Commun. Math. Phys. 105, 675–690 (1986)

Communications in Mathematical Physics

Deconfining Phase Transition in the U(1) Model with Wilson's Action

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Abstract. In dimension $d \geqq 4$, the lattice U(1) gauge theory defined with the Wilson action is shown to have a deconfining phase transition at weak coupling. The proof uses a higher dimensional analogue of the Higgs mechanism and a correlation inequality to remove the massless modes of the theory. The remaining modes are controlled by a simple cluster expansion.

1. Introduction

This paper presents a new proof of the deconfining phase transition in the lattice U(1) gauge theory in four or more dimensions. Although this result is not new, our method of proof is quite novel. Furthermore, the proof uses the Wilson form of the action rather than the Villain form which appeared in earlier proofs [7, 4].

In the lattice formulation of gauge theories promoted by Wilson [10], an element of the gauge group is introduced on each bond of a d-dimensional lattice. The ordered product of group elements on the bonds around each plaquette p is written as $\mathrm{U}(p)$, and the Wilson form of the action is defined by

$$S_W = -\beta \sum_p \mathrm{Re}\,\mathrm{Tr}\,\mathrm{U}(p). \tag{1.1}$$

This lattice theory is now a well-defined model in statistical mechanics, and may be analysed in a non-perturbative way. In this paper we shall examine the abelian model, for which the gauge group is U(1). So we write the group element on bond b as $\exp[iA(b)]$, where $A(b) \in [-\pi, \pi)$. Then $\mathrm{U}(p)$ is $\exp[iF(p)]$, where $F(p)$ is the sum over the oriented bonds in plaquette p of the field $A(b)$. The action (1.1) becomes

$$S_W = -\beta \sum_p \cos F(p). \tag{1.2}$$

We are interested in the phase structure of the theory defined by (1.2). To analyse this, we will consider the expectation of a Wilson loop observable

[†] Research supported in part by U.S. National Science Foundation grant PHY 8117463

$W(L) = \mathrm{Re}\,\mathrm{Tr} \prod_{b \in L} U(b) = \cos\left(\sum_{b \in L} A(b)\right)$, where L is a closed loop on the lattice. For β small, a standard strong coupling cluster expansion [9] can be used to prove that

$$\langle W(L) \rangle \leqq \exp[-C\,\mathrm{Area}(L)],$$

where $\mathrm{Area}(L)$ is the area of the minimal surface with boundary L, and the expectation is defined by the action (1.2). Of course this implies confinement of static charges, which is not a desirable property of electrodynamics. So the interesting question is whether for sufficiently large β the action (1.2) has Coulombic behavior, implying in particular perimeter law decay for $\langle W(L) \rangle$. To understand why this is a difficult problem, we can rescale $A(b) \mapsto \beta^{-1/2} A(b)$ and expand the cosine in (1.2). Then up to a constant,

$$S_W = \frac{1}{2} \sum_p |F(p)|^2 + 0(\beta^{-1}),$$

and we see that for large β the action is a perturbation of the massless gaussian action describing a free electromagnetic field. Technically it is extremely difficult to establish rigorous results in such cases, nor is it obvious what behavior is to be expected. For example it has been proven that in the three-dimensional case the action (1.2) never has a Coulombic phase [6].

The first rigorous result concerning Coulombic behavior of U(1) gauge theories was produced by Guth [7]. He proved that a related model defined with the Villain action had perimeter law decay for large β in dimension four or more. Fröhlich and Spencer produced another proof for the Villain action [4] based on their earlier work on the Kosterlitz-Thouless transition in the two-dimensional $X-Y$ model [5]. They also suggested how their proof could be extended to the U(1) model with the Wilson action (1.2). Both these proofs used a duality transformation to replace integrals of continuous fields by sums over integer-valued fields, thereby getting around the difficulty mentioned before concerning massless modes.

This paper presents a new proof of the deconfining phase transition in the U(1) model with the Wilson action in dimension four or more. Furthermore, the method of proof is quite different from the earlier proofs mentioned before. We will use ideas developed during the analysis of the Higgs mechanism in the abelian Higgs model [8]. We introduce an anti-symmetric tensor field $G_{\mu\nu}$ which is coupled gauge-covariantly to the U(1) field. This new theory has a Higgs mechanism which is completely analogous to the usual Higgs mechanism in the abelian Higgs model. In other words, the U(1) field can be "gauged away" by a gauge transformation of the tensor field $G_{\mu\nu}$, producing a mass for the tensor field. In this way the massless modes are removed from the model and replaced by massive modes.

To be more specific, we introduce a field $G(p)$ defined on plaquettes of the lattice (this is equivalent to defining a tensor field $G_{\mu\nu}(x)$). This field is coupled to the U(1) field in the action (1.2) by replacing $F(p)$ by $F(p) - gG(p)$, where g is the coupling constant. We also need a kinetic term for the tensor field, and for this we choose the gauge-invariant term $\frac{1}{2} \sum_c |dG(c)|^2$. Here dG is the exterior derivative of G, which is naturally represented by a function on cubes c on the lattice (in coordinates, dG is $\partial_\mu G_{\nu\lambda} + \partial_\nu G_{\lambda\mu} + \partial_\lambda G_{\mu\nu}$). Ignoring problems of gauge-fixing, the action for the new abelian tensor gauge field theory is

$$S=\tfrac{1}{2}\sum_{c}|dG(c)|^2-\beta\sum_{p}\cos[F(p)-gG(p)]\,. \tag{1.3}$$

This action is invariant under the simultaneous change of variables [recall that $F(p)=dA(p)$]

$$A(b)\mapsto A(b)+A'(b)\,,\qquad G(p)\mapsto G(p)+\frac{1}{g}\,dA'(p)\,. \tag{1.4}$$

This invariance is the analogue of the usual gauge invariance of the abelian Higgs model. Furthermore, we can use (1.4) to "gauge away" the A-field and produce the action

$$S=\tfrac{1}{2}\sum_{c}|dG(c)|^2-\beta\sum_{p}\cos[gG(p)]\,, \tag{1.5}$$

which for βg^2 large describes a massive tensor field. This is the Higgs mechanism.

In previous work on the abelian Higgs model [8], a "smeared string" observable was used to analyse the phase structure of the model. In our abelian tensor model there is a natural analogue of this, which can be thought of as a "smeared surface" observable. Given any closed loop L on the bonds of the lattice, let S be a surface (a connected set of plaquettes) with boundary L. Then there is an obvious gauge invariant observable supported on S and L:

$$D(S)=\exp\left[i\sum_{b\in L}A(b)-ig\sum_{p\in S}G(p)\right].$$

To see that $D(S)$ is invariant under (1.4), we note that

$$\sum_{p\in S}dA'(p)=\sum_{b\in L}A'(b)\,.$$

We can see this in a slightly different way by writing

$$\sum_{p\in S}G(p)=\sum_{p}G(p)S(p)\,,$$

where $S(p)$ (or $S_{\mu\nu}(x)$) is a tensor field supported on the surface S. The requirement of gauge invariance is then

$$\sum_{p}dA'(p)S(p)=\sum_{b}A'(b)h(b)\,,$$

where $h(b)$ (or $h_\mu(x)$) is a vector field supported on the loop L. In other words, our tensor field S must satisfy

$$\partial_\mu S_{\mu\nu}(x)=-h_\nu(x)\,. \tag{1.6}$$

Now Eq. (1.6) has many solutions for $S_{\mu\nu}(x)$, corresponding to different choices of boundary conditions. In particular, there is a "smeared surface" solution:

$$J_{\mu\nu}(x)=\sum_{y}[\partial_\mu V(x-y)h_\nu(y)-\partial_\nu V(x-y)h_\mu(y)]\,, \tag{1.7}$$

where $V(x-y)$ is the kernel of the inverse of the lattice Laplacian. The reader can check that (1.7) satisfies (1.6) (remember that $\partial_\mu h_\mu=0$), and therefore the following observable is gauge invariant:

$$H(J)=\exp\left[i\sum_{b\in L}A(b)-ig\sum_{p}G(p)J(p)\right].$$

(We have written $J(p)$ instead of $J_{\mu\nu}(x)$).

In order to see the usefulness of $H(J)$, we may repeat the argument leading to (1.5). This leads us to the expectation of the observable $\exp\left[-ig\sum_{p} G(p)J(p)\right]$ in the measure defined by (1.5). Taking βg^2 large and expanding the cosine to quadratic order, we may approximately compute this and get

$$\exp\left[-\tfrac{1}{2}g^2 \sum_{p,p'} J(p)K(p,p')J(p')\right]. \tag{1.8}$$

The covariance $K(p,p')$ has a mass $g\sqrt{\beta}$. When this is large, we may approximate (1.8) by

$$\exp\left[-\frac{1}{2}g^2\frac{1}{\beta g^2}\sum_{p} J(p)^2\right],$$

and using integration by parts we see

$$\sum_{p} J(p)^2 = \sum_{x,y,\mu} h_\mu(x)V(x-y)h_\mu(y). \tag{1.9}$$

In dimension four or more, (1.9) behaves as $\mathrm{Per}(L)$, the length of the loop L, and so $\langle H(J)\rangle$ decays with a perimeter law. Furthermore, when β is small we will see that $\langle H(J)\rangle$ decays as the *area* of the minimal surface spanning L. Therefore, $\langle H(J)\rangle$ gives us a gauge-invariant way of determining the phase structure of the abelian tensor gauge theory.

It is worth pointing out that had we used the observable $D(S)$ instead, the same calculation would have led to

$$\exp\left[-\frac{1}{2}\frac{1}{\beta g^2}\sum_{p\in S} g^2\right].$$

Even in the putative "Coulombic" phase, the expectation $\langle D(S)\rangle$ has area law decay, and is of no immediate use in determining the phase structure of the theory.

At this point the reader may be wondering how these results apply to the U(1) model. Recall that when $g=0$, the abelian tensor gauge theory reduces to the U(1) model. Furthermore, the expectations of observables behave in a particularly simple way when g is sent to zero. In Appendix A we prove that the expectation of $H(J)$ in the measure defined by (1.3) is monotonic decreasing in the coupling g. Therefore, the expectation of the Wilson loop in the U(1) model is bounded from below by $\langle H(J)\rangle$ for all values of β. When β is large, $\langle H(J)\rangle$ should have perimeter law decay, implying the same behavior for the Wilson loop. In the remainder of this paper we will make this argument rigorous.

The introduction of local gauge symmetry is an attractive way of removing the massless modes associated with a continuous symmetry. In principle this idea could be applied to models with non-abelian symmetries. The difficulty is finding an analogue for the correlation inequality which we use in this paper to turn on the coupling constant g.

The paper is organized as follows. In Sect. 2 we define the model and state the results. In Sect. 3 we derive the Higgs mechanism for the tensor gauge field, and in Sect. 4 we use a cluster expansion to prove perimeter law decay. In Appendix A the correlation inequality is proved and Appendix B contains a technical construction needed in the proof of perimeter law decay.

2. Definition of the Model and Results

We consider the model on a finite, closed rectangular lattice Λ with unit spacing in d dimensions. The p-cells on Λ are written Λ_p. So Λ_0 are sites, Λ_1 are bonds, Λ_2 are plaquettes etc., and a p-form is a function on Λ_p. The inner product of two p-forms f and g is written (f, g). The lattice exterior derivative d maps p-forms into $(p+1)$-forms and its adjoint with respect to $(\cdot,\cdot)$ is written d^*.

The U(1) field is a 1-form $A(b)$, $b \in \Lambda_1$, taking values in $[-\pi, \pi)$. The abelian tensor gauge field $G(p)$ is a 2-form defined on plaquettes p in Λ_2, and taking values in $(-\infty, \infty)$. From these fields we form the action

$$\begin{aligned} S(G, A) = \frac{1}{2}(dG, dG) + \frac{1}{2\alpha}(d^*G, d^*G) \\ -\beta \sum_{p \in \Lambda_2} \cos[dA(p) - gG(p)]. \end{aligned} \tag{2.1}$$

The 3-form dG is supported on cubes c in Λ_3, so the first term in (2.1) is shorthand for

$$\tfrac{1}{2}(dG, dG) = \tfrac{1}{2} \sum_{c \in \Lambda_3} |dG(c)|^2.$$

Similarly d^*G is a 1-form defined on bonds b in Λ_1, and dA is a 2-form. β and g are the coupling constants of the theory. The action (2.1) has two distinct symmetries; one is the usual gauge invariance of the U(1) field A, displayed by the change of variables

$$A(b) \mapsto A(b) + d\chi(b), \tag{2.2}$$

where χ is any 0-form. In addition there is a new gauge symmetry, corresponding to the change of variables

$$A(b) \mapsto A(b) + A'(b), \qquad G(p) \mapsto G(p) + \frac{1}{g} dA'(p), \tag{2.3}$$

where A' is any 1-form. The action (2.1) is invariant under the change (2.3), except for the gauge-fixing term $\frac{1}{2\alpha}(d^*G, d^*G)$. This term is necessary to define the functional integral, but expectations of gauge-invariant observables are independent of α. Finally, we point out that when $g = 0$ the action (2.1) separates into a quadratic action for G and the Wilson action for the U(1) field:

$$S_W(A) = -\beta \sum_{p \in \Lambda_2} \cos[dA(p)]. \tag{2.4}$$

We define the expectation of an observable $\mathscr{F}(G, A)$ by

$$\langle \mathscr{F}(G, A) \rangle = Z^{-1} \int DGDA \exp[-S(G, A)] \mathscr{F}(G, A). \tag{2.5}$$

The measure is given by

$$\int DGDA = \prod_{p \in \Lambda_2} \int_{-\infty}^{\infty} dG(p) \prod_{b \in \Lambda_1} \int_{-\pi}^{\pi} dA(b),$$

and Z is chosen to normalise the expectation. The equations $dG=0$ and $d^*G=0$ have the unique solution $G=0$ on Λ_2, so the functional integral is well defined. When $g=0$, the model reduces to the U(1) gauge theory with the Wilson action, for which the expectation of an observable $\mathscr{F}(A)$ is

$$\langle \mathscr{F}(A)\rangle_W = Z_W^{-1} \int DA \exp[-S_W(A)]\mathscr{F}(A).$$

We are interested in an observable with the following two properties: it is gauge-invariant, and when $g=0$ it reduces to the Wilson loop $\exp\left[i \sum_{b\in L} A(b)\right]$, where L is a closed loop on the lattice. By introducing a 1-form h supported on the loop L, we can write

$$\sum_{b\in L} A(b) = (A, h). \tag{2.6}$$

The lattice Laplacian on 1-forms is d^*d+dd^*, and we denote its inverse on Λ_1 by C (this is well-defined since d^*d+dd^* has trivial kernel on 1-forms). We define a 2-form on Λ by

$$J = dCh, \tag{2.7}$$

and our gauge-invariant observable is given by

$$H(J) = \exp[i(A, h) - ig(G, J)]. \tag{2.8}$$

The gauge invariance of $H(J)$ follows from the two equations $d^*h=0$ and $d^*J=h$. This choice of observable is motivated by ideas in [8]. We can now state our main theorem, which is proved in Sect. 4.

Theorem 2.1. *For $d \geqq 4$ and any $0<\gamma<\infty$ there are constants $g_0(\gamma)$, $R(\gamma)$ such that for $g<g_0(\gamma)$ and $\beta g^2 > R(\gamma)$*

$$\langle H(J)\rangle \geqq \exp[-\gamma(h, Ch)]. \tag{2.9}$$

In dimension $d \geqq 4$, (h, Ch) grows like the length of L, so (2.9) implies perimeter law decay for $\langle H(J)\rangle$ when g is small and βg^2 is large. To relate this result to the U(1) model, we use the following proposition.

Proposition 2.2.

$$\frac{d}{dg}\langle H(J)\rangle \leqq 0. \tag{2.10}$$

The correlation inequality (2.10) is almost identical to a result given in [3], and the proof is sketched in Appendix A.

When β is small, we can use Proposition 2.2 to bound $\langle H(J)\rangle$ from above by $\langle \exp[i(A, h)]\rangle_W$, the expectation of the Wilson loop in the U(1) model. We know that this has area law decay for small β, so we deduce the following result.

Corollary 2.3. *In all dimensions for β sufficiently small, $\langle H(J)\rangle \leqq \exp[-c\,\mathrm{Area}(L)]$.*

By combining (2.9) and (2.10) we deduce the desired result concerning the U(1) model.

Theorem 2.4. *For $d \geqq 4$ and any $0<\gamma<\infty$, there is a constant $\beta_0(\gamma) = R(\gamma)/g_0(\gamma)^2$ such that for $\beta > \beta_0(\gamma)$*

$$\langle \exp[i(A, h)]\rangle_W \geqq \exp[-\gamma(h, Ch)]. \tag{2.11}$$

Proof of Theorem 2.4. The left-hand side of (2.11) is $\langle H(J)\rangle$ with $g=0$. By (2.10), this is bounded from below by $\langle H(J)\rangle$ with $g=g_0(\gamma)-\varepsilon$, for any $\varepsilon<g_0(\gamma)$. Since $\beta>\beta_0(\gamma)$, we can choose $\varepsilon>0$ so that

$$\beta g^2 > R(\gamma),$$

and then (2.11) follows from (2.9). □

The remainder of this paper is devoted to proving Theorem 2.1. As we shall see, this involves a simple cluster expansion for a model without any massless modes.

3. Higgs Mechanism for Tensor Gauge Field

In this section we will exhibit the Higgs mechanism for the model defined in Sect. 2. By making a gauge transformation, we can remove the U(1) field from the functional integral and display explicitly the mass for the tensor field. However, because the U(1) field is compact, some care must be taken to carry out this transformation, and so we follow the example of Balaban et al. who considered a similar transformation for the abelian Higgs model [1].

We first choose a maximal tree T on Λ_1, the bonds in Λ. Then since $H(J)$ is gauge-invariant, we can set $A(b)=0$ for all bonds $b\in T$. This gives

$$\langle H(J)\rangle = Z^{-1}\int DGDA \prod_{b\in T}\delta(A(b))\exp\left[-\frac{1}{2}(dG,dG)-\frac{1}{2\alpha}(d^*G,d^*G)\right.$$
$$\left.+\beta\sum_{p\in\Lambda_2}\cos[dA(p)-gG(p)]+i(A,h)-ig(G,J)\right], \tag{3.1}$$

where Z in (3.1) differs from the Z defined in (2.5) by a factor 2π for each bond in T. Next we separate G into a compact piece and an integer-valued 2-form by

$$G(p)=G'(p)+\frac{2\pi}{g}v(p), \tag{3.2}$$

where $G'(p)\in\left[-\frac{\pi}{g},\frac{\pi}{g}\right]$ and $v(p)\in\mathbb{Z}$. This allows us to write

$$\langle H(J)\rangle = Z^{-1}\int DA \prod_{b\in T}\delta(A(b))\int_c DG' \sum_v \exp\left[-\frac{1}{2}\left(dG'+\frac{2\pi}{g}q,\, dG'+\frac{2\pi}{g}q\right)\right.$$
$$-\frac{1}{2\alpha}\left(d^*G'+\frac{2\pi}{g}d^*v, d^*G'+\frac{2\pi}{g}d^*v\right)+\beta\sum_{p\in\Lambda_2}\cos[dA(p)-gG'(p)]$$
$$\left.+i(A,h)-ig(G',J)-2\pi i(v,J)\right], \tag{3.3}$$

where $q=dv$ and the subscript "c" is a remainder that the integral is now compact. Since the integrand of the G'-integral (which includes the sum over v) is periodic in each $G'(p)$ with period $\frac{2\pi}{g}$, we can change variables as follows:

$$G'(p)=G''(p)+\frac{1}{g}dA(p).$$

Hence, using $d^*J=h$,

$$\langle H(J)\rangle = Z^{-1}\int DA \prod_{b\in T}\delta(A(b))\int_c DG''\sum_v \times\exp\Bigg[-\frac{1}{2}\left(dG''+\frac{2\pi}{g}q, dG''+\frac{2\pi}{g}q\right) - \frac{1}{2\alpha}\left(d^*G''+\frac{2\pi}{g}d^*v+\frac{1}{g}d^*dA, d^*G''+\frac{2\pi}{g}d^*v+\frac{1}{g}d^*dA\right) + \beta\sum_{p\in\Lambda_2}\cos[gG''(p)] - ig(G'',J) - 2\pi i(v,J)\Bigg]. \tag{3.4}$$

Given any closed, integer-valued 3-form q we consider an integer-valued 2-form v^q with the property

$$dv^q = q. \tag{3.5}$$

It will be shown in Appendix B that such a 2-form always exists. Furthermore, any other integer-valued 2-form v satisfying (3.5) can be written as $v=v^q+dn$, where n is an integer-valued 1-form. This correspondence can be made unique by fixing $n=0$ on the maximal tree T. Since each 2-form v gives rise to a closed 3-form q, we have the relation

$$\sum_v = \sum_{q:dq=0}\sum_n^T,$$

where the superscript "T" indicates that n is fixed to zero on T. Therefore,

$$\langle H(J)\rangle = Z^{-1}\int DA \prod_{b\in T}\delta(A(b)) \sum_{q:dq=0}\int_c DG''\sum_n^T \times\exp\Bigg[-\frac{1}{2}\left(dG''+\frac{2\pi}{g}q, dG''+\frac{2\pi}{g}g\right) + \beta\sum_{p\in\Lambda_2}\cos[gG''(p)] - \frac{1}{2\alpha}\Big(d^*G''+\frac{2\pi}{g}d^*v^q+\frac{1}{g}d^*d(A+2\pi n), d^*G'' + \frac{2\pi}{g}d^*v^q+\frac{1}{g}d^*d(A+2\pi n)\Big) - ig(G'',J) - 2\pi i(v^q,J)\Bigg]. \tag{3.6}$$

We have used the fact that $(dn, J)=(n,h)$ is an integer. Reordering the integrals and defining $\chi=A+2\pi n$, we get

$$\langle H(J)\rangle = Z^{-1}\sum_{q:dq=0}\int_c DG'' \times\exp\Bigg[\beta\sum_{p\in\Lambda_2}\cos[gG''(p)] - \frac{1}{2}\left(dG''+\frac{2\pi}{g}q, dG''+\frac{2\pi}{g}q\right) - ig(G'',J) - 2\pi i(v^q,J)\Bigg]\int D\chi\prod_{b\in T}\delta(\chi(b)) \times\exp\Bigg[-\frac{1}{2\alpha}\Big(d^*G''+\frac{2\pi}{g}d^*v^q+\frac{1}{g}d^*d\chi, d^*G'' + \frac{2\pi}{g}d^*v^q+\frac{1}{g}d^*d\chi\Big)\Bigg]. \tag{3.7}$$

We now want to perform the χ-integral. This is done by finding a 0-form f such that the following 1-form B vanishes on T:

$$B = df + gCd^*G'' + 2\pi Cd^*v^q .$$

We can compute f by integrating the 1-form $gCd^*G'' + 2\pi Cd^*v^q$ along the tree T. Now we change variables to $\chi' = \chi + B$, and the χ-integral in (3.7) becomes

$$\int D\chi' \prod_{b\in T} \delta(\chi'(b)) \exp\left[-\frac{1}{2\alpha g^2} (d^*d\chi', d^*d\chi') \right] \tag{3.8}$$

which is a finite, positive constant [the gauge-fix ensures the absence of zero modes in (3.8)].

Therefore, our final representation for $\langle H(J)\rangle$ is

$$\begin{aligned}\langle H(J)\rangle = Z^{-1} \sum_{q:dq=0} \int_c DG \exp\Bigg[&\beta \sum_{p\in\Lambda_2} \cos[gG(p)] \\ &-\frac{1}{2}\left(dG + \frac{2\pi}{g} q, dG + \frac{2\pi}{g} q \right) - ig(G,J) - 2\pi i(v^q, J) \Bigg].\end{aligned} \tag{3.9}$$

4. Proof of Perimeter Decay

The proof of Theorem 2.1 which we present in this section is very similar to the proof of long-range order for the abelian Higgs model in [8]. The proof uses a simple cluster expansion taken around a product measure. Some of the technical arguments which can be copied verbatim from [8] are omitted here.

We construct a simple polymer expansion for the functional integral in (3.9). In that representation, it is clear that the tensor field $G(p)$ has a mass of order $g\sqrt{\beta}$, and that for g small non-zero values of q are strongly suppressed. We define a product measure for G by

$$d\mu(G) = N \prod_{p\in\Lambda_2} \{dG(p) \exp[\beta \cos gG(p)]\} \tag{4.1}$$

with N chosen to normalise the measure. The expansion will be made about this product measure.

The zeroth order term in the expansion will be $\int d\mu(G) \exp[-ig(G,J)]$.

This factorizes into a product over plaquettes, and gives the required behavior. The idea of the polymer expansion is to write the remainder as a sum of terms, which contain couplings between some plaquettes. Each of these terms then factorizes into a product over sets of plaquettes. Some of these sets may be single plaquettes, which others may contain many plaquettes coupled together. The mass $g\sqrt{\beta}$ gives a large penalty for couplings of the field G between different plaquettes, and any non-zero q-configurations are damped by their action $\frac{1}{2}\left(\frac{2\pi}{g}\right)^2 (q,q)$. These penalties mean that large coupled sets are unlikely, and that the behavior is dominated by the zeroth order term.

It will be natural to consider the terms in our expansion as supported on sets of sites on the lattice. Two such sets X and Y will be *disjoint* if no site in X is the nearest neighbor of a site in Y. Similarly a set X is *connected* if it does not have two disjoint subsets. We also define the *support* of a p-form as the set of all sites belonging to the p-cells on which the p-form is non-zero, and we write $|X|$ for the number of sites in X.

The expectation (3.9) can be rewritten as

$$\langle H(J)\rangle = Z(0)^{-1}Z(J),$$

with

$$Z(J) = \sum_{q:dq=0} \int d\mu(G)\exp\left[-\frac{1}{2}\left(dG+\frac{2\pi}{g}q, dG+\frac{2\pi}{g}q\right) - ig(G,J) - 2\pi i(v^q,J)\right]. \tag{4.2}$$

For every $c\in\Lambda_3$ and $p\in\Lambda_2$, we define

$$1+\varrho(c) = \exp\left[-\frac{1}{2}|dG(c)|^2 - \frac{2\pi}{g}dG(c)q(c)\right], \tag{4.3}$$

$$1+\sigma(p) = \exp[-igG(p)J(p)]. \tag{4.4}$$

Therefore, we have

$$\begin{aligned} Z(J) &= \sum_{q:dq=0}\exp\left[-\frac{1}{2}\left(\frac{2\pi}{g}\right)^2(q,q) - 2\pi i(v^q,J)\right] \\ &\quad \times \int d\mu(G)\prod_{c\in\Lambda_3}[1+\varrho(c)]\prod_{p\in\Lambda_2}[1+\sigma(p)] \\ &= \sum_{q:dq=0}\sum_{Y\subset\Lambda_3}\sum_{Z\subset\Lambda_2} k(q,Y,Z), \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} k(q,Y,Z) &= \exp\left[-\frac{1}{2}\left(\frac{2\pi}{g}\right)^2(q,q) - 2\pi i(v^q,J)\right] \\ &\quad \times \int d\mu(G)\prod_{c\in Y}\varrho(c)\prod_{p\in Z}\sigma(p). \end{aligned} \tag{4.6}$$

In order to proceed, we must factorize each term in (4.5) over its connected components on the lattice. This requires some additional properties of the 2-forms v^q. These properties, some of which will be needed further on, are stated in Lemma 4.1 and proved in Appendix B. Notice that any closed 3-form q can be rewritten as $q=\sum_i q_i$, where each set $\operatorname{supp} q_i$ is connected and $dq_i=0$ for each i. We define $B(X)$ as the smallest rectangular parallelepiped on Λ which contains X, for any set $X\subset\Lambda$.

Lemma 4.1. *For $d\geq 4$ there is a choice of integer-valued 2-forms v^q such that*

(a) $dv^q = q$,

(b) $v^q = \sum_i v^{q_i}$, *where* $q=\sum_i q_i$, $dq_i=0$, *and each set* $\operatorname{supp} q_i$ *is connected,*

(c) $v^{-q} = -v^q$,

(d) $\operatorname{supp} v^q \subset B(\operatorname{supp} q)$,

(e) $\|v^q\|_\infty \leq c(q,q)^3$, *some constant* c.

We now return to (4.5). Given q, Y and Z, we let $X_1, \ldots, X_n \subset \Lambda$ denote the connected components of the set $\bigcup_i B(\operatorname{supp} q_i) \cup Y \cup Z$, where $q = \sum_i q_i$, $dq_i = 0$ and each $\operatorname{supp} q_i$ is connected. We have abused notation by writing Y, Z for the sites contained in the sets Y, Z. By Lemma 4.1(b) and because $d\mu(G)$ is a product measure, the activity (4.6) factorizes over different sets X_i. Indeed we can collect together all the terms corresponding to a choice of connected components $X_1, \ldots, X_n$, and rewrite (4.5) as a sum over connected, disjoint sets $\{X_1, \ldots, X_n\}$:

$$Z(J) = \sum_{n=0}^{\infty} \sum_{\{X_1, \ldots, X_n\}}' \prod_{i=1}^{n} K(X_i, J). \tag{4.7}$$

The primed sum is taken over connected, disjoint sets, and

$$K(X, J) = \sum_{\substack{Y, Z, q \\ X = \bigcup_i B(\operatorname{supp} q) \cup Y \cup Z}} k(q, Y, Z). \tag{4.8}$$

The point is that now $Z(J)$ is the grand canonical partition function for a gas of particles with activities $K(X_i, J)$ interacting with a hard core repulsion. When the activities are sufficiently small, we can represent $\log Z(J)$ by a Mayer series. We state below the required estimates for $K(X, J)$. These bounds are straightforward and are almost identical to estimates proved for similar activities in [8], and so we refer the reader there for proofs.

Lemma 4.2. *For any $M > 0$ there exist $\varepsilon > 0$ and $\mu < \infty$ such that for $g < \varepsilon$ and $g\sqrt{\beta} > \mu$, $|K(X, J)| \leqq \exp(-M|X|)$.*

Then the Mayer expansion for $\ln Z(J)$ is the following:

$$\ln Z(J) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{X_1, \ldots, X_n} \psi_c(X_1, \ldots, X_n) \prod_{i=1}^{n} K(X_i, J). \tag{4.9}$$

The sum over sets $X_1, \ldots, X_n$ in (4.9) is unrestricted. The function $\psi_c(X_1, \ldots, X_n)$ is the connected part of the hard-core interaction between $X_1, \ldots, X_n$, and it vanishes unless $\bigcup_i X_i$ is connected. The derivation of (4.9) and the proof of convergence of the series, given Lemma 4.2, are standard, and we refer the reader to [2, 9] for details.

We are now ready to prove Theorem 2.1. This is done by proving a bound on $\ln Z(J) - \ln Z(0)$, which will then imply a lower bound on $\langle H(J) \rangle$ of the required form. For $s \in [0, 1]$ define

$$f(s) = \ln Z(sJ). \tag{4.10}$$

It follows from (3.9) and Lemma 4.1 (c) that $f(s) = f(-s)$. Therefore,

$$\ln Z(J) - \ln Z(0) = f(1) - f(0) = \int_0^1 ds (1 - s) f''(s). \tag{4.11}$$

Using (4.9) we get

$$f''(s) = \sum_{p, p'} J(p) J(p') m(p, p'), \tag{4.12}$$

with

$$m(p,p')=\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{i,j=1}^{n}\sum_{\substack{X_1,\ldots,X_n:\\ p\in B(X_i),\,p'\in B(X_j)}}\psi_c(X_1,\ldots,X_n)$$

$$\times\left[\prod_{\substack{k=1\\ k\neq i,j}}^{n}K(X_k,sJ)\right]K_p(X_i,sJ)K_{p'}(X_j,sJ)\,. \tag{4.13}$$

We have written $K_p(X,J)=\frac{\partial}{\partial J(p)}K(X,J)$. Also the terms in (4.13) with $i=j$ contain $K_{p,p'}(X_i,J)$, rather than $K_p(X_i,J)K_{p'}(X_i,J)$. Notice that we have used Lemma 4.1 (d) to impose the constraints $p\in B(X_i)$, $p'\in B(X_j)$. In order to estimate (4.13), we will need the following lemma giving bounds on $K_p(X,J)$ and $K_{p,p'}(X,J)$. For these bounds to hold, it is crucial that we have Lemma 4.1 (e), since a derivative may pull down a factor $v^q(p)$. Again the reader is referred to [8] for the proofs, where almost identical bounds are established.

Lemma 4.3. *For any $M>0$ there exist $\varepsilon>0$ and $\mu<\infty$ such that for $g<\varepsilon$ and $g\sqrt{\beta}>\mu$,*

$$|K_p(X,J)|,\,|K_{p,p'}(X,J)|\leqq\exp(-M|X|)\,.$$

Using Lemmas 4.2, 4.3, we see that a factor $\exp[-\frac{1}{2}M\,\mathrm{dist}(p,p')]$ may be extracted from the right-hand side of (4.13) $\left(\text{recall that }\psi_c\text{ vanishes unless }\bigcup_i X_i\text{ is connected}\right)$. The remainder may then be bounded by the same methods used to prove convergence of the Mayer series (see [8] for details), and therefore we get

$$|m(p,p')|\leqq\delta\exp[-\tfrac{1}{2}M\,\mathrm{dist}(p,p')]\,, \tag{4.14}$$

where δ can be made arbitrarily small by taking M sufficiently large. Combining (4.14) with (4.11) and (4.12) we deduce

$$|\ln Z(J)-\ln Z(0)|\leqq(J,J)\sup_{s,p}\sum_{p'}|m(p,p')|\,.$$

Recall that $(J,J)=(h,Ch)$, so we get $|\ln Z(J)-\ln Z(0)|\leqq\gamma(h,Ch)$, where γ can be made arbitrarily small by taking M sufficiently large. This implies the desired result

$$\langle H(J)\rangle\geqq\exp[-\gamma,(h,Ch)]\,.$$

Appendix A

We present a sketch of the proof of Proposition 2.2 below. The result is very similar to the correlation inequalities in [3] and is proved in the same way. First we notice by rescaling the field G that

$$\frac{d}{dg}\langle H(J)\rangle=\left\langle H(J);\left(\frac{1}{g^3}(dG,dG)+\frac{1}{\alpha g^3}(d^*G,d^*G)\right)\right\rangle\,,$$

where $\langle -;-\rangle$ is the truncated correlation. By using the identity

$$x^2=\lim_{\varepsilon\to 0}\frac{2}{\varepsilon^2}(1-\cos\varepsilon x)\,,$$

Proposition 2.2 reduces to the inequalities

$$\langle H(J); \cos[\varepsilon dG(c)]\rangle \geqq 0\,, \qquad \langle H(J); \cos[\varepsilon d^* G(b)]\rangle \geqq 0 \tag{A.1}$$

for each $c \in \Lambda_3$ and $b \in \Lambda_1$. We will write the expectation $\langle H(J)\rangle$ in the following way:

$$\langle H(J)\rangle = Z^{-1} \int d\mu_C(G) \int DA \exp\left\{\beta \sum_{p\in\Lambda_2} \cos[dA(p) - G(p)]\right\} \times \cos[(A,h)-(G,J)]\,,$$

where $d\mu_C(G)$ is a Gaussian measure with the obvious covariance. We now introduce duplicate variables A' and G', and in the usual way (A.1) reduces to the positivity of the integral

$$\begin{aligned} I = \int d\mu_C(G) d\mu_C(G') \int DADA' \cos[(A,h)-(G,J)] \\ \times \exp\left\{\beta \sum_{p\in\Lambda_2} \cos[dA(p)-G(p)] + \beta \sum_{p\in\Lambda_2} \cos[dA'(p)-G'(p)]\right\} \\ \times \{\cos[\varepsilon dG(c)] - \cos[\varepsilon dG'(c)]\} \end{aligned} \tag{A.2}$$

(or with the last factor replaced by $\cos[\varepsilon d^* G(b)] - \cos[\varepsilon d^* G'(b)]$). Positivity of I is shown by making the following change of variables:

$$\begin{aligned} \alpha = \tfrac{1}{2}(A'-A)\,, \qquad \beta = \tfrac{1}{2}(A'+A)\,, \\ \Gamma = G'-G\,, \qquad \Delta = G'+G\,. \end{aligned} \tag{A.3}$$

Using simple trigonometric identities and expanding the exponential in (A.2), we get a sum of terms with positive coefficients, and each term has the form

$$\begin{aligned} \Bigg[\int d\mu_{C_1}(\Gamma) \int D\alpha f\left((\alpha,h) - \frac{1}{2}(\Gamma,J)\right) \sin\left[\frac{\varepsilon}{2} d\Gamma(c)\right] \prod_{p\in\Lambda_2} \\ \times \left\{\cos^{n(p)}\left[d\alpha(p) - \frac{1}{2}\Gamma(p)\right]\right\}\Bigg]^2\,, \end{aligned} \tag{A.4}$$

where $C_1 = 2C$, and $n(p) \geqq 0$ is an integer. Also the function f is either sine or cosine. Since (A.4) is manifestly positive, the desired result follows.

Appendix B

In this appendix we construct 2-forms v^q which satisfy the properties of Lemma 4.1. Because of the difficulty of visualising objects in four or more dimensions, our construction is quite detailed at every step, although it is a natural generalization of the construction in Appendix B of [8].

We will take the lattice Λ to be a subset of the region $\{x : x_\mu \geqq 0 \forall 1 \leqq \mu \leqq d\}$ which contains the origin $x=0$. Denote by T a maximal tree on the bonds in Λ, specified in terms of the d lattice directions $x_1, \ldots, x_d$. T contains all bonds in the x_d-direction; in the sublattice $x_d = 0$, T contains all bonds in the x_{d-1}-direction; in the sublattice $x_d = x_{d-1} = 0$, T contains all bonds in the x_{d-2}-direction; and so on.

Each lattice site x in Λ is connected to the origin by a unique path Γ_x along T. Suppose that b is any bond on the lattice lying in the μ-direction, so that $b = \langle b_-, b_+\rangle$.

Then we define a closed curve $L(b)$ on the lattice by

$$L(b) = b \cup \Gamma(b_+) \cup -\Gamma(b_-), \tag{B.1}$$

where the minus sign indicates orientation is reversed. So $-\Gamma(x)$ is the path along T connecting 0 to x. This definition (B.1) allows us to introduce an oriented surface $\Sigma(b)$ as the surface of minimal area whose boundary is $L(b)$, that is

$$\partial\Sigma(b) = L(b). \tag{B.2}$$

Notice that if $\mu = d$, $\Sigma(b)$ is empty, since then $\Gamma(b_+) = -b \cup \Gamma(b_-)$. Indeed for any $\mu > 1$, some parts of $\Gamma(b_+)$ and $-b \cup \Gamma(b_-)$ cancel out in this way.

Equipped with (B.2), we now define for each plaquette p on the lattice an oriented surface $S(p)$ whose boundary is ∂p:

$$S(p) = \bigcup_{b \in \partial p} \Sigma(b). \tag{B.3}$$

The union in (B.3) is taken over the oriented bonds comprising ∂p, and from (B.2) and (B.1) it follows that

$$\partial S(p) = \partial p. \tag{B.4}$$

We can now address the problem of constructing the 2-form v^q with the desired properties. Let q be a closed 3-form on the lattice, with $\operatorname{supp} q$ connected. For each plaquette p, choose a 3-volume $V(p)$ such that

$$\partial V(p) = p \cup -S(p). \tag{B.5}$$

Once again, the minus sign indicates that the orientation of the surface $S(p)$ has been reversed. Furthermore, (B.5) makes sense since $\partial^2 = 0$ and $\partial p = \partial S(p)$. Then we define a 2-form w^q on the plaquette p by

$$w^q(p) = \sum_{c \in V(p)} q(c), \tag{B.6}$$

where the sum is over cubes c belonging to $V(p)$. This definition is independent of the choice of 3-volume $V(p)$ because $dq = 0$.

From (B.6) we see that w^q is integer-valued. Also for each p there is a choice of $V(p)$ which contains each cube on the lattice no more than once, and so

$$\|w^q\|_\infty \leqq \|q\|_1 \leqq (q, q),$$

since q is integer-valued. Most importantly, we can see that $dw^q = q$ as follows:

$$dw^q(c) = \sum_{p \in \partial c} \sum_{c' \in V(p)} q(c') = \sum_{c' \in \bigcup_{p \in \partial c} V(p)} q(c'). \tag{B.7}$$

However, we have

$$\partial \bigcup_{p \in \partial c} V(p) = \bigcup_{p \in \partial c} (p \cup -S(p)) = \partial c \cup \bigcup_{p \in \partial c} \left(- \bigcup_{b \in \partial p} \Sigma(b)\right) = \partial c,$$

since $\partial^2 = 0$. Again because $dq = 0$, (B.7) can now be written

$$dw^q(c) = \sum_{c' \in c} q(c') = q(c), \tag{B.8}$$

which is the desired result.

The 2-form w^q does not satisfy part (d) of Lemma 4.1. However, by adding to w^q the derivative of an integer-valued 1-form m^q we will produce the desired 2-form v^q. We will write $B(q)$ for the minimal rectangular box on Λ containing $\operatorname{supp} q$. Let us denote by $\tilde{\Lambda}$ the set $\tilde{\Lambda}=B(q)^c \cup \partial B(q)$, where $B(q)^c$ is the set of sites outside $B(q)$, and $\partial B(q)$ is the boundary of $B(q)$. It follows from (B.8) that w^q is closed on $\tilde{\Lambda}$, and hence exact in dimension four or more. However, we will need some specific information about the 1-form n^q on $\tilde{\Lambda}$ whose derivative equals w^q.

In order to define n^q, we introduce a maximal tree $\tilde{T}$ on $\tilde{\Lambda}$. Choose a hyperplane, $x_\mu=\text{constant}$ for some μ, which intersects $\partial B(q)$, but not $\tilde{\Lambda}^c$ (this can always be done unless $B(q)=\Lambda$, in which case (d) holds for w^q). We can redefine coordinates so that this hyperplane is $x_d=0$. If we write $x=(\vec{x}, x_d)$, then the set $B(q)$ can be specified by

$$B(q)=\{x \in \Lambda : 0 \leqq x_d \leqq L, \vec{x} \in \Delta\},$$

where Δ is a rectangular subset of R^{d-1}, and L is some positive integer.

We now define $\tilde{T}$ on the sublattice $x_d=0$ in the manner explained at the beginning of this appendix (that is taking all bonds in the x_{d-1}-direction etc.), but this time taking the origin to be some site in $\partial B(q)$. The tree is extended to $x_d<0$ by taking all bonds in the x_d-direction. In the region $x_d>0$, we first consider all sites x whose "spatial coordinates" satisfy $\vec{x} \in \Delta^c \cup \partial \Delta$. Each of these sites lies in $\tilde{\Lambda}$, and we include in $\tilde{T}$ all bonds in the x_d-direction attached to such sites.

Finally we consider the sites in the region $x_d \geqq L$, $\vec{x} \in \Delta \backslash \partial \Delta$. This intersects $\partial B(q)$, and may be covered by a maximal tree defined in the usual way. Once again we take the origin of this tree as a site in $\partial B(q)$. By inserting one additional bond in the hyperplane $x_d=L$, connecting $\partial\Delta$ to its interior, we obtain a maximal tree $\tilde{T}$ on the set $\tilde{\Lambda}$.

The foregoing rather complicated construction is just one choice of a tree on $\tilde{\Lambda}$. Unfortunately we need this detailed description to prove our results. Let Γ_x be the unique path in $\tilde{T}$ connecting x to the origin (which is in the hyperplane $x_d=0$), and define $L(b)$ for any bond b by (B.1). Then for any choice of surface $\Sigma(b)$ in $\tilde{\Lambda}$ with boundary $L(b)$, define

$$n^q(b)=\sum_{p \in \Sigma(b)} w^q(p). \tag{B.9}$$

In dimension $d \geqq 4$, the definition (B.9) is independent of the choice of surface $\Sigma(b)$ (this is because $dw^q=0$ on $\tilde{\Lambda}$ and $H^2(S^{d-1})=0$ for $d \geqq 4$). It is easy to check that $dn^q=w^q$ on $\tilde{\Lambda}$. Furthermore, (B.9) shows that n^q is integer-valued. Finally our explicit construction of $\tilde{T}$ shows that the following bound holds:

$$|n^q(b)| \leqq \|w^q\|_\infty A(q) \qquad \forall b \in \partial B(q), \tag{B.10}$$

where $A(q)$ is the sum of the areas of the sides of $B(q)$. Since $\operatorname{supp} q$ is connected, we have

$$A(q) \leqq c(q,q)^2$$

for some constant c.

Therefore, define the 1-form m^q on Λ to equal n^q on $\tilde{\Lambda}$ and zero otherwise, and define

$$v^q=w^q-dm^q. \tag{B.11}$$

Then (B.8), (B.9), and (B.10) imply Lemma 4.1, parts (a), (d), and (e). If $q=\sum_i q_i$ with each suppq_i connected, then $dq_i=0$ for each i, and we can construct v^{q_i} separately and add to get v^q. To satisfy (c), we consider pairs q and $-q$, construct v^q, and define $v^{-q}=-v^q$.

Acknowledgements. The author would like to thank Tom Kennedy for pointing out an error in the original manuscript.

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Communicated by A. Jaffe

Received March 6, 1986; in revised form April 1, 1986