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On the Massive Sine-Gordon Equation in all Regions of Collapse

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Abstract. The ultraviolet stability for the cosine interaction in two dimensions and finite volume Λ is rederived for values $\alpha^2 \in [4\pi, \frac{32}{5}\pi[$ and proven for the remaining $\alpha^2 \in [4\pi, 8\pi[$ by using renormalization group methods developed in [G, GN 1] to portray renormalized effective potentials arising from a multiscale decomposition.

1.1. Introduction

The two dimensional sine-Gordon model has been widely studied as an interesting model in quantum field theory as well as in statistical mechanics. In constructive quantum field theory, its interest is mainly due to the fact that by varying the parameter α in the cosine potential

$$V_0[\Lambda] := \lambda \int_\Lambda :\cos \alpha \varphi_\xi : d\xi\,, \qquad \Lambda \subset \mathbb{R}^2\,, \tag{1.1}$$

we encounter either a finite, superrenormalizable, renormalizable, or nonrenormalizable theory. In fact, Fröhlich showed in [F] that the theory remains finite for $\alpha^2 \in [0, 4\pi[$. Furthermore, for $\alpha^2 \in [4\pi, 8\pi[$, the theory can be renormalized by subtracting an ever-increasing number of field independent counterterms (cf. [G]). Ultraviolet stability was proven by Benfatto et al. in [BGN] for the interval $\alpha^2 \in [4\pi, 2\pi(\sqrt{17}-1)[$, and subsequently extended in [N 1] to the interval $\alpha^2 \in [4\pi, \frac{32}{5}\pi[$. This paper proves the ultraviolet stability for all values $\alpha^2 \in [4\pi, 8\pi[$. The crucial ingredients for the proof are the application of the tree expansion developed by Gallavotti and one of us (F.N.) in [G, GN 1] as well as a general strategy to solve the large fluctuation problem which arises as one analyzes (1.1) in the framework of Euclidean scalar field theory. This strategy is

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*** Supported in part by the Schweizerischer Nationalfonds different from the one used in [N1]; however, as we will show in [RS], the technique developed there can also be extended to prove ultraviolet stability for all values $\alpha^2 \in [4\pi, 8\pi[$. In classical statistical mechanics, on the other hand, the potential (1.1) describes a two dimensional gas of spinless particles with Yukawa interaction in the grand canonical ensemble.

The charges of the particles are $\pm e$, the activity is λ, and the inverse temperature of the gas is such that $\alpha^2 = \beta e^2$. For $\beta e^2 < 4\pi$, the gas is stable, whereas the situation in the interval $\beta e^2 \in [4\pi, 8\pi[$ can be interpreted as a sequence of partial collapses in which infinitesimal neutral clusters composed of an ever increasing number of particles are formed. At $\beta e^2 = 8\pi$ one expects full collapse (cf. [F, BGN, N1, N2]). The methods developed for studying the interaction (1.1) may also be used to study a two dimensional, neutral gas of classical, spinless particles with Coulomb interaction, where a similar interpretation of the phase transitions holds (see [G, N1, N2, GN1, GN2, NRS]).

1.2. The General Strategy

We use a multiscale decomposition of the cutoff field $\varphi^{(\leqq N)}$:

$$\varphi_\xi^{(\leqq N)} := \sum_{h=0}^{N} \varphi_\xi^{(h)}, \qquad \xi \in \Lambda, \tag{1.2}$$

defined as the gaussian field with covariance

$$C^{(\leqq N)} := (1-\Delta)^{-1} - (\gamma^{2(N+1)} - \Delta)^{-1}, \tag{1.3}$$

where $\gamma > 1$ is a scaling factor chosen close to one and Δ is the Laplace operator in $\mathbb{R}^2$.

The fields of frequency h, $\varphi^{(h)}$, are independent gaussian processes whose covariances are given by

$$C_{\xi\eta}^{(h)} = \frac{1}{(2\pi)^2} \int dp\, e^{ip\cdot(\xi-\eta)} [(\gamma^{2h} + p^2)^{-1} - (\gamma^{2(h+1)} + p^2)^{-1}]. \tag{1.4}$$

Writing $\mathscr{E}$ for the expectations with respect to the measure $P(d\varphi^{(\leqq N)})$, we define the truncated expectation of order n,

$$\mathscr{E}^T(f_1, \ldots, f_n) := \frac{\partial^n}{\partial\tau_1 \ldots \partial\tau_n} \log \mathscr{E}(e^{\tau_1 f_1 + \ldots + \tau_n f_n})|_{\tau_1 = \ldots \tau_n = 0}, \tag{1.5}$$

where $f_1, \ldots, f_n$ are n random variables. Note that $\mathscr{E}^T(f;q) := \mathscr{E}^T(f, \ldots, f)$ (q times).

Similarly we define expectation $\mathscr{E}_k$ and truncated expectation $\mathscr{E}_k^T$ with respect to the measure $P(d\varphi^{(k)})$. Finally, we write $\mathscr{E}_{\leqq h}$ for $\mathscr{E}_0 \cdot \ldots \cdot \mathscr{E}_n$. We will prove the following theorem:

Theorem 1.0. *Let*

$$V_0^{(N)}[\Lambda] := \lambda \int_\Lambda :\cos\alpha\varphi_\xi^{(\leqq N)}: d\xi, \tag{1.6}$$

then the renormalized cosine potential $\left(\textit{note here that when } t \textit{ is odd the sum } \sum_{\substack{n=2\\(\text{even})}}^{t} \textit{ runs over the even integers from } 2 \textit{ thru } t-1\right)$

$$V^{(N)}[\Lambda]:=V_0^{(N)}[\Lambda]-\sum_{\substack{n=2\\(\text{even})}}^{t}\frac{1}{n!}\mathscr{E}^T(V_0^{(N)}[\Lambda];n) \tag{1.7}$$

is stable in the ultraviolet limit for t an integer satisfying

$$\left(\frac{\alpha^2}{4\pi}-2\right)(t+1)+2<0. \tag{1.8}$$

That is, there exist two positive constants $E_-(\lambda)$ *and* $E_+(\lambda)$ *independent of the cutoff* N *and the finite volume* $|\Lambda|$, *so that*

$$e^{-E_-(\lambda)|\Lambda|}\leqq\int e^{V^{(N)}[\Lambda]}P(d\varphi^{(\leqq N)})\leqq e^{+E_+(\lambda)|\Lambda|}. \tag{1.9}$$

Moreover, it easily follows from the proof of (1.9) that

$$\lim_{\lambda\to 0}E_\pm(\lambda)\lambda^{-(t+\tau)}=0 \tag{1.10}$$

for some $\tau>0$.

For fixed α^2, inequality (1.8) defines a minimal integer $t_0(\alpha^2)$ such that for $t\geqq t_0$ it holds. On the other hand, we encounter an infinite number of thresholds

$$\alpha_t^2:=8\pi\left(1-\frac{1}{t}\right)\quad(t=2,4,6,\ldots), \tag{1.11}$$

which means that in the interval $[\alpha_t^2,\alpha_{t+2}^2[$, the interaction needs renormalization up to counterterms of order t (even).

In the Yukawa gas interpretation, these thresholds correspond to the critical temperatures $T_t=\frac{e^2}{8\pi k}\left(\frac{t}{t-1}\right)$, $t=2,4,6,\ldots$ (where k is the Boltzmann constant) at which neutral clusters consisting of t particles collapse (see [BGN, N1]).

We now introduce the "effective potential" on "scale k" or at "frequency k" by defining recursively for $k=0,\ldots,N-1$:

$$e^{V^{(k)}[\Lambda](\varphi^{(\leqq k)})}:=\int e^{V^{(k+1)}[\Lambda](\varphi^{(\leqq k+1)})}P(d\varphi^{(k+1)}),$$

and

$$V^{(N)}[\Lambda](\varphi^{(\leqq N)}):=V^{(N)}[\Lambda]\quad\text{for}\quad k=N. \tag{1.12}$$

Moreover, let the "truncated effective potential on scale k" be

$$\tilde{V}^{(k)}[\Lambda]:=[V^{(k)}[\Lambda]]_{\leqq t}\qquad(\text{for } k=0,\ldots,N), \tag{1.13}$$

where $[\cdot]_{\leqq t}$ will stand for truncation of the power series in λ at order t.

The basic idea for showing the inequalities (1.9) is to use the concept of effective potential to perform the integration with respect to $P(d\varphi^{(\leqq k)})$ frequency by frequency. In fact, due to the simple scaling properties of the fields

$$\varphi_\xi^{(k)}\equiv\varphi_{\gamma^k\xi}^{(0)} \tag{1.14}$$

it will be enough to perform a generic step of this interative procedure. In order to do this one step, i.e. to perform an integration of the truncated effective potential at frequency k with respect to $P(d\varphi^{(k)})$, we use a lemma which has been previously proven in [BCGNOPS] and adapted to the sine-Gordon problem in [BGN]. This "Main Lemma", which reduces our problem to one in perturbation theory, will be explained briefly in Sect. 2.3. It turns out that integration with respect to $P(d\varphi^{(k)})$ results in an expression of the following type (see also [BGN])

$$\int e^{\tilde{V}^{(k)}} P(d\varphi^{(k)}) \leqq \exp\left(\left[\sum_{n=1}^{t} \frac{1}{n!} \mathscr{E}_k^T(\tilde{V}^{(k)}; n)\right]_{\leqq t} + R^{(k-1)}(\lambda)\,|\Lambda|\right), \tag{1.15a}$$

$$\int e^{\tilde{V}^{(k)}} P(d\varphi^{(k)}) \geqq \exp\left(\left[\sum_{n=1}^{t} \frac{1}{n!} \mathscr{E}_k^T(\tilde{V}^{(k)}; n)\right]_{\leqq t} - R^{(k-1)}(\lambda)\,|\Lambda|\right), \tag{1.15b}$$

where $R^{(k-1)}(\lambda)$ is a remainder of order $t+1$ in λ independent of the volume $|\Lambda|$ and the ultraviolet cutoff N. Actually, for the sum $\sum_{h=0}^{N-1} R^{(h)}(\lambda)$ to remain finite when $N \to \infty$ [and thus proving inequalities (1.9)], it is necessary to restrict the fields $\varphi^{(\leqq k)}$ to be "smooth", i.e. to be Hölder continuous of given modulus (B_k) and exponent $(1-\varepsilon)$ [cf. Eq. (2.1)]. This is easily achieved in the case of the lower bound by just introducing the appropriate characteristic functions restricting to such "smooth" fields. Then by using the estimates for the effective potential in Sect. 2.3, it is an easy task to prove the lower bound in (1.9) precisely in the spirit of [BGN]. For proving the upper bound, on the other hand, it is a priori not legitimate to introduce any characteristic functions.

Nevertheless, let us first discuss how the integration of the "smooth" part of the effective potential can be performed: Instead of $\tilde{V}^{(k)}[\Lambda]$ we consider $\hat{V}^{(k)}[\mathscr{D}_k^c]$, where $\mathscr{D}_k^c$ is the complement of the region $\mathscr{D}_k$ in which the field $\varphi^{(\leqq k)}$ is "rough" (i.e. not "smooth"). As we will see in Sect. 2.1, the letters $\mathscr{D}_k$, $\mathscr{D}_k^c$, etc. actually symbolize sets of field dependent regions. Writing $\mathscr{D}_k$ or $\mathscr{D}_k^c$ in the argument of $\hat{V}^{(k)}[\cdot]$ is only meant to be a suggestive notation for the "rough" respectively "smooth" part of $\tilde{V}^{(k)}[\Lambda]$. This new effective potential $\hat{V}^{(k)}[\mathscr{D}_k^c]$, being the "smooth" part of the effective potential $\tilde{V}^{(k)}[\Lambda]$, would be the natural object to be integrated with respect to $P(d\varphi^{(k)})$. The field-dependent region $\mathscr{D}_k^c$, however, introduces an additional complicated $\varphi^{(\leqq k)}$-field dependence in the effective potential on scale k. But since the large fluctuations (that is, the "roughness") on scale k are related to the ones on scale $k-1$, we have the following relationship:

$$\mathscr{D}_k \subset \mathscr{D}_{k-1} \cup \mathscr{R}_k, \tag{1.16}$$

where $\mathscr{D}_{k-1}$ is a region in which the field $\varphi^{(\leqq k-1)}$ is "rough" while $\mathscr{R}_k$ is a region in which the field $\varphi^{(k)}$ is "rough" [the precise definitions and the proof of (1.16) will be given in Sect. 2.1].

Inclusion (1.16) motivates the idea to consider $\hat{V}^{(k)}[\mathscr{D}_{k-1}^c \cap \mathscr{R}_k^c]$ instead of $\hat{V}^{(k)}[\mathscr{D}_k^c]$ as the "smooth" part of the effective potential which is to be integrated with respect to $P(d\varphi^{(k)})$, since the $\varphi^{(k)}$-field dependence introduced by the set $\mathscr{R}_k^c$ (the complement of $\mathscr{R}_k$) is now manageable ($\mathscr{R}_k^c$ being a collection of squares on scale k). Thus, instead of (1.15a), we prove the following inequality

$$\int \chi^b_{\mathscr{R}_k^c} \mathring{\chi}^b_{\mathscr{R}_k} e^{\hat{V}^{(k)}[\mathscr{D}^c_{k-1} \cap \mathscr{R}^c_k]} P(d\varphi^{(k)}) \leqq \exp\left(\left[\sum_{n=1}^{t} \frac{1}{n!} \mathscr{E}^T_k(\hat{V}^{(k)}[\mathscr{D}^c_{k-1}]; n)\right]_{\leqq t} + \bar{R}^{(k-1)}(\lambda)\,|\Lambda|\right) \tag{1.17a}$$

and, for the lower bound,

$$\int \chi^b_{Q_k} e^{\tilde{V}^{(k)}[\Lambda]} P(d\varphi^{(k)}) \geqq \chi^b_{Q_{k-1}} e^{\tilde{V}^{(k-1)}[\Lambda] - R^{(k-1)}(\lambda)[\Lambda]}, \tag{1.17b}$$

where Q_k is a pavement using squares of side length γ^{-k} and with suitably defined characteristic functions $\chi^b_{\mathscr{R}_k^c}$, $\mathring{\chi}^b_{\mathscr{R}_k}$ (cf. Sect. 2) and $\bar{R}^{(k-1)}(\lambda)$ a controllable remainder.

In order to set up an iterative procedure for the upper bound, it would be sufficient to show

(i) that it is possible to reduce the expression on the right-hand side of (1.17a) to a term like:

$$e^{\hat{V}^{(k-1)}[\mathscr{D}^c_{k-1}] + \bar{\bar{R}}^{(k-1)}(\lambda)\,|\Lambda|}$$

with, as before, a reasonable remainder $\bar{\bar{R}}^{(k-1)}(\lambda)$.

In other words, it would be sufficient to control

$$\Delta^{(k-1)}[\mathscr{D}_{k-1}] := \left[\sum_{n=1}^{t} \frac{1}{n!} \mathscr{E}^T_k(\hat{V}^{(k)}[\mathscr{D}^c_{k-1}]; n)\right]_{\leqq t} - \hat{V}^{(k-1)}[\mathscr{D}^c_{k-1}], \tag{1.18}$$

provided the passages:

(ii) from $V^{(N-1)}$ to $\hat{V}^{(N-1)}[\mathscr{D}^c_{N-1}]$

and

(iii) from $\hat{V}^{(k)}[\mathscr{D}^c_k]$ to $\hat{V}^{(k)}[\mathscr{D}^c_{k-1} \cap \mathscr{R}^c_k]$, $\forall k \leqq N-1$

are allowed.

The first step is the most difficult one. Unfortunately, one cannot show in general that $\Delta^{(k)}[\mathscr{D}_k]$ is negative as one would like to in order to prove the upper bound. The crucial reason behind this difficulty is that the iterative procedure envisioned by steps (i), (ii), and (iii) only "transports" the smooth part of the effective potential from one frequency to the next. The second step is possible since

$$V^{(N-1)} - \hat{V}^{(N-1)}[\mathscr{D}^c_{N-1}] \leqq 0, \tag{1.19}$$

and the third since

$$\hat{V}^{(k)}[\mathscr{D}^c_k \cap \mathscr{R}^c_k] - \hat{V}^{(k)}[\mathscr{D}^c_{k-1} \cap \mathscr{R}^c_k] \leqq 0, \tag{1.20}$$

and $\hat{V}^{(k)}[\mathscr{D}^c_k \cap \mathscr{R}_k]$ is again a term which can be safely put into the remainder (see [BGN]).

Inequalities (1.19) and (1.20) are properties of the large fluctuation part of the effective potential and are proven in Sect. 2. The crux of a correct iterative procedure, is the "transport" of a large fluctuation part of the effective potential from one frequency to another allowing us to use in step (i) as well a part of the negativity exhibited in (1.19) and (1.20).

Calling $W^{(k)}[\mathscr{D}_k]$ this "transported" large fluctuation part of the effective potential at a generic level k, the iterative mechanism we are going to apply will essentially proceed as follows: Start at a generic level k:

1)
$$\begin{aligned} U^{(k)} &\equiv \hat{V}^{(k)}[\mathscr{D}_k^c] + \Delta^{(k)}[\mathscr{D}_k] + W^{(k)}[\mathscr{D}_k], \\ U^{(N-1)} &\equiv \tilde{V}^{(N-1)}[\Lambda]. \end{aligned} \tag{1.21}$$

2a) Transform it using step (iii) and the "large fluctuation transport" mechanism into

$$\hat{V}^{(k)}[\mathscr{D}_{k-1}^c \cap \mathscr{R}_k^c] + W^{(k-1)}[\mathscr{D}_{k-1}]. \tag{1.22}$$

2b) Apply the Main Lemma: [integration with respect to $P(d\varphi^{(k)})$] obtaining

$$U^{(k-1)} \equiv \hat{V}^{(k-1)}[\mathscr{D}_{k-1}^c] + \Delta^{(k-1)}[\mathscr{D}_{k-1}] + W^{(k-1)}[\mathscr{D}_{k-1}].$$

[The precise version of (1.21) is given in Sect. 3.]

In Sect. 4 we describe as clearly as possible how all the theorems and lemmata proven in the foregoing sections contribute to the proof of the announced result, namely Theorem 1.0. We thus urge the reader to follow this guide attentively while reading the next sections, so that in the midst of so many trees he will not lose sight of the forest.

1.3. The Effective Potential in Tree Notation

We write the cosine interaction $V_0^{(N)}[\Lambda]$ as a sum of exponentials:

$$V_0^{(N)}[\Lambda] = \tfrac{1}{2} \sum_{\sigma = \pm 1} \lambda \int_\Lambda : e^{i\alpha\sigma\varphi_\xi^{(\leq N)}} : d\xi, \tag{1.23}$$

where the Wick-ordering is defined by

$$: e^{i\alpha\sigma\varphi} : := e^{\frac{\alpha^2}{2}\mathscr{E}(\varphi^2)} e^{i\alpha\sigma\varphi}. \tag{1.24}$$

The parameters σ will in the following be referred to as "charges." Using (1.23) and recursively applying the formula (1.12), we get a "tree expansion" for the effective potential defined in terms of the tree notation introduced in [G, GN1, GN2, NRS]:

$$\begin{aligned} \tilde{V}^{(k)} = & \sum_{n=1}^{t} \sum_{\substack{\theta \\ \{k(\theta)=k \\ v(\theta)=n}} \frac{1}{n(\theta)} \sum_{\underline{\sigma}} \int_{\Lambda^n} d\xi_1 \dots d\xi_n \bar{V}(\theta, \varphi^{(\leq k)}, \underline{\sigma}) \\ & - \sum_{n=1}^{t} \sum_{\substack{\theta \\ \{k(\theta)=k \\ v(\theta)=n}}^{*} \frac{1}{n(\theta)} \sum_{\underline{\sigma}} \int_{\Lambda^n} d\xi_1 \dots d\xi_n \bar{V}(\theta, \underline{\sigma}), \end{aligned} \tag{1.25}$$

where θ is a tree with definite frequencies at each bifurcation, $k(\theta)$ denotes its root frequency and $v(\theta)$ is the number of final lines. The number $n(\theta)$ is the usual combinatorial factor (see [G, GN1]) and $\underline{\sigma} = (\sigma_1, \dots, \sigma_n)$, $\sigma_i = \pm 1$, are the charges of the final lines.

The sum $\sum_{\theta, \{}$ runs over all different trees with n final lines and root frequency k. The second sum $\sum_{\theta, \{}^{*}$ differs from the first in that the frequency of the lowest bifurcation, hereafter called $h(\theta)$, runs over frequencies from zero to k instead of frequencies from $k+1$ to N as in the first sum. Note that we will generally reserve the letter k for depicting the root frequency, while h will be a running frequency. The second term in (1.25) corresponds to the parts of the counterterms which had not yet been utilized, having gone from scale N down to scale k. (See [BGN] and [NRS] for an explicit discussion of the counterterms.) We now decompose the sum $\sum_\theta$ in the following way: We fix the shape s of the tree, then we fix the absolute value of the charge at each vertex (bifurcation) v of the tree $Q_v \geqq 0$ (a vertex can also be thought of as a cluster of charges whose average size depends on the frequency h_v of the vertex). Finally, we call $\{Q_v\}_s$ a compatible vertex charge configuration for a tree θ with shape s.

Therefore, we can write

$$\sum_{\substack{\theta \\ k(\theta)=k \\ v(\theta)=n}} \sum_{\underline{\sigma}} = \sum_{\substack{s \\ v(s)=n}} \sum_{\{Q_v\}_s} \sum_{\substack{\theta \\ \left\{\begin{subarray}{l} s(\theta)=s \\ k(\theta)=k \end{subarray}\right.}} \sum_{\substack{\underline{\sigma} \\ \{Q_v\}_s \text{fixed}}}, \tag{1.26}$$

where $s(\theta)=s$ means that the tree θ has the shape s and $\sum_{\substack{\underline{\sigma} \\ \{Q_v\}_s}}$ implies that the absolute value of the charge of the vertex v has the value Q_v. Next we decompose $\sum_{\substack{\underline{\sigma} \\ \{Q_v\}_s}}$ in the following way: We call $\underline{\sigma}=(\sigma_1, \ldots, \sigma_n)$ satisfying the constraint $\{Q_v\}_s$ an "admissible configuration" and associate a label μ_v to each vertex v such that when $Q_v=0$, $\mu_v=\pm 1$, while $\mu_v=1$ when $Q_v>0$. Then we fix an "admissible configuration" for a given $\{Q_v\}_s : \underline{\bar{\sigma}}=(\bar{\sigma}_1, \ldots, \bar{\sigma}_n)$ and define:

$$\sigma_i = \bar{\sigma}_i \prod_{v \ni i} \mu_v, \tag{1.27}$$

where $v \ni i$ means that the i^{th} endpoint of θ is inside the cluster θ_v (i.e. those points pertaining to a vertex v). It is clear that the sum over all admissible configurations of a tree θ with fixed s and $\{Q_v\}_s$ can be decomposed as a sum over a suitable family $\mathscr{S}$ of admissible configurations $\underline{\bar{\sigma}}$ times a sum over the other configurations $\underline{\sigma}$ which can be obtained from a fixed $\underline{\bar{\sigma}}$ by just summing over the set of μ_v-labels $\{\mu_v\}$, and dividing by a factor which takes into account a possible double counting.

Therefore

$$\begin{aligned} \sum_{\substack{\theta \\ k(\theta)=k \\ v(\theta)=n}} \sum_{\underline{\sigma}} &= \left\{ \sum_{\substack{s \\ v(s)=n}} \sum_{\{Q_v\}_s} \sum_{\underline{\bar{\sigma}} \in \mathscr{S}} \frac{c(\underline{\sigma})}{n(s)} \right\} \sum_{\substack{\theta \\ \left\{\begin{subarray}{l} s(\theta)=s \\ k(\theta)=k \end{subarray}\right.}} \sum_{\substack{\{\mu_v\} \\ \underline{\bar{\sigma}} \\ \text{fixed}}} \\ &=: \sum_{(s,\{Q_v\}_s,\underline{\bar{\sigma}})} \sum_{\substack{\theta \\ \{:}} \sum_{\substack{\{\mu_v\} \\ \{:}}, \end{aligned} \tag{1.28}$$

where $n(\theta)=n(s)$ only if when we sum over the frequencies we do not impose any constraints between frequencies of different branches. Now we can write

$$\tilde{V}^{(k)}[\Lambda] = \sum_{n=1}^{t} \sum_{(s,\{Q_v\}_s,\underline{\bar{\sigma}})} \sum_{\substack{\theta \\ \left\{\begin{subarray}{l} s(\theta)=s \\ k(\theta)=k \end{subarray}\right.}} \tilde{V}(\theta, \underline{\bar{\sigma}}) - [\text{counterterms}; k], \tag{1.29}$$

where

$$\tilde{V}(\theta, \underline{\bar{\sigma}}) = \int_{\Lambda^{v(\theta)}} d\xi_1 \ldots d\xi_{v(\theta)} \sum_{\substack{\{\mu_v\} \\ \underline{\bar{\sigma}} \\ \text{fixed}}} \bar{V}(\theta, \varphi^{(\leqq k)}, \{\mu_v\}). \tag{1.30}$$

Remarks. i) This decomposition is such that each term of the sum $\sum_{(s,\{Q_v\}_s,\underline{\bar{\sigma}})}$ satisfies the estimates we need. To prove them we need to use important cancellations provided by the $\sum_{\{\mu_v\}}$.

ii) We do not explicitly write the second term of (1.25) because these parts of the counterterms do not play any role at level k. A piece of them will be extracted and used when we go to the next level $k-1$.

We are still free to change the names of the final lines, changing the names of the coordinates, and therefore to require that $\underset{\sim}{\sigma}$ always appears as

$$\begin{aligned}\bar{\underset{\sim}{\sigma}} &= (+,-,\dots,+,-) \quad \text{when } Q(\theta)=0 \text{ and } n=2m,\\ &= (\bar{\sigma}_1,\dots,\bar{\sigma}_n)\\ \bar{\bar{\underset{\sim}{\sigma}}} &= (+,-,\dots,+,-;\pm,\pm,\dots,\pm) \quad \text{when } |Q(\bar{\theta})|=p>0\\ &= (\bar{\bar{\sigma}}_1,\dots,\bar{\bar{\sigma}}_{2m};\bar{\bar{\sigma}}_{2m+1},\dots,\bar{\bar{\sigma}}_{2m+p}) \quad \text{and } n=2m+p,\end{aligned} \tag{1.31}$$

where $Q=Q(\theta)$ is the total charge of the tree θ. (Hereafter when $Q\neq 0$ we write $\bar{\bar{\underset{\sim}{\sigma}}}$ for $\bar{\underset{\sim}{\sigma}}$.) This can be done by ordering the bifurcations hierarchically as follows.

Definition of Hierarchical Ordering of Vertices (Bifurcations). A vertex v is called of order ℓ if and only if there is at least one subtree pertaining to v whose lowest bifurcation is of order $\ell-1$ and no other subtree has its lowest bifurcation of order $>\ell-1$; furthermore, a vertex is of order zero when its subtrees are all trivial, that is, they have no bifurcations.

We start by considering all the vertices of order zero and we arrange the names of the coordinates in such a way that the charges of the $\underset{\sim}{\sigma}$ configurations associated to that vertex are $(+,-,+,-,\dots,+,-)=\bar{\underset{\sim}{\sigma}}$ if the vertex is neutral. We do the same for the non-neutral vertices, but in that case we label only the neutral part (for example, if in a vertex of order zero three lines merge with charges $(+,-,+)$ we label the first two only). We go on by considering the final lines of the order one vertices which have not yet been labelled and we proceed as before, order by order. With this choice $\bar{\underset{\sim}{\sigma}}$ appears as in (1.31) and in each vertex the lines with opposite charges always have adjacent indices.

Now we decompose: $\cos\alpha\varphi(\theta,\bar{\sigma})-1:=:\cos\alpha\sum_{\ell=1}^{m}(\Delta_\ell\varphi)-1:$ into

$$\begin{aligned}:\cos\alpha\left(\sum_{\ell=1}^{m}\Delta_\ell\varphi\right)-1: &= \tfrac{1}{2}\sum_{\sigma}:e^{i\alpha\sigma\Delta_1\varphi}\cdot\ldots\cdot e^{i\alpha\sigma\Delta_m\varphi}-1:\\ &= \tfrac{1}{2}\sum_{\sigma}:\prod_{\ell=1}^{m}(\cos\alpha\sigma\Delta_\ell\varphi+i\sin\alpha\sigma\Delta_\ell\varphi)-1:=\sum_{|\mathscr{P}|\,\text{even}}\Sigma_{\mathscr{P}}:P_{\mathscr{P}}(\varphi):,\end{aligned} \tag{1.32}$$

where $\mathscr{P}$ is a subset $\{\ell_1,\dots,\ell_q\}\subset\{1,\dots,m\}$, $0\leqq q\leqq m$, $|\mathscr{P}|=q$ and

$$\begin{aligned}:P_{\mathscr{P}}(\varphi): \quad &:= i^{|\mathscr{P}|}:\left(\prod_{j=1}^{q}\sin\alpha\Delta_{\ell_j}\varphi\right)\prod_{s\notin\mathscr{P}}\cos\alpha\Delta_s\varphi:,\\ :P_{\emptyset}(\varphi): \quad &:= :\left(\prod_{\ell=1}^{m}\cos\alpha\Delta_\ell\varphi\right)-1:,\end{aligned} \tag{1.33}$$

using the following conventions

$$\Delta_\ell\varphi:=\varphi_{\xi_{2\ell-1}}-\varphi_{\xi_{2\ell}}, \tag{1.34}$$

for $Q=0$:

$$\varphi(\theta,\bar{\sigma}):=\sum_{i=1}^{2m}\bar{\sigma}_i\varphi_{\xi_i}\equiv\sum_{\ell=1}^{m}\Delta_\ell\varphi, \tag{1.35}$$

and for $|Q|=p>0$:

$$\varphi(\bar{\theta},\bar{\sigma}):=\sum_{i=1}^{2m+p}\bar{\bar{\sigma}}_i\varphi_{\xi_i}=\sum_{\ell=1}^{m}\Delta_\ell\varphi+\operatorname{sign}Q\sum_{j=1}^{p}\varphi_{\xi_{2\bar{m}+j}}. \tag{1.36}$$

We now observe that $P_{\mathscr{P}}(\varphi)$ is odd under the exchange $\xi_{2\ell_j-1}\leftrightarrow\xi_{2\ell_j}$, $\forall\ell_j\in\mathscr{P}$, and even under the exchange $\xi_{2s-1}\leftrightarrow\xi_{2s}$, $\forall s\notin\mathscr{P}$.

Therefore we can define the following operations for $f\in C(\mathbb{R}^n)$:

$$\begin{aligned}S_i(f(\xi_1,\dots,\xi_n)):=&\tfrac{1}{2}\{f(\xi_1,\dots,\xi_{2i-1},\xi_{2i},\dots,\xi_n)\\&+f(\xi_1,\dots,\xi_{2i-2},\xi_{2i},\xi_{2i-1},\dots,\xi_n)\},\\A_i(f(\xi_1,\dots,\xi_n)):=&\tfrac{1}{2}\{f(\xi_1,\dots,\xi_{2i-1},\xi_{2i},\dots,\xi_n)\\&-f(\xi_1,\dots,\xi_{2i-2},\xi_{2i},\xi_{2i-1},\dots,\xi_n)\},\end{aligned}$$

as well as the operation:

$$\mathcal{O}_{\mathscr{P}}(f):=\Big(\prod_{i\in\mathscr{P}}A_i\prod_{j\notin\mathscr{P}}S_j\Big)f. \tag{1.38}$$

Of course,

$$\sum_{\mathscr{P}}\mathcal{O}_{\mathscr{P}}(f)\equiv f, \tag{1.39}$$

and moreover, if f satisfies for all i

$$f(\xi_1,\xi_2,\dots,\xi_{2i-1},\xi_{2i},\dots,\xi_{n-1},\xi_n)\equiv f(\xi_2,\xi_1,\dots,\xi_{2i},\xi_{2i-1},\dots,\xi_n,\xi_{n-1}), \tag{1.40}$$

then

$$\sum_{\substack{\mathscr{P}\\|\mathscr{P}|\text{ even}}}\mathcal{O}_{\mathscr{P}}(f)=f. \tag{1.41}$$

Thus it is possible to prove that we can write

$$\tilde{V}(\theta,\bar{\sigma})=\left(\frac{\lambda}{2}\right)^{2m}\sum_{\substack{\mathscr{P}\\|\mathscr{P}|\text{ even}}}\int_{\Lambda^{2m}}d\xi:P_{\mathscr{P}}(\varphi):F_{\theta,\mathscr{P}}(\underline{\xi},\underline{\sigma};Q=0), \tag{1.42}$$

where the explicit expression for $F_{\theta,\mathscr{P}}(\dots)$ can be inferred from the proof of Theorem 1.1.

For non-neutral trees we proceed in a slightly different way; we divide the coordinates ξ of the final lines of $\bar{\theta}$ in two groups: we call $\bar{\underline{\xi}}$ the $2\bar{m}$ lines which merge at some neutral bifurcation of $\bar{\theta}$ and $\underline{\zeta}$ the $\bar{p}$ remaining ones. Then we decompose $:e^{i\alpha\varphi(\bar{\theta},\bar{\sigma})}:$ as follows:

$$:e^{i\alpha\varphi(\bar{\theta},\bar{\sigma})}:=:e^{i\alpha\sum_{p=1}^{\bar{m}}\Delta_\ell\varphi}\cdot e^{i\alpha\varphi(\mathscr{L}(\bar{\theta}))}:=\sum_{\mathscr{N}}:P_{\mathscr{N}}(\varphi)e^{i\alpha\varphi(\mathscr{L}(\bar{\theta}))}:, \tag{1.43}$$

where $P_{\mathscr{N}=\emptyset}(\varphi):=\prod_{\ell=1}^{\bar{m}}\cos\alpha\Delta_\ell\varphi$, and $\varphi(\mathscr{L}(\bar{\theta})):=\sum_{s=1}^{\bar{p}}\sigma_{2\bar{m}+s}\varphi_{\zeta_s}$ and $\mathscr{N}$ is a subset of $\{1,\dots,\bar{m}\}$. Therefore, analogously

$$\tilde{V}(\theta,\bar{\sigma})=\left(\frac{\lambda}{2}\right)^{2\bar{m}+\bar{p}}\sum_{\mathscr{N}}\int_{\Lambda^{2\bar{m}+\bar{p}}}d\bar{\xi}\,d\underline{\zeta}:P_{\mathscr{N}}(\varphi)e^{i\alpha\varphi(\mathscr{L}(\bar{\theta}))}:F_{\theta\mathscr{N}}(\bar{\underline{\xi}},\underline{\zeta},\bar{\underline{\sigma}};Q\neq0). \tag{1.44}$$

The relations (1.42) and (1.44) follow as a corollary of the next theorem.
The decompositions (1.42) and (1.44) are useful as we have a recursive expression for $F_{\theta,\mathscr{P}}(\underline{\xi},\underline{\sigma};Q=0)$ and $F_{\bar{\theta},\mathscr{N}}(\bar{\underline{\xi}},\underline{\zeta},\bar{\underline{\sigma}};Q\neq0)$:

Theorem 1.1.

$$F_{\theta,\mathscr{P}}(\underline{\xi},\underline{\sigma};Q=0)=\sum_{\substack{\mathscr{P}_1,\dots,\mathscr{P}_s\\|\mathscr{P}_i|\ \mathrm{even}}}\ \sum_{\mathscr{N}_1,\dots,\mathscr{N}_{\bar{s}}}\Bigg[\mathcal{O}_{\mathscr{P}'\Delta(\mathscr{P}_1\cup\dots\cup\mathscr{N}_{\bar{s}})}(W_{(\theta)}(\underline{\xi},\underline{\sigma}))$$
$$\times\prod_{i=1}^{s}F_{\theta_i,\mathscr{P}_i}(\underline{\xi}^{(i)},\underline{\sigma}^{(i)};Q_i=0)\prod_{j=1}^{\bar{s}}F_{\bar{\theta}_j,\mathscr{N}_j}(\underline{\bar{\xi}}^{(j)},\underline{\zeta}^{(i)},\underline{\bar{\sigma}}^{(j)};Q_j\neq 0)\Bigg],\tag{1.45}$$

$$F_{\bar{\theta},\mathscr{N}}(\underline{\bar{\xi}},\underline{\zeta},\underline{\bar{\sigma}};Q\neq 0))=\sum_{\substack{\mathscr{P}_1,\dots,\mathscr{P}_s\\|\mathscr{P}_i|\ \mathrm{even}}}\ \sum_{\mathscr{N}_1,\dots,\mathscr{N}_{\bar{s}}}\Bigg[O_{\mathscr{N}\Delta(\mathscr{P}_1\cup\dots\cup\mathscr{N}_{\bar{s}})}(W_{(\theta)}(\underline{\bar{\xi}},\underline{\zeta},\underline{\bar{\sigma}})$$
$$\times\prod_{i=1}^{s}F_{\theta_i,\mathscr{P}_i}(\underline{\xi}^{(i)},\underline{\sigma}^{(i)};Q_i=0)\prod_{j=1}^{\bar{s}}F_{\bar{\theta}j,\mathscr{N}_j}(\underline{\bar{\xi}}^{(j)},\underline{\zeta}^{(j)},\underline{\bar{\sigma}}^{(j)};Q_j\neq 0)\Bigg],\tag{1.46}$$

where $\mathcal{O}_{\mathscr{N}}$ operates only on the $\bar{\xi}$ coordinates.

We have assumed that at the lowest bifurcation $h(\theta)\,(h(\bar{\theta}))$, s neutral trees $\theta_1,\dots,\theta_s$ and $\bar{s}$ non-neutral trees $\bar{\theta}_1,\dots,\bar{\theta}_{\bar{s}}$ merge as in Fig. 1. The symbol Δ in $\mathscr{P}\Delta\mathscr{G}$ stands for the symmetric difference $(\mathscr{P}\backslash\mathscr{G})\cup(\mathscr{G}\backslash\mathscr{P})$, thus when $\ell\in\mathscr{P}\Delta\mathscr{G}$ we know that the function $\mathcal{O}_{\mathscr{P}\Delta\mathscr{G}}(f)$ is odd under the exchange $\xi_{2\ell-1}\leftrightarrow\xi_{2\ell}$.

The function W is associated to the truncated expectation at the lowest frequency; its explicit expression is given in the proof of the theorem. The set $\mathscr{P}'$ and the operation $\mathcal{O}_{\mathscr{P}'\Delta(\mathscr{P}_1\cup\dots\cup\mathscr{N}_{\bar{s}})}$ will be defined in the course of the proof as well.

Proof. We first consider the case when $Q=0$: starting from the lowest bifurcation, $h(\theta)=k+1$, the tree θ looks like:

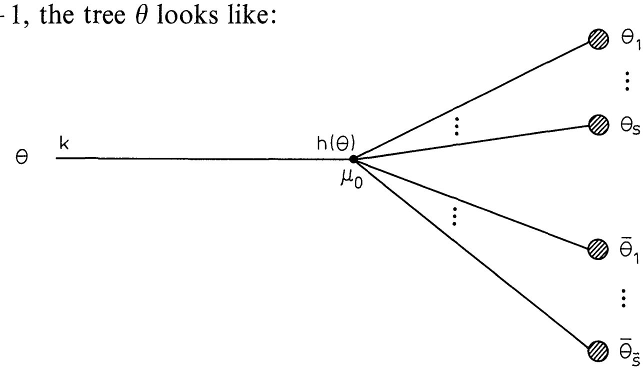

Fig. 1

Given the subtrees $\theta_1,\dots,\theta_s$ and $\bar{\theta}_1,\dots,\bar{\theta}_{\bar{s}}$ we assume the expressions (1.42) and (1.44) for $\tilde{V}(\theta_i,\underline{\sigma}^{(i)})$ and $\tilde{V}(\bar{\theta}_j,\underline{\bar{\sigma}}^{(j)})$:

$$\tilde{V}(\theta,\underline{\sigma})=\sum_{\mu_0=\pm1}\mathscr{E}^T_{k+1}(\tilde{V}(\theta_1,\underline{\sigma}^{(1)}),\dots,\tilde{V}(\theta_s,\underline{\sigma}^{(s)}),\tilde{V}(\bar{\theta}_1,\mu_0\underline{\bar{\sigma}}^{(1)}),\dots,\tilde{V}(\bar{\theta}_{\bar{s}},\mu_0\underline{\bar{\sigma}}^{(\bar{s})}))$$
$$=\sum_{\substack{\mathscr{P}_1\dots\mathscr{P}_s\\|\mathscr{P}_i|\ \mathrm{even}}}\ \sum_{\mathscr{N}_1\dots\mathscr{N}_{\bar{s}}}\left(\frac{\lambda}{2}\right)^{2m}\int d\underline{\xi}^{(1)}\dots d\underline{\xi}^{(s)}\int d\underline{\bar{\xi}}^{(1)}\dots d\underline{\bar{\xi}}^{(\bar{s})}d\underline{\zeta}^{(1)}\dots d\underline{\zeta}^{(\bar{s})}$$
$$\times\sum_{\mu_0}\mathscr{E}^T_{k+1}(:P_{\mathscr{P}_1}(\varphi^{(\leqq k+1)}:,\dots,:P_{\mathscr{P}_s}(\varphi^{(\leqq k+1)}):,$$
$$:P_{\mathscr{N}_1}(\varphi^{(\leqq k+1)})e^{i\alpha\mu_0\varphi(\mathscr{L}_1)}:,\dots,:P_{\mathscr{N}_{\bar{s}}}(\varphi^{(\leqq k+1)})e^{i\alpha\mu_0\varphi(\mathscr{L}_{\bar{s}})})$$
$$\times\left[\prod_{i=1}^{s}F_{\theta_i,\mathscr{P}_i}(\underline{\xi}^{(i)},\underline{\sigma}^{(i)};Q_i=0)\prod_{j=1}^{\bar{s}}F_{\bar{\theta}_j,\mathscr{N}_j}(\underline{\bar{\xi}}^{(j)},\underline{\zeta}^{(j)},\underline{\bar{\sigma}}^{(j)};Q_j\neq 0)\right],\tag{1.47}$$

where $\mathscr{L}_j=\mathscr{L}(\bar{\theta}_j)$ [cf. Eq. (1.43)].

We remark that $F_{\bar{\theta}_j, \mathcal{N}_j}$ does not depend on μ_0 as it is left invariant under the simultaneous change of sign of all charges of $\bar{\theta}_j$.

Moreover, from definition (1.33), we can write:

$$P_{\mathscr{P}}(\varphi^{(\leqq k+1)}) = \frac{1}{2^m} \sum_{\underline{\tau} \in \{-1,+1\}^m} \tau_{\mathscr{P}} e^{i\alpha\mu_0 \underline{\tau} \cdot \underline{\Delta}\varphi^{(\leqq k+1)}} - \delta_{\mathscr{P},\emptyset}, \tag{1.48}$$

where $\underline{\tau} = (\tau_1, \ldots, \tau_m)$, $\tau_{\mathscr{P}} = \tau_{\ell_1} \cdot \tau_{\ell_2} \cdot \ldots \cdot \tau_{\ell_q}$ for $\mathscr{P} = \{\ell_1, \ldots, \ell_q\}$, and

$$\underline{\tau} \cdot \underline{\Delta}\varphi^{(\leqq k+1)} := \sum_{\ell=1}^{m} \tau_\ell \Delta_\ell \varphi^{(\leqq k+1)}, \tag{1.49}$$

and further,

$$P_{\mathcal{N}}(\varphi^{(\leqq k+1)}) = \frac{1}{2^{\bar{m}}} \sum_{\bar{\underline{\tau}}} \bar{\tau}_{\mathcal{N}} e^{i\alpha\mu_0 \bar{\underline{\tau}} \cdot \underline{\Delta}\varphi^{(\leqq k+1)}}, \tag{1.50}$$

where $\bar{\underline{\tau}}$, $\bar{\tau}_{\mathcal{N}}$, $\mathcal{N}$, and $\bar{\underline{\tau}} \cdot \underline{\Delta}$ are defined analogously. The -1 present in $P_\emptyset$ has been neglected since it has no effect in the truncated expectations.

Recalling the relation

$$:e^{i\alpha\varphi^{(\leqq k+1)}}: \equiv :e^{i\alpha\varphi^{(\leqq k)}}::e^{i\alpha\varphi^{(k+1)}}: \tag{1.51}$$

and using (1.48) and (1.50) to compute $\sum_{\mu_0} \mathscr{E}^T_{k+1}(\cdot)$ of (1.47) we get

$$\sum_{\mu_0} \mathscr{E}^T_{k+1}(\cdot) = \sum_{\underline{\tau}} :\cos\alpha\left(\underline{\tau} \cdot \underline{\Delta}\varphi^{(\leqq k)} + \sum_{j=1}^{\bar{s}} \varphi^{(\leqq k)}(\mathscr{L}_j)\right): \tau_{\mathscr{P}_1} \ldots \tau_{\mathscr{P}_s} \tau_{\mathcal{N}_1} \ldots \tau_{\mathcal{N}_{\bar{s}}} W_{(\theta)}(\underline{\xi}, \underline{\tau}), \tag{1.52}$$

where

$$\underline{\tau} := \underline{\tau}^{(1)} \oplus \ldots \oplus \underline{\tau}^{(s)} \oplus \bar{\underline{\tau}}^{(1)} \oplus \ldots \oplus \bar{\underline{\tau}}^{(\bar{s})}, \tag{1.53}$$

and

$$\begin{aligned} W_{(\theta)}(\underline{\xi}, \underline{\tau}) = {} & \frac{1}{2^{c(\underline{m}, \bar{\underline{m}})-1}} \cdot e^{U^{(\leqq k)}(\theta_1, \ldots, \theta_s, \bar{\theta}_1, \ldots, \bar{\theta}_{\bar{s}}; \underline{\tau}^{(1)}, \ldots, \underline{\tau}^{(s)}, \bar{\underline{\tau}}^{(1)}, \ldots, \bar{\underline{\tau}}^{(\bar{s})})} \\ & \times \mathscr{E}^T_{k+1}(:e^{i\alpha\underline{\tau}^{(1)} \cdot \underline{\Delta}\varphi^{(k+1)}}:, \ldots, :e^{i\alpha\underline{\tau}^{(s)} \cdot \underline{\Delta}\varphi^{(k+1)}}:, \\ & :e^{i\alpha(\bar{\underline{\tau}}^{(1)} \cdot \underline{\Delta}\varphi^{(k+1)} + \varphi^{(k+1)}(\mathscr{L}_1))}:, \ldots, :e^{i\alpha(\bar{\underline{\tau}}^{(\bar{s})} \cdot \underline{\Delta}\varphi^{(k+1)} + \varphi^{(k+1)}(\mathscr{L}_{\bar{s}}))}:) \end{aligned}$$

$$c(\underline{m}, \bar{\underline{m}}) := \sum_{i=1}^{s} m_i + \sum_{j=1}^{\bar{s}} \bar{m}_j, \tag{1.55}$$

where $U^{(\leqq k)}(\ldots)$ is the interaction energy between the clusters $\theta_1, \ldots, \theta_s(\bar{\theta}_1, \ldots, \bar{\theta}_{\bar{s}})$ and $\underline{\tau}^{(i)}(\bar{\underline{\tau}}^{(\ell)})$ the vector defining the charges of $\theta_i(\bar{\theta}_\ell)$ with coordinates $\xi_j^{(i)}(\bar{\xi}_j^{(\ell)})$ (j odd).

Finally, we decompose $:\cos\alpha\left(\underline{\tau} \cdot \underline{\Delta}\varphi^{(\leqq k)} + \sum_{j=1}^{\bar{s}} \varphi^{(\leqq k)}(\mathscr{L}_j)\right):$ once more following Eq. (1.32) and obtaining:

$$:\cos\alpha(\underline{\tau} \cdot \underline{\Delta}\varphi^{(\leqq k)} + \underline{1} \cdot \underline{\Delta}\varphi^{(\leqq k)}): = \sum_{|\mathscr{P}|\ \text{even}} :P_{\mathscr{P}}(\varphi^{(\leqq k)}): \cdot \tau_{\mathscr{P}'} + 1\,. \tag{1.56}$$

The $+1$ will be cancelled by the corresponding part of the counterterm, and $\mathscr{P} \subset \{1, \ldots, m\}$. Note that

$$\mathscr{P}' := \mathscr{P} \cap (\text{subset of } \{1, \ldots, m\} \text{ in which } \tau_i \text{ is present}). \tag{1.57}$$

We have

$$\sum_{\mu_0} \mathscr{E}_{k+1}^{\mathscr{P}T}(\cdot) = \sum_{|\mathscr{P}|\text{ even}} {}_{\mathscr{P}} : P_{\mathscr{P}}(\varphi^{(\leq k)}) : \sum_{\underline{\tau}} (\tau_{\mathscr{P}'}, \tau_{\mathscr{P}_1} \dots \tau_{\mathscr{P}_s} \tau_{\mathscr{N}_1} \dots \tau_{\mathscr{N}_{\bar{s}}}) W_{(\theta)}(\underline{\xi}, \underline{\tau}). \quad (1.58)$$

The equalities (1.45) and (1.46) are proven using the explicit expression of $W_{(\theta)}$ given in Eq. (1.55): From this expression we recognize that changing the sign of $\tau_\ell^{(i)}$ is equivalent to exchanging $\xi_{2\ell-1}^{(i)} \leftrightarrow \xi_{2\ell}^{(i)}$ in the function $W_{(\theta)}(\underline{\xi}, \underline{\tau})$. Therefore

$$\sum_{\underline{\tau}} (\tau_{\mathscr{P}}, \tau_{\mathscr{P}_1} \dots \tau_{\mathscr{P}_s} \tau_{\mathscr{N}_1} \dots \tau_{\mathscr{N}_{\bar{s}}}) W_{(\theta)}(\underline{\xi}, \underline{\tau}) \sim \mathcal{O}_{\mathscr{P}'\Delta(\mathscr{P}_1 \cup \dots \cup \mathscr{P}_s \cup \mathscr{N}_1 \cup \dots \cup \mathscr{N}_{\bar{s}})}(W_\theta(\underline{\xi}, \underline{\sigma})) \quad (1.59)$$

and

$$F_{\theta, \mathscr{P}}(\underline{\xi}, \bar{\sigma}; Q=0) = \sum_{\substack{\mathscr{P}_1, \dots, \mathscr{P}_s \\ |\mathscr{P}_i| \text{ even}}} \sum_{\mathscr{N}_1 \dots \mathscr{N}_{\bar{s}}} \Bigg[\mathcal{O}_{\mathscr{P}'\Delta(\mathscr{P}_1 \cup \dots \cup \mathscr{N}_{\bar{s}})}(W_\theta(\underline{\xi}, \underline{\sigma})) \times \prod_{i=1}^{s} F_{\theta_i, \mathscr{P}_i}(\underline{\xi}^{(i)}, \underline{\sigma}^{(i)}; Q_i = 0) \prod_{j=1}^{\bar{s}} F_{\bar{\theta}_j, \mathscr{N}_j}(\underline{\bar{\xi}}^{(j)}, \underline{\zeta}^{(j)}, \underline{\bar{\sigma}}^{(j)}; Q_j \neq 0) \Bigg]. \quad (1.60)$$

Remark. The operator $\mathcal{O}_{\mathscr{P}'\Delta(\mathscr{P}_1 \cup \dots \cup \mathscr{N}_{\bar{s}})}$ is a symmetrization or antisymmetrization operator only with respect to those variables in the $\Delta_\ell \varphi$'s which are multiplied by the τ's [cf. Eq. (1.56) and Eq. (1.59)]. Thus, Eqs. (1.45) is proven; the case when $Q \neq 0$ [Eq. (1.46)] is proven analogously. □

2.1. The Smooth Part of the Effective Potential

The smooth part of the effective potential is defined by subtracting from the regions of integration in $\tilde{V}^{(k)}[\Lambda]$ those parts in which the fields are insufficiently Hölder continuous. We first define the basic field-dependent sets of which these regions consist; they describe in terms of pairs of variables $(\xi_{2\ell-1}, \xi_{2\ell})$, where the fields are insufficiently Hölder continuous (with respect to a given modulus B_k):

$$D_\ell^{(k)}(\varphi^{(\leq k)}) := D_\ell^{(k)} := \left\{ (\xi_{2\ell-1}, \xi_{2\ell}) \in \Lambda^2 \middle| \left| \sin \frac{\alpha}{2} \Delta_\ell \varphi^{(\leq k)} \right| > B_k (\gamma^k |\xi_{2\ell-1} - \xi_{2\ell}|)^{1-\varepsilon} \right\}, \quad (2.1)$$

and

$$R_\ell^{(k)}(\varphi^{(k)}) := R_\ell^{(k)} := \Big\{ \Delta \in Q_k | \exists \xi_{\tilde{\ell}} \in \Delta, \eta \in \Lambda \text{ such that } \gamma^k |\xi_{\tilde{\ell}} - \eta| < 1$$

and

$$\left| \sin \frac{\alpha}{2} (\varphi_{\xi_{\tilde{\ell}}}^{(k)} - \varphi_\eta^{(k)}) \right| > \frac{B_k}{\sigma} (\gamma^k |\xi_{\tilde{\ell}} - \eta|)^{1-\varepsilon} (1 + \gamma^k d(\Delta, \Lambda)) \Big\}, \quad (2.2)$$

where Q_k is a pavement of $\mathbb{R}^2$ on scale k; e.g. a set of squares Δ with side length γ^{-k}. The constant $\sigma > 1$ is to be chosen sufficiently large subsequently [cf. (2.19), (3.6)].

The strictly increasing succession $\{B_k\}$ is chosen as in [G] with $B > 1$ and $a > 2$ arbitrarily fixed:

$$B_k := B \log(e + k + \lambda^{-1}) (1 + k)^a. \quad (2.3)$$

Note that $B_k > 1$ and $B_{k+1}/B_k > 1$ for all $k \in \mathbb{N}$.

In analogy with Eq. (1.29) we write

$$\hat{V}^{(k)}[\mathscr{D}_k^c] := \sum_{n=1}^{t} \sum_{(s,\{Q_v\}_s,\underline{\sigma})} \sum_{\theta \left\{ \substack{s(\theta)=s \\ k(\theta)=k} \right.} \hat{V}(\theta,\underline{\sigma})[\mathscr{D}_k^c], \tag{2.4}$$

with

$$\hat{V}(\theta,\underline{\sigma})[\mathscr{D}_k^c] := \left(\frac{\lambda}{2}\right)^{2m} \sum_{|\mathscr{P}| \text{ even}} \int_{\mathscr{D}_k^c(\mathscr{P})} d\underline{\xi} : P_{\mathscr{P}}(\varphi^{(\leq k)}) : F_{\theta,\mathscr{P}}(\underline{\xi},\underline{\sigma}; Q=0),$$

$$\hat{V}(\bar{\theta},\bar{\underline{\sigma}})[\mathscr{D}_k^c] := \left(\frac{\lambda}{2}\right)^{2\bar{m}+\bar{p}} \sum_{\mathscr{N}} \int_{\mathscr{D}_k^c(\mathscr{N}) \times \Lambda^{\bar{p}}} d\underline{\bar{\xi}}\, d\underline{\zeta} : P_{\mathscr{N}}(\varphi^{(\leq k)}) : F_{\theta,\mathscr{N}}(\underline{\bar{\xi}},\underline{\zeta},\bar{\underline{\sigma}}; Q\neq 0). \tag{2.5}$$

Remember when $Q \neq 0$, we write $\bar{\underline{\sigma}}$ for $\underline{\sigma}$; and remark that when $\mathscr{P}=\emptyset$, $:P_\emptyset:$ is thought of as decomposed in the following way:

$$:P_\emptyset(\varphi): := \sum_{\ell=1}^{n} :P_{\emptyset_\ell}(\varphi): := \sum_{\ell=1}^{n} :(\cos \Delta_\ell \varphi - 1) \prod_{j=\ell+1}^{n} \cos \Delta_j \varphi : .$$

Furthermore, we have

$$\mathscr{D}_k^c(\mathscr{P}) := \Lambda^2 \times \ldots \times \Lambda^2 \backslash D_{\ell_1}^{(k)} \times \ldots \times \Lambda^2 \backslash D_{\ell_q}^{(k)} \times \ldots \times \Lambda^2, \tag{2.6}$$

and a similar definition for $\mathscr{D}_k^c(\mathscr{N})$; $\mathscr{D}_k^c$ stands symbolically for either $\mathscr{D}_k^c(\mathscr{P})$ or $\mathscr{D}_k^c(\mathscr{N})$. Note that when $\mathscr{P}=\emptyset$ we use the above decomposition and define

$$\mathscr{D}_k^c(\emptyset_\ell) := \Lambda^2 \times \ldots \times \Lambda^2 \backslash D_\ell^{(k)} \times \ldots \times \Lambda^2 .$$

From (2.6) it follows for the complement of $\mathscr{D}_k^c(\mathscr{P})$, i.e. for the region of insufficient Hölder continuity

$$\mathscr{D}_k(\mathscr{P}) = \bigcup_{\ell=1}^{q} \Lambda^2 \times \ldots \times D_\ell^{(k)} \times \ldots \times \Lambda^2 . \tag{2.7}$$

Similarly we define:

$$\mathscr{R}_k(\mathscr{P}) := \bigcup_{i=1}^{q} \{(\Lambda \times \ldots \times R_{2\ell_i-1} \times \ldots \times \Lambda) \cup (\Lambda \times \ldots \times R_{2\ell_i} \times \ldots \times \Lambda)\} . \tag{2.8}$$

We thus have:

$$\begin{aligned} \mathscr{R}_k^c(\mathscr{P}) = \Lambda^n \backslash \mathscr{R}_k(\mathscr{P}) = \Lambda^2 \times \ldots \times \{(\Lambda \backslash R_{2\ell_1-1}^{(k)}) \times (\Lambda \backslash R_{2\ell_1}^{(k)})\} \times \ldots \\ \ldots \times \{(\Lambda \backslash R_{2\ell_q-1}^{(k)}) \times (\Lambda \backslash R_{2\ell_q}^{(k)})\} \times \ldots \times \Lambda^2 . \end{aligned} \tag{2.9}$$

Furthermore, we define:

$$\mathscr{R}_k^{(2)}(\mathscr{P}) := \bigcup_{i=1}^{q} \Lambda^2 \times \ldots \times R_{2\ell_i-1}^{(k)} \times R_{2\ell_i}^{(k)} \times \ldots \times \Lambda^2 . \tag{2.10}$$

We note that

$$\mathscr{R}_k^{(2)}(\mathscr{P}) \subset \mathscr{R}_k(\mathscr{P}) . \tag{2.11}$$

Formulae (2.7) thru (2.11) obviously hold for $\mathscr{N}$ as well.

Let us now prove an important lemma.

Lemma 2.1. *For all integers k and all sets of indices $\mathscr{P}=\{\ell_1, \ldots, \ell_q\}$, we have:*

$$\mathscr{D}_k(\mathscr{P}) \subset \mathscr{D}_{k-1}(\mathscr{P}) \cup \mathscr{R}_k^{(2)}(\mathscr{P}) . \tag{2.12}$$

Proof. The proof is easily reduced to that of an analogous inclusion for the sets defined in terms of pairs of variables (omitting subscripts):

$$D^{(k)} \subset D^{(k-1)} \cup R^{(k)2} . \tag{2.13}$$

We show the converse, that is, for $(\xi, \eta) \notin D^{(k-1)} \cup R^{(k)2}$, we show $(\xi, \eta) \notin D^{(k)}$:

$$(\xi, \eta) \notin D^{(k-1)} \Leftrightarrow \left|\sin\frac{\alpha}{2}(\varphi_\xi^{(\leqq k-1)} - \varphi_\eta^{(\leqq k-1)})\right| \leqq B_{k-1}(\gamma^{k-1}|\xi-\eta|)^{1-\varepsilon}, \tag{2.14}$$

and

$$(\xi, \eta) \in \Lambda^2 \backslash R^{(k)2} \Leftrightarrow$$

either

$$\left|\sin\frac{\alpha}{2}(\varphi_\xi^{(k)} - \varphi_\eta^{(k)})\right| \leqq \frac{B_k}{\sigma}(\gamma^k|\xi-\eta|)^{1-\varepsilon} \tag{2.15}$$

or

$$\gamma^k|\xi-\eta| \geqq 1 \Rightarrow B_k(\gamma^k|\xi-\eta|)^{1-\varepsilon} \geqq B_k > 1 . \tag{2.16}$$

The latter, however, immediately implies $(\xi, \eta) \notin D^{(k)}$. For the former we apply (2.14) onto the triangular inequality

$$\begin{aligned} &\left|\sin\frac{\alpha}{2}(\varphi_\xi^{(\leqq k-1)} - \varphi_\eta^{(\leqq k-1)})\right| + \left|\sin\frac{\alpha}{2}(\varphi_\xi^{(k)} - \varphi_\eta^{(k)})\right| \\ &\geqq \left|\sin\frac{\alpha}{2}(\varphi_\xi^{(\leqq k)} - \varphi_\eta^{(\leqq k)})\right| , \end{aligned} \tag{2.17}$$

and thus

$$\begin{aligned} &\frac{B_{k-1}}{B_k \gamma^{1-\varepsilon}} B_k(\gamma^k|\xi-\eta|)^{1-\varepsilon} + \frac{B_k}{\sigma}(\gamma^k|\xi-\eta|)^{1-\varepsilon} \\ &\geqq \left|\sin\frac{\alpha}{2}(\varphi_\xi^{(\leqq k)} - \varphi_\eta^{(\leqq k)})\right| , \end{aligned} \tag{2.18}$$

which implies $(\xi, \eta) \notin D^{(k)}$, since we can pick a finite $\sigma_1(\theta_1)$ large enough so that for any θ_1 such that $\gamma^{-(1-\varepsilon)} < \theta_1 < 1$,

$$\frac{B_{k-1}}{B_k \gamma^{1-\varepsilon}} + \frac{1}{\sigma} \leqq \theta_1 \quad \text{for all } k \in \mathbb{N} \text{ and all } \sigma > \sigma_1(\theta_1) . \tag{2.19}$$

The statement of Lemma 2.1 is illustrated in Fig. 2. □

As already mentioned in Sect. 1.2, Lemma 2.1 suggests that we consider $\hat{V}^{(k)}[\mathscr{D}_{k-1}^c \cap \mathscr{R}_k^c]$ as that part of the effective potential to which the Main Lemma (cf. Sect. 2.3) is applied. $\hat{V}^{(k)}[\mathscr{D}_{k-1}^c \cap \mathscr{R}_k^c]$ is defined in complete analogy to $\hat{V}^{(k)}[\mathscr{D}_k^c]$, just replacing $[\mathscr{D}_k^c]$ by $[\mathscr{D}_{k-1}^c \cap \mathscr{R}_k^c]$. Furthermore, we note that the difference of $\hat{V}^{(k)}[\mathscr{D}_k^c]$ and $\hat{V}^{(k)}[\mathscr{D}_{k-1}^c \cap \mathscr{R}_k^c]$ can be written as [cf. (1.20)]:

$$\hat{V}^{(k)}[\mathscr{D}_k^c] - \hat{V}^{(k)}[\mathscr{D}_{k-1}^c \cap \mathscr{R}_k^c] = \hat{V}^{(k)}[\mathscr{D}_k^c \cap \mathscr{D}_{k-1} \cap \mathscr{R}_k^c] + \hat{V}^{(k)}[\mathscr{D}_k^c \cap \mathscr{R}_k] , \tag{2.20}$$

where $[\mathscr{D}_k^c \cap \mathscr{D}_{k-1} \cap \mathscr{R}_k^c]$ again symbolizes either

$$\{\mathscr{D}_k^c(\mathscr{P}) \cap \mathscr{D}_{k-1}(\mathscr{P}) \cap \mathscr{R}_k^c(\mathscr{P})\}$$

or

$$\{\mathscr{D}_k^c(\mathscr{N}) \cap \mathscr{D}_{k-1}(\mathscr{N}) \cap \mathscr{R}_k^c(\mathscr{N})\} , \tag{2.21}$$

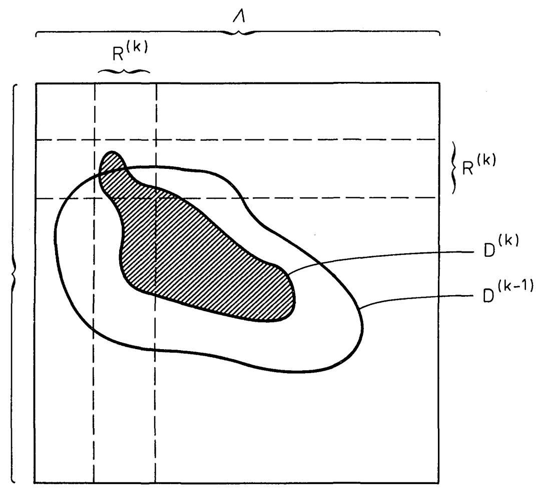

Fig. 2

and $[\mathscr{D}_k^c\cap\mathscr{R}_k]$ is to be understood analogously. Observe that (2.20) is true because, due to Lemma 2.1, $\mathscr{D}_k\cap\mathscr{D}_{k-1}^c\cap\mathscr{R}_k^c$ is the empty set.

Furthermore, we remark that the sets in $[\mathscr{D}_k^c\cap\mathscr{R}_k]$ are of the form:

$$\begin{aligned}\mathscr{D}_k^c(\mathscr{P})\cap\mathscr{R}_k(\mathscr{P})=\bigcup_{i=1}^{q}\{&\Lambda^2\times\dots\times\Lambda^2\backslash D^{(k)}_{\ell_1}\times\dots\times(R^{(k)}_{2\ell_i-1}\times\Lambda)\backslash D^{(k)}_{\ell_i}\times\dots\\ &\dots\times\Lambda^2\backslash D^{(k)}_{\ell_q}\times\dots\times\Lambda^2\cup\Lambda^2\times\dots\times\Lambda^2\backslash D^{(k)}_{\ell_1}\times\dots\\ &\dots\times(\Lambda\times R^{(k)}_{2\ell_i})\backslash D^{(k)}_{\ell_i}\times\dots\times\Lambda^2\backslash D^{(k)}_{\ell_q}\times\dots\times\Lambda^2\}.\end{aligned}\tag{2.22}$$

We finally give an explicit expression for $\Delta^{(k)}[\mathscr{D}_k]$ defined by (1.18),

$$\Delta^{(k)}[\mathscr{D}_k]=\sum_{n=2}^{t}\sum_{(s,\{Q_v\}_s,\bar{\sigma})}\sum_{\theta\,\left\{\substack{s(\theta)=s\\ k(\theta)=k\\ h(\theta)=k+1}\right.}\Delta^{(k)}(\theta,\bar{\sigma})[\mathscr{D}_k],$$

where (for $Q=0$)

$$\begin{aligned}&\Delta^{(k)}(\theta,\bar{\sigma})[\mathscr{D}_k]\\ &=\left(\frac{\lambda}{2}\right)^n\sum_{|\mathscr{P}|\,\mathrm{even}}\Bigg\{\sum_{|\mathscr{P}_i|\,\mathrm{even}}{}_{\mathscr{P}_1,\dots,\mathscr{P}_s}\sum{}_{\mathscr{N}_1,\dots,\mathscr{N}_{\bar{s}}}\int_{\mathscr{D}_k^c(\mathscr{P}_1)\times\dots\times\mathscr{D}_k^c(\mathscr{P}_s)\times\mathscr{D}_k^c(\mathscr{N}_1)\times\dots\times\mathscr{D}_k^c(\mathscr{N}_{\bar{s}})}d\underline{\xi}\\ &\quad:P_{\mathscr{P}}(\varphi^{(\leq k)}):\mathcal{O}_{\mathscr{P}}\Bigg[\mathcal{O}_{\mathscr{P}'\Delta(\mathscr{P}_1\cup\dots\cup\mathscr{N}_{\bar{s}})}(W_\theta(\underline{\xi},\bar{\sigma}))\cdot\prod_{i=1}^{s}F_{\theta_i,\mathscr{P}_i}(\underline{\xi}^{(i)},\bar{\sigma}^{(i)};Q_i=0)\\ &\quad\times\prod_{j=1}^{s}F_{\bar{\theta}_j,\mathscr{N}_j}(\underline{\bar{\xi}}^{(j)},\underline{\zeta}^{(j)},\bar{\bar{\sigma}}^{(j)};Q_j\neq0)\Bigg]\\ &\quad-\int_{\mathscr{D}_k^c(\mathscr{P})}d\underline{\xi}:P_{\mathscr{P}}(\varphi^{(\leq k)}):F_{\theta,\mathscr{P}}(\underline{\xi},\bar{\sigma};Q=0)\Bigg\},\end{aligned}\tag{2.23}$$

and a similar expression for the non-neutral case. Note that the only difference between the two terms in (2.23) consists in their regions of integration.

Recalling definition (1.18) of $\Delta^{(k)}[\mathscr{D}_k]$ and the fact that all trees with root k having their first bifurcation at a frequency $h>k+1$ come from the simple expectation $\mathscr{E}_{k+1}(\hat{V}^{(k+1)}[\mathscr{D}_k^d])$, it becomes clear that $\Delta^{(k)}[\mathscr{D}_k]$ does not contain any of these trees. [For these trees the regions $\mathscr{D}_k(\mathscr{P})$, $\mathscr{D}_k(\mathscr{N})$ already have their correct "position."] In other words, all trees in $\Delta^{(k)}[\mathscr{D}_k]$ have their first bifurcation fixed at the frequency $k+1$. This fact will turn out to be crucial in the following.

2.2. Estimates of the Effective Potential

In this section we give essentially three types of estimates, all of which will be useful for the proof of the iterative procedure discussed in Sect. 3. The first type is mainly used to provide an estimate of the smooth part of the effective potential which shows that the Main Lemma can be applied to it and that the remainder terms are well behaved. The second type of estimate serves to "factorize" certain tree contributions from $\Delta^{(k)}[\mathscr{D}_k]$ and $[\hat{V}^{(k)}[\mathscr{D}_k^c \cap \mathscr{R}_k^c] - \hat{V}^{(k)}[\mathscr{D}_{k-1}^c \cap \mathscr{R}_k^c]]$ of order greater than two into a second order tree and into a remainder onto which the first estimate can be applied. The motivation for this operation becomes evident in the context of the third type of estimate which is the easiest one, since only neutral trees of second order integrated on a large fluctuation region are concerned. The third estimate shows that these terms are in fact negative. In the iterative procedure they will be used to go from (1.21) to (1.22) in the scheme presented in Sect. 1.2. In other words, we will show that these second order trees render the large fluctuation part of the effective potential negative. The second estimate makes sure that the second order trees actually dominate the higher order contributions to the effective potential.

Consider the contribution $\hat{V}(\theta, \underline{\sigma})\,[\mathscr{D}_k^c]$ to the effective potential of frequency k as given by (2.5). All zeroes of the field dependent part are effective. They can be estimated by (for $Q=0$)

$$(\mathrm{const})B_k^{|\mathscr{P}|}\gamma^{k|\mathscr{P}|(1-\varepsilon)}[\text{zeroes}, \mathscr{P}]$$

(and, obviously, we have a similar expression with $\mathscr{P}$ substituted by $\mathscr{N}$ for $Q \neq 0$), where [zeroes, $\mathscr{P}$] is defined by ($\mathscr{P}=\{\ell_1, \ldots, \ell_q\}$):

$$[\text{zeroes}, \mathscr{P}] := \prod_{j=1}^{q} |\xi_{2\ell_j-1} - \xi_{2\ell_j}|^{(1-\varepsilon)}. \tag{2.24}$$

The quotient

$$\left| \frac{:P_{\mathscr{P}}(\varphi^{(\leq k)}):}{\gamma^{k|\mathscr{P}|(1-\varepsilon)}[\text{zeroes}, \mathscr{P}]} \right|$$

does not have any zeroes, and as we consider a region in which the field $\varphi^{(\leq k)}$ is Hölder continuous with modulus B_k, it is in fact bounded by $(\mathrm{const})B_k^{|\mathscr{P}|}$. For $\mathscr{P}=\emptyset$ we define

$$[\text{zeroes}, \emptyset] := \gamma^{2k(1-\varepsilon)}(d(\underline{\xi}))^{2(1-\varepsilon)}, \tag{2.25}$$

where $d(\underline{\xi})$ is the length of the shortest polygonal connecting the points $\xi_1, \ldots, \xi_{v(\theta)}$.

In order to further analyze the factor $\mathcal{O}_{\mathscr{P}}(F_\theta)$ in (1.42) only depending on the covariances, we use (1.45) and (1.55). First, we rewrite (1.55) in the following way:

$$W_{(\theta)}(\underline{\xi}, \underline{\tau}) = e^{U^{(\leqq k)}(\theta_1, \ldots, \bar{\theta}_{\bar{s}}; \underline{\tau}^{(1)}, \ldots, \underline{\tau}^{(\bar{s})})} \cdot \frac{1}{2^{c(m,\bar{m})-1}}$$
$$\times \exp\left(-\frac{\alpha^2}{2}\left(\sum_{i=1}^{s} C^{(\leqq k)}_{\theta_i\theta_i}(0) + \sum_{j=1}^{\bar{s}} C^{(\leqq k)}_{\bar{\theta}_j\bar{\theta}_j}(0)\right)\right)$$
$$\times \exp\left(+\frac{\alpha^2}{2}\left(\sum_{i=1}^{s} C^{(\leqq k)}_{\theta_i\theta_i}(0) + \sum_{j=1}^{\bar{s}} C^{(\leqq k)}_{\bar{\theta}_j\bar{\theta}_j}(0)\right)\right)\mathscr{E}^{\mathcal{O}T}_{k+1}(\ldots). \tag{2.26}$$

We use the definitions:

$$C^{(\leqq k)}_{\theta\theta} := \sum_{i,j\in\theta} \bar{\sigma}_i\bar{\sigma}_j C^{(\leqq k)}(\xi_i, \xi_j), \tag{2.27}$$

and

$$C^{(\leqq k)}_{\theta\theta}(0) := \sum_{i,j\in\theta} \bar{\sigma}_i\bar{\sigma}_j C^{(\leqq k)}(0,0) = Q_\theta^2(k+1)C^{(0)}(0), \tag{2.28}$$

where i and j run over the indices of all the final lines of the tree θ. Let us further define:

$$U_0^{(\leqq k)}(\theta_1, \ldots, \bar{\theta}_{\bar{s}}) := U^{(\leqq k)}(\theta_1, \ldots, \bar{\theta}_{\bar{s}}; \underline{\tau}^{(1)}, \ldots, \underline{\tau}^{(\bar{s})}) \quad \text{(all coordinates equal)}, \tag{2.29}$$

that is

$$U_0^{(\leqq k)}(\theta_1, \ldots, \bar{\theta}_{\bar{s}}) = -\frac{\alpha^2}{2}\sum_{\substack{\ell, s\\ \ell \neq s}} C^{(\leqq k)}_{\theta_\ell\theta_s}(0), \tag{2.30}$$

where ℓ, s both run over $(\ell, \ldots, s)$ and $(\ell, \ldots, \bar{s})$ and θ_ℓ can be either θ_i or $\bar{\theta}_j$ depending on its index.

We also introduce:

$$\delta U^{(\leqq k)}(\theta_1, \ldots, \bar{\theta}_{\bar{s}}; \underline{\tau}^{(1)}, \ldots, \underline{\tau}^{(\bar{s})}) := U^{(\leqq k)}(\theta_1, \ldots, \bar{\theta}_{\bar{s}}; \underline{\tau}^{(1)}, \ldots, \underline{\tau}^{(\bar{s})}) - U_0^{(\leqq k)}(\theta_1, \ldots, \bar{\theta}_s). \tag{2.31}$$

Using these definitions (2.26) becomes:

$$W_{(\theta)}(\underline{\xi}, \underline{\tau}) = e^{\delta U^{(\leqq k)}(\theta_1, \ldots, \bar{\theta}^{\bar{s}}; \underline{\tau}^{(1)}, \ldots, \underline{\tau}^{(\bar{s})})} \cdot e^{-\frac{\alpha^2}{2}C^{(\leqq k)}_{\theta\theta}(0)} \cdot \frac{1}{2^{c(m,\bar{m})-1}}$$
$$\times \prod_{i=1}^{s} e^{\frac{\alpha^2}{2}C^{(\leqq k)}_{\theta_i\theta_i}(0)} \cdot \prod_{j=1}^{\bar{s}} e^{\frac{\alpha^2}{2}C^{(\leqq k)}_{\bar{\theta}_j\bar{\theta}_j}(0)} \cdot \mathscr{E}^{\mathcal{O}T}_{k+1}$$
$$=: e^{-\frac{\alpha^2}{2}C^{(\leqq k)}_{\theta\theta}(0)} \cdot \delta W_{(\theta)}(\underline{\xi}, \underline{\tau}) \prod_{i=1}^{s} e^{\frac{\alpha^2}{2}C^{(\leqq k)}_{\theta_i\theta_i}(0)} \cdot \prod_{j=1}^{\bar{s}} e^{\frac{\alpha^2}{2}C^{(\leqq k)}_{\bar{\theta}_j\bar{\theta}_j}(0)}. \tag{2.32}$$

With this expression for $W_{(\theta)}(\underline{\xi}, \underline{\tau})$ we can rewrite (1.45) (and analogously the non-neutral part):

$$F_{\theta,\mathscr{P}}(\underline{\xi}, \underline{\bar{\sigma}}; Q=0) = e^{-\frac{\alpha^2}{2}C^{(\leqq k)}_{\theta\theta}(0)} \sum_{\substack{\mathscr{P}_1\ldots\mathscr{P}_s\\ |\mathscr{P}_i|\ \text{even}}} \sum_{\mathscr{N}_1\ldots\mathscr{N}_{\bar{s}}} [O_{\mathscr{P}'\Delta(\mathscr{P}_1\cup\ldots\cup\mathscr{N}_{\bar{s}})}(\delta W_\theta(\underline{\xi}, \underline{\sigma}))$$
$$\times \prod_{i=1}^{s} e^{\frac{\alpha^2}{2}C^{(\leqq k)}_{\theta_i\theta_i}(0)} \cdot F_{\theta_i,\mathscr{P}_i}(\underline{\xi}^{(i)}, \underline{\bar{\sigma}}^{(i)}; Q_i=0)$$
$$\times \prod_{j=1}^{\bar{s}} e^{\frac{\alpha^2}{2}C^{(\leqq k)}_{\bar{\theta}_j\bar{\theta}_j}(0)} \cdot F_{\bar{\theta}_j,\mathscr{N}_j}(\underline{\bar{\xi}}^{(j)}, \underline{\zeta}^{(j)}, \underline{\bar{\bar{\sigma}}}^{(j)}; Q_j \neq 0). \tag{2.33}$$

Using formula (2.33), the estimate of

$$\{\gamma^{k|\mathscr{P}|(1-\varepsilon)}[\text{zeroes}, \mathscr{P}]F_{\theta,\mathscr{P}}(\underline{\xi}, \underline{\sigma}; Q=0)\}$$

(respectively $\{\gamma^{k|\mathscr{N}|(1-\varepsilon)}[\text{zeroes}, \mathscr{N}]F_{\bar{\theta},\mathscr{N}}(\underline{\bar{\xi}}, \underline{\zeta}, \underline{\bar{\sigma}}; Q\neq 0)\}$) can be performed recursively provided that we previously prove the following fundamental lemma:

Lemma 2.2. *Let $h(\theta)=h$ be the lowest bifurcation frequency and $k(\theta)=k$ the root of the tree θ, then*

$$\gamma^{h(1-\varepsilon)|\mathscr{P}|}[(\text{zeroes}, \mathscr{P}]F_{\theta,\mathscr{P}}(\underline{\xi}, \underline{\sigma}; Q=0)$$

and

$$\gamma^{h(1-\varepsilon)|\mathscr{N}|}[\text{zeroes}, \mathscr{N}]F_{\bar{\theta},\mathscr{N}}(\underline{\bar{\xi}}, \underline{\zeta}, \underline{\bar{\sigma}}; Q\neq 0)$$

have at each neutral bifurcation $v>v_0$ a "zero" of second order, where by "zero" we mean that in the estimates of these functions at each neutral bifurcation – different from the lowest one – we have a factor $\gamma^{-2(h_v-h_{v'})(1-\varepsilon)}$ or $\gamma^{-2h_{v'}(1-\varepsilon)}|\xi-\xi'$, where v' is the bifurcation immediately following v (going from top to bottom) and $h_v, h_{v'}$ are the associated frequencies.

Proof. The proof will be by induction; we assume it true for the trees with final bifurcation of order n and we prove it for a tree with final bifurcation of order $n+1$. Let θ be such a tree ($Q_\theta=0$) as drawn in Fig. 1. Let us consider a generic term $[(\mathscr{P}_1, \ldots, \mathscr{P}_s, \mathscr{N}_1, \ldots, \mathscr{N}_{\bar{s}}]$ of the sum (1.45) defining $F_{\theta,\mathscr{P}}(\underline{\xi}, \underline{\sigma}; Q=0)$:

$$\begin{aligned}
&\gamma^{h|\mathscr{P}|(1-\varepsilon)}[\text{zeroes}, \mathscr{P}]\,[(\mathscr{P}_1, \ldots, \mathscr{P}_s, \mathscr{N}_1, \ldots, \mathscr{N}_{\bar{s}})] \\
&\quad=\left\{\left(\frac{\gamma^{h|\mathscr{P}|}}{\prod\limits_{i=1}^{s}\gamma^{q_i|\mathscr{P}_i|}\prod\limits_{j=1}^{\bar{s}}\gamma^{\bar{q}_j|\mathscr{N}_j|}}\right)^{1-\varepsilon}\cdot\frac{[\text{zeroes}, \mathscr{P}]}{\prod\limits_{i=1}^{s}[\text{zeroes}, \mathscr{P}_i]\prod\limits_{j=1}^{\bar{s}}[\text{zeroes}, \mathscr{N}_j]}\right. \\
&\qquad\left.\times O_{\mathscr{P}'\Delta(\mathscr{P}_1\cup\ldots\cup\mathscr{N}_{\bar{s}})}(W_\theta(\underline{\xi}, \underline{\sigma}))\right\} \qquad (2.34)\\
&\qquad\times\left[\prod_{i=1}^{s}\gamma^{q_i|\mathscr{P}_i|(1-\varepsilon)}[\text{zeroes}, \mathscr{P}_i]\cdot\prod_{j=1}^{\bar{s}}\gamma^{\bar{q}_j|\mathscr{N}_j|(1-\varepsilon)}[\text{zeroes}, \mathscr{N}_j]\right] \\
&\qquad\times\left[\prod_{i=1}^{s}F_{\theta_i,\mathscr{P}_i}(\underline{\xi}^{(i)}, \underline{\sigma}^{(i)}; Q_i=0)\prod_{j=1}^{\bar{s}}F_{\bar{\theta}_j,\mathscr{N}_j}(\underline{\bar{\xi}}^{(j)}, \underline{\zeta}^{(j)}, \underline{\bar{\sigma}}^{(j)}; Q_j\neq 0)\right],
\end{aligned}$$

where

$$\begin{cases} q_i=h(\theta_i)\,; & i=1, \ldots, s \\ q_j=h(\bar{\theta}_j)\,; & j=1, \ldots, \bar{s}\end{cases}. \qquad (2.35)$$

We have to investigate the $\{\ldots\}$ factor of (2.34). $O_{\mathscr{P}'\Delta(\ldots)}(W_\theta(\underline{\xi}, \underline{\sigma}))$ is symmetric under all the exchanges $\xi_{2\ell-1}\leftrightarrow\xi_{2\ell}$ for $\ell\in\mathscr{P}'\cap(\mathscr{P}_1\cup\ldots\cup\mathscr{N}_{\bar{s}})$, whereas for $\ell\in\mathscr{P}'\Delta(\mathscr{P}_1\cup\ldots\cup\mathscr{N}_{\bar{s}})$ it is antisymmetric, and therefore has first order zeroes in $(\xi_{2\ell-1}-\xi_{2\ell})^{1-\varepsilon}$. Therefore

$$\begin{aligned}\mathscr{O}_{\mathscr{P}'\Delta(\ldots)}(W_\theta(\underline{\xi}, \underline{\sigma}))&=\gamma^{h[|\mathscr{P}'\setminus(\ldots)|+|(\ldots)\setminus\mathscr{P}'|](1-\varepsilon)} \\ &\quad\times[\text{zeroes}, \mathscr{P}'\setminus(\ldots)]\,[\text{zeroes}, (\ldots)\setminus\mathscr{P}']G(\underline{\xi}, \underline{\sigma})\,, \qquad (2.36)\end{aligned}$$

where $G(\underset{\sim}{\xi}, \bar{\sigma})$ does not have first order zeroes anymore. We have:

$$\{(2.34)\} = \frac{\gamma^{h[|\mathscr{P}'\backslash(\dots)|+|\mathscr{P}'\cap(\dots)|](1-\varepsilon)}\gamma^{h[|\mathscr{P}'\backslash(\dots)|+|(\dots)\backslash\mathscr{P}'|](1-\varepsilon)}}{\prod_{i=1}^{s}\gamma^{q_i|\mathscr{P}_i|(1-\varepsilon)}\prod_{j=1}^{\bar{s}}\gamma^{\bar{q}_j|\mathscr{N}_j|(1-\varepsilon)}}$$
$$\times[\text{zeroes}, \mathscr{P}'\backslash(\dots)]^2 G(\underset{\sim}{\xi}, \bar{\sigma})$$
$$\leqq \prod_{i=1}^{s}\gamma^{(h-q_i)|\mathscr{P}_i|(1-\varepsilon)}\prod_{j=1}^{\bar{s}}\gamma^{(h-\bar{q}_j)|\mathscr{N}_j|(1-\varepsilon)}\times[\gamma^{h|\mathscr{P}'\backslash(\dots)|(1-\varepsilon)}$$
$$[\text{zeroes}, \mathscr{P}'\backslash(\dots)]]^2 G(\underset{\sim}{\xi}, \bar{\sigma})$$
$$\leqq \prod_{i=1}^{s}\gamma^{(h-q_i)|\mathscr{P}_i|(1-\varepsilon)}(\gamma^{h|\mathscr{P}'\backslash(\dots)|(1-\varepsilon)}[\text{zeroes}, \mathscr{P}'\backslash(\dots)])^2\cdot G(\underset{\sim}{\xi}, \bar{\sigma}). \quad (2.37)$$

If $\mathscr{P}_i \neq \emptyset$, $|\mathscr{P}_i| \geqq 2$, and we have produced s second order "zeroes" associated to the neutral bifurcations $h(\theta_1), \dots, h(\theta_s)$. If some $\mathscr{P}_i = \emptyset$, there are two possibilities: either a zero associated to the bifurcation $h(\theta_i)$ is in $(\gamma^{h|\mathscr{P}'\backslash(\ )|(1-\varepsilon)}[\text{zeroes}, \mathscr{P}'\backslash(\dots)])^2$ and is again of second order, or neither in $\mathscr{P}'$ nor in $(\mathscr{P}_1 \cup \dots \cup \mathscr{N}_{\bar{s}})$, there are indices associated to the coordinates of θ_i in which case [recalling the definition of $W_\theta(\underset{\sim}{\xi}, \bar{\sigma})$ and Eq. (1.47)] there must be a zero of second order $(\gamma^h d(\xi^{(i)}))^{2(1-\varepsilon)}$ in $G(\underset{\sim}{\xi}, \bar{\sigma})$.

This completes the proof of the lemma in the neutral case (the argument for the non-neutral trees is completely equivalent), provided we prove the inductive assumption for the trees with only one bifurcation. This is trivial as they do not have any zeroes at all. $\square$

The final estimate is now an easy consequence of the next lemma.

Lemma 2.3. *The following estimates hold for a generic tree $\theta(h(\theta)=h)$,*

$$\gamma^{h|\mathscr{F}|(1-\varepsilon)}[\text{zeroes}, \mathscr{F}]\mathscr{O}_{\mathscr{F}}(F_\theta(\underset{\sim}{\xi}, \bar{\sigma}; Q))$$
$$\leqq (\text{const}) e^{-\frac{\alpha^2}{2}C^{(\leqq k)}_{\theta\theta}(0)}\cdot\prod_{v>v_0} e^{-\frac{\alpha^2}{2}[C^{(\leqq h_v)}_{\theta_v\theta_v}(0)-C^{(\leqq h_{v'})}_{\theta_v,\theta_{v'}}(0)]}$$
$$\times\prod_{i=1}^{n} e^{\frac{\alpha^2}{2}C^{(\leqq h_{v_i})}(0)}\cdot\prod_{v\geqq v_0}\gamma^{-2h_v(s_v-1)}\cdot\prod_{v>v_0}\gamma^{-2(1-\varepsilon)(h_v-h_{v'})\delta_{Q_v,0}}$$
$$\times\prod_{v\geqq v_0}\left(\frac{\text{exp. decay factor at } h_v}{\gamma^{-2h_v(s_v-1)}}\right), \quad (2.38)$$

where $\mathscr{F}$ stands for both $\mathscr{P}$ and $\mathscr{N}$.

Following the notations of [GN1, GN2], we have:

i) θ_v is the subtree whose lowest bifurcation is v.

ii) $\prod_{v\geqq v_0}\gamma^{-2h_v(s_v-1)}$ are the volume factors due to the exponentially decaying factor present at each bifurcation; s_v is the number of lines entering into v (from right to left).

iii) $\prod_{v>v_0}\gamma^{-2(1-\varepsilon)(h_v-h_{v'})\delta_{Q_v,0}}$ are the "zeroes" (of second order) discussed in Lemma 2.2.

iv) The exponentially decaying factor at a generic bifurcation is $\exp(\kappa\gamma^{h_v}d^*(x_v))$, where $d^*(x_v)$ is the length of the shortest path connecting the clusters (bifurcations) $v^{(1)}, \ldots, v^{(s_v)}$, which come immediately before v (from the right to left), $\kappa > 0$.

v) v_i is the vertex where the i^{th} final line merges.

Proof. The proof follows immediately from (2.33) noting that Lemma 2.2 is still valid with $\delta W_{(\theta)}(\underline{\xi}, \underline{\tau})$, instead of $W_{(\theta)}(\underline{\xi}, \underline{\tau})$, such that in $\delta W_{(\theta)}(\underline{\xi}, \underline{\tau})$ of Eq. (2.32) there is an exponentially decaying factor $\exp(-\kappa\gamma^{h(\theta)} d^*(x_{v_0}))$ with $\kappa > 0$. □

Now we are ready to state and prove the first estimate.

Theorem 2.1. *Let $\Delta_i \in Q_k$, $(k = k(\theta), h = h(\theta))$; then for all shapes s, for all $\{Q_v\}_s$, and for all $\bar{\sigma}$, we have for $Q(\theta) = Q = 0$,*

$$\sum_{\substack{s(\theta)=s \\ k(\theta)=k \\ v(\theta)=n}}{}_{\theta} \left(\frac{\lambda}{2}\right)^n \cdot \int_{\Delta_1 \times \ldots \times \Delta_n} d\xi_1 \ldots d\xi_n \{\gamma^{|\mathscr{P}|(1-\varepsilon)k} [\text{zeroes}, \mathscr{P}] \times |F_{\theta,\mathscr{P}}(\xi, \bar{\sigma}; Q=0)|\} \leqq C_n \lambda_{\text{eff}}^n(k) e^{-\gamma^k d(\Delta_1, \ldots, \Delta_n)}, \tag{2.39}$$

where, as in [BGN], $\lambda_{\text{eff}}(k) = \lambda\gamma^{\left(\frac{\alpha^2}{4\pi} - 2\right)k}$; for $Q(\bar{\theta}) = Q \neq 0$,

$$\sum_{\substack{s(\theta)=s \\ k(\theta)=k \\ v(\theta)=n}}{}_{\theta} \left(\frac{\lambda}{2}\right)^n \int_{\Delta_1 \times \ldots \times \Delta_n} d\xi_1 \ldots d\xi_n \gamma^{|\mathscr{N}|(1-\varepsilon)k} [\text{zeroes}, \mathscr{N}] \, |F_{\bar{\theta},\mathscr{N}}(\bar{\underline{\xi}}, \underline{\zeta}, \bar{\bar{\sigma}}; Q \neq 0)|$$

$$\leqq C_n \lambda_{\text{eff}}^n(k) \gamma^{-\frac{\alpha^2}{4\pi} k Q^2} e^{-\gamma^k d(\Delta_1, \ldots, \Delta_n)}. \tag{2.40}$$

Proof (sketch, for details see [GN2]). We first observe that

$$C_{\theta\theta}^{(\leqq k)}(0) = Q_\theta^2 C^{(\leqq k)}(0) = Q_\theta^2 (k+1) C^{(0)}(0) = Q_\theta^2 (k+1) \frac{1}{2\pi} \log\gamma, \tag{2.41}$$

and

$$\sum_{v \geqq v_0} (s_v - 1) = n - 1 = n_{v_0} - 1, \tag{2.42}$$

where v_0 is the lowest bifurcation and n_v the number of final lines which finally merge into the vertex v.

It is simple to realize, choosing $n_{v_0} = n$, $Q_\theta = 0 = Q_{v_0}$:

$$\begin{aligned}
&\left(\frac{\lambda}{2}\right)^n \sum_{\substack{s(\theta)=s \\ k(\theta)=k \\ v(\theta)=n}}{}_{\theta} \gamma^{2(k-h)(1-\varepsilon)} \int_{\Delta_1 \times \ldots \times \Delta_n} d\xi_1 \ldots d\xi_n \\
&\quad \times \gamma^{(1-\varepsilon)|\mathscr{P}|h} [\text{zeroes}, \mathscr{P}] \cdot |F_{\theta,\mathscr{P}}(\xi, \bar{\sigma}, Q=0)| \\
&\leqq (\text{const}) \left(\frac{\lambda}{2}\right)^n \sum_{\{h_v\}}{}' \gamma^{2(k-h)(1-\varepsilon)} \cdot \prod_{v > v_0} \gamma^{-\left[\left(2 - \frac{\alpha^2}{4\pi}\right)(n_v - 1) - \frac{\alpha^2}{4\pi} + \frac{\alpha^2}{4\pi} Q_v^2 + 2(1-\varepsilon)\delta_{Q_v,0}\right](h_v - h_{v'})} \\
&\quad \times \gamma^{-\left[\left(2 - \frac{\alpha^2}{4\pi}\right)(n_{v_0} - 1) - \frac{\alpha^2}{4\pi} + \frac{\alpha^2}{4\pi} Q_{v_0}^2\right] h_{v_0}} \left\{ \int_{\Delta_1 \times \ldots \times \Delta_n} d\xi_1 \ldots d\xi_n \prod_{v \geqq v_0} \left(\frac{e^{-\kappa\gamma^{h_v} d^*(x_v)}}{\gamma^{-2h_v(s_v - 1)}} \right) \right\},
\end{aligned} \tag{2.43}$$

and, as discussed in [G, GN2],

$$\{(2.43)\} \leqq (\mathrm{const})\,|\Delta| e^{-\bar{\kappa}\gamma^k d(\Delta_1,\ldots,\Delta_n)} \leqq (\mathrm{const})\gamma^{-2k} e^{-\bar{\kappa}\gamma^k d(\Delta_1,\ldots,\Delta_n)}.$$

Thus, we get:

$$\begin{aligned}\text{l.h.s. of (2.43)} \leqq\; & C_n \lambda^n \sum_{\{h_v\}} \gamma^{2(k-h)(1-\varepsilon)} |\Delta| \\ & \times \left[\prod_{v>v_0} \gamma^{-\left[\left(2-\frac{\alpha^2}{4\pi}\right)(n_v-1) - \frac{\alpha^2}{4\pi} + \frac{\alpha^2}{4\pi} Q_v^2 + 2(1-\varepsilon)\delta_{Q_v,0}\right](h_v - h_{v'})} \right] \\ & \times \gamma^{-\left[\left(2-\frac{\alpha^2}{4\pi}\right)\cdot(n_{v_0}-1) - \frac{\alpha^2}{4\pi} + \frac{\alpha^2}{4\pi} Q_{v_0}^2\right] h} \times e^{-\bar{\kappa}\gamma^k d(\Delta_1,\ldots,\Delta_n)},\end{aligned} \tag{2.44}$$

where C_n is a constant depending only on the number of final lines and $\sum_{\{h_v\}}$ is the sum over all possible frequencies associated to the bifurcations of θ. Since $\alpha^2 < 8\pi$, we have $\frac{\alpha^2}{4\pi} - 2(1-\varepsilon)\delta_{Q_v,0} - \frac{\alpha^2}{4\pi} \cdot Q_v^2 < 0$ as $\varepsilon > 0$ is arbitrary. Therefore, we finally obtain:

$$\text{l.h.s. of (2.43)} \leqq (\mathrm{const}) \lambda_{\mathrm{eff}}^n(k) e^{-\bar{\kappa}\gamma^k d(\Delta_1,\ldots,\Delta_n)}. \tag{2.45}$$

The proof in the non-neutral case is completely equivalent. □

As already mentioned, the second estimate we are going to perform is used to control the expressions $\hat{V}^{(k)}[\mathscr{D}_k^c \cap \mathscr{D}_{k-1} \cap \mathscr{R}_k^c]$ and $\Delta^{(k)}[\mathscr{D}_k]$ arising from passages (1.21) and (1.22) of the iterative procedure. Both of these terms can be dominated by certain second order contributions to the effective potential. In the case of $\hat{V}^{(k)}[\mathscr{D}_k^c \cap \mathscr{D}_{k-1} \cap \mathscr{R}_k^c]$, they are already present, whereas in the case of $\Delta^{(k)}[\mathscr{D}_k]$ they have to be partially brought down from higher frequencies. In any case, we now restrict our attention to the contributions of order higher than two. We will estimate them as second order terms multiplied by a suitable finite constant depending on λ. It is sufficient to explicitly consider only the case concerning $\Delta^{(k)}[\mathscr{D}_k]$, as it is the more complicated one due to the more intricate dependence of $\Delta^{(k)}[\mathscr{D}_k]$ in its regions of integration.

We recall that we can go from $\left[\sum_{n=1}^{t} \frac{1}{n!} \mathscr{E}_{k+1}^T(\hat{V}^{(k+1)}[\mathscr{D}_k^c]; n)\right]_{\leqq t}$ to $\hat{V}^{(k)}[\mathscr{D}_k^c]$ in two steps [cf. (2.23)]:

a) We eliminate the dangerous regions of $:P_{\mathscr{P}}:$, where the "zeroes" are not effective because they lack sufficient Hölder continuity.

b) We put back those regions $\mathscr{D}_k(\mathscr{P}_1), \ldots, \mathscr{D}_k(\mathscr{P}_s)$, $\mathscr{D}_k(\mathscr{N}_1), \ldots, \mathscr{D}_k(\mathscr{N}_{\bar{s}})$, which are not useful anymore.

Thus, the desired estimate of $\Delta^{(k)}[\mathscr{D}_k]$ is attained if we prove Theorems 2.2a) and 2.2b):

Theorem 2.2a). *Let $\mathscr{P}_1, \ldots, \mathscr{P}_s$, $\mathscr{N}_1, \ldots, \mathscr{N}_{\bar{s}}$ and $\mathscr{P} = \{\ell_1, \ldots, \ell_q\}$ be arbitrarily fixed. We write (in the case $Q=0$):*

$$\tilde{\mathscr{I}} := I_1^2 \times \ldots \times I_m^2 := (\Lambda^{2m_1} \backslash \mathscr{D}_k(\mathscr{P}_1)) \times \ldots \times ((\Lambda^{2m_{\bar{s}}} \times \Lambda^{(p_1 + \cdots + p_{\bar{s}})}) \backslash \mathscr{D}_k(\mathscr{N}_{\bar{s}})), \tag{2.46}$$

where I_i can be Λ^2 or $\Lambda^2 \backslash D_{s_i}^{(k)}$ depending on i.

$$\tilde{\mathscr{I}} \backslash \mathscr{D}_k(\mathscr{P}) = I_1^2 \times \ldots \times I_{\ell_1}^2 \backslash D_{\ell_1}^{(k)} \times \ldots \times I_{\ell_q}^2 \backslash D_{\ell_q}^{(k)} \times \ldots \times I_m^2. \tag{2.47}$$

Therefore,

$$\tilde{\mathscr{I}} \cap \mathscr{D}_k(\mathscr{P}) = \bigcup_{j=1}^{q} (I_1^2 \times \ldots \times I_{\ell_j-1}^2 \times D_{\ell_j}^{(k)} \times I_{\ell_j+1}^2 \times \ldots \times I_m^2). \tag{2.48}$$

Then we have:

$$\begin{aligned}
&\sum_{\substack{\left\{\begin{array}{l} s(\theta)=s \\ k(\theta)=k \\ h(\theta)=k+1 \\ v(\theta)>2 \end{array}\right.}} \theta \left(\frac{\lambda}{2}\right)^n \int_{I_1^2 \times \ldots \times I_{\ell,-1}^2 \times D_{\ell_j}^{(k)} \times \ldots \times I_{\ell,+1}^2 \times \ldots \times I_m^2} d\xi_1 \ldots d\xi_n : P_{\mathscr{P}}(\varphi^{(\leqq k)}) : \\
&\quad \times \mathcal{O}_{\mathscr{P}} \left[\mathcal{O}_{\mathscr{P}'\Delta(\ldots)}(W_\theta(\underline{\xi}, \underline{\sigma})) \prod_{i=1}^{s} F_{\theta_i, \mathscr{P}_i}(\underline{\xi}^{(i)}, \underline{\sigma}^{(i)}; Q_i = 0) \right. \\
&\quad \left. \times \prod_{j=1}^{\bar{s}} F_{\bar{\theta}_j, \mathscr{N}_j}(\underline{\bar{\xi}}^{(j)}, \underline{\zeta}^{(j)}, \underline{\bar{\sigma}}^{(i)}; Q_j \neq 0) \right] \\
&\quad \leqq (\text{const}) \lambda^2 \int_{D_\ell^{(k)}} d\xi_1 \, d\xi_2 \sin^2 \frac{\alpha}{2} (\varphi_{\xi_1}^{(\leqq k)} - \varphi_{\xi_2}^{(\leqq k)}) \\
&\quad \times \lambda_{\text{eff}}^{n-2}(k) \sum_{q=k+1}^{N} e^{\alpha^2 C_{12}^{(<q)}} (e^{\alpha^2 C_{12}^{(q)}} - 1) \gamma^{-2(q-k)(1-\varepsilon)}.
\end{aligned}$$

(the case $Q \neq 0$ is analogous).

Theorem 2.2b). *For any $\ell \in (\mathscr{P}_1 \cup \ldots \cup \mathscr{N}_{\bar{s}}) \backslash \mathscr{P}'$ (in the case $Q = 0$), we have (the case $Q \neq 0$ is analogous).*

$$\begin{aligned}
&\sum_{\substack{s(\theta)=s \\ k(\theta)=k \\ h(\theta)=k+1 \\ v(\theta)>2}} \theta \left(\frac{\lambda}{2}\right)^n \int_{\mathscr{L}^2 \times \ldots \times \mathscr{L}_{\ell-1} \times {}^{(k)} \times \mathscr{L}_{\ell+1} \times \ldots \times \mathscr{L}_m} d\xi_1 \ldots d\xi_n : P_{\mathscr{P}}(\varphi^{(\leqq k)}) : \\
&\quad \times \mathcal{O}_{\mathscr{P}} \left[\mathcal{O}_{\mathscr{P}'\Delta(\ldots)}(W_\theta(\underline{\xi}, \underline{\sigma})) \prod_{i=1}^{s} F_{\theta_i, \mathscr{P}_i}(\underline{\xi}^{(i)}, \underline{\sigma}^{(i)}; Q_i = 0) \right. \\
&\quad \left. \times \prod_{j=1}^{\bar{s}} F_{\bar{\theta}_j, \mathscr{N}_j}(\underline{\bar{\xi}}^{(j)}, \underline{\zeta}^{(j)}, \underline{\bar{\sigma}}^{(j)}; Q_j \neq 0) \right] \\
&\leqq (\text{const}) \lambda^2 \int_{D_\ell^{(k)}} d\xi_1 \, d\xi_2 (\gamma^k |\xi_1 - \xi_2|)^{2(1-\varepsilon)} \\
&\quad \times \lambda_{\text{eff}}^{n-2}(k) \sum_{q=k+1}^{N} e^{\alpha^2 C_{12}^{(<q)}} (e^{\alpha^2 C_{12}^{(q)}} - 1),
\end{aligned}$$

where now the regions $\mathscr{L}_j^2$ are implicitly defined and such that the zeroes of $: P_{\mathscr{P}}(\varphi^{(\leqq k)}) :$ are all effective.

Proof of Theorem 2.2a). We observe, first of all, that the zeroes of $: P_{\mathscr{P}}(\varphi^{(\leqq k)}) :$ associated to the indices $\ell \in \mathscr{P} \backslash (\mathscr{P}_1 \cup \ldots \cup \mathscr{N}_{\bar{s}})$ are not "effective" as the field is not forced to be sufficiently Hölder continuous in the corresponding regions. Therefore we estimate $: P_{\mathscr{P}}(\varphi^{(\leqq k)}) :$ in the following way: $\left(\begin{array}{l} \mathscr{P} \neq \emptyset; \text{ the case } \mathscr{P} = \emptyset \\ \qquad \text{is simpler} \end{array}\right)$.

Assume $\ell_j=1$, then in $D_1^{(k)}$:

$$
\begin{aligned}
|:P_{\mathscr{P}}(\varphi^{(\leqq k)}):| &\leqq C(k)\left|\frac{\sin\alpha\Delta_1\varphi^{(\leqq k)}:\bar{P}_{\mathscr{P}\cap(\dots)}(\varphi^{(\leqq k)}):}{\gamma^{k(1-\varepsilon)|\mathscr{P}\cap(\dots)|}[\text{zeroes},\mathscr{P}\cap(\dots)]}\right| \\
&\quad\times\gamma^{k(1-\varepsilon)|\mathscr{P}\cap(\dots)|}[\text{zeroes},\mathscr{P}\cap(\dots)]\leqq\bar{C}\cdot\sin^2\frac{\alpha}{2}\Delta_1\varphi^{(\leqq k)} \\
&\quad\times\left|\frac{:\bar{P}_{\mathscr{P}\cap(\dots)}(\varphi^{(\leqq k)}):}{\gamma^{k(1-\varepsilon)|\mathscr{P}\cap(\dots)|}[\text{zeroes},\mathscr{P}\cap(\dots)]}\right|\cdot\frac{\gamma^{k(1-\varepsilon)|\mathscr{P}\cap(\dots)|}[\text{zeroes},\mathscr{P}\cap(\dots)]}{\gamma^{k(1-\varepsilon)}|\xi_1-\xi_2|^{1-\varepsilon}} \\
&\leqq\bar{\bar{C}}\cdot\sin^2\frac{\alpha}{2}\Delta_1\varphi^{(\leqq k)}\cdot\frac{\gamma^{k(1-\varepsilon)|\mathscr{P}\cap(\dots)|}[\text{zeroes},\mathscr{P}\cap(\dots)]}{\gamma^{k(1-\varepsilon)}|\xi_1-\xi_2|^{1-\varepsilon}},
\end{aligned}
\tag{2.51}
$$

where $:\bar{P}_{\mathscr{P}\cap(\dots)}:$ is that part of $:P_{\mathscr{P}}:$ which gives effective zeroes.

The left-hand side of (2.49) can be estimated by

$$
\begin{aligned}
&\bar{\bar{C}}\cdot\lambda^2\int_{D_1^{(k)}}d\xi_1\,d\xi_2\sin^2\frac{\alpha}{2}\Delta_1\varphi^{(\leqq k)}\Bigg\{\sum_{\theta}\int_{\{:\,l_1^2\times\dots\times l_j\times\dots\times l_n^2}d\underline{\xi}\backslash d\xi_1\,d\xi_2 \\
&\quad\times[\gamma^{k(1-\varepsilon)|\mathscr{P}\cap(\dots)|}[\text{zeroes},\mathscr{P}\cap(\dots)]]\,\mathcal{O}_{\mathscr{P}}\left(\left[\frac{\mathcal{O}_{\mathscr{P}'\Delta(\mathscr{P}_1\cup\dots\cup\mathscr{N}_{\bar{s}})}(W_\theta(\underline{\xi},\underline{\sigma}))}{\gamma^{k(1-\varepsilon)}|\xi_1-\xi_2|^{1-\varepsilon}}\right]\right. \\
&\quad\times\prod_{i=1}^{s}F_{\theta_i,\mathscr{P}_i}(\underline{\xi}^{(i)},\underline{\sigma}^{(i)};Q_i=0)\prod_{j=1}^{\bar{s}}F_{\theta_j,\mathscr{N}_j}(\underline{\bar{\xi}}^{(j)},\underline{\zeta}^{(j)},\underline{\bar{\sigma}}^{(j)};Q_j\neq0)\Bigg)\Bigg\}.
\end{aligned}
\tag{2.52}
$$

There are now different possibilities to investigate:

i) The final lines associated to ξ_1,ξ_2 merge into a neutral bifurcation before the lowest one.

ii) The only neutral bifurcation the final lines ξ_1,ξ_2 meet is the lowest one (we are assuming the tree to be neutral).

Case i). Here there are two different possible situations,

either: a) (ξ_1,ξ_2) belong to a neutral subtree θ_i,

or b) (ξ_1,ξ_2) belong to a non-neutral subtree $\bar{\theta}_j$.

We start by considering the case i) a): We rewrite $\{(2.52)\}$ in the following way:

$$
\begin{aligned}
\{(2.52)\}&=\sum_{\substack{\theta_i\\ \left\{\begin{smallmatrix}s(\theta_i)=s_i\\ k(\theta_i)=k+1\\ h(\theta_i)=q>k+1\end{smallmatrix}\right.}}\lambda^{n_i-2}\int_{l_1^{(i)2}\times\dots\times l_{\ell_j}^{(i)2}\times\dots\times l_{n_i}^{(i)2}}d\underline{\xi}^{(i)}\backslash d\xi_1\,d\xi_2 \\
&\quad\times(\gamma^{k(1-\varepsilon)|\mathscr{P}\cap\mathscr{P}_i|}[\text{zeroes},\mathscr{P}\cap\mathscr{P}_i]\gamma^{k(1-\varepsilon)|\mathscr{P}\backslash\mathscr{P}_i|}[\text{zeroes},\mathscr{P}\backslash\mathscr{P}_i]) \\
&\quad\times F_{\theta_i,\mathscr{P}_i}(\underline{\xi}^{(i)},\underline{\sigma}^{(i)};Q_i=0)\cdot\Bigg\{\sum_{\substack{\theta\backslash\theta_i\\ \left\{\begin{smallmatrix}s(\theta\backslash\theta_i)=s\backslash s_i\\ k(\theta)=k\\ h(\theta)=k+1\end{smallmatrix}\right.}}\lambda^{n-n_i}\int_{l_1^2\times\dots\times(l_1^{(i)2}\times\dots\times l_{n_i}^{(i)2})\times\dots\times l_n^2}d\underline{\xi}\backslash d\underline{\xi}^{(i)} \\
&\quad\times\mathcal{O}_{\mathscr{P}}\left(\left[\frac{\mathcal{O}_{\mathscr{P}'\Delta(\mathscr{P}_1\cup\dots\cup\mathscr{N}_{\bar{s}})}(W_\theta(\underline{\xi},\underline{\sigma}))}{\gamma^{k(1-\varepsilon)|\mathscr{P}_1\cup\dots\cup\mathscr{N}_{\bar{s}}\backslash\mathscr{P}'|}[\text{zeroes},(\dots)\backslash\mathscr{P}']\gamma^{k(1-\varepsilon)}|\xi_1-\xi_2|^{1-\varepsilon}}\right]\right. \\
&\quad\times\prod_{\tilde{s}\neq i}^{s}[\gamma^{k(1-\varepsilon)|\mathscr{P}_{\tilde{s}}|}[\text{zeroes},\mathscr{P}_{\tilde{s}}]F_{\theta_{\tilde{s}},\mathscr{P}_{\tilde{s}}}(\underline{\xi}^{(\tilde{s})},\underline{\sigma}^{(\tilde{s})};Q_{\tilde{s}}=0)] \\
&\quad\times\prod_{j=1}^{\bar{s}}[\gamma^{k(1-\varepsilon)|\mathscr{N}_j|}[\text{zeroes},\mathscr{N}_j]\cdot F_{\bar{\theta}_j,\mathscr{N}_j}(\underline{\bar{\xi}}^{(j)},\underline{\zeta}^{(j)},\underline{\bar{\sigma}}^{(j)};Q_j\neq0)]\Bigg)\Bigg\},
\end{aligned}
$$

where one has to remember that (ξ_1, ξ_2) are associated to an index ℓ_j which does not belong to $\mathscr{P}_i$. The expression $\{(2.53)\}$ can now be estimated as in Theorem 2.1 obtaining

$$\{(2.53)\} \leqq (\text{const})\,(\lambda_{\text{eff}}(k))^{n-n_i}, \tag{2.54}$$

and the first part of (2.53) as

$$\begin{aligned}
&\sum_{\substack{\theta_i\\ \begin{cases} s(\theta_i)=s_i\\ k(\theta_i)=k+1\\ h(\theta_i)=q_i>k+1\end{cases}}} \gamma^{-(q_i-k)(1-\varepsilon)|\mathscr{P}_i|}\cdot\lambda^{n_i-2} \int\limits_{(l_1^{(i)2}\times\ldots\times l_{\ell_j}^{(i)}\times\ldots\times l_{n_i}^{(i)2})} d\underline{\xi}^{(i)}\backslash d\xi_1 d\xi_2\\
&\quad\times\gamma^{q_i(1-\varepsilon)|\mathscr{P}_i|}[\text{zeroes},\mathscr{P}_i]\cdot F_{\theta_i,\mathscr{P}_i}(\underline{\xi}^{(i)},\underline{\sigma}^{(i)};Q_i=0)\\
&\leqq(\text{const})\sum_{q_i=k+2}^{N}\gamma^{-(q_i-k)(1-\varepsilon)|\mathscr{P}_i|}\cdot e^{\alpha^2 C_{12}^{(<q_i)}}(e^{\alpha^2 C_{12}^{(q_i)}}-1)(\lambda_{\text{eff}}(k))^{n_i-2},
\end{aligned}\tag{2.55}$$

which globally gives, as $|\mathscr{P}_i|$ is even, [for $\mathscr{P}=\emptyset$ remember definition (2.25)]

$$\begin{aligned}
\{(2.52)\} &\leqq (\text{const})\lambda^2 \int\limits_{D_1^{(k)}} d\xi_1 d\xi_2 \sin^2\frac{\alpha}{2}\Delta_1\varphi^{(\leqq k)}\\
&\quad\times\sum_{q=k+1}^{N} e^{\alpha^2 C_{12}^{(<q)}}(e^{\alpha^2 C_{12}^{(q)}}-1)\cdot\gamma^{-2(q-k)(1-\varepsilon)}\cdot(\lambda_{\text{eff}}(k))^{2(\bar{m}-1)}.
\end{aligned}\tag{2.56}$$

Remark. If $|\mathscr{P}_i|=0$ ($\mathscr{P}_i=\emptyset$), remembering the remark following Eq. (2.5) it is easy to recognize that in $\mathscr{O}_{\mathscr{P}'\Delta(\mathscr{P}_1\cup\ldots\cup\mathscr{N}_{\tilde{s}})}(W_\theta)$, there is a second order zero which can again be estimated by

$$\gamma^{-2(q-k)(1-\varepsilon)} \tag{2.57}$$

due to the $(\cos\Delta\varphi-1)$ factor present in each term $:P_{\emptyset_\ell}:$ of $:P_\emptyset(\varphi^{(\leqq k+1)}):$, producing the same estimates.

Case i) b). In this case we rewrite $\{(2.52)\}$ in the following way:

$$\begin{aligned}
\{(2.52)\} = &\sum_{\substack{\bar{\theta}_j\\ \begin{cases} s(\bar{\theta}_j)=\bar{s}_j\\ h(\bar{\theta}_j)=\bar{q}_j\geqq k+1\end{cases}}} \lambda^{n_j-2} \int\limits_{l_1^{(j)2}\times\ldots\times l_{l_j}^{(j)2}\times\ldots\times l_{n_i}^{(j)2}} d\underline{\bar{\xi}}^{(j)} d\underline{\zeta}^{(j)}\backslash d\xi_1 d\xi_2\\
&\times\mathscr{O}_{\mathscr{P}}\,(\gamma^{k(1-\varepsilon)|\mathscr{N}_j|}[\text{zeroes},\mathscr{N}_j])F_{\bar{\theta}_j,\mathscr{N}_j}(\underline{\bar{\xi}}^{(j)},\underline{\zeta}^{(j)},\underline{\bar{\sigma}}^{(j)};Q_j\neq 0)\\
&\times\left\{\sum_{\substack{\theta\backslash\bar{\theta}_j\\ \begin{cases} s(\theta\backslash\bar{\theta}_j)=s\backslash s_j\\ k(\theta)=k\\ h(\theta)=k+1\end{cases}}} \lambda^{n-n_j} \int\limits_{l_1^2\times\ldots\times(l_1^{(\bar{j})}\times\ldots\times l_{\mu_j}^{(j)})\times\ldots\times l_\mu^2} d\xi\backslash d\underline{\bar{\xi}}^{(j)} d\underline{\zeta}^{(j)}\right.\\
&\times\left[\frac{\mathscr{O}_{\mathscr{P}'\Delta(\mathscr{P}_1\cup\ldots\cup\mathscr{N}_s)}(W_{\bar{\theta}}(\xi,\bar{\sigma}))}{\gamma^{k(1-\varepsilon)|(\ldots)\backslash\mathscr{P}'|}[\text{zeroes},(\ldots)\backslash\mathscr{P}']\gamma^{k(1-\varepsilon)}|\xi_1-\xi_2|^{1-\varepsilon}}\right]\\
&\times\prod_{i=1}^{s}\gamma^{k(1-\varepsilon)|\mathscr{P}_i|}[\text{zeroes},\mathscr{P}_i]F_{\theta_i,\mathscr{P}_i}(\xi^{(i)},\underline{\sigma}^{(i)};Q_i=0)\prod_{\tilde{s}\neq j}^{\tilde{s}}\gamma^{k(1-\varepsilon)|\mathscr{N}_j|}\\
&\left.\times[\text{zeroes},\mathscr{N}_{\tilde{s}}]F_{\bar{\theta}_{\tilde{s}},\mathscr{N}_{\tilde{s}}}(\xi^{(\tilde{s})},\zeta^{(\tilde{s})},\underline{\bar{\sigma}}^{(\tilde{s})};Q_{\tilde{s}}\neq 0)\right\}\Bigg).
\end{aligned}$$

The $\{\ldots\}$ factor is again estimated using Theorem 2.1. The first part can be estimated considering now $\bar{\theta}_j$ as the whole tree and iterating the previous proof if the lines $(1,2)$ are in one of the neutral subtrees which merge in the lowest bifurcation of $\bar{\theta}_j$, otherwise one iterates the procedure as many times as needed to arrive at such a situation.

Case ii). In this case (ξ_1, ξ_2) merge together only in the lowest bifurcation (this case is trivial) or in a non-neutral bifurcation and then the line going out from this bifurcation still merges in a non-neutral one and so on. Therefore, neutrality is only restored at the lowest bifurcation. This is an easier case; in fact, there will be at least one line of coordinate (say ξ_3) merging into the same non-neutral bifurcation of frequency q as ξ_1, ξ_2 which therefore is at a distance $|\xi_3 - \xi_2| \sim \gamma^{-q}$ from (ξ_1, ξ_2). The integration over ξ_3 gives a factor γ^{-2q}.

But if ξ_3 is associated to ξ_4 in $\Delta_3 \varphi$ only at the lowest bifurcation it will be enough – in order to get the usual estimates – that ξ_3 be at a distance $\gamma^{-(k+1)}$ from ξ_4 and that the integration over it gives a factor $\gamma^{-2(k+1)}$; that is, a factor

$$\gamma^{-2(q-k)} \tag{2.59}$$

has been gained. This argument can easily be generalized to all possible situations of case ii). The non-neutral case can be worked out in a similar way both for the case i) and for the case ii), and we do not report it here.

This completes the proof of Theorem 2.2a). □

Proof of Theorem 2.2b). (The regions $\mathscr{L}_j^2$ are such that the zeroes of $:P_{\mathscr{P}}:$ are all effective.) We mimick the proof of Theorem 2.2a) and call ξ_1, ξ_2 the coordinates associated to the index ℓ which we assume to belong to the subtree θ_i. Again, for $\mathscr{P} \neq \emptyset$,

$$\begin{aligned}
&\tilde{C} \sum_{\substack{\theta \\ \left\{\substack{s(\theta)=s \\ k(\theta)=k \\ h(\theta)=k+1}\right.}} \left(\frac{\lambda}{2}\right)^n \int_{\mathscr{L}_1^2 \times \ldots \times \mathscr{L}_{\ell-1}^2 \times D_\ell^{(k)} \times \mathscr{L}_{\ell+1}^2 \times \ldots \times \mathscr{L}_m^2} d\xi : P_{\mathscr{P}}(\varphi^{(\leqq k)}): \\
&\times \mathcal{O}_{\mathscr{P}} \left[\mathcal{O}_{\mathscr{P}'\Delta(\ldots)}(W_\theta(\underset{=}{\xi}, \underset{=}{\sigma})) \prod_{i=1}^{s} F_{\theta_i, \mathscr{P}_i}(\underset{=}{\xi}^{(i)}, \underset{=}{\sigma}^{(i)}; Q_i = 0) \right. \\
&\left. \times \prod_{j=1}^{\bar{s}} F_{\bar{\theta}_j, \mathscr{N}_j}(\underset{=}{\xi}^{(j)}, \underset{=}{\zeta}^{(j)}, \underset{=}{\bar{\sigma}}^{(j)}; Q_j \neq 0) \right] \\
&\leqq (\text{const}) \lambda^2 \int_{D_1^{(k)}} d\xi_1 \, d\xi_2 (\gamma^k |\xi_1 - \xi_2|)^{2(1-\varepsilon)} \\
&\times \left\{ \sum_\theta \int_{\{: \mathscr{L}_1^2 \times \ldots \times \mathscr{L}_\ell^2 \times \ldots \times \mathscr{L}_m^2} d\xi \backslash d\xi_1 d\xi_2 \right. \\
&\times \left| \frac{:P_{\mathscr{P}}(\varphi^{(\leqq k)}):}{\gamma^{k(1-\varepsilon)}[\text{zeroes}, \mathscr{P}]} \cdot \frac{\mathcal{O}_{\mathscr{P}'\Delta(\ldots)}(W_\theta(\underset{=}{\xi}, \underset{=}{\sigma}))}{\gamma^{k(1-\varepsilon)|\mathscr{P}'\Delta(\ldots)|}[\text{zeroes}, \mathscr{P}'\Delta(\ldots)]} \right| \\
&\times (\gamma^{k(1-\varepsilon)|\mathscr{P}'\backslash(\ldots)|}[\text{zeroes}, \mathscr{P}'\backslash(\ldots)])^2 \gamma^{k(1-\varepsilon)|\mathscr{P}_1 \cup \ldots \cup \check{\mathscr{P}} \cup \ldots \cup \mathscr{N}_s|} \\
&\times [\text{zeroes}, \mathscr{P}_1 \cup \ldots \cup \check{\mathscr{P}} \cup \ldots \cup \mathscr{N}_s] \sum_{l \neq i}^{s} F_{\theta, \mathscr{P}_\ell}(\ldots) \prod_{j=1}^{s} F_{\theta_j, \mathscr{N}_i}(\ldots) \\
&\times \left. \left(\frac{\gamma^{k(1-\varepsilon)|\mathscr{P}_i|}[\text{zeroes}, \mathscr{P}_i]}{(\gamma^k |\xi_1 - \xi_2|)^{2(1-\varepsilon)}} \cdot \gamma^{-2(q-k)(1-\varepsilon)} F_{\theta_i, \mathscr{P}_i}(\ldots) \right) \gamma^{2(q-k)(1-\varepsilon)} \right\}.
\end{aligned} \tag{2.60}$$

At this point we proceed as before. But the last factor $\gamma^{2(q-k)(1-\varepsilon)}$ cancels a factor $\gamma^{-2(q-k)(1-\varepsilon)}$ of (2.59) and we get

$$\{(2.58)\} \leqq (\text{const})\lambda^2 \int_{D_1^{(k)}} d\xi_1 d\xi_2 (\gamma^k|\xi_1-\xi_2|)^{2-\varepsilon} \times \lambda_{\text{eff}}^{n-2}(k) \sum_{q=k+1}^{N} e^{\alpha^2 C_{12}^{(<q)}} (e^{\alpha^2 C_{12}^{(q)}} - 1). \tag{2.61}$$

The other cases in which $\mathscr{P}_i = \emptyset$ or in which θ is not neutral are treated in a similar way and we do not discuss them here.

This completes the proof of Theorem 2.2b). □

Remarks. 1) The method we use is slightly different if $\mathscr{P}_i = \emptyset$, due to the fact that we have split: $P_{\mathscr{P}_i=\emptyset}$: into a sum of different terms [cf. remark after Eq. (2.5)]. We leave the details to the reader.

2) It is completely trivial to extract from $F_{\theta_i, \mathscr{P}_i}$ a factor $\gamma^{\frac{\alpha^2}{4\pi} q_i} e^{-\gamma^{q_i}|\xi_1-\xi_2|}$ to extract a factor $e^{\alpha^2 C^{(<q_i)}} (e^{\alpha^2 C^{(q_i)}} - 1)$ requires a little more work.

3) If θ is a non-neutral tree the last factors in (2.57) and (2.61) appear multiplied by an extra factor $\gamma^{-\frac{\alpha^2}{4\pi} k Q^2}$ according to the estimate (2.40) of Theorem 2.1.

The third and last estimate of this section is concerned with second order trees integrated over a large fluctuation region.

Theorem 2.3. *For a region $I \subset D^{(k)}$ or $\{(\Lambda \backslash R^{(k)})^2 \cap D^{(k-1)} | D^{(k)}\}$; $h, k \in \mathbb{N}$; $h > k$, we have:*

$$\begin{aligned}
&\left(\frac{\lambda}{2}\right)^2 \int_I d\xi_1 d\xi_2 e^{\alpha^2 C_{12}^{(<h)}} (e^{\alpha^2 C_{12}^{(h)}} - 1) : \cos\alpha \Delta_1 \varphi^{(\leqq k)} - 1 : \\
&\leqq -\left(\frac{\lambda}{2}\right)^2 \int_I d\xi_1 d\xi_2 e^{\alpha^2 C_{12}^{(<h)}} (e^{\alpha^2 C_{12}^{(h)}} - 1) \sin^2 \frac{\alpha}{2} \Delta_1 \varphi^{(\leqq k)} \\
&\leqq -\left(\frac{\lambda}{2}\right)^2 \int_I d\xi_1 d\xi_2 e^{\alpha^2 C_{12}^{(<h)}} (e^{\alpha^2 C_{12}^{(h)}} - 1) B_k^2 (\gamma^k |\xi_1 - \xi_2|)^{2(1-\varepsilon)} \\
&\quad \times \begin{cases} 1 & \text{for } \; I \subset D^{(k)} \\ \theta^2(\sigma) & \text{for } \; I \subset (\Lambda \backslash R^{(k)})^2 \cap D^{(k-1)} \backslash D^{(k)}, \end{cases}
\end{aligned} \tag{2.62}$$

where

$$\theta(\sigma) := \min_{k \geqq 1} \left\{ \frac{B_{k-1}}{B_k \gamma^{1-\varepsilon}} - \frac{1}{\sigma} \right\} > 0 .$$

Proof. In $D^{(k)}$, it follows

$$(\cos\alpha \Delta_1 \varphi^{(\leqq k)} - 1) = -2\sin^2 \frac{\alpha}{2} \Delta_1 \varphi^{(\leqq k)} \leqq -2B_k^2 (\gamma^k |\xi_1 - \xi_2|)^{2(1-\varepsilon)}. \tag{2.63}$$

In $\{(D^{(k-1)} \backslash D^{(k)}) \cap (\Lambda \backslash R^{(k)})^2\}$, we have:

$$(\cos\alpha \Delta_1 \varphi^{(\leqq k)} - 1) = -2\sin^2 \frac{\alpha}{2} \Delta_1 \varphi^{(\leqq k)} \leqq -2\theta^2(\sigma) B_k^2 (\gamma^k |\xi_1 - \xi_2|)^{2(1-\varepsilon)}. \tag{2.64}$$

The relation (2.64) is attained as follows: For

$$(\xi_1,\xi_2)\in D^{(k-1)}, \qquad (\gamma^k|\xi_1-\xi_2|)^{1-\varepsilon}<\gamma^{1-\varepsilon}B_{k-1}^{-1}<1,$$

and therefore

$$\left|\sin\frac{\alpha}{2}\Delta_1\varphi^{(k)}\right|\leqq\frac{B_k}{\sigma}(\gamma^k|\xi_1-\xi_2|)^{1-\varepsilon}$$

must follow for $(\xi_1,\xi_2)\in(I\backslash R^{(k)})^2$. Applying the triangular inequality

$$\left|\sin\frac{\alpha}{2}\Delta_1\varphi^{(\leqq k)}\right|\geqq\left|\sin\frac{\alpha}{2}\Delta_1\varphi^{(\leqq k-1)}\right|-\left|\sin\frac{\alpha}{2}\Delta_1\varphi^{(k)}\right| \tag{2.65}$$

immediately gives

$$\left|\sin\frac{\alpha}{2}\Delta_1\varphi^{(\leqq k)}\right|\leqq\left\{\frac{B_{k-1}}{B_k\gamma^{1-\varepsilon}}-\frac{1}{\sigma}\right\}B_k(\gamma^k|\xi_1-\xi_2|)^{1-\varepsilon}\geqq\theta(\sigma)B_k(\gamma^k|\xi_1-\xi_2|)^{1-\varepsilon}.$$

We now write

$$\begin{aligned}
:\cos\alpha\Delta_1\varphi^{(\leqq k)}-1: &= e^{\alpha^2(C_{00}^{(\leqq k)}-C_{12}^{(\leqq k)})}(\cos\alpha\Delta_1\varphi^{(\leqq k)}-1)\\
&+(e^{\alpha^2(C_{00}^{(\leqq k)}-C_{12}^{(\leqq k)})}-1)=e^{\alpha^2(C_{00}^{(\leqq k)}-C_{12}^{(\leqq k)})}\left(-\sin^2\frac{\alpha}{2}\Delta_1\varphi^{(\leqq k)}\right)\\
&+e^{\alpha^2(C_{00}^{(\leqq k)}-C_{12}^{(\leqq k)})}\left[-\sin^2\frac{\alpha}{2}\Delta_1\varphi^{(\leqq k)}+(1-e^{-\alpha^2(C_{00}^{(\leqq k)}-C_{12}^{(\leqq k)})})\right].
\end{aligned} \tag{2.66}$$

For B large enough we have in I:

$$1-e^{-\alpha^2(C_{00}^{(\leqq k)}-C_{12}^{(\leqq k)})}\leqq\alpha^2(C_{00}^{(\leqq k)}-C_{12}^{(\leqq k)}).$$

Using also

$$\begin{aligned}
C_{00}^{(\leqq k)}-C_{12}^{(\leqq k)}&=\sum_{h=0}^{k}(C_{00}^{(h)}-C_{12}^{(h)})\leqq\sum_{h=0}^{k}\tilde{C}_\varepsilon(\gamma^h|\xi_1-\xi_2|)^{2(1-\varepsilon)}\\
&\leqq\tilde{C}(\gamma^k|\xi_1-\xi_2|)^{2(1-\varepsilon)},
\end{aligned} \tag{2.67}$$

we see that the square bracket in (2.66) is negative for B sufficiently large. Furthermore, $e^{\alpha^2(C_{00}^{(\leqq k)}-C_{12}^{(\leqq k)})}>1$, therefore we finally get

$$:\cos\alpha\Delta_1\varphi^{(\leqq k)}-1:\leqq-\sin^2\frac{\alpha}{2}\Delta_1\varphi^{(\leqq k)},$$

from which the theorem follows. □

2.3. The Main Lemma

In this subsection we restrict ourselves to a brief presentation of the central inequality needed to prove the Main Lemma. How one is reduced to it is explained in complete detail in [BGN] and [NRS]. The proof of the lemma can be found in [BCGNOPS].

Consider a function $H[\Lambda]$ of the fields $\varphi^{(k)}$ at fixed $\varphi^{(\leqq k-1)}$ for an arbitrary set $\Lambda \subset \Lambda$,

$$H[\Lambda](\varphi^{(k)}) := H^{(k)}[\Lambda] := \sum_{n=1}^{t} \sum_{m=1}^{n} \sum_{\substack{p_i, q_i, m_i \\ 0 \leqq \sum_{i=1}^{m} q_i \leqq m}} \times \left\{ \lambda^n \int_{\Lambda^n} v_{pqm}(\xi_1, \ldots, \xi_n) \times \prod_{j=1}^{m} \left(\cos\frac{\alpha}{2} \varphi_{\xi_j}^{(\leqq k)} \right)^{p_j} \left(\sin\frac{\alpha}{2} \varphi_{\xi_j}^{(\leqq k)} \right)^{q_j} \times \sum_{\substack{j'=1 \\ j' \neq j}}^{m} \left(\frac{\sin\frac{\alpha}{2}(\varphi_{\xi_j}^{(\leqq k)} - \varphi_{\xi_{j'}}^{(\leqq k)})}{(\gamma^k |\xi_j - \xi_{j'}|)^{1-\varepsilon}} \right)^{m_j} \right\}, \tag{2.68}$$

letting $0 < \varepsilon < \frac{1}{2}$ and $\gamma > 1$ be the arbitrarily fixed parameters of Hölder continuity and scaling. Assuming that Q_k is an exact pavement of $\Lambda \subset \Lambda$ on scale k, e.g. a set of squares Δ with side length γ^{-k} so that $\bigcup_{\Delta \in Q_k} \Delta = \Lambda$ and letting $d(\Delta_1, \ldots, \Delta_n)$ be the length of the shortest path connecting the squares $\Delta_1, \ldots, \Delta_n$, we require that the v-functions satisfy bounds of the following form (A, κ are arbitrary, positive constants)

$$\begin{aligned} &\lambda^n \int_{\Delta_1 \times \ldots \times \Delta_n} |v_{pqm}(\xi_1, \ldots, \xi_n)| d\xi_1 \ldots d\xi_n \\ &\leqq A e^{-\kappa \gamma^k d(\Delta_1, \ldots, \Delta_n)} B_k^2 \lambda_{\mathrm{eff}}^n(k) \leqq \bar{H}_k e^{-\kappa \gamma^k d(\Delta_1, \ldots, \Delta_n)}, \end{aligned} \tag{2.69}$$

where $\bar{H}_k$ may be picked independent of n as $\bar{H} \cdot \lambda_{\mathrm{eff}}(k) \cdot B_k^2$; (with $\bar{H}$ a positive constant) for $\lambda > 0$ sufficiently small.

We now further introduce the $P(d\varphi^{(k)})$-measurable events

$$E_\Delta^B := \left\{ \varphi^{(k)} \,\middle|\, \sup_{\xi, \eta \in \Delta} \left[\frac{|\varphi_\xi^{(k)} - \varphi_\eta^{(k)}|}{(\gamma^k |\xi - \eta|)^{1-\varepsilon}} \right] \leqq B(1 + \gamma^k d(\Delta, \Lambda)) \right\} \tag{2.70}$$

whose characteristic functions we call χ_Δ^B. Defining

$$\mathring{\chi}_\Delta^B := 1 - \chi_\Delta^B,$$

and

$$\chi_{Q_k \setminus G}^B := \prod_{\Delta \in Q_k \setminus G} \chi_\Delta^B; \qquad \mathring{\chi}_G := \prod_{\Delta \in G} \mathring{\chi}_\Delta, \tag{2.71}$$

we have the following decomposition of the identity

$$1 \equiv \sum_{G \subset Q_k} \mathring{\chi}_G^B \cdot \chi_{Q_k \setminus G}^B. \tag{2.72}$$

We are now ready to formulate the Main Lemma:

Lemma 2.2. *For every integer $t \geqq 0$ there exist constants B^*, D, g, g' depending on ε, γ, t, and κ, so that for $B > B^*$*

$$\begin{aligned} &\int \mathring{\chi}_G^B \chi_{\Lambda \setminus G}^B e^{H[\Lambda \setminus G](\varphi^{(k)})} P(d\varphi^{(k)}) \\ &\leqq \exp\{\delta(B, \bar{H}_k) \gamma^{2k} |\Lambda| + \delta'(B, \bar{H}_k) \gamma^{2k} |G \cap \Lambda|\} \\ &\quad \times \exp\left[\sum_{p=1}^{t} \frac{1}{p!} \mathscr{E}_k^T(H[\Lambda](\varphi^{(k)}); p) \right]_{\leqq t} \cdot \left(\int \mathring{\chi}_G^B P(d\varphi^{(k)})\right)^{1/2}, \end{aligned} \tag{2.73a}$$

and

$$\int \mathring{\chi}_{\emptyset}^{B} \chi_{/}^{B} e^{H[/](\varphi^{(k)})} P(d\varphi^{(k)}) \geqq \exp\{-\delta(B, \bar{H}_k)\gamma^{2k}|/|\} \times \exp\left[\sum_{p=1}^{t} \frac{1}{p!} \mathscr{E}_k^T(H[/](\varphi^{(k)}); p)\right]_{\leqq t}, \tag{2.73b}$$

where

$$\delta(B, \bar{H}_k) := D\{(H_k B^g e^{g\bar{H}_k B g})^{t+1} + e^{-g'B^2 + g\bar{H}_k B g}\}, \tag{2.74}$$

and

$$\delta'(B, \bar{H}_k) := D\{\bar{H}_k B^g\}. \tag{2.75}$$

Furthermore, for all $\varepsilon > 0$ there exist positive constants B', a, and b so that for $B > B'$,

$$\int \mathring{\chi}_G^B P(d\varphi^{(k)}) \leqq \prod_{\Delta \in G} e^{a - bB^2(1 + d(\Delta, /))} \tag{2.76}$$

for any $G \subset Q_k$. The latter has also been called the "Tail Lemma."

Remark. Clearly, one can drop the ^-signs on the sets which designate a coating on scale γ^{-k} in [BCGNOPS] by readjusting the multiplicative constants; that is, a set G on scale k is essentially the same as $\hat{G}$.

We now observe that we can use the Main Lemma in our case since we can identify $H^{(k)}[/]$ with $\hat{V}^{(k)}[\mathscr{D}_{k-1}^c \cap \mathscr{R}_k^c]$ as Theorem 2.1 shows that $\hat{V}[\mathscr{D}_{k-1}^c \cap \mathscr{R}_k^c]$ satisfies the estimates (2.69) appropriately identifying the constant factors.

3. Proof of Ultraviolet Stability

In this section we completely present the iterative mechanism used for "transporting" the effective potential from frequency to frequency. As we have seen in Sect. 1, the main difficulty in setting up this mechanism consists in treating the large fluctuation part of the effective potential, while the smooth part is taken care of by the Main Lemma. Since it is not possible to exclusively base the iterative procedure on this smooth part, one has to create an instrument, analogous to the Main Lemma, for the transportation of the large fluctuation part as well. It turns out that the mechanism required for this purpose is actually based on a very simple lemma which makes use of the relationship of large fluctuations of different frequencies. The true object of the iteration, in the following called $U^{(k)}$ for a given frequency k, consists of a smooth part as well as of a large fluctuation part:

$$U^{(k)} = \hat{V}^{(k)}[\mathscr{D}_k^c] + \Delta^{(k)}[\mathscr{D}_k] + W^{(k)}[\mathscr{D}_k]. \tag{3.1a}$$

For $k = N-1$ we have:

$$U^{(N-1)} = \hat{V}^{(N-1)}[\mathscr{D}_{N-1}^c] + \Delta^{(N-1)}[\mathscr{D}_{N-1}] = \tilde{V}^{(N-1)}, \tag{3.1b}$$

where $\hat{V}^{(k)}[\mathscr{D}_k^c]$ and $\Delta^{(k)}[\mathscr{D}_k]$ are defined in Sect. 2. The term $\Delta^{(k)}[\mathscr{D}_k]$ is a large fluctuation part generated at frequency k. By $W^{(k)}[\mathscr{D}_k]$ we denote the large fluctuation part brought down from higher frequencies via the "large fluctuation transport mechanism;" it will be defined in a moment.

When going from $\hat{V}^{(k)}[\mathscr{D}_k^c]$ to $\hat{V}^{(k)}[\mathscr{D}_{k-1}^c \cap \mathscr{R}_k^c]$ (cf. Sect. 1.2) another large fluctuation part of the effective potential is generated; namely

$$\hat{V}^{(k)}[\mathscr{D}_{k-1} \cap \mathscr{D}_k^c \cap \mathscr{R}_k^c].$$

All of the three large fluctuation parts present at the frequency k will be split into a fraction which is "used," i.e. which makes the iterative mechanism work at frequency k and into another fraction, denoted by the subscript "s", which will be saved and transported to lower frequencies. For instance:

$$W^{(k)}[\mathscr{D}_k] = (W^{(k)}[\mathscr{D}_k] - W_s^{(k)}[\mathscr{D}_k]) + W_s^{(k)}[\mathscr{D}_k]. \tag{3.2}$$

The precise meaning of the operation "s" as well as the symbol $W^{(k)}$ will be defined in the central theorem of this section. Let us now state the iterative mechanism:

$$\begin{aligned}
&\text{1) a)} && U^{(k)} = \hat{V}^{(k)}[\mathscr{D}_k^c] + W^{(k)}[\mathscr{D}_k] + \Delta^{(k)}[\mathscr{D}_k] \\
&&& \leqq \hat{V}^{(k)}[\mathscr{D}_k^c] + W_s^{(k)}[\mathscr{D}_k] + \Delta_s^{(k)}[\mathscr{D}_k] \\
&&& = \hat{V}^{(k)}[\mathscr{D}_{k-1}^c \cap \mathscr{R}_k^c] + W_s^{(k)}[\mathscr{D}_k] + \Delta_s^{(k)}[\mathscr{D}_k] \\
&&& \quad + \hat{V}^{(k)}[\mathscr{D}_{k-1} \cap \mathscr{D}_k^c \cap \mathscr{R}_k^c] + \hat{V}^{(k)}[\mathscr{D}_k^c \cap \mathscr{R}_k] \\
&\text{1) b)} && \leqq \hat{V}^{(k)}[\mathscr{D}_{k-1}^c \cap \mathscr{R}_k^c] + W_s^{(k)}[\mathscr{D}_k] + \Delta_s^{(k)}[\mathscr{D}_k] \\
&&& \quad + \hat{V}_s^{(k)}[\mathscr{D}_{k-1} \cap \mathscr{D}_k^c \cap \mathscr{R}_k^c] + W^{(k-1)}[\mathscr{D}_{k-1} \cap \mathscr{R}_k] \\
&&& \quad + [\text{other terms}].
\end{aligned}$$

2) a) (large fluctuation transport) $\Rightarrow$

$$\leqq \hat{V}^{(k)}[\mathscr{D}_{k-1}^c \cap \mathscr{R}_k^c] + W^{(k-1)}[\mathscr{D}_{k-1}] + [\text{other terms}].$$

b) (Main Lemma) $\Rightarrow$

$$U^{(k-1)} = \hat{V}^{(k-1)}[\mathscr{D}_{k-1}^c] + W^{(k-1)}[\mathscr{D}_{k-1}] + \Delta^{(k-1)}[\mathscr{D}_{k-1}].$$

Theorem 3.1. *We introduce the following definitions:*

$$W^{(k)}[I] := \sum_{\tilde{h}=k+2}^{N} W^{(k,\tilde{h})}[I], \tag{3.3}$$

$$W^{(k,\tilde{h})}[I] := C \sum_{q=0}^{k} \{p_{p1}^{(k+1-q)} \bar{Y}_{1k}^{\tilde{h}}[I] + p_2^{(k+1-q)} \bar{Y}_{2k}^{\tilde{h}}[I]\}, \tag{3.4}$$

$$p_1 := \varrho^{-1} \gamma^{-(2-2\varepsilon)}; \qquad p_2 := \gamma^{-(2-2\varepsilon)}; \qquad (0 < p_i < 1), \tag{3.5}$$

$$\varrho := \left(\min_{k \geqq 1} \left\{ 1 - \frac{B_k}{B_{k-1}} \cdot \frac{\gamma^{(1-\varepsilon)}}{\sigma} \right\} \right)^2, \tag{3.6}$$

where we have the following conditions on p_1, p_2, and C:

$$0 < C < \frac{(1-p_2)(1-p_2)}{2-(p_i+p_2)}.$$

(Note that these conditions can always be satisfied for σ sufficiently large which is compatible with the condition on σ coming from the lower bound (cf. [BGN]))

$$\bar{Y}^{\tilde{h}}_{1k}[I]:=-c_1\lambda^2\gamma^{-2(1-\varepsilon)(\tilde{h}-k)}$$

$$\times\int_I d\xi_1 d\xi_2 e^{\alpha^2 C_{12}^{(<\tilde{h})}}(e^{\alpha^2 C_{12}^{(\tilde{h})}}-1)\sin^2\frac{\alpha}{2}\Delta_1\varphi^{(\leqq k)}, \tag{3.7}$$

$$\bar{Y}^{\tilde{h}}_{2k}[I]:=-c_2\lambda^2\int_I d\xi_1 d\xi_2 e^{\alpha^2 C_{12}^{(<\tilde{h})}}(e^{\alpha^2 C_{12}^{(\tilde{h})}}-1)$$

$$\times[B_k(\gamma^k|\xi_1-\xi_2|)^{1-\varepsilon}]^2, \tag{3.8}$$

$$W_s^{(k)}[I]:=\sum_{\tilde{h}=k+2}^{N} W_s^{(k,\tilde{h})}[I], \tag{3.9}$$

$$W_s^{(k,\tilde{h})}[I]:=C\sum_{q=0}^{k-1}\{p_1^{(k+1-q)}\bar{Y}^{\tilde{h}}_{1k}[I]+p_2^{(k+1-q)}\bar{Y}^{\tilde{h}}_{2k}[I]\}, \tag{3.10}$$

$$\Delta_s^{(k)}[I]:=W_s^{(k,k+1)}[I], \tag{3.11}$$

$$\hat{V}_s^{(k)}[I]:=\sum_{\tilde{h}=k+1}^{N} W_s^{(k,\tilde{h})}[I], \tag{3.12}$$

$$[\text{other terms}]\equiv -W^{(k-1)}[\mathscr{D}_{k-1}\cap\mathscr{R}_k]+\hat{V}^{(k)}[\mathscr{D}_k^c\cap\mathscr{R}_k]. \tag{3.13}$$

Theorem 3.1. *With these definitions the iterative mechanism as described by the steps* 1) a), b) *and* 2) a), b) *above works in the sense that there is a finite* $k_0=k_0(\lambda)$ *such that for all frequencies* k *with* $N-1\geqq k\geqq k_0$ *the following properties hold:*

(step 1) a))

$$W^{(k)}[\mathscr{D}_k]-W_s^{(k)}[\mathscr{D}_k]+\Delta^{(k)}[\mathscr{D}_k]-\Delta_s^{(k)}[\mathscr{D}_k]\leqq 0, \tag{3.14}$$

(step 1) b))

$$V^{(k)}[\mathscr{D}_{k-1}\cap\mathscr{D}_k^c\cap\mathscr{R}_k^c]-V_c^{(k)}[\mathscr{D}_{k-1}\cap\mathscr{D}_k^c\cap\mathscr{R}_k^c]\leqq 0, \tag{3.15}$$

(step 2) a))

$$W_s^{(k)}[\mathscr{D}_k]+\Delta_s^{(k)}[\mathscr{D}_k]+V_s^{(k)}[\mathscr{D}_{k-1}\cap\mathscr{D}_k^c\cap\mathscr{R}_k^c]\leqq W^{(k-1)}[\mathscr{D}_{k-1}\cap\mathscr{R}_k^c], \tag{3.16}$$

(step 2) b))

$$[\text{other terms}]\ \textit{can be safely put into the remainder}. \tag{3.17}$$

Before proving the Theorem 1.0 we state a lemma which is crucial for the "large fluctuation transport:"

Lemma 3.1. *In* $D^{(k-1)}\cap(I\backslash R^{(k)})^2$, *we have:*

$$\left|\sin\frac{\alpha}{2}(\varphi_{\xi_1}^{(\leqq k)}-\varphi_{\xi_2}^{(\leqq k)})\right|\geqq\sqrt{\varrho}\left|\sin\frac{\alpha}{2}(\varphi_{\xi_1}^{(\leqq k-1)}-\varphi_{\xi_2}^{(\leqq k-1)})\right|, \tag{3.18}$$

with ϱ *defined by* (3.6).

Proof of the Lemma. We start from the triangular inequality:

$$\left|\sin\frac{\alpha}{2}\Delta_1\varphi^{(\leqq k-1)}\right|\leqq\left|\sin\frac{\alpha}{2}\Delta_1\varphi^{(\leqq k)}\right|+\left|\sin\frac{\alpha}{2}\Delta_1\varphi^{(k)}\right|. \tag{3.19}$$

Let $\xi_1 \in \Delta$ and $\xi_2 \in \Delta'$ (Δ and Δ' are squares of the pavement Q_k), we then have:

(i) $(\xi_1, \xi_2) \in D^{(k-1)}: (\gamma^{k-1}|\xi_1 - \xi_2|)^{1-\varepsilon} B_{k-1} < 1 \Rightarrow |\xi_1 - \xi_2| < \gamma^{-k}$

for B_{k-1} large enough ($\gamma^{1-\varepsilon} B_{k-1}^{-1} < 1$),

(ii) $(\Delta, \Delta') \notin R^{(k)^2}$, let $\Delta \notin R^{(k)} \Rightarrow \forall \xi_1, \xi_2$ with $\xi_1 \in \Delta$: either $\gamma^k|\xi_1 - \xi_2| > 1$ or

$$\left|\sin\frac{\alpha}{2}\Delta_1\varphi^{(k)}\right| \leqq \frac{B_k}{\sigma}(\gamma^k|\xi_1 - \xi_2|)^{1-\varepsilon}.$$

(i), (ii) $\Rightarrow$ in $D^{(k-1)} \cap (I \backslash R^{(k)})^2$:

$$\left|\sin\frac{\alpha}{2}\Delta_1\varphi^{(k)}\right| \leqq \frac{B_k}{\sigma}(\gamma^k|\xi_1 - \xi_2|)^{1-\varepsilon}. \tag{3.20}$$

On the other hand, we have in $D^{(k)}$:

$$\left|\sin\frac{\alpha}{2}\Delta_1\varphi^{(\leqq k)}\right| > B_k(\gamma^k|\xi_1 - \xi_2|)^{1-\varepsilon},$$

and in $(D^{(k-1)} \backslash D^{(k)}) \cap (I \cap R^{(k)})^2$:

$$\left|\sin\frac{\alpha}{2}\Delta_1\varphi^{(\leqq k)}\right| \geqq \theta(\sigma) B_k(\gamma^k|\xi_1 - \xi_2|)^{1-\varepsilon}. \tag{3.21}$$

Taking (3.19), (3.20), and (3.21) together, the lemma follows. □

Proof of Theorem 3.1.

Proof of (3.14).

$$\begin{aligned}
W^{(k)}[\mathscr{D}_k] - W_s^{(k)}[\mathscr{D}_k] &= \sum_{\tilde{h}=k+2}^{N} C\{p_1 \bar{Y}_{1k}^{\tilde{h}}[\mathscr{D}_k] + p_2 \bar{Y}_{2k}^{\tilde{h}}[\mathscr{D}_k]\} \\
&= -\sum_{\tilde{h}=k+2}^{N} C\left\{p_1 c_1 \lambda^2 \gamma^{-2(1-\varepsilon)(\tilde{h}-k)} \int_{D^{(k)}} d\xi_1 d\xi_2 e^{\alpha^2 C_{12}^{(<\tilde{h})}} (e^{\alpha^2 C_{12}^{(\tilde{h})}} - 1)\right. \\
&\quad \times \sin^2\frac{\alpha}{2}\Delta_1\varphi^{(\leqq k)} + p_2\lambda^2 \int_{D^{(k)}} d\xi_1 d\xi_2 e^{\alpha^2 C_{12}^{(<\tilde{h})}} \\
&\quad \left. \times (e^{\alpha^2 C_{12}^{(\tilde{h})}} - 1)\,[B_k(\gamma^k|\xi_1 - \xi_2|)^{1-\varepsilon}]^2\right\}
\end{aligned} \tag{3.22}$$

comparing this expression with the estimates provided by Theorem 2.2 and taking into account that $\Delta^{(k)}[\mathscr{D}_k] - \Delta_s^{(k)}[\mathscr{D}_k]$ itself contains a negative second order tree with bifurcation at the frequency $k+1$, we obtain the desired result for $\lambda_{\text{eff}}(k)$ sufficiently small (i.e. for k large enough).

Proof of (3.15). This follows immediately from the fact that

$$\hat{V}^{(k)}[\mathscr{D}_{k-1} \cap \mathscr{D}_k^c \cap \mathscr{R}_k^c] - V_s^{(k)}[\mathscr{D}_{k-1} \cap \mathscr{D}_k^c \cap \mathscr{R}_k^c]$$

contains negative second order trees with bifurcations at all frequencies $h \geqq k+1$ and from the estimates of Sect. 2.2.

Proof of (3.16). We have

$$\bar{Y}_{2k}^{\tilde{h}}[I] \leqq p_2^{-1} \bar{Y}_{2k-1}^{\tilde{h}}[I], \tag{3.23}$$

and for $I \subset D^{(k-1)} \cap (/\backslash R^{(k)})^2$ from Lemma 3.1:

$$\bar{Y}_{1k}^{\tilde{h}}[I] \leqq p_1^{-1} \bar{Y}_{1k-1}^{\tilde{h}}[I]. \tag{3.24}$$

Thus,

$$\begin{aligned} W_s^{(k)}[\mathscr{D}_k] + \Delta_s^{(k)}[\mathscr{D}_k] &\leqq W_s^{(k)}[\mathscr{R}_k^c \cap \mathscr{D}_{k-1} \cap \mathscr{D}_k] \\ &\quad + \Delta_s^{(k)}[\mathscr{R}_k^c \cap \mathscr{D}_{k-1} \cap \mathscr{D}_k] \\ &\leqq W^{(k-1)}[\mathscr{R}_k^c \cap \mathscr{D}_{k-1} \cap \mathscr{D}_k], \end{aligned} \tag{3.25}$$

and

$$\hat{V}_s^{(k)}[\mathscr{R}_k^c \cap \mathscr{D}_{k-1} \cap \mathscr{D}_k^c] \leqq W^{(k-1)}[\mathscr{R}_k^c \cap \mathscr{D}_{k-1} \cap \mathscr{D}_k^c]. \tag{3.26}$$

Proof of (3.17). First we observe that in $\hat{V}^{(k)}[\mathscr{D}_k^c \cap \mathscr{R}_k]$ all zeroes are effective. But also $-W^{(k-1)}[\mathscr{D}_{k-1} \cap \mathscr{R}_k]$, in spite of being a large fluctuation term, has – by construction – an "artificial" zero so that it essentially obeys the same estimate as the first contribution to (other terms): In fact, using the results of Sect. 2, both terms can be estimated by

$$(\text{const}) \lambda_{\text{eff}}^2(k) B_k \gamma^{2k} |R^{(k)} \times \Lambda|,$$

which can be safely put into the remainder.

Since from (3.14) $W^{(k_0)} + \Delta^{(k_0)} = 0$, and applying the estimates given by the Main Lemma for the remainder terms, we are left at the finite frequency k_0 with an interaction which doesn't have ultraviolet problems anymore, i.e. it is ultraviolet stable in the sense of (1.9). □

The frequency k_0 at which we end the recursive procedure had been chosen for given λ such that $\lambda_{\text{eff}}(k_0)$ was very small. Yet, by choosing λ sufficiently, small we could even force k_0 to be zero. Nevertheless, the only need of requiring λ small is to obtain the property that [cf. (1.10)]:

$$\lim_{\lambda \to 0} E_{\pm}(\lambda) \cdot \lambda^{-(t+\tau)} = 0 \qquad (\tau > 0).$$

4

Let us briefly recollect the central results of the most important sections:

Section 1.3. This section is devoted to the derivation of an expansion for the effective potential $\tilde{V}^{(k)}$ (a "tree expansion," see [G, GN1, GN2]) which is appropriate for defining "smooth" and "rough" parts of $\hat{V}^{(k)}$ in such a way as to have good control of each of these terms. The central results are the expansions (1.29), the explicit expressions for $\tilde{V}(\theta, \underline{\sigma})$ and $\tilde{V}(\underline{\theta}, \underline{\sigma})$ [Eqs. (1.42) and (1.44)] as well as the recursive relation for their coefficients given in Theorem 1.1.

Section 2.1. This section is devoted to the study of the smooth part of the effective potential (see the discussion in Sect. 1.2 as well). The effective potential $\hat{V}^{(k)}[\mathscr{D}_k^c]$ is defined in Eqs. (2.4) and (2.5) and the composition of the "rough" regions $\mathscr{D}_k$ is given explicitly. Finally, $\hat{V}^{(k)}[\mathscr{D}^c_{k-1} \cap \mathscr{R}^c_k]$ is defined (following Fig. 2). These definitions allow us to write down explicit expressions for

$$[\hat{V}^{(k)}[\mathscr{D}^c_k \cap \mathscr{R}^c_k] - \hat{V}^{(k)}[\mathscr{D}^c_{k-1} \cap \mathscr{R}^c_k]] \quad \text{[Eqs. (2.20) and (2.21)]}$$

and

$$\Delta^{(k)}[\mathscr{D}_k] = \left[\sum_{n=1}^{t} \frac{1}{n!} \mathscr{E}^T_{k+1}(V^{(k+1)}[\mathscr{D}^c_k]; n)\right]_{\leq t} - V^{(k)}[\mathscr{D}^c_k] \quad \text{[Eq. (2.23)]}.$$

The control of these expressions is essential to perform steps (i) and (iii) as discussed in Sect. (1.2).

Section 2.2. In this section the estimates for $\Delta^{(k)}[\mathscr{D}_k]$ and $[\hat{V}^{(k)}[\mathscr{D}^c_k \cap \mathscr{R}^c_k] - \hat{V}^{(k)}[\mathscr{D}^c_{k-1} \cap \mathscr{R}^c_k]]$ are performed. The first one is the more difficult one, once it is proven the second one is proven similarly but with fewer complications. The estimate for the $O(\lambda^{>2})$ part of $\Delta^{(k)}[\mathscr{D}_k]$ is obtained in Theorems (2.2a) and (2.2b), that is $|\Delta^{(k)}[\mathscr{D}_k]|$ is bounded by the sum of the right-hand side of inequalities (2.49) and (2.50). The estimate for the $O(\lambda^{>2})$ part of $[\hat{V}^{(k)}[\mathscr{D}^c_k \cap \mathscr{R}^c_k] - \hat{V}^{(k)}[\mathscr{D}^c_{k-1} \cap \mathscr{R}^c_k]]$ is not given but may be obtained proceeding as in the proof of Theorem (2.2a) with the few relevant modifications discussed following that proof. The result is that this part is again bounded by the right-hand side of (2.49) with $D^{(k)}_1$ substituted by $(\Lambda \backslash R_k)^2 \cap D^{(k-1)} \backslash D^{(k)}$.

Section 2.3. Here the "Main Lemma" is briefly discussed. It has been proven in every detail in [BGN, BCGNOPS]. This lemma provides inequality (1.17), that is it allows us to perform the integration of the "smooth" part of the effective potential $\hat{V}^{(k)}[\mathscr{D}^c_{k-1} \cap \mathscr{R}^c_k]$ with respect to the fields $\varphi^{(k)}$ on a generic scale k. The two parts of this lemma (i.e. the two inequalities) are one of the main ingredients used to prove the upper and lower bounds of ultraviolet stability. The way one reduces the proof of the upper and lower bounds (1.9) to the proof of this "Main Lemma" together with estimates of the "rough" parts of the effective potential (for the upper bound) is described in detail in [BGN] and [NRS]. Observe that the estimate of $\hat{V}^{(k)}[\mathscr{D}^c_k \cap \mathscr{R}_k]$ is incorporated in $\delta'(B, \bar{H}_k)\gamma^{2k}|G \cap I|$ of Eq. (2.73) with appropriate notations.

As discussed in Sect. (1.2), the proof of the upper bound of Theorem 1.0 is completed if at each level k we can reduce it to the proof of (1.17a) which in turn followed from the "Main Lemma" in a straightforward way. To do so we must be able to control the terms $O(\lambda^{>2})$ of $[\hat{V}^{(k)}[\mathscr{D}^c_k \cap \mathscr{R}^c_k] - \hat{V}^{(k)}[\mathscr{D}^c_{k-1} \cap \mathscr{R}^c_k]]$ and $\Delta^{(k)}[\mathscr{D}_k]$, which are produced at each level. But as was remarked in Sect. 1.2, (see also Theorem 2.3) the $O(\lambda^2)$ parts of these terms are negative and thus natural candidates for dominating the $O(\lambda^{>2})$ parts which are smaller due to their dependence on $\lambda_{\text{eff}}(k)$ [see Eqs. (2.49) and (2.50)] and recalling that $|\lambda_{\text{eff}}(k)|$ is very small when k is large. This idea is correct but the difficult task is to keep track of all the negative $O(\lambda^2)$ parts which were produced at previous levels. How these negative terms can be brought down to a generic level k without integrating them by using the relationships between the large fluctuations on different scales is the content of Sect. 3. Therein a new type of effective potential $U^{(k)}$ is defined ($= \tilde{V}^{(N-1)}$ for $k = N-1$), and it is shown that while the smooth part of $U^{(k)}$ on each scale k is transported to scale $k-1$ by integration with respect to $P(d\varphi^{(k)})$ through application of the "Main Lemma" [see Eq. (1.17)], the other terms $\Delta^{(k)}$ and $W^{(k)}$ associated to the "rough" parts are transported to level $k-1$ using in a delicate way the negativity of their $O(\lambda^2)$ parts and the relationship between large fluctuations coming from different levels (see Lemmata 2.1 and 3.1).

To summarize, the proof of the upper bound of Theorem 1.0 is attained in the following way:

a) We reduce the integration with respect to $P(d\varphi^{(k)})$ to a sum over the regions R_k composed of terms like on the left-hand side of inequality (1.17) where the $\exp \hat{V}^{(k)}$ is substituted by $U^{(k)}$ [Eq. (3.1a)]. This requires some manipulations which are given in detail in [BGN] and in a more general setting in [NRS].

b) Before performing the integration we transport to scale $k-1$ the "rough" parts using the transport mechanism discussed in Sect. 3.1, hence one is left only with the integration of the "smooth" parts.

c) The integration with respect to the smooth part is performed using the "Main Lemma."

d) We reconstruct from the right-hand side of (1.17a) $\hat{V}^{(k-1)}[\mathscr{D}_{k-1}^c]$, and prove (again in Sect. 3.1) that the terms newly produced together with the ones transported down reproduce $W^{(k-1)}+\Delta^{(k-1)}$.

The iterative procedure is now complete; at the lowest level, the "rough" part is dropped because of its negativity. This completes the proof of the upper bound. The lower bound is much easier as only step c) above has to be performed. In fact, in this case we can introduce appropriate characteristic functions preventing the fields $\varphi^{(\leqq k)}$ and $\varphi^{(k)}$ from being "rough" for all k. This was again discussed in detail in [BGN] and [NRS]. Finally we remark that the proof of the "Main Lemma" can be used to prove (1.17a) and (1.17b) since $\hat{V}^{(k)}[\mathscr{D}_k^c \cap \mathscr{R}_k^c]$ satisfies the appropriate estimates which allow us to identify it with the $H[I]$ of the "Main Lemma." These estimates are the ones provided by Theorem 2.1 in Sect. 2.2.

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