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Calabi-Yau Manifolds – Motivations and Constructions*

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Abstract. The possible ways of compactification of the $E_8 \otimes E_8$ Superstring theory to four dimensions are reviewed. The phenomenological need for $N=1$ supersymmetry is argued (on quite general grounds) to favour the choice of a Calabi-Yau manifold for the compact internal manifold. The massless spectrum after compactification is derived in full detail revealing, beside the usual particles, others that may have great phenomenological impact. The technical aspects of the construction of such manifolds are examined and the methods of calculation of the relevant topological properties are given. A big family of such constructions, giving rise to many new Calabi-Yau manifolds, is presented and its relevance to the search of a phenomenologically acceptable solution is discussed.

1. Introduction

The idea that the dimensionality of the space-time might be greater than $3+1$ was realized already 65 years ago by Kaluza and Klein [1]. Dormant for about half a century, it got revived in the past decade [2] and proved to be extremely appealing in theories that attempt to provide a framework for a consistent unified description of both the gravitational and non-gravitational interactions [3, 4]. Originally, the only known non-gravitational interaction being the electromagnetic one, a $4+1$ dimensional (hereafter $D4+1$) space-time seemed to provide a suitable framework. The "extra" spatial coordinate is assumed to be highly curved and periodic (with radius of curvature $\sim M_{Pl}^{-1}$), ensuring its non-observability in experiments performed at energies below M_{Pl}. This gives rise to a new compact symmetry of the theory and thus a new charge, quantized and related to M_{Pl}, – interpreted as the electric charge.

Modern versions of this program [4, 5] try to accomplish for the weak and strong interactions, too, which are, together with the electromagnetic one and ignoring gravity, quite successfully described by the "Standard" $SU(3)_C \otimes SU(2)_L$

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$\otimes U(1)_Y$ Yang-Mills theory [6]. To incorporate these in the program, more "extra" dimensions are introduced to span the internal space. In most approaches, however, the internal space was considered to have high continuous symmetries, giving rise to at least part of the non-gravitational interactions [4, 5].

The most recent models of such unification [7] rely on Superstring theories [8] which are believed to incorporate, for the first time, consistent finite quantum gravity. The one that has recently attracted most attention [9] (known as "heterotic") describes, in the limit when higher excitations of the superstring decouple, an $N=1$ Supergravity with $E_8 \otimes E_8$ Yang-Mills interactions in $D9+1$ space-time, modified to account for anomaly cancellation [10]. Even though it is only a theory of closed strings it has a Yang-Mills sector that emerges from its peculiar construction. On the other hand, it has been shown [11] that this theory may be a ground state of the $D25+1$ bosonic closed string theory. The $D9+1$ Yang-Mills sector is related only partially to the continuous symmetries of the $D16$ torus. The completion to the full massless $E_8 \otimes E_8$ or $SO(32)$ Yang-Mills sector is obtained from a class of massless solitons which would not exist had one dealt with a point-field theory [9, 11]. Thus, these theories provide a true unification of gravitational and non-gravitational interactions in a (generalized) Kaluza-Klein manner, providing a space-time-geometrical origin for all of them.

To make contact with the $D3+1$ world of the present day's experiments (real world, hereafter), it was shown [12] that the internal six dimensions may be chosen to span one of the complex manifolds with $SU(3)$ holonomy, known in the literature as Calabi-Yau manifolds ($\mathcal{M}_{\mathrm{CY}}$). It is also attempted in the literature [13] to construct heterotic theories in dimensions lower than $9+1$. Whether these theories can yield phenomenologically acceptable models is still an open question, and they are not going to be discussed here. Instead, with the above remarks in mind, we return to the $D9+1$ heterotic theory.

The choice of Calabi-Yau internal manifold was motivated by the requirement of a surviving $N=1$ Supergravity in $3+1$ dimensions, since that is the only framework known to be able to maintain [14] the huge mass ratio of the Planck-mass vs. the weak mass-scale $(M_{Pl}/M_W) \sim 10^{17}$. On the other hand, it also provides a solution to the quadratic relation of the curvature and the Yang-Mills field strength [12]:

$$\tfrac{1}{30} F^a_{[mn} F^a_{pq]} = R_{[mn}{}^{rs} R_{pq]rs}\,, \tag{1.1}$$

which is a consequence of the choice of a vanishing modified field strength of the supergravity antisymmetric tensor (identified with the torsion of the internal manifold). Even though it is not the most general solution, it is going to be argued in Sect. 2 that it seems to be the most desirable one, at least.

The other aim of this paper is to show the most necessary technical details of the construction of these manifolds and to indicate some generalizations of the examples presented in the literature [15, 16]. That these generalizations may indeed be very helpful is suggested by the fact that the few manifolds with Euler character $\chi_E = -6$ have been constructed by embedding not in simple CP^n, but rather in $CP^n \times CP^m$. Thus a generalization of these constructions is done, yielding many more possibilities at hand. Even though every construction has subtleties on its own, some general features are noted that may provide considerable help.

The paper is organized as follows: Further motivations for the choice of the Calabi-Yau manifolds are found by reexamining the possibilities of compactification to $D3+1$ and their prospects, presented in Sect. 2. The analysis of the massless spectrum is given in Sect. 3. The construction of the Calabi-Yau manifolds given in the literature is reviewed and the technical details of the necessary calculations are given in Sect. 4. Section 5 provides some generalizations of the known constructions, presents some new classes of Calabi-Yau manifolds and their possible application in construction of phenomenologically acceptable models is discussed. Appendix A presents a detailed construction of a Calabi-Yau manifold, illustrating most of the techniques sufficient for such constructions and familiar to most physicists. Appendix B provides an algorithm for obtaining generalizations described in Sect. 5 and tables of some them. Appendix C provides a few explicit examples of Calabi-Yau manifolds constructed along the descriptions of Sect. 5 and Appendix B. Throughout the paper, "topological" is used to denote independence of smooth deformations, rather than topological in the mathematically rigorous sense.

2. $N=1$ Supersymmetry

The surviving $N=1$ supersymmetry of the $D3+1$ theory was motivated by the argument that all the dynamically generated mass-scales of the theory are bound to be closely related to the Planck-mass (being the only mass-scale of the theory), and the only known mechanism for maintaining a huge hierarchy needs $N=1$ Supergravity. This argument was, however, built on the implicit assumption that the lowest lying states of the matter vector of the Superstring theory are to be interpreted as the quarks and leptons of the real world. In this approach, the topological properties of the internal manifold link very tightly to the phenomenology of the real world, like the charges, the masses, the couplings of the observed particles – and are all calculable at least in principle [17, 18]. Various analyses [7, 19] have been made with the aim to construct a Calabi-Yau manifold that would reproduce all these parameters of the real world, but to the best of my knowledge not one has been found.

A slightly different approach has also been attempted in the literature [20], where the massless particle spectrum of the Superstring theory is identified not with the observed particles, but rather with their hypothesized constituents – preons [21]. It has been shown that a phenomenologically successful model [22] of this kind can be obtained with assumptions equivalent to those necessary in the usual "quark-lepton" approach. It should, however, be noted that all preonic models, **in general**, incorporate at least one stage of confinement, thereby preventing a straightforward connection of the parameters of the theory above and below the confinement scale. This may seem to lessen the predictability of such theories due to difficulties in understanding the dynamics of confinement; on the contrary, however, existing results show quite a good agreement [23] with the phenomenology of the real world, provide testable predictions [24] and open intriguing possibilities of understanding its structure. In order to avoid certain no-go theorems [25] one is naturally led to the requirement of local supersymmetry [22, 26], which is also very strongly suggested by tying together these and various

Table 1. Possible holonomies and corresponding decompositions

$[\text{Holonomy}]^{(\text{reducibility})}$	Tangent vector	Tangent spinor
$[SO(6)]^{(6)}$	$\mathbf{6}$	$\mathbf{4}+\mathbf{4}^*$
$[U(3)]^{(6)}$	$\mathbf{3}_4+\text{c.c.}$	$\mathbf{3}_{-2}+\mathbf{1}_6+\text{c.c.}$
$[SU(3)]^{(6)}$	$\mathbf{3}+\text{c.c.}$	$\mathbf{3}+\mathbf{1}+\text{c.c.}$
$[SO(4)]^{(4)}\otimes[U(1)]^{(2)}$	$(\mathbf{2},\mathbf{2})_0+[(\mathbf{1},\mathbf{1})_2+\text{c.c.}]$	$(\mathbf{2},\mathbf{1})_1+(\mathbf{1},\mathbf{2})_{-1}+\text{c.c.}$
$[SU(2)]^{(4)}\otimes[U(1)]^{(2)}$	$\mathbf{2}_0+\mathbf{2}_0+[\mathbf{1}_2+\text{c.c.}]$	$\mathbf{2}_1+\mathbf{1}_{-1}+\mathbf{1}_{-1}+\text{c.c.}$
$[SU(2)\otimes U(1)]^{(4)}$ $\otimes[U(1)]^{(2)}$	$(\mathbf{2};3)_0+(\mathbf{1};0)_2+\text{c.c.}$	$(\mathbf{2};0)_1+(\mathbf{1};3)_{-1}$ $+(\mathbf{1};-3)_{-1}+\text{c.c.}$
$[U(1)]^{(2)}\otimes[U(1)]^{(2)}$ $\otimes[U(1)]^{(2)}$	$(\mathbf{2},0,0)+(0,\mathbf{2},0)$ $+(0,0,\mathbf{2})+\text{c.c.}$	$(1,1,1)+(-1,-1,1)+(1,-1,-1)$ $+(-1,1,-1)+\text{c.c.}$

other phenomenological facts that a preonic theory has to reproduce [27]. On the other hand, by analysis of the β-function for different possible preon-confining (meta-colour) forces, [28] it is possible to argue that the supersymmetry ought to be simple. Hence the conclusion that $N=1$ Supergravity is strongly favoured by composite models as well.

It has been noted already that the choice of Calabi-Yau manifolds is not the most general one, and a search for other possible solutions has been attempted in the literature [29–31]. In particular, Eq. (1.1) has shown to be a fruitful starting point, yielding a large list of candidates [31]. This list is reproduced in Table 1, here with the decomposition of the spinor as well (which will be helpful in the analysis below).

In order to understand the importance of the results presented in Table 1, one ought to recall [32–35] that the holonomy group is the group of all transformations mapping any vector, parallel transported along any closed curve, into the original one. (Naturally, if there are spinors on the manifold, they are to be included in the above definition, since they cannot be constructed from vectors.) Observe now that it is **only the case of $SU(3)$ holonomy** with respect to which the spinor contains a holonomy-invariant component.

The latter four cases correspond to product spaces of $D4$ and $D2$ factors (as indicated by the reducibility). They are shown to lead to instabilities [36]; also, at least one of them would have to be S^2 with a monopole configuration to ensure not more than $N=1$ supersymmetry, yielding a positive contribution to the Gauss curvature of the internal space. Since no $D9+1$ supergravity[1] can have a cosmological term [5] to provide cancellation, one expects the non-vanishing curvature of the internal space to yield anti-de Sitter rather then Minkowski $D3+1$ space-time. The known examples of the first two cases (S^6 and CP^3) share this fate. Here we confine our attention to Minkowski $D3+1$ space-time only.

Cases with non-vanishing torsion with a Freund-Rubin type of ansatz have been found [30] to require $D9+1$ cosmological term and lead to anti-de Sitter $D3+1$ space-time. Recently, possible solutions with non-vanishing torsion have

[1] Here we assume that the superstring theory has a meaningful point-field theory limit with perturbative corrections [37]

been studied [38] (revealing much greater complexity than the vanishing torsion case), but whether phenomenologically acceptable solutions can be constructed is still an open question.

The criteria for surviving $N=1$ supersymmetry were analyzed in [37] as well. There a manifestly supersymmetric formulation of string corrections to the point-field theory is achieved. Along this analysis, one obtains equations that generalize those of [12] in a way that identically vanishing torsion ($\tilde{N}_{pqr}\equiv 0$) is a solution stable under first order string corrections. Vanishing torsion, however, implies Calabi-Yau manifolds for the internal space, since all the equations to be solved reduce to those of [12]. For what it is worth, note also that $\tilde{N}_{pst}$, having no $D3+1$ indices and dependence on space-time coordinates, represents a scalar with internal degrees of freedom. The problem of its background value is then essentially a Landau-Higgs problem and for the renormalizable part of the effective potential was shown [39] to favour $SU(3)$ for the internal space in half of the region of the coupling parameters (treating $\tilde{N}_{pst}$ as an independent field, rather then the modified field strength of B_{st}, since the latter never appears explicitly). We shall thus proceed assuming that the internal manifold is of the Calabi-Yau type.

3. Massless Spectrum

Let us now see how to determine the massless spectrum upon compactification. The massless bosons of the $D9+1$ theory are: $g_{(MN)}$, $B_{[MN]}$, Φ and A^{Θ}_{M}, while the fermions are: ψ_M, λ and χ^{Θ}, where M and N are $D9+1$ vector indices, spinor indices are suppressed, and Θ is the index of the adjoint representation of $E_8\otimes E_8$ or $SO(32)$.

Upon compactification, the $SO(9,1)$ transformation properties split into parts [40] describable by $SO(3,1)\otimes[SO(6)\sim SU(4)]$. The Yang-Mills group is also assumed to be broken to $\mathscr{G}\otimes SU(3)_{\mathrm{YM}}$, where $\mathscr{G}$ is $E_8\otimes E_6$ or $SO(26)\otimes U(1)_{\mathrm{global}}$. Since the fields to be interpreted as matter are invariant under the E_8 in the first case, one may safely drop it from further consideration. The fate of $U(1)_{\mathrm{global}}$ is discussed in [12, 41]. We also assume that the holonomy of the internal manifold is spanned by the diagonal subgroup $SU(3)_H$ of the space-time $SU(3)_{D6}\subset SO(6)$ part and $SU(3)_{\mathrm{YM}}$ part [42]. This will lead to the same results as the identification of the corresponding connections which is suggested by Eq. (1.1), but is more general. One now has to analyze the entire field content and determine its transformation properties under $SO(3,\ 1)\otimes\mathscr{G}\otimes SU(3)_H$. The fact that the transformation properties under the holonomy originate from both the space-time and the Yang-Mills sector is indicated in Table 3 by the Kronecker product $SU(3)$ representations. In discussing the fate of the states coming from the $D9+1$ supergravity multiplet, the Yang-Mills labels are trivial and thus omitted.

Using the fact that the fields transform in $D3+1$ and the internal $D6$ space as $(\mathbf{R},\mathbf{r})$, finding $D3+1$ massless states that transform as $\mathbf{R}$ translates into looking for $D6$ harmonic forms[2] transforming as $\mathbf{r}$. This is what we now turn to discuss. Actually, the existence of a covariantly constant spinor (necessary for $N=1$ supersymmetry) implies [43] that the internal manifold is Kähler, Ricci-flat and

[2] An s-form ω_s may be represented by its tensor coefficient antisymmetric in all s indices

with the holonomy in $SU(3)$. Also, the spinors are equivalent to $(0,q)$-forms (see below), and so one can use the theorem [34]:

If $F(\omega_s)$ is positive semidefinite, a harmonic ω_s is covariantly constant.
If $F(\omega_s)$ is positive definite, ω_s cannot be harmonic.

$F(\omega_s)$ is defined to be:

$$F(\omega_s) := R_{ij}\omega^{i}{}_{i_2\ldots i_s}\omega^{ji_2\ldots i_s} + \frac{s-1}{2} R_{ijkl}\omega^{ij}{}_{i_3\ldots i_s}\omega^{kli_3\ldots i_s}, \tag{3.1}$$

where R_{ij} and R_{ijkl} are the Ricci and the Riemann tensors of the internal manifold, and for Calabi-Yau manifolds the Ricci tensor vanishes. It is useful to note that using $SO(6) \sim SU(4) \to SU(3)_{D6}$ the $D6$ forms decompose [35] as:

$$\begin{aligned}
&\omega_0 \sim 1 && \to \omega_{0,0} && \sim \mathbf{1}, \\
&\omega_1 \sim \mathbf{6} && \to \omega_{1,0} + \omega_{0,1} && \sim \mathbf{3} + \mathbf{3}^*, \\
&\omega_2 \sim \mathbf{15} && \to \omega_{2,0} + \omega_{1,1} + \omega_{0,2} && \sim \mathbf{3}^* + (\mathbf{1} + \mathbf{8}) + \mathbf{3}, \\
&\omega_3 \sim \mathbf{10} + \mathbf{10}^* && \to \omega_{3,0} + \omega_{2,1} + \omega_{1,2} + \omega_{0,3} && \sim \mathbf{1} + (\mathbf{3} + \mathbf{6}^*) + (\mathbf{3}^* + \mathbf{6}) + \mathbf{1}.
\end{aligned} \tag{3.2}$$

Note also that, since the $SU(3)$ holonomy implies vanishing background Ricci tensor, the background Riemann tensor is traceless and therefore transforms as **27** of $SU(3)_H$. Thus from Eq. (3.1), $F(\omega) \propto (p+q-1)\,\mathrm{Tr}\{\mathbf{27} \otimes \omega \otimes \omega^*\}$, and it is clear that it is zero for every term in Eqs. (3.2) except for the **8** and **6**(**6***). One cannot, however, apply the above theorem for **8** and **6**, because positivity is not ensured in general. In fact it will indeed turn out that there may be harmonic forms of this type.

Since the $D6$ tangent vector splits into:

$$\begin{aligned}
&V^m = v^\mu \oplus v^{\bar\mu}, && v^\mu = V^\mu + iV^{\mu+3}, && v^{\bar\mu} = \bar v^\mu = V^\mu - iV^{\mu+3}, \\
&m = 1, \ldots, 6, && \mu = 1, 2, 3, && \bar\mu = \bar 1, \bar 2, \bar 3,
\end{aligned} \tag{3.3}$$

and by complex conjugation the number of harmonic fields transforming as **r** is the same as those transforming as **r***, the number of harmonic s-forms of the $D6$ real space (known as Betti numbers) decompose:

$$\begin{aligned}
b_0 &= b_{0,0} &&= b^{\mathbf{1}}_{0,0}, \\
b_1 &= b_{1,0} + b_{0,1} &&= 2b^{\mathbf{3}}_{1,0}, \\
b_2 &= b_{2,0} + b_{1,1} + b_{0,2} &&= 2b^{\mathbf{3}}_{1,0} + b^{\mathbf{1}}_{1,1} + b^{\mathbf{8}}_{1,1}, \\
b_3 &= b_{3,0} + b_{2,1} + b_{1,2} + b_{0,3} &&= 2b^{\mathbf{1}}_{0,0} + 2b^{\mathbf{3}^*}_{1,2} + 2b^{\mathbf{6}}_{1,2},
\end{aligned} \tag{3.4}$$

where the boldface superscripts denote the irreducible representation (irrep) of $SU(3)_H$. Here $b_{p,q}$ are the Hodge numbers (in the mathematical literature usually $h^{p,q}$), the refinement of b_s, satisfying $p+q=s$. Their existence and invariance under smooth deformations of the internal manifold is guaranteed by Hodge's theorem [35].

However, the $(1,1)$-form can be decomposed as the **1** part, corresponding to the Kähler form ($\mathscr{J}$) itself, which is covariantly constant, and the **8** part is its orthogonal complement. This, in turn, is used to decompose higher forms (known as Lefschetz decomposition) into "$\mathcal{J}$-trace" and $\mathcal{J}$-traceless parts[3] (the latter ones usually called primitive [35] or effective [33, 34]). This is done by defining [33, 35] $L(\omega) := \omega \wedge \mathcal{J}$ and Λ, the adjoint of L. Writing out the tensor coefficients explicitly, it is manifest that in the product of forms by $\mathcal{J}$, L, antisymmetrizes all indices while Λ contracts two of those of the form with those of $\mathcal{J}$. This decomposition is essentially topological (in its rigorous sense [35]), so it is also independent of smooth deformations of the internal manifold. This refinement of Hodge numbers, denoted by $b^{\mathbf{r}}_{p,q}$, can be defined, denoting the number of (p,q)-forms transforming under the holonomy as $\mathbf{r}$ [hereafter $(p,q)^{\mathbf{r}}$] that are harmonic. That this refined decomposition of forms is equivalent to the decomposition under the action of $U(n)$ [in our case $SU(3)$] was shown in [44].

Using the facts that: $b^{\mathbf{1}}_{0,0} = b_{0,0} = b_0$, by Poincaré duality $b_0 = b_6$ and that $b_6 = 1$, since b_6 is the number of compact and connected $D6$ linearly independent submanifolds, we rederive $b^{\mathbf{1}}_{0,0} = 1$. Further, by the arguments given in the previous section, $b^{\mathbf{3}}_{1,0} = b_{1,0} = b_{0,1} = 0$. Now, since $\varepsilon^{\mu\nu\varrho}$ and $\varepsilon^{\bar{\mu}\bar{\nu}\bar{\varrho}}$ are invariants of $SU(3)_H$, it follows that: $b^{\mathbf{1}}_{0,0} = b_{0,0} = b_{3,0} = b_{0,3} = b_{3,3} = 1$ and $b^{\mathbf{3}}_{1,0} = b_{1,0} = b_{2,0} = b_{0,1} = b_{0,2} = b_{1,3} = b_{2,3} = b_{3,1} = b_{3,2} = b_1 = 0$. One can now obtain:

$$b^{\mathbf{8}}_{1,1} = b_2 - b^{\mathbf{1}}_{1,1} = b_2 - 1\,, \qquad b^{\mathbf{6}}_{1,2} = \tfrac{1}{2} b_3 - 1 - b^{\mathbf{3}^*}_{1,2} = \tfrac{1}{2} b_3 - 1\,. \tag{3.5}$$

The second equalities can be derived by using the fact that [33–35] $b_s - b_{s-2}$ equals the number of $\mathcal{J}$-traceless s-forms (e_s) and applying both Hodge and Lefschetz decomposition of forms. Then:

$$b^{\mathbf{r}}_{p,q} - b^{\mathbf{r}}_{p-1,q-1} = e^{\mathbf{r}}_{p,q}\,. \tag{3.6}$$

Since there can be no $(0,1)^{\mathbf{6}}$-forms, all $(1,2)^{\mathbf{6}}$-forms are effective, and precisely the opposite is true for $\mathbf{3}^*$, thus:

$$b^{\mathbf{3}^*}_{1,2} = b^{\mathbf{3}^*}_{0,1} = 0 \;\Rightarrow\; b_{1,2} = b^{\mathbf{6}}_{1,2}\,. \tag{3.7}$$

The argument for $b^{\mathbf{1}}_{1,1}$ is the same as the one for $b^{\mathbf{3}^*}_{1,2}$, yielding of course, $b^{\mathbf{1}}_{1,1} = b^{\mathbf{1}}_{0,0} = 1$, for the unique Kähler form and for the unique covariantly constant spinor implying $N = 1$ supersymmetry in $D3+1$ as discussed in Sect. 2. Moreover, note that $b^{\mathbf{r}}_{p,q}$ changes only with $\mathbf{r}$, rendering thus the Hodge decomposition [middle column of Eq. (3.2)] inessential. In order to keep the contact with the literature, we shall not drop the p, q labels.

The $D9+1$ gravity supermultiplet can now be decomposed and its field content is given in Table 2. Apart from the $N=1$, $D3+1$ gravity and two scalar supermultiplets, one finds $b_2 + \frac{1}{2} b_3 - 2$ massless scalar supermultiplets of both chiralities, the number of which can be also recast in the form $2b_2 - \frac{1}{2}\chi_E - 1$. Note that at the first massive level all the states in Table 2 are present, and $D3+1$, $N=4$ supergravity is manifest.

When dealing with the Yang-Mills sector, the convention of [17] is followed, treating the $\mathbf{3}$ as a $(1,0)^{\mathbf{3}}$-form[4]. Owing to the fact that the transformation property

[3] Note that $\mathcal{J}$ is defining the Kähler metric used to "take traces"

[4] This just defines the embedding of $SU(3)_H \in SU(3)_{\mathrm{YM}} \otimes SU(3)_{D6}$

Table 2. Decomposition of the $D9+1$ gravity supermultiplet

Superfield type	$SU(3)_H$	N° of massless states
$(2,3/2)_\pm$	$(0,0)^1$	$b^1_{0,0}=1$
$(3/2,1)_\pm$	$(0,1)^{3^*}$	$b^3_{1,0}=0$
$(1,1/2)_\pm$	$(1,0)^3$	$b^3_{1,0}=0$
	$(0,1)^{3^*}$	$b^3_{1,0}=0$
	$(0,1)^{3^*}$	$b^3_{1,0}=0$
$(1/2,0)_+ +$h.c.	$(1,2)^6$	$b^6_{1,2}=\frac{1}{2}b_3-1$
$(1/2,0)_\pm$	$(0,0)^1$	$b^1_{0,0}=1$
	$(1,1)^1$	$b^1_{1,1}=1$
	$(1,1)^8$	$b^8_{1,1}=b_2-1$
	$(0,2)^3$	$b^3_{1,0}=0$

The superfields are denoted by the spins of the physical particles, and the $\pm$ subscript denotes $\pm$ helicity

Table 3. Decomposition of the $E_8 \otimes E_8$ Yang-Mills $D9+1$ supermultiplet

Superfield type	E_6	$SU(3)_H$	N° of massless fields
$(1,1/2)_\pm$	**78**	$(0,0)\otimes\mathbf{1}=(0,0)^1$	$b^1_{0,0}=1$
	27	$(0,0)\otimes\mathbf{3}=(1,0)^3$	$b^3_{1,0}=0$
	27*	$(0,0)\otimes\mathbf{3}^*=(0,1)^{3^*}$	$b^3_{1,0}=0$
	1	$(0,0)\otimes\mathbf{8}=(1,1)^8$	$b^8_{1,1}=b_2-1$
$(1/2,0)_+ + h.c.$	**78**	$(1,0)\otimes\mathbf{1}=(1,0)^3$	$b^3_{1,0}=0$
	27	$(1,0)\otimes\mathbf{3}=(1,2)^{3^*}+(1,2)^6$	$b^6_{1,2}=\frac{1}{2}b_3-1$
	27*	$(1,0)\otimes\mathbf{3}^*=(1,1)^1+(1,1)^8$	$b^1_{1,1}+b^8_{1,1}=b_2$
	1	$(1,0)\otimes\mathbf{8}=(2,1)^3+(2,1)^{6^*}+\mathbf{15}$	$(b^6_{1,2}=\frac{1}{2}b_3-1)+N^\circ(\mathbf{15})$

The superfields are denoted by spins of the physical particles and the $\pm$ subscript denotes $\pm$ helicity

under the holonomy of these fields is found in the tensor (rather than the wedge) product of their $SU(3)_{\rm YM}$ irrep and the $SU(3)_{D6}$ irrep, the result need not be representable by a form. This shows up in Table 3, where one finds fields transforming as **15** of $SU(3)$ as well. Since $\mathbf{15}\otimes\mathbf{15}^*\otimes\mathbf{27}\ni\mathbf{1}$, $F(\mathbf{15})\neq 0$, and one cannot conclude $N^\circ(\mathbf{15})=0$ for **15** not being holonomy-invariant [again, positivity of $F(\mathbf{15})$ is not ensured]. **15** cannot be represented by a form, consequently, this representation does not occur in Eq. (3.2), and one cannot relate $N^\circ(\mathbf{15})$ to any of the Betti (or Hodge) numbers.

Utilizing $D3+1$ hermitian conjugation, all the scalar superfields are recast into positive-chiral (left handed) ones, obtained the known result:

$$\begin{aligned} N(\mathbf{27}) := \quad & N^\circ(\mathbf{27})-N^\circ(\mathbf{27}^*)=1+b_2-\tfrac{1}{2}b_3\equiv=\tfrac{1}{2}\chi_E\,,\\ \delta(\mathbf{27}) := & \min\{N^\circ(\mathbf{27}),N^\circ(\mathbf{27}^*)\}=(\text{usually})N^\circ(\mathbf{27}^*)=b_2\,. \end{aligned} \tag{3.8}$$

In addition, there appear to be $N_S(\mathbf{1})$ massless Yang-Mills singlet chiral superfields and $N_V(\mathbf{1})$ massless Yang-Mills singlet vector superfields:

$$\begin{aligned} N_S(\mathbf{1}) &= 3b_2 - \chi_E + N^\circ(\mathbf{15}) + 1 = \tfrac{3}{2}b_3 + \tfrac{1}{2}\chi_E + N^\circ(\mathbf{15}) - 2\,, \\ N_V(\mathbf{1}) &= b_2 - 1 \qquad\qquad\qquad\quad = \tfrac{1}{2}(b_3 + \chi_E) - 2 \end{aligned} \tag{3.9}$$

as well. The fate of some of these was discussed in [18] and we return to this question after briefly examining the $SO(32)$ case.

In the $SO(32)$ case the analysis carries over almost completely. One can simply replace $\mathbf{27}$ by $(\mathbf{26}_{+2} + \mathbf{1}_{-4})$ and correspondingly the complex conjugate states in Eq. (3.8):

$$\begin{aligned} N(\mathbf{26}_{+2}) &= N(\mathbf{1}_{-4}) = 1 + b_2 - \tfrac{1}{2}b_3 \equiv -\tfrac{1}{2}\chi_E\,, \\ \delta(\mathbf{26}_{+2}) &= \delta(\mathbf{1}_{-4}) \;= (\text{usually}) N^\circ(\mathbf{26}_{-2} + \mathbf{1}_4) = b_2\,. \end{aligned} \tag{3.10}$$

This verifies the discussion of massless chiral superfields in the $SO(32)$ case in [20]. The number of Yang-Mills invariant vector superfields in Eq. (3.9) increases by one since there is now a would-be gauge superfield of the $U(1)_{\text{global}}$ in $\mathscr{G}$ for the $SO(32)$ case. It was shown in [41] that this vector superfield acquires mass via the Chern-Simons form and the same mechanism works again making the $U(1)$ into a Peccei-Quinn like global symmetry.

Note, however, that this superfield transforms under holonomy as $(0,0)^{\mathbf{1}}$, while the other $b_2 - 1$ Yang-Mills invariant vector superfields transform as $(1,1)^{\mathbf{8}}$. For most of the Calabi-Yau manifolds considered in the literature, $b_2 = 1$ and so this problem did not occur. For manifolds with $b_2 > 1$ these superfields would imply a $[U(1)]^{b_2-1}$ new-born gauge symmetry contradicting the fact that Calabi-Yau manifolds have no continuous symmetries. Fortunately, $B_{[MN]}$ contains $b_2 - 1$ $(1,1)^{\mathbf{8}}$ chiral superfields as well and the mechanism of [41] combines them into massive vector superfields and only the vector superfields gauging $E_6 \otimes E_8$ ($SO(26)$) appear in the massless spectrum. (That there are no massless superfields transforming as $\mathbf{8}$ under the Yang-Mills $SU(3)$ and having no $D6$ indices was concluded in [18] on grounds of vanishing of $\dim H^0(\operatorname{End} T)$.)

The chiral superfields that remain massless are (in the $E_6 \otimes E_8$ case) as follows: U, V, and W transforming as $\mathbf{27}$, $\mathbf{27}^*$, and $\mathbf{27}^*$ under E_6 and as $(1,2)^{\mathbf{6}}$, $(1,1)^{\mathbf{1}}$, and $(1,1)^{\mathbf{8}}$ under $SU(3)_H$; Yang-Mills invariant $C \sim (1,2)^{\mathbf{6}}$, $\phi \sim (0,0)^{\mathbf{1}}$, $\sigma \sim (1,1)^{\mathbf{1}}$ from the $D9+1$ graviton supermultiplet and $\varphi \sim (2,1)^{\mathbf{6}^*}$, $\varphi' \sim \mathbf{15}$ from the $D9+1$ Yang-Mills supermultiplet. Their number can be read off from Table 2 and Table 3 that agrees with the earlier results [12, 17, 18, 42]. In addition, using the fact that the φ fields found in:

$$[(1,0) \otimes \mathbf{8}]_{\text{antisymm.}} \ni (2,1)^{\mathbf{6}^*}\,, \tag{3.11}$$

where $(1,0)$ stands for a holomorphic $D6$ index and $\mathbf{8}$ for the transformation property under $SU(3)_{\text{YM}}$, can be represented by a form, the number of harmonic $D6$ fields of this type was related to $b_{1,2}$. As noted earlier, the number of φ' chiral superfields cannot be related to the Betti numbers. (This result is consistent with that of [18], where φ, φ' and the $(2,1)^{\mathbf{3}}$ fields were treated jointly, relating their number to $\dim H^1(\operatorname{End} T)$ that cannot be related further to any Betti number.)

Originating partially from $B_{[MN]}$, which appears only with derivative couplings, ϕ and σ cannot appear in the superpotential to any order in perturbation theory [18]. Whether they could successfully play the role of the inflation field or some other "dark matter," remains as a subject for future study. The most general superpotential is now easily determined since it has to be Yang-Mills and Holonomy invariant. To the lowest order, it contains the following terms: U^3, V^3, W^3, C^3, φ^3, φ'^3, $UV\varphi$, $UW\varphi$, $UW\varphi'$, WWV, $\varphi'^2\varphi$, and $\varphi^2\varphi'$. To summarize, the number of U, C, and φ is $b^{\mathbf{6}}_{1,2}=(b_2-\frac{1}{2}\chi_E)$, there are one V, ϕ, and σ, $b^{\mathbf{8}}_{1,1}=b_2-1\,W$ and $N^\circ(\mathbf{15})\,\varphi'$ chiral superfields.

4. The $CP^n|_{\{I^a=0\}}$ Constructions

We now review some of the constructions presented in the literature [15] studying the details of the calculations that will prove to be necessary for generalizing these results. It is useful to realize that most of these examples are complete intersections of hypersurfaces in CP^n defined as non-singular intersections of solutions of $n-3$ homogeneous analytic polynomial constraints[5], denoted by $\mathscr{Y}(n;q_a)$, where $q_1,\ldots,q_{n-3}$ are the degrees of homogeneities of the constraints on CP^n. Some examples [16] are constructed which are analogous subvarieties in $CP^n\times CP^m$, and it is therefore natural to explore the possibility of similar constructions in spaces of the type $CP^{n_1}\times\ldots\times CP^{n_k}$, which is going to be done later on. It is, however, useful to note that these constructions, and in particular the one of $\mathscr{Y}(4;5)$ in [12] is closely following the analysis of the K_3 surface as given in [32].

Since the knowledge of Chern classes is of utmost importance, note that while constructing $\mathscr{M}_{CY}$ as a submanifold of some complex manifold ($\mathscr{M}$), one has the relation [32]:

$$\mathscr{T}(\mathscr{M}|_{\mathscr{M}_{CY}})=\mathscr{T}(\mathscr{M}_{CY})\oplus\mathscr{N}(\mathscr{M}_{CY})\,,$$

where $\mathscr{T}$ and $\mathscr{N}$ denote the tangent and the normal bundles, respectively. It follows that the total Chern class is:

$$c[\mathscr{T}(\mathscr{M})|_{\mathscr{M}_{CY}}]=c[\mathscr{T}(\mathscr{M}_{CY})]\cdot c[\mathscr{N}(\mathscr{M}_{CY})]\,.$$

For CP^n and for analytic, homogeneous polynomial constraints $I(z)=0$, where $z\in CP^n$:

$$c[\mathscr{T}(CP^n)]=(1+x)^{n+1}\,, \tag{4.1}$$

$$c[\mathscr{T}(I)]=(1+qx)\,, \tag{4.2}$$

where I is homogeneous of degree q and x represents the Kähler 2-form. Thus:

$$c[\mathscr{M}_{CY}]=\frac{(1+x)^{n+1}}{(1+qx)}\,. \tag{4.3}$$

The powers of x represent its wedge-products, and so it should be regarded nilpotent of degree 3 (i.e. $x^m\equiv 0$, $\forall m>3$). As it stands, Eq. (4.3) is applicable to $\mathscr{Y}(4;5)$ (also K_3 with $n=3$, $q=4$, and x being nilpotent of degree 2) only:

$$c[\mathscr{Y}(4;5)]=1+c_1x+c_2x^2+c_3x^3=1+0x+10x^2-40x^3\,. \tag{4.4}$$

[5] Usually called analytic subvarieties

The vanishing of c_1 (the first Chern class) is guaranteed since $q=(n+1)$. For $n>4$, and consequently, a set of $n-3$ constraints, this is easily generalized [12]:

$$c[\mathscr{Y}(n;q_1,\ldots,q_{n-3})]=\frac{(1+x)^{n+1}}{(1+q_1x)\ldots(1+q_{n-3}x)}, \tag{4.5}$$

and the vanishing first Chern-class restriction for the q's reads: $\sum_{a=1}^{n-3} q_a=(n+1)$.

Apart from being analytic and homogeneous, the set of constraints need to satisfy also:

$$dI^1\wedge\ldots\wedge dI^{n-3}\neq 0\,,\qquad \forall z\in CP^n|_{\{I^a=0\}}, \tag{4.6}$$

where the superscript a enumerates the constraints. This inequality ensures non-singularity of the hypersurface (non-vanishing volume of the normal bundle at every point). For the case of $\mathscr{Y}(4;5)$, one of the simplest choices, $I=\sum_{i=1}^{5} z_i^5=0$ fulfills the requirements, since $dI=\frac{1}{5}\sum_{i=1}^{5} z_i^4 dz^i=0$ would require $z_i=0\notin CP^4$. [$I'=z_1z_2z_3z_4z_5=0$ would e.g. not satisfy relation (4.6), since the solution of $dI'=0$ can be parametrized by $z_5=z_4=0$ and $z_{1,2,3}$ that easily satisfy $I'=0$ and the projectiveness of CP^4, leaving thus a singular complex line in the would-be Calabi-Yau manifold.]

The volume element in relation (4.6) is:

$$\begin{aligned}dI^1\wedge\ldots\wedge dI^{n-3}&=\partial_i I^1\ldots\partial_j I^{n-3}\cdot dz^i\wedge\ldots\wedge dz^j\\&=\partial_{[i}I^1\ldots\partial_{j]}I^{n-3}\cdot dz^i\wedge\ldots\wedge dz^j\,,\end{aligned} \tag{4.7a}$$

by the antisymmetry of the wedge product, where $\partial_i=\partial/(\partial z^i)$. This is equivalent to:

$$\partial_{[i}I^1\ldots\partial_{j]}I^{n-3}=0\,, \tag{4.7b}$$

since $dz^i\wedge\ldots\wedge dz^j$ are independent (taking into account that indices are totally antisymmetrized). Thus Eq. (4.7b) contains $\binom{n+1}{n-3}=\binom{n+1}{4}=\frac{n}{24}(n^2-1)(n-2)$ independent equations in addition to the $(n-3)$ constraints themselves and the n projectiveness equivalence relations, whereas there are $2n$ homogeneous coordinates. Thus, one generally expects that the two systems, $\{I^a=0\}_{a=1,\ldots,n-3}$ **and** $\left\{\bigwedge_{a=1}^{n-3} dI^a=0\right\}$, have no common solution in CP^n, for being overdetermined. However, when constructing an explicit example [where the set of constraints is typically invariant under some discrete group, turning some of the equations in Eq. (4.7b) equivalent] one has to verify this important property.

As is shown in Sect. 3, the Euler character (χ_E) of the manifold is twice the difference of the number of massless "left-handed" and "right-handed" (matter) chiral superfields in $3+1$ dimensions, thus very important for building physical models. Note, however, that the derivation of this result relied on the assumptions:

1. vanishing torsion: $\tilde{N}_{pqr}=0$ (actually, this condition can even be weakened as shown in [30] still leading to Calabi-Yau manifolds, but anti-de Sitter $D3+1$; see however [38] also) and

2. the holonomy group is identified as the diagonal subgroup of the groups spanned by the background Yang-Mills and $D6$ spin-connection.

A more general analysis in this respect can be found in [29] (without, however, requiring supersymmetry), but we adopt the above assumptions throughout this paper.

In order to calculate the Euler character of the manifold, one needs to integrate $c_3 x^3$, taking into account the contributions of all Riemann-sheets of the set $I^a = 0$. This becomes simple if one notes that, embedded in the initial CP^n, x is nilpotent of degree n, and the integration becomes weighted by the Chern class of the set of constraints:

$$\chi_E = \int_{\mathcal{M}_{CY}} c[\mathcal{M}_{CY}] = \int_{CP^n} \xi \cdot c[\mathcal{M}_{CY}] = [\xi \cdot c[\mathcal{M}_{CY}]]_{x^5}, \tag{4.8}$$

where ξ denotes the Chern class of the set of constraints:

$$\xi = \prod_{a=1}^{n-3} (1 + q_a x), \tag{4.9}$$

and $[f(x)]_{x^5}$ denotes the coefficient of x^5 in the expansion of $f(x)$. Equation (4.8) can be manipulated into a more suitable form:

$$\chi_E = \frac{1}{3}\left[(n+1) - \sum_{a=1}^{n-3} q_a^3\right] \prod_{a=1}^{n-3} q_a, \tag{4.10}$$

recovering easily all the values in [12].

In order to know the exact number of both the left- and right-handed massless chiral superfields under the above assumptions, one has to compute the Betti number b_2 or b_3 (actually $b^{\mathbf{8}}_{1,1}$ and $b^{\mathbf{6}}_{1,2}$) and $N^\circ(\mathbf{15})$. The latter is important only for determining the total number of Yang-Mills invariant scalar superfields as well. Unfortunately I do not know of a general method of computing these numbers.

The Euler characters of the manifolds considered so far are too big to yield any phenomenologically acceptable model. However, it is possible to generate new manifolds via dividing out the free action of a discrete group $\mathcal{D}$. The resulting manifold will have:

$$\chi_E[\mathcal{M}_{CY'}] = \frac{\chi_E[\mathcal{M}_{CY}]}{\text{number of elements of } \mathcal{D}}, \tag{4.11}$$

and may be more realistic. In addition, $\mathcal{M}_{CY'}$ is multiply connected, making a simultaneous Yang-Mills symmetry-breaking possible, and changing the content of massless chiral superfields as well [12, 17, 7]. In order to illustrate the procedure of constructing a Calabi-Yau manifold an example is given in Appendix A. All the necessary technical details are given there.

5. Generalizations

Guided by the constructions of [16], let us examine the possibility to define analytic subvarieties in $CP^{n_1} \times \ldots \times CP^{n_m}$. In particular, if all the n_r are the same, an interesting possibility emerges. The commonly utilized discrete symmetries (shown in Appendix A) are either the (cyclic) permutation of the homogeneous coordinates or their multiplication by different roots of 1. Having $\mathcal{M}_{CY}$ embedded into a space like the m-fold direct product of copies of CP^n makes it possible to find suitable discrete symmetries in S_m, the group of permutations of these copies.

It is clear in this case, Eq. (4.3) generalizes to:

$$c[\mathcal{M}_{CY}]=\xi^{-1}\cdot \prod_{r=1}^{m} (1+x_r)^{n_r+1} \tag{5.1a}$$

$$\rightarrow \xi^{-1}\cdot \prod_{r=1}^{m} (1+x_r)^{n+1}, \tag{5.2a}$$

with the total Chern class of the constraints being:

$$\xi=\prod_{a=1}^{n_1+\ldots+n_r-3} \left(1+\sum_{r=1}^{m} q_a^r x_r\right) \tag{5.1b}$$

$$\rightarrow \prod_{a=1}^{mn-3} \left(1+\sum_{r=1}^{m} q_a^r x_r\right). \tag{5.2b}$$

Here the indices r and a enumerate the copies of CP^n and constraints, respectively. The degree of homogeneity of the a^{th} constraint with respect to the homogeneous coordinates of the r^{th} copy of CP^n is q_a^r. From now on, only m-fold direct products of copies of the same CP^n are going to be considered.

To ensure vanishing of the first Chern class, this matrix has to satisfy:

$$c_1^r=(n+1)-\sum_{a=1}^{mn-3} q_a^r=0, \qquad \forall r. \tag{5.3}$$

On the other hand, to achieve irreducibility of $\mathcal{M}_{CY}$, for every copy of CP^n there has to be at least one constraint that couples its coordinates with the coordinates of another copy of CP^n. Also, a linear constraint would reduce the corresponding CP^n to CP^{n-1}, bringing us out of the scope of the present paper. Thus:

$$\sum_{r=1}^{m} q_a^r \geqq 2, \qquad \forall a. \tag{5.4}$$

Summation over the free index in both Eqs. (5.3) and (5.4) leads to the inequality:

$$m(n-1)\leqq 6. \tag{5.5}$$

Substituting $m=1$ it follows that $n\leqq 7$, which is the limit already obtained in [12]. Note the fact that Eq. (5.5) is satisfied for any $m>3$ when $n=1$; it can, however, be

Table 4. Number of necessary constraints for a Calabi-Yau manifold. The dashes denote no possibility existing, and m take an arbitrary large value (see however text below)

	m								
n↓	1	2	3	4	5	6	7	8	9, and so on...
1	–	–	–	1	2	3	4	5	6, and so on...
2	–	1	3	5	7	9	–	–	–
3	–	3	6	–	–	–	–	–	–
4	1	5	–	–	–	–	–	–	–
5	2	–	–	–	–	–	–	–	–
6	3	–	–	–	–	–	–	–	–
7	2	–	–	–	–	–	–	–	–

proven that for $m>9$, $n=1$ and $m>6$, $n=2$ no new Calabi-Yau manifold can be constructed in this way[6]. This analysis is beyond the scope of the present paper and will be presented elsewhere [45]. Nevertheless, some of the constructions may be technically more advantageous and Table 4 displays the number of constraints for various m and n.

For all cases in Table 4, the number of homogeneous coordinates is $m(n+1)$, the number of constraints $mn-3$ and the number of projectiveness equivalence relations n. Thus it is always possible to solve these equations. On the other hand, the equation $dI^1 \wedge \ldots \wedge dI^{mn-3}=0$ is equivalent to:

$$\partial_{[i}I^1 \wedge \ldots \partial_{j]}I^{mn-3}=0\,, \tag{5.6}$$

with ∂_i being the derivative with respect to any of the $m(n+1)$ homogeneous variables. The number of independent equations in Eq. (5.6) is $\binom{m(n+1)}{mn-3} = \binom{m(n+1)}{m+3}$. This gives an $(m+3)^{\text{th}}$-order polynomial in $(mn+1)$ that, for growing m, grows much faster than the number of homogeneous variables. Thus we conclude that there should be no obstruction[7] in general to the constructions listed in the Table 4, but note that – as mentioned earlier – the $m>9$, $n=1$ and $m>6$, $n=2$ cases are just different embeddings of those obtained for lower m, same n.

To explicitly construct a particular manifold, one chooses certain m and n and a corresponding q_a^r matrix from the Appendix B (or constructs one for higher m according to the algorithm given there). The symmetries of q_a^r imply a set of discrete symmetries with respect to which the constraints can be chosen to be invariant. Factorizing the value of the Euler character, one obtains possible values for the order of the discrete group the free action of which is to be divided out (bearing in mind that the resulting Euler character must be an even integer). Checking Eq. (5.6) is usually a tedious but straightforward task, where the recognition of the discrete symmetries may provide short-cuts. Instead of explicit verification it is possible to use some theorems [16], but this requires much more familiarity with the underlying mathematics.

As noted above, usually one looks for cyclic permutations or multiplication by growing powers of a nontrivial root of unity. This will yield (products of) cyclic groups Z_n. Instead of just cyclic permutations one may of course try quotienting any subgroup of (or even the full) permutation group. To verify non-singularity of the resulting manifold one enlists the fixed points and makes sure that they do not satisfy $\{I^a=0\}$. If some of the fixed points do satisfy $\{I^a=0\}$, one can proceed the construction in the manner of [19], leaving the singularities and obtaining "orbifolds" instead of manifolds. To obtain manifolds, one has to apply the "blowing up" procedure [12, 15, 16, 35]. The latter not only removes the singularity but also annihilates the multiple connectedness caused by dividing out the action of a discrete symmetry; thus one would apply this method only combined with dividing out free actions [16]. The counting of massless fields however appears to be the same for both procedures.

[6] I wish to thank P. Green for pointing this out to me

[7] This question is addressed again in Appendix C

For some special numbers of variables, one may utilize other discrete groups, in particular some non-abelian ones [15]. Finally, in the case $m > 1$ one may combine these symmetries with (a subgroup of) the group of permutation of the copies of CP^n.

The Euler character of these manifolds can be calculated as before, relying merely on the knowledge of the matrix q_a^r. Expanding Eq. (5.2a) in powers of x_r and making use of Eq. (5.3) one readily obtains:

$$c_2^{ij} = \eta^{ij}\left[\sum_{a=1}^{mn-3} q_a^i q_a^j - (n+1)\delta^{ij}\right], \quad c_3^{ijk} = 2\eta^{ijk}\left[(n+1)\delta^{ijk} - \sum_{a=1}^{mn-3} q_a^i q_a^j q_a^k\right], \tag{5.7}$$

with the numerical factors:

$$\delta^{ijk} := \begin{cases} 1 & \text{if } i=j=k, \\ 0 & \text{otherwise}; \end{cases}$$
$$\eta^{ij} := \begin{cases} 1/2 & \text{if } i=j, \\ 1 & \text{otherwise}; \end{cases} \tag{5.8}$$
$$\eta^{ijk} := \begin{cases} 1/6 & \text{if } i=j=k, \\ 1/2 & \text{if only two are equal}, \\ 1 & \text{if all three are different}. \end{cases}$$

To calculate the Euler character, one needs

$$\begin{aligned} \xi_{ijk} &\equiv \left[x_i x_j x_k \prod_{a=1}^{mn-3}\left(1 + \sum_{r=1}^{m} q_a^r x_r\right)\right]_{x_1^n \dots x_m^n}, \\ &= \left[x_i x_j x_k \prod_{a=1}^{mn-3}\left(\sum_{r=1}^{m} q_a^r x_r\right)\right]_{x_1^n \dots x_m^n}, \end{aligned} \tag{5.9}$$

since only the highest terms contribute in Eq. (4.8). It is clear that a closed formula of the type Eq. (4.10) is not available in general, since one ought to interchange the summation with the product.

Now we come to the question how to choose the degrees in the q_a^r matrices. It is clear that two matrices that differ by a permutation of rows and/or columns are equivalent. So, to obtain all the possible constructions in a particular choice of m and n, one has to enlist all the corresponding inequivalent matrices and thus every entry in Table 4 determines a class of constructions. It should be cautioned that different constructions do not imply necessarily topologically distinct Calabi-Yau manifolds. A way to prove that two particular constructions yield distinct manifolds would be to compute one of their topological invariants to be different.

Listings of inequivalent q_a^r matrices are given in Appendix B for the few low-m classes together with their Euler characters. For the higher-m classes one might want to generate q_a^r by a computer, since their number tends to grow uncomfortably.

To summarize, very strong motivations in favour of the compactification on Calabi-Yau manifolds are found both in schemes where quarks and leptons are identified with elementary fields of the $D9+1$ supergravity theory believed to be the massless level of the "heterotic" superstring theory, as well in schemes where these elementary fields are identified with subconstituents of quarks and leptons.

The full spectrum of states appearing in the low-energy limit after compactification is derived. Beside the fields analyzed in the literature, compactification yields some Yang-Mills invariant chiral and vector superfields, only a subset of these chiral superfields remaining massless. The influence of these fields on the phenomenology requires more study.

The constructions of Calabi-Yau manifolds embedded into CP^n for $n=4, 5, 6, 7$ are reviewed and the technical details shown in order to provide a background for generalizations. A big family of classes of such generalizations is constructed, possibly useful for do-it-yourself Unified model building. These constructions may be helpful in finding the manifold that would lead to a phenomenologically successful Unified model with either quark-lepton or preon identification of the fundamental fields. On the other hand, they might shed some light on the (as yet non-existent) classification of Calabi-Yau manifolds. At this stage one still does not know how the dynamics of the $D9+1$ theory chooses the ground state, and hence the feeling that by enlisting different possibilities one may gain a better insight.

Appendix A

Here an explicit example is given, showing the procedure of finding a set of constraints that are invariant under a discrete symmetry and checking the non-singularity of the set of solutions to the vanishing of the constraints. Throughout the construction the symmetries are employed to reduce the problem.

Let us consider $\mathscr{Y}(5;3,3)$ with $\chi_E = -144$ in this illustration. One defines this $\mathscr{Y}$ by imposing the vanishing of two cubic polynomials in the homogeneous complex variables $(z_1, \ldots, z_6)$ of CP^5 which are subject to following relations:

$$|z| \neq 0, \qquad z_k \cong \lambda z_k \qquad \forall \lambda \in C \backslash \{0\}. \tag{A.1}$$

Six coordinates in Eq. (A.1) suggest that it is natural to look for order-6 symmetric polynomials. A straightforward choice is:

$$S: z_k \to z_{k+1} \;\Rightarrow\; Z_6^S, \qquad T: z_k \to z_k \beta^k \;\Rightarrow\; Z_6^T, \quad \text{for} \quad \beta^6 = 1. \tag{A.2}$$

Since the polynomials are cubic, it will be impossible to make them invariant under S or T, but that is actually not necessary as it is sufficient for them to transform with an overall phase or into each other (up to an overall phase), the solution – the Calabi-Yau manifold – will remain invariant.

To find suitable polynomials it is useful to classify the cubic terms in z_k according to the power of β with which the whole term transforms, and then assembling all terms transforming with the same power into one polynomial. This, of course yields three pairs which are interchanged by S and thus offer three possibilities. They are:

$$\begin{aligned} I^{(0)} &= z_1 z_3 z_5 + \alpha_1 (z_1^3 + \ldots) + \alpha_2 (z_2^2 z_5 + \ldots) + \alpha_3 (z_2 z_3 z_4 + \ldots), \\ J^{(0)} &= z_2 z_4 z_6 + \alpha_1 (z_2^3 + \ldots) + \alpha_2 (z_3^2 z_6 + \ldots) + \alpha_3 (z_3 z_4 z_5 + \ldots), \end{aligned} \tag{A.3}$$

$$\begin{aligned} I^{(+)} &= \alpha_4 (z_2^2 z_3 + \ldots) + \alpha_5 (z_3^2 z_1 + \ldots) + \alpha_6 (z_2 z_4 z_1 + \ldots), \\ J^{(+)} &= \alpha_4 (z_3^2 z_4 + \ldots) + \alpha_5 (z_4^2 z_2 + \ldots) + \alpha_6 (z_3 z_5 z_2 + \ldots), \end{aligned} \tag{A.4}$$

$$\begin{aligned} I^{(-)} &= \alpha_7 (z_2^2 z_1 + \ldots) + \alpha_8 (z_3^2 z_5 + \ldots) + \alpha_9 (z_2 z_3 z_6 + \ldots), \\ J^{(-)} &= \alpha_7 (z_3^2 z_2 + \ldots) + \alpha_8 (z_4^2 z_6 + \ldots) + \alpha_9 (z_3 z_4 z_1 + \ldots), \end{aligned} \tag{A.5}$$

where "..." are two more terms obtained from the preceding one by applying S^2 and S^4 [note that **all** cubic terms are listed in Eq. (A.3–5)]. It is easy to verify:

$$S: I^{(\mu)} \leftrightarrow J^{(\mu)} \quad \text{for} \quad \mu = -, 0, +,$$

and

$$T: \begin{cases} I^{(\mu)} \to I^{(\mu)}(-\beta^{\mu}), \\ J^{(\mu)} \to J^{(\mu)}\beta^{\mu}, \end{cases} \quad \text{with} \quad \beta^6 = 1. \tag{A.6}$$

Any of the three sets $\{I^{(\mu)} = J^{(\mu)} = 0\}$ is a candidate. To proceed, one needs to examine the fixed points of $Z_6^S \otimes Z_6^T$.

A fixed point is defined by the relation:

$$z_k = \lambda \mathcal{O}(z_k), \qquad \forall \lambda \in C: |\lambda| = 1, \tag{A.7}$$

where $\mathcal{O}$ is any operator of the respective symmetry group. It is easy to verify that $S \cdot T - T \cdot S$ is identity [up to an overall phase which is irrelevant because of Eq. (A.1)] and thus S and T commute on any complex projective space. For this reason they span a direct product $Z_6^S \otimes Z_6^T$ and $\mathcal{O}$ can be written in the form $S^p \cdot T^q$; for the corresponding fixed point the notation $\langle pq \rangle$ shall be used. Now since $S^6 = T^6 = \mathrm{Id}$, the order n of $\mathcal{O}$ (for which $\mathcal{O}^n = \mathrm{Id}$) can be $1, \ldots, 6$, and the corresponding $\langle pq \rangle$ is a $6/n$-parameter subspace of $\mathcal{Y}$ (noting that $n = 4, 5 \cong n = -2, -1$).

This leads to 36 fixed points, but not all of them have to be examined. It follows that 1-parameter subspaces have to be subspaces of 2-, and 3-parameter ones and therefore do not require analysis on their own. So the following ones remain: $\langle 30 \rangle$, $\langle 33 \rangle$, $\langle 03 \rangle$, $\langle 20 \rangle$, $\langle 40 \rangle$, $\langle 02 \rangle$, $\langle 04 \rangle$, $\langle 22 \rangle$, $\langle 24 \rangle$, $\langle 44 \rangle$, $\langle 42 \rangle$. One can verify that $\langle (6-p)(6-q) \rangle$ are the same as $\langle pq \rangle$ explicitly, or showing that there is an $\mathcal{O}$ that transforms one into another. By straightforward calculation one finds e.g.:

$$\begin{aligned} \langle 20 \rangle &= (z_1, z_2, z_1\beta^{4n}, z_2\beta^{4n}, z_1\beta^{2n}, z_2\beta^{2n}) \cong \langle 40 \rangle|_{m=-n}, \\ \langle 40 \rangle &= (z_1, z_2, z_1\beta^{2m}, z_2\beta^{2m}, z_1\beta^{4m}, z_2\beta^{4m}) \cong \langle 20 \rangle|_{n=-m}, \end{aligned} \tag{A.8}$$

and similarly for the other pairs. (When deriving the expressions for $\langle pq \rangle$ it becomes manifest that $|\lambda| = 1$.) Thus the relevant fixed points are:

$$\begin{aligned} \langle 30 \rangle_{\pm} &= (z_1, z_2, z_3, \pm z_1, \pm z_2, \pm z_3), \\ \langle 03 \rangle_{1,2} &= (z_1, 0, z_3, 0, z_5, 0), (0, z_2, 0, z_4, 0, z_6), \\ \langle 33 \rangle_{\pm} &= (z_1, z_2, z_3, \pm iz_1, \mp iz_2, \pm iz_3), \\ \langle 20 \rangle_n &= (z_1, z_2, z_1\beta^{4n}, z_2\beta^{4n}, z_1\beta^{2n}, z_2\beta^{2n}), \\ \langle 02 \rangle_n &= (z_1, 0, 0, z_4, 0, 0), (0, z_2, 0, 0, z_5, 0), (0, 0, z_3, 0, 0, z_6), \\ \langle 22 \rangle_n &= (z_1, z_2, z_1\beta^{-2n}, z_2\beta^{4-2n}, z_1\beta^{2-4n}, z_2\beta^{4-4n}), \\ \langle 42 \rangle_n &= (z_1, z_2, z_1\beta^{2-4n}, z_2\beta^{4-4n}, z_1\beta^{2-2n}, z_2\beta^{-2n}), \qquad n = 1, 2, 3. \end{aligned} \tag{A.9}$$

One can simplify the task even more by noting that if $\langle 03 \rangle_+$ is excluded from $\mathcal{Y}$ by the most general S- and T-invariant constraint-polynomials, so is $\langle 03 \rangle_-$. This, however, means that one has to examine only one of each type of the fixed points given above. This becomes obvious after observing that there is always a linear transformation:

$$\zeta_j = \Omega_j^k z_k : \begin{cases} T(z) \to S(\zeta), & \text{or} \\ T(z) \to S \cdot T(\zeta). \end{cases} \tag{A.10}$$

But then note also that $\langle 22 \rangle_2 = \langle 42 \rangle_{-1}$, and so only six fixed points would need to be examined.

By Eq. (A.10), if it is possible to avoid $\langle 02 \rangle$ and $\langle 03 \rangle$ all the others are avoidable. There is, however, a subtlety to this conclusion, namely that the linear combination of the coordinates induces linear combinations of the parameters α_i, and one wishes to verify that all the fixed points are avoidable for a certain choice of α_i. In general, one can argue that this happens since the α-space is ten complex-dimensional; for avoiding every fixed point there will be one relation to be avoided, yielding two ten complex-dimensional subspaces of which at least one is open; the allowed region of the α-space is then the intersection of these pairs of subspaces, quite unlikely to be empty. We shall, however, explicitly verify this.

Note that $\langle 03 \rangle$ cannot be avoided in any of the choices of Eq. (A.3–5) and one is forced to take their sum:

$$I := I^{(0)} + I^{(+)} + I^{(-)}, \qquad J := J^{(0)} + J^{(+)} + J^{(-)}. \tag{A.11}$$

This reduces Z_6^T to Z_2^T and the relevant fixed points are: $\langle 30 \rangle_+$, $\langle 03 \rangle_1$, $\langle 33 \rangle_-$, and $\langle 20 \rangle_3$ ($\langle 23 \rangle$ and $\langle 43 \rangle$ are subspaces of $\langle 20 \rangle$). Substituting these into Eq. (A.11) one obtains relations of the same form for the first two:

$$\begin{aligned} I(\text{f.p.}) &= A z_1 z_2 z_3 + B(z_1^3 + z_2^3 + z_3^3) \\ &\quad + C(z_1 z_3^2 + z_3 z_2^2 + z_2 z_1^2) + D(z_1 z_2^2 + z_2 z_3^2 + z_3 z_1^2), \end{aligned} \tag{A.12}$$

and slightly different for the third one:

$$\begin{aligned} I(\text{f.p.}) &= A z_1 z_2 z_3 + B(z_1^3 + i z_2^3 + z_3^3) \\ &\quad + C(z_1 z_3^2 - z_3 z_2^2 - i z_2 z_1^2) + D(z_1 z_2^2 + i z_2 z_3^2 - z_3 z_1^2), \end{aligned} \tag{A.13}$$

and J are proportional to I or vanish identically. The coefficients A, B, C, D are given as:

$$\begin{array}{llll} & \langle 30 \rangle_+ & \langle 03 \rangle_1 & \langle 33 \rangle_- \\ A = & (1 + 3\alpha_3) & 1 & -i(1 - 3\alpha_3), \\ B = & (\alpha_1 + \alpha_2) & \alpha_1 & -(\alpha_1 - \alpha_2), \\ C = & (\alpha_4 + \alpha_5 + \alpha_6) & \alpha_5 & -(\alpha_4 - \alpha_5 + \alpha_6), \\ D = & (\alpha_7 + \alpha_8 + \alpha_9) & \alpha_8 & \quad (\alpha_7 - \alpha_8 + \alpha_9). \end{array} \tag{A.14}$$

Using the fact that α_i are z-independent and imposing the non-vanishing of A, B, C, D, Eq. (A.13, 14) can be fullfilled only if each z-term vanishes. That, however, leads to $z = 0$, $\notin CP^n$. [It should be cautioned that "irrelevant" fixed points in Eq. (A.9) lead to expressions similar but not necessarily same as the "relevant" ones. $\langle 30 \rangle_-$ would lead, e.g., to a relation of the type of Eq. (A.12, 13), with $(1 + 3\alpha_3)$ for A

and some different signs but the same argument will apply as for the $\langle 30 \rangle_+$. The $(1-3\alpha_3)$ term is obtained already in the analysis of $\langle 33 \rangle_-$ and hence manifestly irrelevant. Note, however, that even a new relation would only split the allowed region of α_3, leaving still at least one open subregion.]

The fourth fixed point yields:

$$\begin{aligned} &I(\langle 20\rangle_0)=3[(\alpha_1+\alpha_5+\alpha_8+\tfrac{1}{3})z_1^2+(\alpha_2+\alpha_3+\alpha_4+\alpha_6+\alpha_7+\alpha_9)z_2^2]z_1\,, \\ &J(\langle 20\rangle_0)=I(\langle 20\rangle_0)|_{z_1\leftrightarrow z_2}\,. \end{aligned} \tag{A.15}$$

If say $z_2=0$ but $z_1 \neq 0$, then the vanishing of I is avoided by imposing:

$$\alpha_1+\alpha_5+\alpha_8 \neq -\tfrac{1}{3}\,. \tag{A.16a}$$

If both z_1 and z_2 are nonzero, one needs to impose:

$$(\alpha_1+\alpha_5+\alpha_8+\tfrac{1}{3})^2 \neq (\alpha_2+\alpha_3+\alpha_4+\alpha_6+\alpha_7+\alpha_9)^2\,. \tag{A.16b}$$

Thus, for generic α_i such that no A, B, C, D in Eq. (A.14) vanishes and that Eq. (A.15, 16) hold, $\mathscr{Y}$ contains no fixed points of $Z_6^S \otimes Z_2^T$.

By using $\partial^k := \partial/\partial z_k$, Eqs. (A.2) and (A.6) one shows:

$$S:(\partial^i I \partial^j J=\partial^j I \partial^i J) \;\Rightarrow\; (\partial^{i+1} I \partial^{j+1} J=\partial^{j+1} I \partial^{i+1} J)\,. \tag{A.17}$$

To prove non-singularity of $\mathscr{Y}$ [i.e. that Eq. (4.7b) is fullfilled for no point of $\mathscr{Y}$] one needs to verify only that $\partial^{[1} I \partial^{i]} J \neq 0$, $i=2,3,4$; the rest of verification of non-singularity of $\mathscr{Y}$ is guaranteed by Eq. (A.17). To do so, one can even reduce the parameter space to α_1, α_5, α_8 only. Their values are constrained only by:

$$\alpha_1 \neq 0\,, \qquad \alpha_5 \neq 0\,, \qquad \alpha_8 \neq 0\,, \qquad \alpha_1+\alpha_5+\alpha_8 \neq -\tfrac{1}{3}\,.$$

With this reduction, $\partial^{[1} I \partial^{3]} J \equiv 0$, but the other two yield:

$$\begin{aligned} &[z_3z_5+3\alpha_1z_1^2+\alpha_5(z_3^2+2z_1z_5)+\alpha_8(z_5^2+2z_1z_3)] \\ &\quad\times[z_4z_6+3\alpha_1z_2^2+\alpha_5(z_4^2+2z_2z_6)+\alpha_8(z_6^2+2z_2z_4)]=0\,, \\ &[z_3z_5+3\alpha_1z_1^2+\alpha_5(z_3^2+2z_1z_5)+\alpha_8(z_5^2+2z_1z_3)] \\ &\quad\times[z_2z_6+3\alpha_1z_4^2+\alpha_5(z_6^2+2z_2z_4)+\alpha_8(z_2^2+2z_4z_6)]=0\,. \end{aligned} \tag{A.18}$$

Choosing say the right-hand side (odd-subscripted z) bracket to vanish, this implies $z_1=z_3=z_5=0$ for generic α_i and thus $z=\langle 03\rangle_2$, which was proven to be out of $\mathscr{Y}$. Similar argument implies for any other solution to Eq. (A.18), showing non-singularity of $\mathscr{Y}$. Dividing out the free action of $\mathscr{D}=Z_6^S \otimes Z_2^T$ from $\mathscr{Y}$, a $\mathscr{D}$-multiply connected $\chi_E=-12$ Calabi-Yau manifold is obtained. Its $b_2=b_{1,1}^{\mathbf{1}}+b_{1,1}^{\mathbf{8}}$ is known [12] to be one, and so apart from the gravity and $E_6 \otimes E_8$ gauge $N=1$ supermultiplets there are chiral multiplets: seven **27**'s, one **27***, and seven C, seven φ, one ϕ, and $N^\circ(\mathbf{15})\varphi'$ gauge-singlets.

The fact that non-singularity was provable for a very restricted set of α's is suggesting that further discrete symmetries may be found with free action and thus lead to a Calabi-Yau manifold with $\chi_E = -4$ or -6.

Appendix B

This appendix presents an algorithm for generating the matrices of the degrees of homogeneity of the constraints that define the algebraic subvarieties discussed in Sect. 5. They are explicitly constructed for cases of embeddings in the products of up to four CP^n's.

Matrices of degrees of homogeneity for the constructions are generated most easily by using the observations that led to Eq. (5.3, 4). Arranging degrees with respect to a constraint in a column, the rows can be $(mn-3)$-digit numbers (hereafter simply "numbers"), with the sum of digits being $n+1$. To obtain inequivalent matrices one can start assigning the top row the highest number, the next-to-top row the next-to-highest number and so forth, provided that the sums of entries in every column satisfy Eq. (5.4). To prevent doubling that comes from permutations of columns, after all possibilities of a certain number in the first row (by changing numbers in lower rows) have been exhausted, one prevents using any permutation of that number in any of the further cases (obviously a computer may be of great help). Some doubling may still occur, but that can easily be resolved upon calculating χ_E and checking cases where it is equal.

Now we tabulate the degrees of homogeneity for the few low-m cases. The notation will be:

$$\left(\begin{array}{c\|ccc} n & q_1^1 & \cdots & q_{mn-3}^1 \\ \vdots & \vdots & \iddots & \vdots \\ n & q_1^m & \cdots & q_{mn-3}^m \end{array}\right)_{\chi_E}.$$

With this notation, the $\mathscr{Y}(n; q_a)$ manifolds of [12] become:

$$(4 \,\|\, 5)_{-200}, \quad (5 \,\|\, 4 \quad 2)_{-176}, \quad (5 \,\|\, 3 \quad 3)_{-144},$$
$$(6 \,\|\, 3 \quad 2 \quad 2)_{-144}, \quad (7 \,\|\, 2 \quad 2 \quad 2 \quad 2)_{-128},$$

and those of [16]:

$$\left(\begin{array}{c\|c} 2 & 3 \\ 2 & 3 \end{array}\right)_{-162}, \quad \left(\begin{array}{c\|ccc} 3 & 3 & 0 & 1 \\ 3 & 0 & 3 & 1 \end{array}\right)_{-18}, \quad \left(\begin{array}{c\|cccc} 3 & 3 & 0 & 0 & 1 \\ 4 & 0 & 2 & 2 & 1 \end{array}\right)_{-24}.$$

Note that the last example is embedded in $CP^3 \times CP^4$, which is not covered by the generalizations presented here. Embeddings in direct products of unequal-dimensional CP's can be listed and analyzed using similar methods as here. In particular, note that a linear constraint on CP^n reduces it to CP^{n-1}. With this in mind, it is clear that:

$$\left(\begin{array}{c\|cccc} 3 & 3 & 0 & 0 & 1 \\ 4 & 0 & 2 & 2 & 1 \end{array}\right)_{-24} \equiv \left(\begin{array}{c\|ccccc} 4 & 3 & 1 & 0 & 0 & 1 \\ 4 & 0 & 0 & 2 & 2 & 1 \end{array}\right)_{-24}.$$

This generalization may lead to, at least partial, classification of the Calabi-Yau manifolds – a task far too ambitious to be addressed here.

In the case of two CP's, we find nine more possibilities:

$$\left(\begin{array}{c\|ccc} 3 & 3 & 0 & 1 \\ 3 & 1 & 2 & 1 \end{array}\right)_{-120}, \quad \left(\begin{array}{c\|ccc} 3 & 3 & 0 & 1 \\ 3 & 0 & 2 & 2 \end{array}\right)_{-48}, \quad \left(\begin{array}{c\|ccc} 3 & 2 & 1 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right)_{-128},$$

$$\begin{pmatrix} 3 \| 2 & 1 & 1 \\ 3 \| 1 & 2 & 1 \end{pmatrix}_{-106}, \quad \begin{pmatrix} 3 \| 2 & 1 & 1 \\ 3 \| 0 & 2 & 2 \end{pmatrix}_{-104}, \quad \begin{pmatrix} 3 \| 2 & 0 & 2 \\ 3 \| 0 & 2 & 2 \end{pmatrix}_{-128},$$

$$\begin{pmatrix} 4 \| 2 & 2 & 0 & 0 & 1 \\ 4 \| 0 & 0 & 2 & 2 & 1 \end{pmatrix}_{-32}, \quad \begin{pmatrix} 4 \| 2 & 0 & 1 & 1 & 1 \\ 4 \| 0 & 2 & 1 & 1 & 1 \end{pmatrix}_{-88},$$

$$\begin{pmatrix} 4 \| 1 & 1 & 1 & 1 & 1 \\ 4 \| 1 & 1 & 1 & 1 & 1 \end{pmatrix}_{-100}.$$

In the case of three CP's we find:

$$\begin{pmatrix} 2 \| 3 & 0 & 0 \\ 2 \| 2 & 1 & 0 \\ 2 \| 0 & 1 & 2 \end{pmatrix}_{-144}, \quad \begin{pmatrix} 2 \| 3 & 0 & 0 \\ 2 \| 1 & 2 & 0 \\ 2 \| 1 & 0 & 2 \end{pmatrix}_{-144}, \quad \begin{pmatrix} 2 \| 3 & 0 & 0 \\ 2 \| 1 & 2 & 0 \\ 2 \| 0 & 1 & 2 \end{pmatrix}_{-36},$$

$$\begin{pmatrix} 2 \| 3 & 0 & 0 \\ 2 \| 1 & 1 & 1 \\ 2 \| 1 & 1 & 1 \end{pmatrix}_{-108}, \quad \begin{pmatrix} 2 \| 3 & 0 & 0 \\ 2 \| 1 & 1 & 1 \\ 2 \| 0 & 2 & 1 \end{pmatrix}_{-36}, \quad \begin{pmatrix} 2 \| 2 & 1 & 0 \\ 2 \| 2 & 1 & 0 \\ 2 \| 1 & 0 & 2 \end{pmatrix}_{-132},$$

$$\begin{pmatrix} 2 \| 2 & 1 & 0 \\ 2 \| 2 & 1 & 0 \\ 2 \| 0 & 1 & 2 \end{pmatrix}_{-96}, \quad \begin{pmatrix} 2 \| 2 & 1 & 0 \\ 2 \| 2 & 0 & 1 \\ 2 \| 1 & 1 & 1 \end{pmatrix}_{-120}, \quad \begin{pmatrix} 2 \| 2 & 1 & 0 \\ 2 \| 2 & 0 & 1 \\ 2 \| 0 & 2 & 1 \end{pmatrix}_{-84},$$

$$\begin{pmatrix} 2 \| 2 & 1 & 0 \\ 2 \| 1 & 2 & 0 \\ 2 \| 1 & 0 & 2 \end{pmatrix}_{-96}, \quad \begin{pmatrix} 2 \| 2 & 1 & 0 \\ 2 \| 1 & 1 & 1 \\ 2 \| 1 & 1 & 1 \end{pmatrix}_{-96}, \quad \begin{pmatrix} 2 \| 2 & 1 & 0 \\ 2 \| 1 & 1 & 1 \\ 2 \| 0 & 2 & 1 \end{pmatrix}_{-84},$$

$$\begin{pmatrix} 2 \| 2 & 1 & 0 \\ 2 \| 1 & 1 & 1 \\ 2 \| 0 & 1 & 2 \end{pmatrix}_{-66}, \quad \begin{pmatrix} 2 \| 2 & 1 & 0 \\ 2 \| 1 & 0 & 2 \\ 2 \| 0 & 2 & 1 \end{pmatrix}_{-48}, \quad \begin{pmatrix} 2 \| 1 & 1 & 1 \\ 2 \| 1 & 1 & 1 \\ 2 \| 1 & 1 & 1 \end{pmatrix}_{-90},$$

and

$$\begin{pmatrix} 3 \| 1 & 1 & 1 & 1 & 0 & 0 \\ 3 \| 1 & 1 & 0 & 0 & 1 & 1 \\ 3 \| 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}_{-72}.$$

In the case of four CP's we find:

$$(1 \quad 1 \quad 1 \quad 1 \,\|\, 2 \quad 2 \quad 2 \quad 2)_{-128},$$

where the transposed form is used for convenience, and:

$$\begin{pmatrix} 2 \| 2 & 1 & 0 & 0 & 0 \\ 2 \| 1 & 1 & 1 & 0 & 0 \\ 2 \| 0 & 0 & 1 & 2 & 0 \\ 2 \| 0 & 0 & 0 & 1 & 2 \end{pmatrix}_{-8}, \quad \begin{pmatrix} 2 \| 2 & 1 & 0 & 0 & 0 \\ 2 \| 1 & 1 & 1 & 0 & 0 \\ 2 \| 0 & 0 & 1 & 1 & 1 \\ 2 \| 0 & 0 & 0 & 1 & 2 \end{pmatrix}_{-8},$$

$$\begin{pmatrix} 2 \| 2 & 1 & 0 & 0 & 0 \\ 2 \| 1 & 1 & 1 & 0 & 0 \\ 2 \| 0 & 0 & 1 & 1 & 1 \\ 2 \| 0 & 0 & 1 & 1 & 1 \end{pmatrix}_{-24}, \quad \begin{pmatrix} 2 \| 2 & 1 & 0 & 0 & 0 \\ 2 \| 1 & 0 & 2 & 0 & 0 \\ 2 \| 0 & 1 & 0 & 1 & 1 \\ 2 \| 0 & 1 & 0 & 1 & 1 \end{pmatrix}_{-24},$$

$$\begin{pmatrix} 2 & \| & 2 & 1 & 0 & 0 & 0 \\ 2 & \| & 1 & 1 & 1 & 0 & 0 \\ 2 & \| & 0 & 0 & 2 & 1 & 0 \\ 2 & \| & 0 & 0 & 0 & 1 & 2 \end{pmatrix}_{-32}, \quad \begin{pmatrix} 2 & \| & 2 & 1 & 0 & 0 & 0 \\ 2 & \| & 1 & 1 & 1 & 0 & 0 \\ 2 & \| & 0 & 0 & 1 & 2 & 0 \\ 2 & \| & 0 & 0 & 1 & 0 & 2 \end{pmatrix}_{-32},$$

$$\begin{pmatrix} 2 & \| & 2 & 1 & 0 & 0 & 0 \\ 2 & \| & 1 & 0 & 2 & 0 & 0 \\ 2 & \| & 0 & 2 & 0 & 1 & 0 \\ 2 & \| & 0 & 0 & 0 & 1 & 2 \end{pmatrix}_{-32}, \quad \begin{pmatrix} 2 & \| & 2 & 1 & 0 & 0 & 0 \\ 2 & \| & 1 & 0 & 2 & 0 & 0 \\ 2 & \| & 0 & 1 & 0 & 2 & 0 \\ 2 & \| & 0 & 1 & 0 & 0 & 2 \end{pmatrix}_{-32},$$

$$\begin{pmatrix} 2 & \| & 3 & 0 & 0 & 0 & 0 \\ 2 & \| & 1 & 1 & 1 & 0 & 0 \\ 2 & \| & 0 & 1 & 1 & 1 & 0 \\ 2 & \| & 0 & 0 & 0 & 1 & 2 \end{pmatrix}_{-36}, \quad \begin{pmatrix} 2 & \| & 2 & 1 & 0 & 0 & 0 \\ 2 & \| & 1 & 0 & 2 & 0 & 0 \\ 2 & \| & 0 & 1 & 0 & 1 & 1 \\ 2 & \| & 0 & 0 & 1 & 1 & 1 \end{pmatrix}_{-36},$$

$$\begin{pmatrix} 2 & \| & 2 & 1 & 0 & 0 & 0 \\ 2 & \| & 0 & 1 & 2 & 0 & 0 \\ 2 & \| & 0 & 1 & 0 & 1 & 1 \\ 2 & \| & 0 & 0 & 1 & 1 & 1 \end{pmatrix}_{-48}, \quad \begin{pmatrix} 2 & \| & 2 & 1 & 0 & 0 & 0 \\ 2 & \| & 1 & 1 & 1 & 0 & 0 \\ 2 & \| & 0 & 1 & 0 & 1 & 1 \\ 2 & \| & 0 & 0 & 1 & 1 & 1 \end{pmatrix}_{-54},$$

$$\begin{pmatrix} 2 & \| & 2 & 1 & 0 & 0 & 0 \\ 2 & \| & 1 & 0 & 2 & 0 & 0 \\ 2 & \| & 0 & 1 & 1 & 1 & 0 \\ 2 & \| & 0 & 0 & 0 & 1 & 2 \end{pmatrix}_{-56}, \quad \begin{pmatrix} 2 & \| & 1 & 1 & 1 & 0 & 0 \\ 2 & \| & 1 & 1 & 1 & 0 & 0 \\ 2 & \| & 1 & 0 & 0 & 1 & 1 \\ 2 & \| & 0 & 1 & 0 & 1 & 1 \end{pmatrix}_{-60},$$

$$\begin{pmatrix} 2 & \| & 2 & 1 & 0 & 0 & 0 \\ 2 & \| & 0 & 1 & 1 & 1 & 0 \\ 2 & \| & 0 & 1 & 1 & 0 & 1 \\ 2 & \| & 0 & 0 & 0 & 2 & 1 \end{pmatrix}_{-60}, \quad \begin{pmatrix} 2 & \| & 2 & 1 & 0 & 0 & 0 \\ 2 & \| & 1 & 1 & 1 & 0 & 0 \\ 2 & \| & 0 & 1 & 1 & 1 & 0 \\ 2 & \| & 0 & 0 & 0 & 1 & 2 \end{pmatrix}_{-68},$$

$$\begin{pmatrix} 2 & \| & 2 & 1 & 0 & 0 & 0 \\ 2 & \| & 0 & 1 & 1 & 1 & 0 \\ 2 & \| & 0 & 1 & 1 & 0 & 1 \\ 2 & \| & 0 & 0 & 1 & 1 & 1 \end{pmatrix}_{-68}, \quad \begin{pmatrix} 2 & \| & 2 & 1 & 0 & 0 & 0 \\ 2 & \| & 0 & 1 & 1 & 1 & 0 \\ 2 & \| & 0 & 0 & 2 & 0 & 1 \\ 2 & \| & 0 & 0 & 0 & 2 & 1 \end{pmatrix}_{-68},$$

$$\begin{pmatrix} 2 & \| & 1 & 1 & 1 & 0 & 0 \\ 2 & \| & 1 & 1 & 0 & 1 & 0 \\ 2 & \| & 1 & 0 & 1 & 0 & 1 \\ 2 & \| & 0 & 1 & 0 & 1 & 1 \end{pmatrix}_{-68}, \quad \begin{pmatrix} 2 & \| & 1 & 1 & 1 & 0 & 0 \\ 2 & \| & 1 & 1 & 0 & 1 & 0 \\ 2 & \| & 1 & 1 & 0 & 0 & 1 \\ 2 & \| & 0 & 0 & 1 & 1 & 1 \end{pmatrix}_{-72},$$

$$\begin{pmatrix} 2 & \| & 2 & 1 & 0 & 0 & 0 \\ 2 & \| & 1 & 1 & 1 & 0 & 0 \\ 2 & \| & 1 & 0 & 0 & 1 & 1 \\ 2 & \| & 0 & 0 & 1 & 1 & 1 \end{pmatrix}_{-72}, \quad \begin{pmatrix} 2 & \| & 2 & 1 & 0 & 0 & 0 \\ 2 & \| & 0 & 1 & 1 & 1 & 0 \\ 2 & \| & 0 & 1 & 1 & 1 & 0 \\ 2 & \| & 0 & 0 & 1 & 0 & 2 \end{pmatrix}_{-72},$$

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}_{-72}, \quad \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}_{-80},$$

$$\begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 2 & 0 \\ 2 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}_{-80}, \quad \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}_{-80},$$

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}_{-84}, \quad \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}_{-84},$$

$$\begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}_{-84}, \quad \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}_{-92},$$

$$\begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}_{-92}, \quad \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 2 & 0 & 1 \\ 2 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}_{-92},$$

$$\begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}_{-96}, \quad \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}_{-96},$$

$$\begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 & 2 & 0 \\ 2 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}_{-104}, \quad \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 & 2 & 0 \\ 2 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}_{-128},$$

and

$$\begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}_{-128}, \quad \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}_{-128}.$$

It should be noted that in general the procedure of Appendix A is applicable to every example and that each degree of homogeneity matrices may correspond to a set of different Calabi-Yau manifolds, provided one finds inequivalent sets of polynomials that have the same degrees. Also, dividing out free actions of different discrete subgroups one obtains distinct Calabi-Yau manifolds, the universal covering space of which is the original manifold.

Finally, let it be remarked that examples with the same Euler characters may in fact be the same manifold, embedded in different bigger spaces; in such a case all other topological invariants would have to be the same. However, from the point of view of a Unified theory model builder this is not so relevant, since the new embedding could make easier the analysis of dividing out the free action of a desired discrete group and thus simplifying the construction of a certain manifold.

Appendix C

In the lack of a conclusive proof of existence of Calabi-Yau manifolds for every choice of m and n in Table 4 and every of the corresponding matrices of degrees of homogeneity, a few brief examples are presented. Note that the first two examples have Euler characters not obtained before, to the best of my knowledge.

1. We start with $CP^3 \times CP^3$:

$$\begin{pmatrix} 3 & \| & 3 & 0 & 1 \\ 3 & \| & 0 & 2 & 2 \end{pmatrix}_{-48} : \; I := \frac{1}{3}\sum_{r=0}^{3} x_r^3 , \qquad J := \frac{1}{2}\sum_{r=0}^{3} y_r^2 , \qquad L := \sum_{r=0}^{3} x_r y_r^2 . \tag{C.1}$$

For notational convenience we define $X := (x_r, y_r)$, i.e.:

$$X^R := \begin{cases} x_r & R = 0,1,2,3 \quad r = R , \\ y_r & R = 4,5,6,7 \quad r = R-4 , \end{cases} \qquad \partial_R := \partial/\partial X^R . \tag{C.2}$$

Then

$$dI = (x_r^2, 0) , \qquad dJ = (0, y_r) , \qquad dL = (y_r^2, 2x_r y_r^2) , \tag{C.3}$$

and the volume-form tensor $V_{RST} := \partial_{[R} I \partial_S J \partial_{T]} L$ becomes:

$$\begin{matrix} -(x_r^2 y_s^2 - x_s^2 y_r^2) y_t , & R,S = 0,1,2,3 , & T = 4,5,6,7 , \\ -2x_r y_s y_t (x_s - x_t) , & R = 0,1,2,3 , & S,T = 4,5,6,7 . \end{matrix} \tag{C.4}$$

Observe now that at least two x's and two y's have to be nonzero for a point in $CP^3 \times CP^3$ to be a solution of Eq. (C.1), i.e. in the manifold we are constructing. Thus we have to examine three cases:

(a) $y_r \neq 0 \; \forall r$. But then, by Eq. (C.4):

$$(x_r/y_r)^2 = (x_s/y_s)^2 \;\Rightarrow\; x_r \neq 0 \;\forall r,s , \qquad x_r = x_s =: x . \tag{C.5}$$

But then $I = \frac{3}{4}x^3 = 0 \Rightarrow x = 0$, that cannot be in CP^3.

(b) $y_0 = 0$, y_1, y_2, $y_3 \neq 0$. Using Eq. (C.4) one gets:

$$\begin{gathered} (x_1/y_1)^2 = (x_2/y_2)^2 = (x_3/y_3)^2 \;\Rightarrow\; x_1, x_2, x_3 \neq 0 , \\ x_0^2 y_1^2 = x_0^2 y_2^2 = x_0^2 y_3^2 = 0 \;\Rightarrow\; x_0 = 0 , \\ x_1 = x_2 = x_3 =: x . \end{gathered} \tag{C.6}$$

But then, similarly, $I = x^3 = 0 \Rightarrow x = 0$, that cannot be in CP^3.

(c) $y_0 = y_3 = 0$, y_1, $y_2 \neq 0$. Analogously as before, one obtains:

$$\begin{gathered} (x_1/y_1)^2 = (x_2/y_2)^2 \;\Rightarrow\; x_1, x_2 \neq 0 , \\ x_0^2 y_1^2 = x_0^2 y_2^2 = x_3^2 y_1^2 = x_3^2 y_2^2 = 0 \;\Rightarrow\; x_0 = x_3 = 0 , \\ x_1 = x_2 =: x . \end{gathered} \tag{C.7}$$

Again, $I=\frac{2}{3}x^3=0 \Rightarrow x=0$, that cannot be in CP^3. This completes the proof.

2. For the next example let us select:

$$\begin{pmatrix} 2 & \| & 3 & 0 & 0 \\ 2 & \| & 1 & 2 & 0 \\ 2 & \| & 0 & 1 & 2 \end{pmatrix}_{-36} : I := \frac{1}{2}\sum_{r=0}^{2} x_r^2 , \quad J := \sum_{r=0}^{2} x_r y_r^2 , \quad L := \sum_{r=0}^{2} y_r z_r^3 . \tag{C.8}$$

In a notation similar to that above, we have:

$$dI=(x_r,0,0)\,, \qquad dJ=(y_r^2, 2x_r y_r, 0)\,, \qquad dL=(0, z_r^3, 3y_r z_r^2)\,. \tag{C.9}$$

Now the volume tensor becomes:

$$\begin{aligned} &-(x_r y_s^2 - x_s y_r^2) z_t\,, \quad R,S=0,1,2\,, \quad T=3,4,5\,,\\ &3(x_r y_s^2 - x_s y_r^2) y_t z_t\,, \quad R=0,1,2\,, \quad S,T=3,4,5\,,\\ &2x_r(x_s y_s z_t^3 - x_t y_t z_s^3)\,, \quad R=0,1,2\,, \quad S,T=3,4,5\,,\\ &6x_r x_s y_s y_t z_t^2\,, \quad R=0,1,2\,, \quad S=3,4,5\,, \quad T=6,7,8\,. \end{aligned} \tag{C.10}$$

Note again that at most one x_r may be nonzero in order for $I=0$ to be solvable in CP^2. Again, by enlisting the possible forms of solutions to $V_{RST}=0$, and it can be easily shown that not one of them satisfies $I=J=K=0$ in $CP^2 \times CP^2 \times CP^2$. This concludes the proof and thus defines a Calabi-Yau manifold with $\chi_E = -36$.

3. The third example:

$$\begin{pmatrix} 2 & \| & 3 & 0 & 0 \\ 2 & \| & 1 & 2 & 0 \\ 2 & \| & 1 & 0 & 2 \end{pmatrix}_{-144} : I := \frac{1}{2}\sum_{r=0}^{2} x_r^2 , \quad J := \frac{1}{2}\sum_{r=0}^{2} y_r^2 , \quad L := \sum_{r=0}^{2} x_r y_r z_r^3 \tag{C.11}$$

is in fact so similar to 2. that the proof of non-singularity even coincides in some steps and can be quickly obtained. This however defines a Calabi-Yau manifold of $\chi_E = -144$.

In all these examples the constraints exhibit huge symmetries. Example 1 possesses a discrete symmetry S_4 (all simultaneous permutations of x_r and y_r) and a Z_2: $y_r \to (-)^r y_r$. Clearly, the action of these cannot be free, since if it was, dividing the original manifold by the free action would yield a multiple connected one with $|\chi_E| \leqq 1$! (This is manifest since the elements of S_4 do not commute with those of Z_2 and the combined group has more than $4! \cdot 2 = 48$ elements.) The constraints of Example 2 are invariant under a Z_{12} defined as follows:

$$Z\colon\ x_r \to x_r e^{ir\pi}\,, \qquad y_r \to y_r e^{ir\pi/2}\,, \qquad z_r \to z_r e^{-ir\pi/6}\,. \tag{C.12}$$

But if it acted freely, dividing out its free action would lead to a manifold of $\chi_E = -3$, which is odd and leads to contradiction. Adding terms to the above constraints may ensure free action of a subgroup and thus a consistent construction of a Calabi-Yau manifold with a phenomenologically interesting χ_E. For the purposes of the present paper the demonstration of constructions suffices.

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Note added in proof. The terms of the $D3+1$ superspotential discussed in Sect. 3 are all volume-integrals over the internal Calabi-Yau manifold. In addition to these, there will in general also exist analogues of surface-terms, integrated over submanifolds of the internal manifold.

The argument concerning the fixed points of order-3 operators in the Appendix A is not complete and the space of solutions indeed contains these fixed points. However, these are isolated tori and are easily "blown-up"; the corresponding analysis will be presented elsewhere.