(\psi_i)$ be the scalarization of ψ_i, both with respect to the frame $\{e_i\}$ obtained by lifting γ. From the Feynman-Kac formula for tensor fields [8], the above equals

$$\int d^r T d^r T'\left[\prod_{i=1}^{r} f_i(T_i) g_i(T_i')\right]\int dm \int d\mu_{m,m,\beta}(\gamma) e^{-\int_\gamma \frac{1}{8}R}$$
$$\times \sum_{\psi}\langle \mathrm{Sc}(\psi_1)|\gamma_{2n+1}\prod_{i=1}^{r} F_i(\gamma(T_i))\,\mathrm{Sc}(\mathrm{Cl}(dG_i))(\gamma(T_i'))|\mathrm{Sc}(\psi_{r+1})\rangle.$$

Now $|\mathrm{Sc}(\psi_{r+1})\rangle\otimes\langle\mathrm{Sc}(\psi_1)| = T_\gamma$, and one obtains

$$\int d^r T d^r T'\left[\prod_{i=1}^{r} f_i(T_i) g_i(T_i')\right]\int dm \int d\mu_{m,m,\beta}(\gamma) e^{-\int_\gamma \frac{1}{8}R}\,\mathrm{Tr}\,\gamma_{2n+1}$$
$$\times\left[\prod_{i=1}^{r} F_i(\gamma(T_i))\,\mathrm{Sc}(\mathrm{Cl}(dG_i))(\gamma(T_i'))\right] T_\gamma. \quad \square$$

Lemma 9.

$$(d+i_{\dot\gamma})\int_0^\beta f(T)F(\gamma(T))dT=\int_0^\beta f(T)\langle dF(\gamma(T)),\psi(T)\rangle dT$$

and

$$(d+i_{\dot\gamma})\int_0^\beta g(T)\langle dG(\gamma(T)),\psi(T)\rangle dT=-\int_0^\beta \frac{dg}{dT}G(\gamma(T))dT.$$

Proof. For $V\in\Gamma([S^1,TM])$, at a curve γ we have

$$\left\langle (d+i_{\dot\gamma})\int_0^\beta f(T)F(\gamma(T))dT,\,V\right\rangle(\gamma)=V\int_0^\beta f(T)F(\gamma(T))dT=\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\int_0^\beta f(T)$$

$$F((\gamma+\varepsilon V_\gamma)(T))dT=\int_0^\beta f(T)\langle dF,V_\gamma\rangle dT=\left\langle\int_0^\beta f(T)\langle dF(\gamma(T)),\psi(T)\rangle dT,\,V\right\rangle(\gamma).$$

Then

$$\begin{aligned}&(d+i_{\dot\gamma})\int_0^\beta g(T)\langle dG(\gamma(T)),\psi(T)\rangle dT\\&=(d+i_{\dot\gamma})^2\int_0^\beta g(T)G(\gamma(T))dT=\mathscr{L}_{\dot\gamma}\int_0^\beta g(T)G(\gamma(T))dT\\&=\int_0^\beta g(T)\frac{d}{dt}G(\gamma(T))dT=-\int_0^\beta\frac{dg}{dT}G(\gamma(T))dT.\end{aligned}$$

Proposition 10. *For $b\in\mathscr{B}$ of the form of Proposition 9, $\langle(d+i_{\dot\gamma})b\rangle_\beta=0$.*

Proof. As in the proof of Proposition 7, we have that $Q=\not{D}$ commutes with H and anticommutes with γ_{2n+1}. Thus for any bounded operator $\tilde{\mathcal{O}}$, $0=\mathrm{Tr}[Q,\gamma_{2n+1}e^{-\beta H}\tilde{\mathcal{O}}]=\mathrm{Tr}\,\gamma_{2n+1}e^{-\beta H}\{Q,\tilde{\mathcal{O}}\}$. Now $[Q,M_f]=-i\,\mathrm{Cl}(dF)$ and $\{Q,\mathrm{Cl}(dG)\}=i\{Q,[Q,M_G]\}=2i[H,M_G]$. The proof then follows as in Proposition 7. □

To compute the Index of $\not{D}$, one can introduce an explicit $\hbar$ dependence into the supermeasure to obtain $\mathrm{Index}\,\not{D}=\langle I\rangle_{\beta,\hbar}=(\text{formally})\int\left(\exp-\frac{1}{\hbar}L\right)\mathscr{D}\gamma\mathscr{D}\psi$. Because the Lagrangian is quadratic in the fermion field, the integration can be carried out explicitly to give

$$\mathrm{Index}\,\not{D}=\int dm\int d\mu_{m,m,\beta,\hbar}(\gamma)e^{-\frac{1}{8}\hbar\int_\gamma R}\,\mathrm{Tr}\,\gamma_{2n+1}T_\gamma.$$

From the large deviations theorem [14],

$$\mathrm{Index}\,\not{D}=\lim_{\hbar\to0}\int dm\int d\mu_{m,m,\beta,\hbar}(\gamma)f(\gamma)e^{-\frac{1}{8}\hbar\int R}\,\mathrm{Tr}\,\gamma_{2n+1}T_\gamma$$

for any continuous function f on ΩM which is identically one in a neighborhood of the constant loops. Thus the index density becomes concentrated near the constant loops and can be evaluated in a quadratic approximation as in [4, 5]. From the Feynman-Kac formula,

$$\mathrm{Index}\,\not{D}=\mathrm{Tr}\,\gamma_{2n+1}e^{-\frac{\beta}{\hbar}H_\hbar}\quad\text{with}\quad H_\hbar=\frac{1}{2}\hbar^2\not{D}^2.$$

Thus

$$\operatorname{Index} \not{D} = \lim_{\hbar \to 0} \operatorname{Tr} \gamma_{2n+1} e^{-\frac{1}{2}\beta\hbar \not{D}^2} = \lim_{\beta \to 0} \operatorname{Tr} \gamma_{2n+1} e^{-\frac{1}{2}\beta \not{D}^2},$$

which shows that in this case, the $\hbar \to 0$ limit is the same as the $\beta \to 0$ limit of [4, 5].

VI. Gauge Fields

Let $\begin{smallmatrix} E \\ \downarrow \\ M \end{smallmatrix}$ be a $\mathbb{R}^{2n'}$ vector bundle over M with an $SO(2n')$ connection A which lifts to a Spin$(2n')$ connection. There is a natural connection $\tilde{A}$ on the vector bundle $\begin{smallmatrix} [S^1, E]^\infty \\ \downarrow \\ [S^1, M]^\infty \end{smallmatrix}$ given by $\tilde{D}_V Z|_\gamma = D_{V_\gamma} Z_\gamma$, which induces a connection on $\Lambda^*[S^1, E]^\infty$.

Definition. Define $\omega_0 \in \Gamma(\Lambda^0[S^1, T^*M] \otimes \Lambda^2[S^1, E^*])$ by

$$\omega_0(Z_1, Z_2)|_\gamma = \int_\gamma \langle D_{\dot{\gamma}} Z_{1\gamma}, Z_{2\gamma} \rangle \quad \text{for} \quad Z_1, Z_2 \in \Gamma([S^1, E])$$

and define $\omega_2 \in \Gamma(\Lambda^2[S^1, T^*M] \otimes \Lambda^2[S^1, E^*])$ by

$$\omega_2(Z_1, Z_2; V_1, V_2)|_\gamma = \int_\gamma \langle F(V_{1\gamma}, V_{2\gamma}) Z_{1\gamma}, Z_{2\gamma} \rangle \quad \text{for} \quad V_1, V_2 \in \Gamma([S^1, TM]).$$

Proposition 11. *Let $\tilde{d}$ denote the covariant exterior derivative using the connection $\tilde{A}$. Then $(\tilde{d} + i_{\dot{\gamma}})(E + \omega + \omega_0 + \omega_2) = 0$.*

Proof. Because $(\tilde{d} + i_{\dot{\gamma}})(E + \omega) = (d + i_{\dot{\gamma}})(E + \omega) = 0$, it suffices to look at $(\tilde{d} + i_{\dot{\gamma}})(\omega_0 + \omega_2) = (\tilde{d}\omega_0 + i_{\dot{\gamma}}\omega_2) + (\tilde{d}\omega_2)$. Let $\gamma(\varepsilon)$ be a 1-parameter family of curves with $\gamma(0) = \gamma$ and $\frac{d}{d\varepsilon}\gamma = V$. Then $[V, \dot{\gamma}] = 0$ and at γ,

$$\begin{aligned}
(\tilde{d}\omega_0)(Z_1, Z_2; V) &= V\omega_0(Z_1, Z_2) - \omega_0(\tilde{D}_V Z_1, Z_2) - \omega_0(Z_1, \tilde{D}_V Z_2) \\
&= V \int_\gamma \langle D_{\dot{\gamma}} Z_1, Z_2 \rangle - \int_\gamma \langle D_{\dot{\gamma}} D_V Z_1, Z_2 \rangle - \int_\gamma \langle D_{\dot{\gamma}} Z_1, D_V Z_2 \rangle \\
&= \int_\gamma [\langle D_V D_{\dot{\gamma}} Z_1, Z_2 \rangle + \langle D_{\dot{\gamma}} Z_1, D_V Z_2 \rangle \\
&\quad - \langle D_{\dot{\gamma}} D_V Z_1, Z_2 \rangle - \langle D_{\dot{\gamma}} Z_1, D_V Z_2 \rangle] = \int_\gamma \langle F(V, \dot{\gamma}) Z_1, Z_2 \rangle .
\end{aligned}$$

Also $(i_{\dot{\gamma}}\omega_2)(Z_1, Z_2; V) = \omega_2(Z_1, Z_2; \dot{\gamma}, V) = \int_\gamma \langle F(\dot{\gamma}, V) Z_1, Z_2 \rangle$.

Thus $\tilde{d}\omega_0 + i_{\dot{\gamma}}\omega_2 = 0$. For the other term,

$$\begin{aligned}
&3(\tilde{d}\omega_2)(Z_1, Z_2; V_1, V_2, V_3) \\
&= V_1 \omega_2(Z_1, Z_2; V_2, V_3) - \omega_2(D_{V_1} Z_1, Z_2; V_2, V_3) \\
&\quad - \omega_2(Z_1, D_{V_1} Z_2; V_2, V_3) - \omega_2(Z_1, Z_2; [V_1, V_2], V_3) \\
&\quad - \omega_2(Z_1, Z_2; V_2, [V_1, V_3]) + \text{cyclic permutations} \\
&= V_1 \int_\gamma \langle F(V_2, V_3) Z_1, Z_2 \rangle - \int_\gamma \langle F(V_2, V_3) D_{V_1} Z_1, Z_2 \rangle \\
&\quad - \int_\gamma \langle F(V_2, V_3) Z_1, D_{V_1} Z_2 \rangle - \int_\gamma \langle F([V_1, V_2], V_3) Z_1, Z_2 \rangle \\
&\quad - \int_\gamma \langle F(V_2, [V_1, V_3]) Z_1, Z_2 \rangle + \text{cyclic permutations} \\
&= 3 \int_\gamma \langle (DF)(V_1, V_2, V_3) Z_1, Z_2 \rangle = 0 \quad \text{by the Bianchi identity.} \quad \square
\end{aligned}$$

For a supersymmetric Lagrangian, we use $L=\frac{1}{2}(E+\omega+\omega_0+\omega_2)$. Then $\langle\eta\rangle_\beta$ can be defined as before for $\eta\in\Gamma(\Lambda^*[S^1, T^*M\oplus E])$ such that $\langle(\tilde{d}+i_{\dot\gamma})\eta\rangle_\beta=0$. The kinetic terms of L are E, ω and ω_0, and ω_2 enters as a potential term. In particular,

$$\begin{aligned}\langle 1\rangle_\beta &= \mathrm{Tr}(\gamma_{2n+1}\otimes\gamma_{2n'+1})e^{-\beta \not{D}_A^2}\\ &= \mathrm{Index}\,\not{D}_A:(S^+\otimes S'^+)\oplus(S^-\otimes S'^-)\to(S^-\otimes S'^+)\oplus(S^+\otimes S'^-).\end{aligned}$$

If $E=T^*M$ and A is the Riemannian connection then

$$\begin{aligned}&\mathrm{Index}\,\not{D}_A:(S^+\otimes S^+)\oplus(S^-\otimes S^-)\to(S^-\otimes S^+)\oplus(S^+\otimes S^-)\\ &\quad=\mathrm{Index}\,d+d^*:\Lambda^*_{\mathrm{even}}\to\Lambda^*_{\mathrm{odd}}=\chi(M).\end{aligned}$$

The formal Lagrangian for this case is that of $N=1$ supersymmetry:

$$L=\int_\gamma\left[\tfrac{1}{2}\langle\dot\gamma,\dot\gamma\rangle-\tfrac{1}{2}\langle\psi_1,\nabla_{\dot\gamma}\psi_1\rangle-\tfrac{1}{2}\langle\psi_2,\nabla_{\dot\gamma}\psi_2\rangle-\tfrac{1}{4}R_{ijkl}\psi_1^i\psi_1^j\psi_2^k\psi_2^l\right].$$

To see more explicitly that this gives $\chi(M)$, one can show that the corresponding Hamiltonian is $\frac{1}{2}(d^*d+dd)$. The first three terms of L will contribute $\frac{1}{2}\nabla^+\nabla+\frac{1}{8}R$ to the Hamiltonian, the $\frac{1}{8}R$ coming from the fact that the first two terms give the Dirac operator squared on $S(M)$. The contribution of the fourth term will be its image under the canonical map.

$$\begin{aligned}\mathrm{Gr}(T^*M\oplus T^*M)&=\mathrm{Gr}(T^*M)\otimes\mathrm{Gr}(T^*M)\to\mathrm{Hom}(S,S)\otimes\mathrm{Hom}(S,S)\\ &=\mathrm{Hom}(S^*\otimes S,S^*\otimes S)=\mathrm{Hom}(\Lambda^*M,\Lambda^*M)\end{aligned}$$

generated by $\psi_1(e_i)\to\frac{1}{\sqrt{2}}(E+I)(e_i)$ and $\psi_2(e_i)\to\frac{1}{\sqrt{2}}i(E-I)(e_i)$.

Proposition 12. *The image of* $-\frac{1}{4}R_{ijkl}\psi_1^i\psi_1^j\psi_2^k\psi_2^l\in\mathrm{Gr}(T^*M\oplus T^*M)$ *is*

$$-\tfrac{1}{2}R_{ijkl}E^iI^jE^kI^l-\tfrac{1}{8}R\in\mathrm{Hom}(\Lambda^*M,\Lambda^*M).$$

Proof. The image of $-\frac{1}{4}R_{ijkl}\psi_1^i\psi_1^j\psi_2^k\psi_2^l$ is $\frac{1}{16}R_{ijkl}(E^i+I^i)(E^j+I^j)(E^k-I^k)(E^l-I^l)$, which can be expanded into terms of various degrees. From the Bianchi identity, those of nonzero degree vanish. This leaves

$$\tfrac{1}{16}R_{ijkl}(E^iE^jI^kI^l-E^iI^jE^kI^l-E^iI^jI^kE^l-I^iE^jE^kI^l-I^iE^jI^kE^l+I^iI^jE^kE^l).$$

Permuting to the form $EIEI$ gives

$$\begin{aligned}&\tfrac{1}{16}R_{ijkl}[-E^iI^kE^jI^l-E^iI^jE^kI^l+E^iI^jE^lI^k+E^jI^iE^kI^l-E^jI^iE^lI^k-E^kI^iE^lI^j]\\ &\quad-\tfrac{1}{8}R+\tfrac{1}{8}R_{ij}E^iI^j\\ &=\tfrac{1}{16}[2R_{acdb}+4R_{abdc}]E^aI^bE^cI^d-\tfrac{1}{8}R+\tfrac{1}{8}R_{ab}E^aI^b\\ &=\tfrac{1}{16}[2R_{adcb}-6R_{abcd}]E^aI^bE^cI^d-\tfrac{1}{8}R+\tfrac{1}{8}R_{ab}E^aI^b\\ &=\tfrac{1}{8}R_{abcd}E^aI^dE^cI^b-\tfrac{3}{8}R_{abcd}E^aI^bE^cI^d-\tfrac{1}{8}R+\tfrac{1}{8}R_{ab}E^aI^b\\ &=(-\tfrac{1}{8}R_{ab}E^aI^b-\tfrac{1}{8}R_{abcd}E^aI^bE^cI^d)-\tfrac{3}{8}R_{abcd}E^aI^bE^cI^d\\ &\quad-\tfrac{1}{8}R+\tfrac{1}{8}R_{ab}E^aI^b=-\tfrac{1}{2}R_{abcd}E^aI^bE^cI^d-\tfrac{1}{8}R.\quad\square\end{aligned}$$

Thus the Hamiltonian is $H=\frac{1}{2}\nabla^\dagger\nabla-\frac{1}{2}R_{abcd}E^aI^bE^cI^d$, acting on Λ^*M. On the other hand, using normal coordinates,

$$d^*d+dd^*=-(I^i\nabla_iE^j\nabla_j+E^j\nabla_jI^i\nabla_i)=-(I^iE^j\nabla_i\nabla_j+E^jI^i(\nabla_i\nabla_j+[\nabla_j,\nabla_i]))$$
$$=\nabla^\dagger\nabla-E^aI^bR(e_a,e_b)=\nabla^\dagger\nabla-E^aI^bE^cI^dR_{abcd},$$

giving $H=\frac{1}{2}(d^*d+dd^*)$.

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References

1. Green, M., Schwarz, J.: Anomaly cancellations in supersymmetric $D=10$ gauge theory and superstring theory. Phys. Lett. **149** B, 117 (1984)
2. Witten, E.: Constraints on supersymmetry breaking. Nucl. Phys. B **202**, 253 (1982)
3. Alvarez-Gaumé, L.: Supersymmetry and the Atiyah-Singer index theorem. Commun. Math. Phys. **90**, 161 (1983)
4. Getzler, E.: A short proof of the local Atiyah-Singer index theorem. Harvard University preprint
5. Bismut, J.-M.: The Atiyah-Singer theorems: a probabilistic approach. I. J. Funct. Anal. **57**, 56 (1984)
6. Osterwalder, K., Schader, R.: Euclidean fermi fields and a Feynman-Kac formula for Boson-Fermion models. Helv. Phys. Acta **46**, 277 (1973)
7. Berezin, F.: The method of second quantization. New York: Academic Press 1966
8. Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes. Amsterdam: North-Holland 1981
9. Simon, B.: Functional integration and quantum physics. New York: Academic Press 1979
10. Leites, D.: Introduction to the theory of supermanifolds. Russ. Math. Surv. **35**, 1 (1980)
11. M. Atiyah: Circular symmetry and stationary phase approximation (to appear)
12. Malliavin, P.: Géométrie différentielle stochastique. Montréal: University of Montréal Press 1978
13. Palais, R.: Foundations of global nonlinear analysis. New York: Benjamin 1968
14. McKean, H.: Stochastic integrals. New York: Academic Press 1969
15. Witten, E.: Supersymmetry and Morse theory. J. Differ. Geom. **17**, 661 (1982)

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