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The Nonlinear Schrödinger Limit of the Zakharov Equations Governing Langmuir Turbulence

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Abstract. We consider the initial value problem for the Zakharov equations

$$\text{(Z)}\qquad \begin{array}{ll} \frac{1}{\lambda^2} n_{tt} - \Delta(n + |E|^2) = 0 & n(x,0) = n_0(x) \\ & n_t(x,0) = n_1(x) \\ iE_t + \Delta E - nE = 0 & E(x,0) = E_0(x) \end{array}$$

($x \in \mathbb{R}^k, k = 2, 3, t \geqq 0$) which model the propagation of Langmuir waves in plasmas. For suitable initial data solutions are shown to exist for a time interval independent of λ, a parameter proportional to the ion acoustic speed. For such data, solutions of (Z) converge as $\lambda \to \infty$ to a solution of the cubic nonlinear Schrödinger equation

$$\text{(CSE)}\qquad iE_t + \Delta E + |E|^2 E = 0.$$

We consider both weak and strong solutions. For the case of strong solutions the results are analogous to previous results on the incompressible limit of compressible fluids.

I. Introduction

The Zakharov equations [Z, GTWT],

$$\frac{1}{\lambda^2} n_{tt} - \Delta(n + |E|^2) = 0, \tag{1.1}$$

$$iE_t + \Delta E - nE = 0, \tag{1.2}$$

$E: \mathbb{R}_x^k \times \mathbb{R}_t^+ \to \mathbb{C}^k, n: \mathbb{R}_x^k \times \mathbb{R}_t^+ \to \mathbb{R}$, describe the propagation of Langmuir waves in plasmas. The complex vector E denotes the slowly varying envelope of the highly oscillatory electric field, and n is the fluctuation in the ion-density about its equilibrium value. The parameter λ is proportional to the ion acoustic speed. Other physical parameters have been removed by scaling.

Formally letting λ tend to infinity in (1.1) yields the equation $\Delta(n + |E|^2) = 0$, which implies $n = -|E|^2$ if n and $|E|^2$ are square-integrable. Substitution of this

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expression for n into (1.2) yields the cubic nonlinear Schrödinger equation,

$$iE_t + \Delta E + |E|^2 E = 0 \tag{1.3}$$

which has been used to model phenomena when λ is large [Z, GJ]. The limit $\lambda \to \infty$ corresponds to the assumption that the plasma responds instantaneously to variations in the electric field.

Our goal in this paper is to present a justification of this reduction. We show that, for certain initial data, a) solutions of the IVP for (1.1–1.2) exist on a time interval [0, T], where T is independent of λ, and b) solutions of the IVP for (1.1–1.2) converge to solutions of the IVP for (1.3), as $\lambda \to \infty$. This is carried out for two types of solutions. First, for suitable initial data in the Sobolev space H^m with m sufficiently large, the classical solutions of (1.1–1.2) exist for a time T independent of λ and converge pointwise together with some number of derivatives to a classical solution of (1.3) (Sect. 3); this case follows from the Klainerman–Majda theory of singular limits [KM, M]. Second, when k, the number of spatial dimensions, is less than or equal to 3, for suitable (sufficiently small) initial data $n(0,x,\lambda)$ in L^2, $n_t(0,x,\lambda)$ in H^{-1} and $E(0,x,\lambda)$ in H^1, weak solutions to (1.1–1.2) exist globally in time and converge weakly to the H^1 solution of (1.3) (Sect. 4).

Computer simulations of solutions to the Zakharov equations [SZ] suggest that solutions may develop singularities in finite time. This is believed to correspond to the collapse of Langmuir waves and onset of turbulence. In fact, solutions to the "approximating" cubic nonlinear Schrödinger equation develop singularities in finite time for a large class of initial data [GRT]. The hypotheses in our theorems ensure that solutions do not develop such singularities for the time intervals considered.

Notation:

$L^p = L^p(\mathbb{R}^k) = \{f \mid \int |f|^p < \infty\}$

$H^s = H^s(\mathbb{R}^k) = \{f \mid (1+|\xi|^2)^{s/2}\hat{f}(\xi) \in L^2\}$

$<f, g> = \int fg$

$H^s_{\text{loc}} = \{f \mid \phi f \in H^s \text{ for all } \phi \in C_0^\infty\}$

$AC([0,T];X)$ = space of absolutely continuous functions on $[0,T]$ with values in X.

$\bar{f}$ = complex conjugate of f.

$[\alpha]$ = largest integer smaller than or equal to α.

${}^T U$ = the transpose of the matrix U.

Throughout this paper integrals are assumed to be taken over $\mathbb{R}^k$, and all constants $c, c_1, c_2, \ldots$ are independent of λ unless otherwise explicitly indicated.

II. Zakharov Equations as a Dispersive Perturbation of a Symmetric Hyperbolic System

To write (1.1–1.2) as a perturbation of a first order system we define:

$$V \equiv -\frac{1}{\lambda}\Delta^{-1}\nabla n_t, \tag{2.1a}$$

$$Q \equiv n + |E|^2. \tag{2.1b}$$

Then, (1.1–1.2) become

$$Q_t + \lambda \nabla \cdot V - (|E|^2)_t = 0, \tag{2.2}$$

$$V_t + \lambda \nabla Q = 0, \tag{2.3}$$

$$iE_t + \Delta E + |E|^2 E - QE = 0. \tag{2.4}$$

In Sect. four, Eqs. (2.2–2.4) will be used to study weak solutions. To study classical solutions, it is convenient to write (1.1–1.2) as a dispersive perturbation of a quasilinear symmetric hyperbolic system. The remainder of this section will be concerned with deriving this alternative system.

We first multiply (2.4) by $\bar{E}$ and take the imaginary part of the resulting equation to get

$$(|E|^2)_t = (\bar{E}\Delta E - E\Delta \bar{E}). \tag{2.5}$$

Next, we take the gradient of (2.4) and get

$$i\nabla E_t + \Delta \nabla E + |E|^2 \nabla E + (E\nabla \bar{E} + \bar{E}\nabla E)E - Q\nabla E - E\nabla Q = 0. \tag{2.6}$$

Now let $\sqrt{2}E \equiv F + iG$ and $\sqrt{2}\nabla E \equiv H + iL$. Then, use of (2.5) in (2.2) leads to the following system equivalent to (2.2–2.4, 2.6):

$$Q_t + \lambda \nabla \cdot V + F\nabla \cdot L - G\nabla \cdot H = 0, \tag{2.7}$$

$$V_t + \lambda \nabla Q = 0, \tag{2.8}$$

$$F_t + \tfrac{1}{2}(F^2 + G^2)G - QG = -\Delta G, \tag{2.9}$$

$$G_t - \tfrac{1}{2}(F^2 + G^2)F + QF = \Delta F, \tag{2.10}$$

$$H_t - G\nabla G + \tfrac{1}{2}(F^2 + G^2)L + (FH - GL)G - QL = -\Delta L, \tag{2.11}$$

$$L_t + F\nabla Q - \tfrac{1}{2}(F^2 + G^2)H - (FH - GL)F + QH = \Delta H. \tag{2.12}$$

Introducing the $(3k+3)$-component vector function $U = {}^T(Q, V, F, G, H, L)$, Eqs. (2.7–2.12) can be written in the form:

$$U_t + \sum_{j=1}^{k} (A^j(U) + \lambda C^j) U_{x_j} + B(U)U = K\Delta U. \tag{2.13}$$

Here A^j and C^j are symmetric $(3k+3) \times (3k+3)$ matrices, and K is an antisymmetric $(3k+3) \times (3k+3)$ matrix. $B(\cdot)$ and $A^j(\cdot)$ are C^∞, and K and C^j are constant matrices. Note that the factors of $\sqrt{2}$ in the definition of F, G, H and L were introduced so that the matrices $A^j(U)$ are symmetric. The antisymmetric operator $K\Delta$ in (2.13) reflects the dispersive nature of the equations. The special structure of (2.13) will be exploited in the following section on classical solutions.

III. Classical Solutions

We shall consider the IVP for (1.1–1.2) with initial data given by

$$\begin{aligned} n(x, 0, \lambda) &= n_0(x, \lambda), \\ n_t(x, 0, \lambda) &= \nabla_x \cdot f_0(x, \lambda), \\ E(x, 0, \lambda) &= E_0(x, \lambda). \end{aligned} \tag{3.1}$$

Theorem 1. (*Existence of solutions of* (*Z*) *for each* λ *and* á priori *estimates.*) *Let* k *denote the spatial dimension and* $m \geqq [k/2] + 3$.
Suppose

$$\| n_0(\lambda) \|_{H^m} + \frac{1}{\lambda} \| f_0(\lambda) \|_{H^m} + \| E_0(\lambda) \|_{H^{m+1}} \leqq C_1 . \tag{3.2}$$

Then the IVP (*1.1–1.2, 3.1*) *has a unique classical on a time interval* $[0, T]$. *Here,* T *depends on the bound* C_1 *in* (*3.2*) *and in particular, not on* λ. *In addition, the solution* (E, n) *satisfies the estimate*

$$\begin{aligned} &\| E(t,\lambda) \|_{H^{m+1}} + \| E_t(t,\lambda) \|_{H^{m-1}} + \| n(t,\lambda) \|_{H^m} \\ &+ \frac{1}{\lambda} \| n_t(t,\lambda) \|_{H^{m-1}} + \frac{1}{\lambda^2} \| n_{tt}(t,\lambda) \|_{H^{m-2}} \leqq C_2 \end{aligned} \tag{3.3}$$

for all $t \in [0, T]$.

Classical solutions of (1.1–1.2, 3.1), for fixed λ, were constructed in [SS] by different methods than the ones we employ.

Due to the rapid oscillations in solutions that arise as $\lambda \to \infty$, we have, *without* an hypothesis ensuring that we are "near" the nonlinear Schrödinger equation initially, the following convergence

Theorem 2. *Assume the hypotheses of Theorem 1. In addition, suppose* $E_0(\lambda) \to \tilde{E}_0$ *weakly in* H^1. *Then, the solutions* $(n(\lambda), (E(\lambda))$ *of* (*1.1–1.2, 3.1*) *converge weakly to the unique* $AC([0, T]; H^1)$ *weak solution of* (*1.3*) *with initial data* $\tilde{E}_0$.

This result can be proved by the method of proof used for Theorem 5, (see Sect. 4). To ensure strong convergence of $(n(\lambda), E(\lambda))$ to a classical solution of (1.3) we require additional hypotheses, as is seen in

Theorem 3. (*Strong convergence to solutions of* (*1.3*) *as* $\lambda \to \infty$) *Suppose in addition to the hypotheses of Theorem* 1 *that*

$$\lambda \| \nabla (n_0(\lambda) + |E_0(\lambda)|^2) \|_{H^{m-1}} + \| \nabla \cdot f(\lambda) \|_{H^{m-1}} \leqq C_3 , \tag{3.4}$$

and that

$$E_0(\lambda) \to \tilde{E}_0 \text{ as } \lambda \to \infty \text{ strongly in } H^{m+1}. \tag{3.5}$$

Let T *denote the time of existence of the solution constructed in Theorem 1. Then, as* $\lambda \to \infty$

$$n(\lambda) + |E(\lambda)|^2 \to 0 \text{ in } C^0([0, T) \times \mathbb{R}^k), \tag{3.6a}$$

$$\nabla [n(\lambda) + |E(\lambda)|^2] \to 0 \text{ in } C^0([0, T]; H^{m-2}), \text{ and} \tag{3.6b}$$

$$E(\lambda) \to \tilde{E}, \tag{3.6c}$$

the unique solution of (*1.3*) *with initial data* $\tilde{E}_0$ *in*

$$C^1([0, T] \times \mathbb{R}^k) \cap C^1([0, T); C^2).$$

III.A Existence of Solutions to (*1.1–1.2, 3.1*) *for Fixed* λ *and á priori Estimates*

To prove Theorem 1, we first write (1.1–1.2) in the form (2.13) as in Sect. two. The existence proof proceeds along the lines of the existence proof for the IVP for quasilinear symmetric hyperbolic systems (see [KM, M]) with modifications which we shall now outline.

Local existence in time can be established for (2.13) with initial data implied by (3.1) via the following iteration scheme. Define $U^0(x,t)=U_0(x)$, where U_0 denotes the data for (2.13) constructed from (3.1) and

$$\frac{\partial U^{p+1}}{\partial t}+\sum_{j=1}^{K}(A^j(U^p)+\lambda C^j)U^{p+1}_{x_j}+B(U^p)U^{p+1}$$
$$=K\Delta U^{p+1},\quad U^{p+1}(x,0)=U_0(x)\quad p=0,1,2,\cdots. \tag{3.7}$$

The existence of solutions to (3.7) for each p follows from a natural extension of the existence theory for linear symmetric hyperbolic systems.

To prove that the iteration scheme is well defined and to prove convergence of the iterates $\{U^p\}$ to a unique classical solution, we need á priori estimates on the space derivatives of the following type

$$\| U^p(t,\lambda)\|_{H^m}\leqq C_4 \tag{3.8}$$

for all $t\in[0,T]$. This would imply by (3.7) that $\| U^p_t(t,\lambda)\|_{H^{m-2}}\leqq C_5(\lambda)$. This ensures that for fixed λ, we have convergence of $\{U^p\}$ in $C^0([0,T];H^{m-2}_{\text{loc}})$ by the Ascoli–Arzela theorem, and therefore in $C^0([0,T];H^{m-\delta}_{\text{loc}})$ by interpolation. It now follows from (3.7) that $\{\partial U^p/\partial t\}$ also converges as $p\to\infty$, and the limit U satisfies (3.8).

That T and C_4 can be chosen independent of λ is seen as follows. First, (3.8) holds at time $t=0$, by (3.2) applied to the initial data in (3.7). Since $V_0=-(1/\lambda)\Delta^{-1}\nabla\nabla\cdot f_0(\lambda)$, we have used here the fact that the operator $\Delta^{-1}\partial^2/\partial x_i\partial x_j$ is bounded on H^m. That (3.8) holds on $(0,T)$ for some T independent of λ follows from *energy estimates* obtained by taking the L^2 inner product of U^{p+1} with (3.7). As noted in [KM, M], the λ dependence contributes nothing to the estimate since $\lambda^T U^{p+1}C^jU^{p+1}_{x_j}$ is a perfect derivative because C^j is symmetric and constant.

Also, the term $\int^T UK\Delta U=-\int^T(\nabla U)K\nabla U=0$ contributes nothing to the estimate, by the antisymmetry of K. Derivative estimates of U^p are obtained by differentiating (3.7), taking the inner product of the resulting equation with the corresponding derivative of U^p, and using the above observations.

Note also that we do not require estimates for $U^p_t(\cdot,\lambda)$ independent of λ since the term U^{p+1}_t is not multiplied by a matrix $A^0(U)$ as in [KM, M].

The result is a solution U on a time interval $[0,T]$, with T independent of λ satisfying

$$\| U(t,\lambda)\|_{H_m}\leqq C_4 \tag{3.9}$$

for $t\in[0,T]$. Although (3.9) implies $U\in L^\infty([0,T];H^m)$, it can be shown [M] that in fact $U\in C^0([0,T];H^m)\cap C^1([0,T];H^{m-2})$.

This result is pulled back to the system (1.1–1.2) as follows: First, Eqs. (2.3–2.4) follows directly from (2.8–2.10), where we defined $E=(1/\sqrt{2})(F+iG)$. Next, an L^2 energy estimate for $W\equiv(\nabla F-H,\nabla G-L)$ implies $\| W(t)\|^2_{L^2}\leqq e^{CT}\| W(0)\|^2_{L^2}=0$, and therefore that $\nabla E=(1/\sqrt{2})(H+iL)$. This implies that the sum of the last two terms in (2.7) is equal to the right-hand side of (2.5). Since (2.5) follows from (2.4), (2.2) holds. Setting $n \equiv Q - |E|^2$, we have that

$$n_t + \lambda \nabla \cdot V = 0. \tag{3.10}$$

Q and W are then smooth enough for (2.3) and (3.10) to imply (1.1). Clearly, (2.4) implies (1.2). Estimate (3.3) follows similarly.

III.B. The Cubic Schrödinger Limit ($\lambda \to \infty$) for Classical Solutions

To prove Theorem 3, we shall first prove (3.6a, b) using (2.8). We require a bound on $U_t(t, \lambda)$ independent of λ. To obtain this bound we differentiate the system (2.13) with respect to t and derive energy estimates for $\| U_t(t, \lambda) \|_{H^{m-2}}$ analogous to those of Sect. 3.A (again using that the C^j are symmetric and constant, and that K is antisymmetric). We obtain a differential inequality for $\| U_t(t, \lambda) \|_{H^{m-2}}$ that has no explicit λ-dependence. Also the initial data satisfies the estimate

$$\| U_t(0, \lambda) \|_{H^{m-2}} \leqq C \tag{3.11}$$

by hypothesis (3.4). It follows that

$$\| U_t(t, \lambda) \|_{H^{m-2}} \leqq C_5$$

for $t \in [0, T]$. Note that unlike solutions to the systems considered in [KM, M], U_t is only in H^{m-2} and not H^{m-1} due to the presence of the higher order $K \Delta U$ term in (2.13). Use of (3.11) in Eq. (2.8) implies that

$$\lambda \| \nabla Q(t, \lambda) \|_{H^{m-2}} \leqq C_6 \tag{3.12}$$

for $t \in [0, T]$, and hence that

$$\| Q \|_{C^0([0,T] \times \mathbb{R}^k)} \leqq C \| D^{m-1} Q \|_{L^2}^{k/2(m-1)} \| Q \|_{L^2}^{1-k/2(m-1)} \leqq C_1 \lambda^{-k/2(m-1)} \tag{3.13}$$

for $t \in [0, T]$ by the Gagliardo–Nirenberg inequalities (see for example [F]), (3.12), and (3.2).

Now, since (F, G, H, L) is bounded in $C^0([0, T]; H^m) \cap C^1([0, T]; H^{m-2})$, by the Arzela–Ascoli theorem (applied in the time variable), the Rellich compactness theorem (applied in the space variables) and interpolation, we have that for every sequence of λ's tending to infinity $(F(\lambda), G(\lambda), H(\lambda), L(\lambda))$ has a subsequence that converges in $C^0([0, T]; H^{m-\varepsilon}_{\mathrm{loc}})$, for $\varepsilon > 0$, to $(\tilde{F}, \tilde{G}, \tilde{H}, \tilde{L})$. By (3.13) and Eqs. (2.9–2.12), the convergence takes place as well in $C^1([0, T]; H^{m-2-\varepsilon}_{\mathrm{loc}})$. Thus, the subsequence converges to a solution of the system obtained from (2.9–2.12) by setting $Q = 0$. By (3.5) this limiting solution has initial data $(\mathrm{Re}\tilde{\mathrm{E}}_0, \mathrm{Im}\tilde{\mathrm{E}}_0, \nabla \mathrm{Re}\tilde{\mathrm{E}}_0, \nabla \mathrm{Im}\tilde{\mathrm{E}}_0)$. Also, perhaps after passing to a second subsequence, the above limit converges weakly in H^m (as a bounded sequence in a Hilbert space). Therefore, by the identity of weak and strong limits, $(\tilde{F}, \tilde{G}, \tilde{H}, \tilde{L}) \in L^\infty([0, T]; H^m) \cap AC([0, T]; H^{m-2})$.

Now, uniqueness of $L^\infty([0, T]; H^m) \cap AC([0, T]; H^{m-2})$ solutions to the IVP for (2.7–2.12) follows from a straightforward energy estimate for the difference of two solutions. It follows that the convergence to $(\tilde{F}, \tilde{G}, \tilde{H}, \tilde{L})$ takes place without passing to subsequences. Theorem 3 now follows upon interpreting this result for the system (1.1–1.2)

IV. Weak Solutions

The main tool in establishing the existence of global weak solutions are the conserved integrals of (1.1–1.2). We define

$$N = \| E \|_{L^2}^2, \text{ and} \tag{4.1}$$

$$H = \| \nabla E \|_{L^2}^2 - \tfrac{1}{2} \| E \|_{L^4}^4 + \tfrac{1}{2} \| Q \|_{L^2}^2 + \tfrac{1}{2} \| V \|_{L^2}^2. \tag{4.2}$$

N and H can be shown, using (2.2–2.4) the definitions (2.1), to be constant on sufficiently smooth solutions of (1.1–1.2) (see for example [GTWT]).

Let $\psi(x)$ denote the spatial profile of the ground state solution of (1.3), i.e. $u = \psi$ is the unique positive H^1 solution of $\Delta u - u + u^3 = 0$ (see [W]). We denote by k the number of spatial dimensions, $1 \leqq k \leqq 3$. We first have for each fixed $\lambda \neq 0$, the following weak existence

Theorem 4. *Consider the initial value problem for (1.1–1.2) with data given by (3.1) for which the functionals N and H are finite,* i.e. $E_0 \in H^1$, $n_0 \in L^2$ *and* $n_t|_{t=0} \in H^{-1}$. *Suppose*

$$\| E_0(\lambda) \|_{L^2} \leqq c' < \| \psi \|_{L^2} \approx (2\pi)^{1/2} (1.8662)^{1/2}, \quad \text{if } k = 2, \text{ and} \tag{4.3}$$

$$\| \nabla E_0(\lambda) \|_{L^2} \leqq c'' \leqq |H| \text{ and } N|H| < \frac{4}{27(24)^2} \text{ if } k = 3. \tag{4.4}$$

Then, for any fixed λ the system (1.1–1.2, 3.1) has a global weak solution for which $n \in L^\infty(\mathbb{R}^+; L^2)$, *and* $E \in L^\infty(\mathbb{R}^+, H^1)$. *If, in addition,*

$$\| n_0(\lambda) \|_{L^2} + \frac{1}{\lambda} \| f_0(\lambda) \|_{L^2} + \| E_0(\lambda) \|_{H^1} \leqq c_1, \tag{4.5}$$

then

$$\| n(t, \lambda) \|_{L^2} + \frac{1}{\lambda} \| n_t(t, \lambda) \|_{H^{-1}} + \frac{1}{\lambda^2} \| n_{tt}(t, \lambda) \|_{H^{-1}}$$
$$+ \| E(t, \lambda) \|_{H^1} + \| E_t(t, \lambda) \|_{H^{-1}} \leqq c_2. \tag{4.6}$$

Weak solutions to (1.1–1.2, 3.1) were constructed for fixed λ in [SS]. The criterion (4.3) was shown in [W] to ensure global existence of H^1 solutions to (1.3).

As for classical solutions, due to rapid oscillations as $\lambda \to \infty$, we have the following weak convergence

Theorem 5. *Assume in addition to the hypothesis of Theorem 4 that*

$$\| \nabla \cdot f_0(\lambda) \|_{H^{-1}} \leqq c_3, \textit{ and} \tag{4.7}$$

$$E_0(\lambda) \to \tilde{E}_0 \textit{ strongly in } H^1 \textit{ as } \lambda \to \infty. \tag{4.8}$$

Then, as $\lambda \to \infty$

$$n(\lambda) + |E(\lambda)|^2 \to 0 \textit{ weak-* in } L^\infty(\mathbb{R}^+; L^2)$$

and for any $\varepsilon > 0$ and $1 < p < \infty$,

$$E(\lambda) \to \tilde{E} \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}^+; H^{1-\varepsilon}_{\mathrm{loc}}),$$

where $\tilde{E}$ is the unique $AC(\mathbb{R}^+; H^1)$ solution of (1.3) with initial data $\tilde{E}_0$.

IV A. Existence of Weak Solutions for Fixed λ and á priori Estimates

We now prove Theorem 4. As in [SS], for the case of fixed λ, we use the Galerkin method. Let $\{w^j\}_{j=1}^{\infty}$ denote a smooth orthonormal basis for $L^2(\mathbb{R}^k)$, and denote by P^m the L^2 projection operator onto span $\{w^j\}_{j=1}^{m}$. We shall seek approximate solutions in the form:

$$Q^m = \sum_{j=1}^{m} q_{mj}(t)w^j, \quad V^m = \sum_{j=1}^{m} v_{mj}(t)w^j, \quad \text{and } E^m = \sum_{j=1}^{m} e_{mj}(t)w^j,$$

where Q^m, V^m and E^m solve the system

$$Q_t^m + P^m(\lambda\nabla\cdot V^m - (|E^m|^2)_t) = 0, \tag{4.9$_m$}$$

$$V_t^m + P^m(\lambda\nabla Q^m) = 0, \tag{4.10$_m$}$$

$$iE_t^m + P^m(\Delta E + |E^m|^2 E^m - Q^m E^m) = 0, \tag{4.11$_m$}$$

with initial data

$$\begin{aligned} Q^m(0) &= P^m(n_0(\lambda) + |E_0(\lambda)|^2), \\ V^m(0) &= P^m\left(-\frac{1}{\lambda}\Delta^{-1}\nabla\nabla\cdot f_0(\lambda)\right), \\ E^m(0) &= P^m(E_0(\lambda)). \end{aligned} \tag{4.12$_m$}$$

These are simply P^m applied to the system (2.2–2.4, 3.1). The system $(4.9\text{–}4.12)_m$ can be solved for $\{dq_{mj}/dt, dv_{mj}/dt, de_{mj}/dt\}_{j=1}^{m}$ to yield a system of $(2k+1)m$ nonlinear ordinary differential equations in time, which have a solution on some maximal time interval $[0, T_m]$.

To pass to the limit $m \to \infty$ we shall need á priori estimates on the sequence of solutions to the finite dimensional problems $(4.9\text{–}4.12)_m$, independent of m. These are obtained by observing that the system $(4.9\text{–}4.12)_m$ inherits the conserved integrals N and H of the exact system (2.2–2.4). To see this, we take the (real) L^2 inner product of $(4.11)_m$ with $\bar{E}^m$ and then take the imaginary part of the resulting identity. Using that P^m is self-adjoint and equal to the identity on span $\{w^j\}_{j=1}^{m}$ we conclude that the functional N is constant on E^m. Next, adding twice the real part of the inner product of $-\bar{E}_t^m$ with $(4.11)_m$ to the sum of the inner products of V^m with $(4.10)_m$ and Q^m with $(4.9)_m$ implies that H is constant.

As shown in [SS], the constancy of N and H imply

$$\|Q^m(t,\lambda)\|_{L^2} + \|V^m(t,\lambda)\|_{L^2} + \|E^m(t,\lambda)\|_{H^1} \leqq c(\lambda) \tag{4.13}$$

without additional conditions if $k = 1$, and provided (4.4) holds, when $k = 3$. When $k = 2$ [SS] shows that (4.13) holds if $N < 1/\sqrt{2}$; the sharper criterion (4.3) follows from the calculation of the optimal constant in a Gagliardo–Nirenberg interpolation inequality [W]. The criterion (4.3) is a sharp condition ensuring that the functional formed from sum of the first two terms in H is positive definite.

When (4.13) holds $T_m = \infty$, for all m. For, if $T_m < \infty$, the local existence theory for

ODE's would imply

$$\limsup_{t \uparrow T_m} \sup_{1 \leqq j \leqq m} \{ |q_{m_j}(t)|, |v_{m_j}(t)|, |e_{m_j}(t)| \} = \infty .$$

This would contradict (4.13).

We shall now establish the sense in which the solution of $(4.9–4.12)_m$ approaches a solution of (2.1–2.4, 3.1) as $m \to \infty$. Fix a positive integer M, and multiply $(4.9–4.11)_m$ respectively by test functions $\phi^{(1)} w^{(1)}, \phi^{(2)} w^{(2)}$ and $\phi^{(3)} w^{(3)}$, where $\phi^{(i)}(t)$ vanishes for all sufficiently large t, and $w^{(i)} \in \operatorname{span} \{w^j\}_{j=1}^M$. Upon integrating the resulting equation over $\mathbb{R}_t^+ \times \mathbb{R}_x^k$ we obtain a *weak form of* $(4.9–4.11)_m$ which we call $(4.9–4.11)_m^M$ where we can omit the projections P^m for $m \geqq M$.

Passing to the limit $m \to \infty$ requires some observations on compactness of (Q^m, V^m, E^m), which we now make. First, studying an equation for the time-Fourier transform of $\chi_{[0,T]} E_m(t)$, using that $\Delta E^m + |E^m|^2 E^m - Q^m E^m$ is uniformly bounded in $L^\infty(\mathbb{R}^+, H^{-1})$, it can be shown that $\{E^m\}$ is bounded in $H^s_{\mathrm{loc}}(\mathbb{R}^+; L^2)$ for $s < \frac{1}{4}$. (See the argument used to prove (3.32) in chapter 3 of [T]. We cannot solve $(4.11)_m$ for E_t^m to obtain regularity of E^m in time, since P^m is not bounded independently of m on H^{-1}.) It follows from compactness Theorem 2.2 of chapter 3 in [T] that there exists a subsequence, still denoted (Q^m, V^m, E^m), and functions $Q, V \in L^\infty(\mathbb{R}^+; L^2)$ and $E \in L^\infty(\mathbb{R}^+; H^1)$, such that $(Q^m, V^m, E^m, \nabla E^m) \to (Q, V, E, \nabla E)$ weak-* in $L^\infty(\mathbb{R}^+; L^2)$, and that $E^m \to E$ in $L^2_{\mathrm{loc}}(\mathbb{R}^+; H^{1-\varepsilon}_{\mathrm{loc}})$ for any $\varepsilon > 0$. Since E^m is bounded in $L^\infty(\mathbb{R}^+; H^{1-\varepsilon})$, E^m converges as well in $L^p_{\mathrm{loc}}(\mathbb{R}^+; H^{1-\varepsilon}_{\mathrm{loc}})$ for any $2 \leqq p < \infty$, by interpolation.

To pass to the limit in the above-defined weak form $(4.9–4.11)_m^M$ we remark that for each M, $\{w^j\}_{j=1}^M$ have uniform decay at infinity. Due to this uniform decay of the test functions we can pass to the limit $m \to \infty$, using the weak-* convergence for the terms linear in Q^m, V^m and E^m, and the $L^p_{\mathrm{loc}}(\mathbb{R}^+; H^{1-\varepsilon})$ convergence of E^m plus the weak-* convergence of Q^m for the nonlinear terms. The limit along this subsequence (Q, V, E) satisfies

$$\int dt \int dx [-\phi_t^{(1)} w^{(1)} Q - \lambda \phi^{(1)} \nabla w^{(1)} \cdot V - \phi_t^{(1)} w^{(1)} |E|^2] = \int dx \, \phi^{(1)}(0) w^{(1)} n_0(\lambda), \tag{4.14}$$

$$\int dt \int dx [-\phi_t^{(2)} w^{(2)} V(\lambda) - \lambda \nabla \cdot w^{(2)} Q(\lambda)] = -\int dx \phi^{(2)}(0) w^{(2)} - \frac{1}{\lambda} \Delta^{-1} \nabla \nabla \cdot f_0(\lambda), \tag{4.15}$$

$$\begin{aligned} &\int dt \int dx [-i\phi_t^{(3)} w^{(3)} E(\lambda) - \phi^{(3)} \nabla w^{(3)} \nabla E(\lambda) + \phi^{(3)} w^{(3)} (|E(\lambda)|^2 E(\lambda) - Q(\lambda) E(\lambda))] \\ &\quad = \int dx \phi^{(3)} w^{(3)} E_0(\lambda). \end{aligned} \tag{4.16}$$

Since the span of the set of functions $\phi(t) w(x)$ with the above properties is dense in $C_0^1(\mathbb{R}^+, H^1)$, and since the functionals in (4.14–4.16) are continuous in the topology of $C_0^1(\mathbb{R}^+; H^1)$, (4.14–4.16) hold if $\phi^{(i)} w^{(i)}$ are replaced by any functions in $C_0^1(\mathbb{R}^+; H^1)$, i.e. (Q, V, E) is a weak solution of the IVP (1.1–1.2, 3.1). Furthermore if (4.5) holds, then N and H are bounded independently of λ, and hence the constant $C(\lambda)$ in (4.13) can be taken to be independent of λ.

Since the H^1 norm is lower-semicontinuous under weak convergence, we have from (4.13) with c independent of λ,

$$\|Q(t)\|_{L^2} + \|V(t)\|_{L^2} + \|E(t)\|_{H^1} \leqq C. \tag{4.17}$$

The á priori estimate (4.6) now follows from (4.17) and the differential equations for n and E. This completes the proof of Theorem 4.

IVB. Passage to the Limit $\lambda \to \infty$ for Weak Solutions

To prove Theorem 5 we shall now make some preliminary observations on compactness of sequences $(Q(\lambda), V(\lambda), E(\lambda))$ as $\lambda \to \infty$. By (4.17) and (2.4), $E_t(\lambda)$ is bounded uniformly in λ in $L^\infty(\mathbb{R}^+, H^{-1})$, and therefore $E(\lambda)$ is uniformly bound in $H^1_{\mathrm{loc}}(\mathbb{R}^+, H^{-1})$. As in Sect. IVA, applying the compactness Theorem 2.2 of Chap. 3 in [T], for every sequence λ_j tending to infinity there is a subsequence (still denoted λ_j), and $(\tilde{Q}, \tilde{V}) \in L^\infty(\mathbb{R}^+; L^2), \tilde{E} \in L^\infty(\mathbb{R}^+, H^1)$, and $\tilde{E}_0 \in H^1$ such that $(Q(\lambda), V(\lambda), E(\lambda), \nabla E(\lambda)) \to (\tilde{Q}, \tilde{V}, \tilde{E}, \nabla\tilde{E})$ weak-* in $L^\infty(\mathbb{R}^+, L^2), E(\lambda) \to \tilde{E}$ in $L^p_{\mathrm{loc}}(\mathbb{R}^+, H^{1-\varepsilon})$ for all $p < \infty$, and $E_0(\lambda) \to \tilde{E}_0$ in $H^{1-\varepsilon}_{\mathrm{loc}}$.

We can now take this limit $\lambda_j \to \infty$ along a subsequence in (4.16) with $\phi^{(3)}w^{(3)}$ replaced by an arbitrary function in $C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^k)$. Using the weak-* convergence to control the linear terms and the $L^p_{\mathrm{loc}}(\mathbb{R}^+; H^{1-\varepsilon})$ convergence of $E(\lambda)$ and the weak-* convergence of $Q(\lambda)$ to control the nonlinear terms we obtain

$$\int dt \int dx [-i\psi_t \tilde{E} + \Delta\psi \tilde{E} + \psi |\tilde{E}|^2 \tilde{E} - \psi \tilde{Q}\tilde{E}] = \int dx\, \psi(0)\tilde{E}_0. \tag{4.18}$$

(We cannot take the limit of the Q or V equations in general, because of the factors of λ.)

Now, if $\tilde{Q} \equiv 0$, then by (4.18) $\tilde{E}$ is a weak solution of (1.3) with initial data $\tilde{E}_0$. As we shall see below, $L^\infty([0,T], H^1)$ solutions of (1.3) are unique. Therefore, by (4.8) and the above convergence along subsequences, $E(\lambda) \to \tilde{E}$, the solution of (1.3) with initial data $\tilde{E}_0$, without passing to subsequences. We now conclude the proof of Theorem 5 by showing (1) $\tilde{Q} = 0$ and (2) $L^\infty(0, T; H^1)$ solutions of (1.3) are unique.

First let $\Lambda^\Theta \equiv (I - \Delta)^{\Theta/2}$, and define

$$R(x, t, \lambda) = \int_0^t d\tau \int_0^\tau d\sigma \Lambda^{-s} Q(x, \sigma, \lambda). \tag{4.19}$$

(We work with this time-integrated form so that we have sufficient regularity in time to derive a differential inequality.) Then we have

$$R_{tt} - \lambda^2 \Delta R = \Lambda^{-s} Q_0 - \lambda t \Lambda^{-s} \nabla \cdot V_0 \equiv F(\lambda), \tag{4.20}$$

$$R|_{t=0} = 0, \quad R_t|_{t=0} = 0. \tag{4.21}$$

Now, by (4.7) and (2.1a)

$$\|F(\lambda)\|_{L^\infty(0,T;H^{s-1})} \leqq c, \tag{4.22}$$

where c is independent of λ.

Lemma. *Let u be the solution of*

$$u_{tt} - \lambda^2 \Delta u = g, \quad u(x,0) = u_t(x,0) = 0,$$

where $\|g\|_{L^\infty([0,T];L^2)} \leqq c$. *Then,*

$$\sup_{0 \leqq t \leqq T} \|\nabla u(t,\lambda)\|_{L^2} = O\left(\frac{1}{\lambda}\right) \text{ as } \lambda \to \infty.$$

Proof. Multiplication of the equation for u by u_t implies

$$\frac{d}{dt}[\|u_t(t)\|_{L^2}^2 + \lambda^2 \|\nabla u(t)\|_{L^2}^2] \leqq [\|u_t(t)\|_{L^2}^2 + \lambda^2 \|\nabla u(t)\|_{L^2}^2] + \|g(t)\|_{L^2}^2.$$

The result now follows from Gronwall's inequality.

Applying the lemma to $R(x,t,\lambda)$ with $s=2$, to satisfy the hypothesis, we have

$$\left\|\int_0^t d\tau \int_0^\tau d\sigma \nabla Q(\sigma,\lambda)\right\|_{H^{-2}} \leqq \frac{C(T)}{\lambda} \tag{4.23}$$

for $t \in [0,T]$. Therefore, for any $\psi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^k)$, with time support in some fixed $[0,T]$,

$$\left|\int dt \int dx (\nabla \cdot \psi) Q(\lambda)\right| = \left|\int dt \int dx \left[\nabla \cdot \psi_{tt} \int_0^t d\tau \int_0^\tau d\sigma Q(\sigma,\lambda)\right]\right|$$
$$\leqq T \|\psi\|_{C^2(\mathbb{R}^+;H^2)} \left\|\int_0^t d\tau \int_0^\tau d\sigma \nabla Q(\sigma,\lambda)\right\|_{L^\infty([0,T];H^{-2})} \leqq \frac{C(T)}{\lambda}.$$

Hence, $\int dx \int dt (\nabla \cdot \psi) Q(\lambda) \to 0$ for all such ψ. Now the set $\{\nabla \cdot \psi \mid \psi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^k)$ is dense in $L^1(\mathbb{R}^+, L^2)$, so $Q(\lambda) \to 0$ weak-* in $L^\infty(\mathbb{R}^+; L^2)$.

It remains to prove that solutions to the initial value problem for (1.3) of the same regularity as $\tilde{E}$, are unique. If $\tilde{E} \in L^\infty(0,T;H^1)$ solves (1.3), then $\tilde{E}_t = i\Delta\tilde{E} + i|\tilde{E}|^2\tilde{E} \in L^\infty(\mathbb{R}^+; H^{-1})$. Therefore $\tilde{E} \in AC(\mathbb{R}^+, H^{-1})$, and by interpolation $\tilde{E} \in C(\mathbb{R}^+, H^{1-\varepsilon})$.

Now define $F(t)$ by

$$F(t) = e^{-i\Delta t}\tilde{E}_0 - i\int_0^t e^{-i\Delta(t-\sigma)}|\tilde{E}(\sigma)|^2 \tilde{E}(\sigma) d\sigma. \tag{4.24}$$

$F(t)$ is differentiable with respect to t with values in $H^{-k/2-2-\varepsilon}$ since $|\tilde{E}|^2\tilde{E} \in L^1$. Also, $F_t = -i\Delta F - i|\tilde{E}|^2\tilde{E}$ and $F(0) = \tilde{E}_0$. Therefore, $\tilde{E} - F$ satisfies the linear Schrödinger equation with zero initial data and we conclude $F \equiv \tilde{E}$. So $\tilde{E}$ satisfies (4.24) with the left-hand side equal to $\tilde{E}(t)$. The local existence theory in [GV] implies that $C^0([0,T];L^4)$ solutions of (4.24) are unique. Since for ε sufficiently small $H^{1-\varepsilon}$ is embedded in L^4, we have uniqueness.

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Note added in proof. Using techniques similar to those employed in [CP] we have shown that the regularity of solutions of the nonlinear Schrödinger equation controls that of solutions to the Zakharov equations in the following sense:

Theorem. *Let* $T>0$ *be arbitrary and* $\tilde{E}_0$ *be such that the* IVP *for the nonlinear Schrödinger equation (1.3) has a solution,* $\tilde{E}$, *in* $C^1([0,T)\times\mathbb{R}^k)\cap C^1([0,T);C^2)$. *Let* $m\geqq[k/2]+3$. *Then, there is a critical value of* λ,λ_c, *and a constant* $C>0$ *such that under the hypotheses*

(i) $E_0(\lambda)\to\tilde{E}_0$ *as* $\lambda\to\infty$ *strongly in* H^{m+1}

(ii) $\lambda\|\nabla(n_0(\lambda)+|E_0(\lambda)|^2)\|_{H^{m-1}}+\|\nabla\cdot f(\lambda)\|_{H^{m-1}}\leqq C$, *and*

(iii) $\lambda>\lambda_c$,

the IVP *for the Zakharov equations has a unique classical solution on the time interval* $[0,T]$.

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This is the analogue of a result of P. Constantin showing that in three space dimensions, so long as the solution of the Euler equations for an ideal fluid are smooth, the solution of the *slightly* viscous incompressible Navier Stokes equations are smooth.

(See [CP] P. Constantin, "Note on loss of regularity for solutions of the 3-D incompressible Euler and related equations", Commun. Math. Phys. 1986; and [SSH] S. Schochet, "Hyperbolic–Hyperbolic Singular Limits", preprint. We wish to thank P. Constantin for discussions on this point.)