Global Validity of the Boltzmann Equation for a Two-Dimensional Rare Gas in Vacuum

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Abstract. We consider a system of N hard disks in $\mathbb{R}^2$ in the Boltzmann–Grad limit (i.e. $N \to \infty$, $d \searrow 0$, $N \cdot d \to \lambda^{-1} > 0$, where d is the diameter of the disks). If λ is sufficiently small and if the joint distribution densities factorize at time zero, we prove that the time-evolved one-particle distribution converges for all times to the solution of the Boltzmann equation with the same initial datum.

1. Introduction

It is generally believed that in certain limit situations the dynamics of a gas of particles can be described by the Boltzmann equation. One of the basic problems in the foundations of kinetic theory is to prove the validity of this statement in a rigorous way, assuming, as a starting point, the laws of classical mechanics. The difficulty and appeal of this problem stem from the necessity that one has to relate two evolutions with very different natures: Newtonian dynamics, which are deterministic and reversible, and Boltzmann dynamics, which have a stochastic character and are irreversible.

A first result in this direction was obtained by Lanford [1] who deduced, in a rigorous way, the validity of the Boltzmann equation for short times (on the order of magnitude of a fraction of the mean free time).

In this paper we consider a two-dimensional system of hard disks and prove, by following the general strategy proposed in [1], the validity of the Boltzmann equation for all times in the case of a gas allowed to expand into free space and for large enough mean free paths (in comparison with the initial datum). In doing this we use the idea developed in [2] for solving the Cauchy problem for the Boltzmann equation in the corresponding three-dimensional situation-namely, that free flow has good contracting properties on functions rapidly decreasing at infinity.

We discuss the limitations of our result. The method applies to cases in which the density decays at infinity, so thermodynamical situations elude our analysis. Furthermore, we are not able to treat more realistic three-dimensional systems. This is probably a technical difficulty. Finally, we assume that the gas be rarefied in the

sense that the mean free path has to be large enough. Although our estimates are not optimal, this constraint could be really essential, also from a physical point of view. In any case: In order to answer the validity question for the Boltzmann equation, one first has to solve the associated Cauchy problem, and a positive answer for this problem, globally-in-time, is not yet known in the general case.

2. Formulation of the Problem and Main Results

We consider the time evolution of N identical disks in $\mathbb{R}^2$, interacting by means of elastic collisions. We denote by $\phi_t^d X = (x_1^d(t), \ldots, x_N^d(t), v_1^d(t), \ldots, v_N^d(t))$ the positions and velocities of the particles at time t, where $X = (x_1, \ldots, x_N, v_1, \ldots, v_N)$ denotes the phase point at time 0 and d is the diameter of the particles. The flow $\phi_t^d X$ is defined almost everywhere with respect to the Liouville measure $dX = dx_1 \cdots dx_N dv_1 \cdots dv_N$.

Let

$$\mu^d(dX) = \mu^d(X)dX \tag{2.1}$$

be an absolutely continuous probability measure on the phase space of the system and

$$f_k^d(x_1, \ldots, x_k, v_1, \ldots, v_k) = \int \mu^d(X)dx_{k+1} \cdots dx_N dv_{k+1} \cdots dv_N \tag{2.2}$$

the joint distribution densities. We assume μ^d (and hence the f_k^d) to be symmetric with respect to the variables $q_i = (x_i, v_i)$, because the particles are indistinguishable.

Let

$$\mu_t^d(X)dX = \mu^d(\phi_{-t}^d X)dX \tag{2.3}$$

be the time evolved measure and let $f_{k,t}^d$ be the associated joint distribution densities. Then

$$f_{k,t}^d = S^d(t)f_k^d + \int_0^t ds \, S^d(t-s)C_{k,k+1}^d f_{k+1,s}^d, \tag{2.4}_1$$

where

$$f_{j,t}^d \equiv 0 \quad \text{if} \quad j > N, \tag{2.5}$$

$$(S^d(t)f_k^d)(x_1 \cdots x_k, v_1 \cdots v_k) = f_k^d(\phi_{-t}^d(x_1 \cdots x_k, v_1 \cdots v_k)), \tag{2.6}$$

$$C_{k,k+1}^d f_{k+1}^d(x_1 \cdots x_k, v_1 \cdots v_k)$$
$$= \sum_{j=1}^k (N-k)d \int dn \int dv_{k+1} \, n \cdot (v_j - v_{k+1}) f_{k+1}^d(x_1 \cdots x_k x_j + d \cdot n, v_1 \cdots v_k v_{k+1}), \tag{2.7}$$

and n denotes a unit vector in $\mathbb{R}^2$.

The derivation of the hierarchy of Eqs. (2.4) can, e.g., be found in [3], but the discussion there is largely formal. The hierarchy corresponds to the well-known BBGKY-hierarchy, usually established for particles interacting via smooth potentials, and is a consequence of the laws of classical mechanics. From a rigorous point of view the meaning of Eqs. $(2.4)_1$ is not completely clear, because the flow ϕ_t^d is only almost everywhere defined. We face the problem of giving sense to the collision operator $C_{k,k+1}^d$ in Eq. (2.7) for f_{k+1}^d, which is only a.e. defined. However, we can give

a meaning to the following series expansion associated with $(2.4)_1$:

$$S^d(-t)f^d_{k,t} = f^d_k + \sum_{n=1}^{N-k\,t} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n S^d(-t_1)C^d_{k,k+1}S^d(t_1-t_2)\cdots S^d(t_n)f^d_{k+n}.$$

$$(2.4)_2$$

In fact, we denote by Γ_k the k-particle phase space and choose $g_{k+1} \in L^\infty(\Gamma_{k+1})$. Consider the configurations $X_k \in \Gamma_k$ and $y_{k+1} \in \mathscr{F}_{k+1}$, where $\mathscr{F}_{k+1} := \{X_{k+1} \in \Gamma_{k+1} | |x_i - x_{k+1}| = d$ for some $i\}$ denotes the set of all $k+1$-phase points obtained from an X_k by adjoining a colliding particle, and study $\phi^d_{-t}(Y_{k+1})$. If we vary Y_{k+1} over $\mathscr{F}_{k+1}$ and $t \in [0, T]$, $\phi^d_{-t}(Y_{k+1})$ spans a tube flow which is a set of positive measure in Γ_{k+1}. Therefore $g_{k+1}(\phi^d_{-t}(Y_{k+1}))$ is defined almost everywhere in $\mathscr{F}_{k+1} \times [0, T]$. Integrating with respect to $\sum_i dv_{k+1}\, dn\, n \cdot (v_i - v_{k+1})$, we obtain a function, denoted by $C_{k,k+1}S^d(t)g_{k+1}$, which is defined a.e. in $\Gamma_k \times [0, T]$. The same argument shows that $S^d(-t_1)C^d_{k,k+1}S^d(t_1-t_2)\cdots S^d(t_n)f^d_{k+n}$ is defined for almost all $t_n < t_{n-1} < t_{n-2} < \cdots < t_1 < t$ and almost all x_k. Thus the right-hand side of $(2.4)_2$ makes sense and "defines," for all t, a function $S^d(-t)f^d_{k,t}$ which is a.e. defined in Γ_k. From now on, $S^d(-t)f^d_{k,t}$ will always be interpreted as the right-hand side of $(2.4)_2$. The equivalence between the $S^d(-t)f^d_{k,t}$ defined by $(2.4)_2$ and the joint distribution densities given by (2.3) will be established in the appendix.

Besides Eqs. (2.4) we also consider the so-called Boltzmann hierarchy:

$$f_{k,t} = S(t)f_k + \int_0^t ds\, S(t-s)C_{k,k+1}f_{k+1,s}, \tag{2.8}$$

where

$$S(t)f_k(x_1 \cdots x_k, v_1 \cdots v_k) = f_k(x_1 - v_1 t \cdots x_k - v_k t, v_1 \cdots v_k) \tag{2.9}$$

describes free flow and

$$\begin{aligned}
C_{k,k+1}&f_{k+1}(x_1 \cdots x_k, v_1 \cdots v_k) \\
&= \sum_{j=1}^k \lambda^{-1} \int dv_{k+1} \int_{n(v_j - v_{k+1}) \geq 0} dn\, n(v_j - v_{k+1}) \\
&\quad \cdot \{f_{k+1}(x_1 \cdots x_k x_j, v_1 \cdots v'_j \cdots v_k v'_{k+1}) \\
&\quad - f_{k+1}(x_1 \cdots x_k x_j, v_1 \cdots v_j \cdots v_k v_{k+1})\}.
\end{aligned} \tag{2.10}$$

Here, v'_j, v'_{k+1} denote the post-collisional velocities belonging to the incoming velocities v_j, v_{k+1} and the collision parameter n. λ is a positive constant proportional to the mean free path between collisions.

If $f_t(x, v)$ is a solution of the Boltzmann equation for hard disks, then the functions

$$f_{k,t}(x_1 \cdots x_k, v_1 \cdots v_k) = \prod_{j=1}^k f_t(x_j, v_j) \tag{2.11}$$

satisfy the hierarchy (2.8). This explains the name of the hierarchy.

The similarity between Eqs. (2.4) and (2.8) becomes more transparent by

rewriting the collision operator from Eqs. (2.7) in the equivalent form

$$C^d_{k,k+1} f_{k+1,t}(x_1 \cdots x_k, v_1 \cdots v_k)$$

$$= \sum_{j=1}^{k} (N-k)d \int dn \int_{n \cdot (v_j - v_{k+1}) \geq 0} dv_{k+1} n(v_j - v_{k+1})$$

$$\cdot \{ f_{k+1,t}(x_1 \cdots x_j \cdots x_k x_j - nd, v_1 \cdots v'_j \cdots v'_{k+1})$$

$$- f_{k+1,t}(x_1 \cdots x_j \cdots x_k x_j + nd, v_1 \cdots v_j \cdots v_{k+1}) \} \tag{2.12}$$

(see [1] for details). In spite of the similarity, the hierarchies are very different. The first (finite) hierarchy corresponds to the time evolution of a reversible Hamiltonian system. The second one has a stochastic nature and displays irreversibility—we know that the H-theorem holds rigorously for short times. Nevertheless, Eqs. (2.12) are the formal limit as $d \to 0$, $N \to \infty$, $N \cdot d \to \lambda^{-1}$ (this is known as the Boltzmann–Grad limit) of Eqs. (2.7), and therefore, in this limit situation, a particle gas is believed to behave according to (2.8).

We now formulate our result. Consider a sequence of statistical states (μ^d) associated with hard disk systems with $d > 0$ and N such that $N \to \infty$, $d \to 0$ and $\lambda^{-1} = N \cdot d$ fixed. We assume that the joint distribution densities f^d_j at time zero satisfy

i) if $(\mathbb{R}^2 \times \mathbb{R}^2)^{j,d}_{\neq} = \{ X \,|\, |x_i - x_k| > d \text{ for } i \neq k, \ i, k = 1, \ldots, j, \ d \geq 0 \}$, then the f^d_j are continuous on $(\mathbb{R}^2 \times \mathbb{R}^2)^{j,d}_{\neq}$. Furthermore, there exists an infinite sequence $\{f_j\}_j$, continuous on $(\mathbb{R}^2 \times \mathbb{R}^2)^{j,0}_{\neq}$, such that

$$\lim_{d \to 0} f^d_j = f_j \tag{2.13}$$

uniformly on compact subsets of $(\mathbb{R}^2 \times \mathbb{R}^2)^{j,0}_{\neq}$.

ii) the estimates

$$\sup_{\substack{x_1 \ldots x_j \\ v_1 \ldots v_j}} |f^d_j(x_1 \cdots x_j, v_1 \cdots v_j)| \exp \beta_0 \left(\sum_{i=1}^{j} (x_i^2 + v_i^2) \right) \leq \text{const } z_0^j \tag{2.14}$$

hold for some $\beta_0 \geq 2 \exp(\lambda^{-1})$ and $z_0 > 0$.

Theorem 2.1. *Assume* i), ii) *and let $z_0 \cdot \lambda^{-1}$ be sufficiently small. Then, for all $t > 0$ and $j > 0$,*

$$\lim_{d \to 0} f^d_{j,t}(x_1 \cdots x_j, v_1 \cdots v_j) = f_{j,t}(x_1 \cdots x_j, v_1 \cdots v_j) \text{ a.e.,} \tag{2.15}$$

where $f_{j,t}$ solves uniquely the Boltzmann hierarchy.

If the initial state factorizes, i.e. if

$$f_j(x_1 \cdots x_j, v_1 \cdots v_j) = \prod_{k=1}^{j} f_1(x_k, v_k), \tag{2.16}$$

then this factorization is preserved in time for $f_{j,t}$, and $f_{1,t}$ is a unique solution of the Cauchy problem for the Boltzmann equation.

Proof. Following [1] we write $(2.4)_2$,

$$f^d_t = S^d(t) f^d + \sum_{n \geq 1} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n S^d(t-t_1) C^d \cdots S^d(t_n) f^d, \tag{2.17}$$

where $f_t^d = \{f_{k,t}^d\}_{k=1}^\infty$, $S^d(t)f^d = \{S^d(t)f_k^d\}_{k=1}^\infty$, $C^df = \{(C^df)_k\}_{k=1}^\infty = \{C_{k,k+1}^d f_{k+1}\}_{k=1}^\infty$ (we recall that we are using the convention $f_{k,t}^d = 0$ for $t \geq 0$ and $k > N$, thus the series appearing on the right of (2.17) is only a finite sum). Analogously we write a similar (formal) expansion for the Boltzmann hierarchy, with initial datum given by the limit (2.13):

$$f_t = S(t)f + \sum_{n \geq 1} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n S(t-t_1)C \cdots S(t_n)f, \tag{2.18}$$

with $S(t)f = \{S(t)f_k\}_{k=1}^\infty$ and $Cf = \{(Cf)_k\}_{k=1}^\infty = \{C_{k,k+1}f_{k+1}\}_{k=1}^\infty$. In [1] it is shown that under the hypotheses of Theorem 2 each term of the series (2.17) converges a.e. to the corresponding term of the perturbation series (2.18). Therefore, to prove (2.15) it is enough to show convergence of the Boltzmann hierarchy series and to bound the series (2.17) by a convergent series of nonnegative terms not depending on d.

To approach the last problem, we rewrite the series under consideration in a slightly different way:

$$S^d(-t)f_t^d = f^d + \sum_{n \geq 1} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$
$$\cdot \Omega^d(t_1)V^d(t_1)\Omega^d(t_1)^{-1} \cdots \Omega^d(t_n)V^d(t_n)\Omega^d(t_n)^{-1}f^d, \tag{2.19}$$

where

$$\Omega^d(t) = S^d(-t)S(t) \tag{2.20}$$

and

$$V^d(t) = S(-t)C^dS(t). \tag{2.21}$$

For convenience, we assume that the f_j^d are defined on all of $(\mathbb{R}^2 \times \mathbb{R}^2)^j$, but are zero on sets which lead to the overlapping of spheres. Moreover, we define $\Omega^d(t)^{-1}$ to be the identity operator whenever $\phi_t(X)$ is not in the domain of definition of ϕ_{-t}^d. This is reasonable because in this case $\Omega^d(t)^{-1}$ is *not* defined by

$$(\Omega^d(t))^{-1}f(X) = f(\phi_{-t}^d(\phi_t(X))).$$

The proof of Theorem 2.1 is based on the key Proposition 2.1 below which gives estimates on the operators $\Omega^d(t)V^d(t)\Omega^d(t)^{-1}$. We now introduce the spaces needed for the formulation of this result.

For $\beta > 0$, let

$$\| f_j \|_{\beta,j} = \operatorname*{ess\,sup}_{\substack{x_1 \ldots x_j \\ v_1 \ldots v_j}} |f_j(x_1 \cdots x_j, v_1 \cdots v_j)| \exp \beta \sum_{i=1}^j (x_i^2 + v_i^2), \tag{2.22}$$

and for $z > 0$ we define a norm on the space of all sequences of measurable functions $f = \{f_j\}_{j=1,2\ldots}$ by

$$\| f \|_\beta^z = \sup_j \| f_j \|_{\beta,j} \cdot z^{-j}. \tag{2.23}$$

We denote by $X_{\beta,j}$ and X_β^z the Banach spaces induced by the norms (2.22) and (2.23). Then we have

Proposition 2.1. *Let* $\tau > 0$ *be arbitrary but fixed and let* $z < z'$, $(1+d)\ \beta' < \beta$,

$d < 1/\sqrt{2}\tau^2$, and $g \in X_\beta^z$. Then there exists a positive constant $C_0 > 0$ (not depending on τ, d, β, β', z, z') such that for almost all $t \in [0, \tau]$,

$$\| \Omega^d(t) V^d(t) \Omega^d(t)^{-1} g \|_{\beta'}^{z'} \leq C_0 z \lambda^{-1} \left(\frac{1}{1 + t^3} \right) l(\beta) e^{2\sqrt{2}\beta}.$$

$$\cdot \left(\frac{1}{\left| \ln \dfrac{z'}{z} \right|} + \frac{1}{\sqrt{\left| \ln \dfrac{z'}{z} \right|} (\beta - \beta'(1 + d))} \right) \| g \|_\beta^z, \tag{2.24}$$

where

$$l(\beta) = \max(\beta^{-3/2}, \beta^{-1}). \tag{2.25}$$

For the proof see the next section.

By Proposition 2.1, we have for suitable sequences, $z = z_n > z_{n-1} > \cdots > z_0$, $\beta = \beta_n < \beta_{n-1} < \cdots < \tilde{\beta}_0$, such that $\beta \geq 1$ and $(1 + d)\beta_j < \beta_{j-1}$, and for $t_j < \tau$ arbitrary but fixed, and $d < 1/\sqrt{2}\tau^2$, that

$$\| \Omega^d(t_1) V^d(t_1) \Omega^d(t_1)^{-1} \cdots \Omega^d(t_n) V^d(t_n) \Omega^d(t_n)^{-1} f^d \|_\beta^z$$

$$\leq \| f^d \|_{\beta_0}^{z_0} (C_0 \lambda^{-1})^n \cdot \prod_{j=1}^n \left(\frac{1}{\ln \left| \dfrac{z_j}{z_{j-1}} \right|} + \frac{1}{\sqrt{\ln \dfrac{z_j}{z_{j-1}}} (\beta_{j-1} - \beta_j(1 + d))} \right)$$

$$\cdot \left(\frac{z_{j-1} l(\beta_{j-1}) \exp(2\sqrt{2}\beta_j)}{1 + t_j^3} \right).$$

To estimate this, we choose the z_j and β_j such that

$$\ln \frac{z_j}{z_{j-1}} = \frac{1}{n} \text{ (i.e. } z_j = z_{j-1} e^{1/n} = z_0 e^{j/n}),$$

and $\beta_{j-1} - \beta_j(1 + d) = 1/n$ (and hence $\tilde{\beta}_0 = 1/n + (1 + d)/n + \cdots + (1 + d)^n \beta_n$). We conclude that $\tilde{\beta}_0 \leq 2 \cdot (1 + d)^n \beta \leq 2(1 + \lambda^{-1}/n)^n \beta \leq 2\beta \cdot e^{\lambda^{-1}}$, because $n \leq N = \lambda^{-1}/d$. (Note that here the dimension enters in a crucial way: In three dimensions the Boltzmann–Grad limit is $N \cdot d^2 = \lambda^{-1}$, which only leads to the (insufficient) estimate $(1 + d)^n \leq (1 + \sqrt{\lambda^{-1}/n})^n$.) Let $\beta_0 := 2\beta \cdot e^{\lambda^{-1}}$. Inserting all this in the above expression, it follows that there is a constant $A_0 > 0$ such that,

$$\| \Omega^d(t_1) V^d(t_1) \cdots V^d(t_n) \Omega^d(t_n)^{-1} f^d \|_\beta^z$$

$$\leq \| f^d \|_{\beta_0}^{z_0} (A_0 n \lambda^{-1} l(\beta) [\exp(\beta_0 2\sqrt{2})] z_0)^n \prod_{j=1}^n \left(\frac{1}{1 + t_j^3} \right).$$

Therefore, if $f^d \in X_{\beta_0}^{z_0}$, we obtain after reordering and integration from zero to infinity

$$\| f_{j,t}^d \|_\infty = \| S^d(-t) f_{j,t}^d \|_\infty \leq z^j \| S^d(-t) f_{j,t}^d \|_\beta^z$$

$$\leq z^j \left(\| f_j^d \|_\beta^z + \sum_{n \geq 1} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \| \Omega^d(t_1) V^d(t_1) \Omega^d(t_1)^{-1} \cdots f^d \|_\beta^z \right)$$

$$\leq e^j z_0^j \sum_{n \geq 0} \left(\int_0^\infty \frac{ds}{1 + s^3} \right)^n (A_0 \lambda^{-1} z_0 l(\beta_0 e^{-\lambda^{-1}}) e^{2\sqrt{2}\beta_0})^n \frac{n^n}{n!} \| f^d \|_{\beta_0}^{z_0}. \tag{2.26}$$

If $z_0 \cdot \lambda^{-1}$ is sufficiently small, we see that the series (2.19) is estimated by a convergent series not depending on d.

For the Boltzmann hierarchy, let $V(t) = S(-t)CS(t)$. Then, following the proof of Proposition 3.2 in Sect. 3, one obtains an estimate corresponding to (2.24):

$$\| V(t)f \|_{\beta'}^{z'} \leqq z \cdot Cl(\beta)\lambda^{-1} \left(\frac{1}{1+t^3} \right) \left\{ \frac{1}{\left| \ln \dfrac{z}{z'} \right|} + \frac{1}{\sqrt{(\beta-\beta') \left| \ln \dfrac{z}{z'} \right|}} \right\} \| f \|_{\beta}^{z},$$

where $\beta > \beta' \geqq 1$, $z' > z > 0$. Here, C can be chosen independently of z, z', β, β', and the estimate holds for all $t > 0$. Using this result, we can mimick the above convergence proof and obtain convergence for the Boltzmann hierarchy. The limit (2.15) follows by the analysis of the term by term convergence, as in [1]. Equation (2.16) follows by elementary but lengthy algebraic manipulations.

Remarks

1. The limit f_t is a unique solution of the Boltzmann hierarchy in the sense that (2.18) makes sense. Equation (2.18) implies that f_t solves

$$f_t = S(t)f + \int_0^t S(t-s)Cf_s. \tag{2.27}$$

   In the factorization case, $f_{1,t}$ is then a mild solution of the Boltzmann equation (see, e.g., [2]).
2. With the techniques of this paper, the series (2.18) can be shown to converge also in the three dimensional case. The difficulty in proving the Boltzmann–Grad limit is related to an estimate of $\Omega^d(t)$ (see Proposition 3.1 below) which diverges in time in 3 dimensions. We believe that different topologies are needed to handle this problem.
3. As pointed out by Spohn [4], solutions of the Boltzmann hierarchy can be interpreted as statistical solutions of the Boltzmann equation, i.e. they describe the statistical evolution of individual solutions of the equation.
4. In our approach, as in [2], it is essential to work in all $\mathbb{R}^2$ (or $\mathbb{R}^3$) in order to use the contractive character of free flow on a special class of functions. Compact domains require other ideas.
5. The result is not as strong as it may appear at first sight. Although time does not appear on the right-hand side of (2.26), the convergence is uniform only for $t < \tau$, because we have assumed $d \leqq 1/\sqrt{2\tau^2}$.
6. As pointed out in [1], the gap between the topologies at time zero (2.13) and at time t, (2.15) cannot be completely removed because of the irreversibility of the limit equation (see [5] for a discussion).

3. Technical Details

In this section we prove Proposition 2.1. We need two other propositions to do so.

Proposition 3.1. *For $d \leq 1/\sqrt{2}t^2$, the following inequalities hold:*

$$\| S^d(t)S(-t)f \|_{\beta,j} \leq e^{2\sqrt{2}\beta} \| f \|_{\beta(1+d),j}, \tag{3.1}$$

$$\| S(t)S^d(-t)f \|_{\beta,j} \leq \| f \|_{\beta,j} \tag{3.2}$$

Proof. First we establish two inequalities. Using the shorthand notation

$$I(X) := \tfrac{1}{2}\Sigma x_i^2, \quad T(X) := \tfrac{1}{2}\Sigma v_i^2,$$

we have

$$I(\phi_t^d X) + T(\phi_t^d X) \geq I(\phi_t X) + T(\phi_t X), \tag{3.3}$$

$$I(\phi_t^d X) + T(\phi_t^d X) \leq I(\phi_t X) + (1 + \sqrt{2}t^2 d^2)T(\phi_t X) + \sqrt{2}. \tag{3.4}$$

To verify (3.3) and (3.4), choose a phase point X arbitrary but fixed. Then, for $t > 0$, denote by $\{t_i\}_{i=1}^k$, $0 \leq t_1 < t_2 \cdots < t_k \leq t$ the instants at which binary collisions take place (we exclude measure zero situations leading to multiple or simultaneous binary collisions). Let $y_1, y_1' \cdots y_k, y_k', u_1, u_1' \cdots u_k, u_k'$ and $p_1, p_1' \cdots p_k, p_k'$ be the positions and the incoming and outgoing velocities respectively of the pairs of particles colliding at the times $t_1, \ldots, t_k$. Finally, we use the notations $\{x_i^d(t), v_i^d(t)\}_{i=1}^j$, $\{x_i(t), v_i(t)\}_{i=1}^j$ for the positions and velocities of $\phi_t^d X$ and $\phi_t X$ respectively, and put

$$(\phi_{t_i}^d X)^- := \lim_{t \nearrow t_i} \phi_t^d X, \quad (\phi_{t_i}^d X)^+ := \lim_{t \searrow t_i} \phi_t^d X, \quad i = 1, \ldots, k.$$

Then, by energy conservation

$$I(\phi_t^d X) = I((\phi_{t_k}^d X)^+) + (t - t_k) \sum_{i=1}^j x_i^d(t_k) \cdot v_i^d(t_k)^+ + (t - t_k)^2 T(X). \tag{3.5}$$

Conservation of momentum implies

$$y_k p_k + y_k' p_k' = y_k(p_k + p_k') + (y_k' - y_k)p_k' = y_k(u_k + u_k') + (y_k' - y_k)p_k'$$
$$= y_k u_k + y_k' u_k' + (y_k' - y_k)(p_k' - u_k'), \tag{3.6}$$

therefore

$$I(\phi_t^d X) = I((\phi_{t_k}^d X)^-) + (t - t_k) \sum_{i=1}^j x_i^d(t_k) v_i^d(t_k)^-$$
$$+ (t - t_k)^2 T(X) + (t - t_k)(y_k' - y_k)(p_k' - u_k')$$
$$= I(\phi_{t-t_{k-1}}(\phi_{t_{k-1}}^d X)^-) + (t - t_k)(y_k' - y_k)(p_k' - u_k'). \tag{3.7}$$

By iterating this procedure of elimination of collisions we obtain

$$I(\phi_t^d X) = I(\phi_t X) + \sum_{i=1}^k (t_{i+1} - t_i)(y_i' - y_i)(p_i' - u_i'), \tag{3.8}$$

where $t_{k+1} := t$. Because the last term on the right of (3.8) is positive, we obtain (3.3). On the other hand, the same term is estimated by

$$2\sqrt{2} \sum_{i=1}^k (t_{i+1} - t_i)d\sqrt{T(X)} \leq \sqrt{2}(1 + t^2 T(X)d^2), \tag{3.9}$$

and this proves (3.4). The statement of the Proposition now follows from the following chain of estimates, where we use that $\sqrt{2}\,t^2 d < 1$:

$$
\begin{aligned}
\| S^d(t)S(-t)f \|_{\beta,j} &= \operatorname{ess\,sup}_{x} e^{2\beta(T(X)+I(X))} f(\phi_t \phi_{-t}^d X) \\
&= \operatorname{ess\,sup}_{y} e^{2\beta(T(\phi_t^d y)+I(\phi_t^d y))} f(\phi_t y) \\
&\leq \operatorname{ess\,sup}_{y} e^{2\beta(I(\phi_t y)+(1+d)T(\phi_t y))} f(\phi_t y) e^{2\sqrt{2\beta}} \\
&\leq e^{2\sqrt{2\beta}} \| f \|_{\beta(1+d),j}.
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
\| S(t)S^d(-t)f \|_{\beta,j} &= \operatorname{ess\,sup}_{y} e^{2\beta(T(\phi_t y)+I(\phi_t y))} f(\phi_t^d y) \\
&\leq \operatorname{ess\,sup}_{y} e^{2\beta(T(\phi_t^d y)+I(\phi_t^d y))} f(\phi_t^d y) \\
&= \| f \|_{\beta,j}.
\end{aligned}
\tag{3.11}
$$

Proposition 3.2. *For $1 > d \geq 0$, $\beta > \beta' > 0$, $z' > z > 0$, there exists a positive constant C_0, not depending on β, β', z, z', d and t such that*

$$
\| V^d(t)f \|_{\beta'}^{z'} \leq C_0 \lambda^{-1} z l(\beta) \left(\frac{1}{1+t^3} \right) \\
\cdot \left\{ \frac{1}{\left| \ln \dfrac{z}{z'} \right|} + \frac{1}{\sqrt{(\beta-\beta') \left| \ln \dfrac{z}{z'} \right|}} \right\} \| f \|_{\beta}^{z}, \text{ for almost all } t,
\tag{3.12}
$$

where $l(\beta)$ is given by (2.25).

Proof. By definition

$$
\begin{aligned}
(V_{j,j+1}^d(t)f_{j+1})(x_1 \cdots x_j, v_1 \cdots v_j) \\
= (N-j)d \sum_{k=1}^{j} \int dn \int dv_{j+1} n \cdot (v_k - v_{j+1}) f_{j+1} \\
\cdot (x_1 \cdots x_j, x_k + nd + (v_k - v_{j+1})t, v_1, \ldots, v_{j+1}).
\end{aligned}
\tag{3.13}
$$

Then

$$
\begin{aligned}
\exp\left(\beta' \left(\sum_{i=1}^{j} v_i^2 + \sum_{i=1}^{j} x_i^2 \right) \right) \left| (V_{j,j+1}^d(t)f_{j+1})(x_1 \cdots x_j, v_1 \cdots v_j) \right| \\
\leq \lambda^{-1} \sum_{k=1}^{j} \int dn \int dv_{j+1} |v_k - v_{j+1}| \, \| f_{j+1} \|_{\beta,j+1}. \\
\cdot \exp\left[-(\beta-\beta') \sum_{i=1}^{j} (x_i^2 + v_i^2) \right] \exp[-\beta v_{j+1}^2] \\
\cdot \exp[-\beta(x_k + nd + (v_k - v_{j+1})t)^2].
\end{aligned}
\tag{3.14}
$$

For $t \leqq 1$, we have

$$\int dv_{j+1} |v_k - v_{j+1}| \exp[-\beta(x_k + nd + (v_k - v_{j+1})t)^2] \exp[-\beta v_{j+1}^2]$$
$$\leqq \int dv_{j+1}(|v_k| + |v_{j+1}|)e^{-\beta v_{j+1}^2}$$
$$\leqq \pi\beta^{-1}|v_k| + C_1\beta^{-3/2}$$
$$\leqq C(\beta)(1 + |v_k|) \cdot \frac{1}{1 + i^3}, \tag{3.15}$$

with

$$C_1 = \int_{\mathbb{R}^2} |x| e^{-x^2} dx, \quad C(\beta) = 2\max(C_1\beta^{-3/2}, \pi\beta^{-1}).$$

For $t > 1$, the same integral is bounded by

$$\int dv_{j+1} |v_k - v_{j+1}| e^{-\beta(x_k + nd + (v_k - v_{j+1})t)^2} \leqq t^{-3} \int d\xi\{|\xi| + |x_k + nd|\} e^{-\beta\xi^2}$$
$$\leqq t^{-3}(C_1\beta^{-3/2} + (|x_k| + d)\beta^{-1}) \leqq 2 \cdot C(\beta)(2 + |x_k|)\frac{1}{1 + t^3} \tag{3.16}$$

(here we have used that $d < 1$).

Inserting (3.15) and (3.16) into (3.14) we obtain

$$\|V_{j,j+1}^d(t)f_{j+1}\|_{\beta',j} \leqq C(\beta)\lambda^{-1}2\pi\left(\frac{1}{1 + t^3}\right) \tag{3.17}$$
$$\sup_{\substack{x_1 \ldots x_j \\ v_1 \ldots v_j}} \sum_{k=1}^{j} \{3 + |x_k| + |v_k|\} \cdot e^{-(\beta-\beta')\sum_{i=1}^{j}(x_i^2 + v_i^2)} \|f_{j+1}\|_{\beta,j+1}.$$

There is a constant $C_2 > 0$ such that

$$\sum_{i=1}^{j} |x_i| e^{-\gamma \sum_{i=1}^{j} x_i^2} \leqq C_2\sqrt{\frac{j}{\gamma}}, \tag{3.18}$$

and it follows that

$$\|V_{j,j+1}^d(t)f_{j+1}\|_{\beta',j} \leqq C_3 \cdot C(\beta)\lambda^{-1}\left(\frac{1}{1 + t^3}\right)\left(j + \sqrt{\frac{j}{\beta - \beta'}}\right)\|f_{j+1}\|_{\beta,j+1} \tag{3.19}$$

with some $C_3 > 0$. Therefore

$$\|V^d(t)f\|_{\beta'}^{z'} \leqq C_3 C(\beta)\|f\|_\beta^z z\lambda^{-1}\left(\frac{1}{1 + t^3}\right) \cdot \left[\sup_j\left(j + \sqrt{\frac{j}{\beta - \beta'}}\right)\left(\frac{z}{z'}\right)^j\right]. \tag{3.20}$$

It is a simple exercise to verify that

$$\sup_j\left(j + \sqrt{\frac{j}{\beta - \beta'}}\right)\left(\frac{z}{z'}\right)^j \leqq \sup_j j\left(\frac{z}{z'}\right)^j + \sup_j \sqrt{\frac{j}{\beta - \beta'}}\left(\frac{z}{z'}\right)^j$$
$$\leqq \frac{1}{\left|\ln\frac{z'}{z}\right|} + \frac{1}{\sqrt{\left|\ln\frac{z'}{z}\right|(\beta - \beta')}},$$

and this completes the proof of Proposition 3.2.

Proposition 2.1 follows trivially from Propositions 3.1 and 3.2.

Appendix

In this appendix we sketch a derivation of Eqs. (2.4) for hard sphere particle systems. A more detailed version of this derivation will be published elsewhere. A more intricate derivation was given in [6].

Our starting point is the flow ϕ_t^N describing the dynamics of N spheres of diameter $d > 0$ in $\mathbb{R}^\nu$ ($\nu = 2, 3$). We denote by

$$\Gamma_N = \{X \mid X = (x_i, v_i)_{i=1}^N, x_i, v_i \in \mathbb{R}^\nu, |x_i - x_j| \geq d \, \forall i \neq j\}$$

the phase space of the system. It has been proved in ref. [7] that the set of all orbits $\phi_t^N(X)$, $X \in \Gamma_N$, $t \in \mathbb{R}$, which lead to either grazing or multiple collisions, has measure 0 in Γ_N. We denote the set of all other (the "good") phase points by $\mathring{\Gamma}_N$.

For fixed $k < N$, let

$$\mathscr{F}_{ij}^{+(-)} = \{X \in \mathring{\Gamma}_N \mid |x_i - x_j| = d, (x_i - x_j)\cdot(v_i - v_j) \leq 0 \, (> 0)\}$$
$$\text{for some} \quad i \in \{1, \ldots, k\}, j \in \{k+1, \ldots, N\} \tag{A.1}$$

be the set of all points in $\mathring{\Gamma}_N$ which display an outgoing (ingoing) collision between the i-particle and the j-particle. Furthermore, we put

$$\mathscr{F}^{+(-)} = \bigcup_{i=1}^k \bigcup_{j=k+1}^N \mathscr{F}_{ij}^{+(-)}, \quad \text{and} \quad \mathscr{F} = \mathscr{F}^+ \cup \mathscr{F}^-. \tag{A.2}$$

We fix the origin of the time axis and split $\mathring{\Gamma}_N$ into two disjoint sets:

$$\mathring{\Gamma}_N = \Gamma^\infty \cup \tilde{\Gamma}, \tag{A.3}$$

where

$$\Gamma^\infty = \{X = (X^k, X^{N-k}) \mid X^k \in \Gamma_k, X^{N-k} \in \Gamma_{N-k}, \phi_{-t}^N X = (\phi_{-t}^k X^k, \phi_{-t}^{N-k} X^{N-k}), t \geq 0\} \tag{A.4}$$

is the set of all phase points for which the particles from the group $1, \ldots, k$ never interacted in the past with the rest of the particles, and

$$\tilde{\Gamma} = \mathring{\Gamma}_N \backslash \Gamma^\infty. \tag{A.5}$$

For $X \in \tilde{\Gamma}$, the following representation (known as special flow representation) is natural. Let

$$s = s(X) = \min\{t \mid \phi_{-t}^N(X) = Y \in \mathscr{F}^+\}, \quad \varphi(Y) = \min\{t \mid \phi_t^N(Y) \in \mathscr{F}^-, Y \in \mathscr{F}^+\}. \tag{A.6}$$

The mapping $\psi: \tilde{\Gamma} \rightarrow \{(Y, s) \mid Y \in \mathscr{F}^+, 0 \leq s \leq \varphi(Y)\}$, $X \rightarrow (Y, s)$, is then one to one and bimeasurable. For $g \in L^1(\tilde{\Gamma})$, we have

$$\int_{\tilde{\Gamma}} g(X) dX = \int_{\mathscr{F}^+} d\sigma^+(y) \int_0^{\varphi(Y)} ds \, g(\psi^{-1}(Y, s)), \tag{A.7}$$

where $d\sigma^+$ denotes the following measure on $\mathscr{F}^+$:

$$d\sigma^+ = dX^k \sum_{j=k+1}^{N} dx_{k+1} \cdots dx_{j-1} dx_{j+1} \cdots dx_N dv_{k+1} \cdots dv_N dy_j \cdot n_{ij} \cdot (v_j - v_i). \quad \text{(A.8)}$$

Here, dX^k is the Lebesgue measure on Γ_k, and dy_j stands for the Lebesgue measure on the sphere with radius d and center x_i, and $n_{ij} = (x_j - x_i)/|x_j - x_i|$. The right-hand side of (A.8) defines also a (negative) measure on $\mathscr{F}^-$, denoted by $d\sigma^-$. For the proof of the above statements, see e.g. [8].

The advantage of the special flow representation consists of the fact that

$$\phi_t^N(X) \overset{\psi}{\to} (Y, s+t) \quad \text{if} \quad X \overset{\psi}{\to} (Y, s) \quad \text{and} \quad s+t < \varphi(Y). \quad \text{(A.9)}$$

Moreover, there exists a map $T: \mathscr{F}^+ \to \mathscr{F}^+$ (the "displacement function" for the Poincaré mapping associated with the section $\mathscr{F}^+$), which is $d\sigma^+$-preserving and for which

$$\phi_t^N(X) \overset{\psi}{\to} (TY, \tau) \quad \text{(A.10)}$$

if $\varphi(Y) \leqq s + t \leqq \varphi(TY)$ and $\tau = s + t - \varphi(Y)$. The flow ϕ_t^N, restricted to $\tilde{\Gamma}$, gives rise to a strongly continuous group of unitary operators whose generator we denote by $\tilde{\mathscr{L}}$. We denote its image on $L^2(\psi(\tilde{\Gamma}), d\sigma^+ dt)$ by D. For a sufficiently smooth function $v = v(Y, s)$, we have

$$(Dv)(Y, s) = \frac{\partial}{\partial s} v(Y, s). \quad \text{(A.11)}$$

The domain $\mathscr{D}(D)$ can be completely characterized:

Lemma A. $\mathscr{D}(D)$ *consists of all functions of the type*

$$v(Y, t) = v_0(Y) + \int_0^t ds h(Y, s), \quad \text{(A.12)}$$

where

$$v_0 \cdot \sqrt{\varphi} \in L^2(\mathscr{F}^+, d\sigma^+),$$
$$v_0(Y) + \int_0^{\varphi(Y)} h(Y, s) ds = v_0(TY) \quad \text{(A.13)}$$

and

$$h = Dv \in L^2(\psi(\tilde{\Gamma}), d\sigma^+ dt).$$

The proof of Lemma A will be given at the end of the appendix.

We now consider a probability density $\mu_0 \in L^2(\Gamma_N)$ which is differentiable along the trajectories, i.e. with $\mu_t(X) = \mu_0(\phi_{-t}^N X)$, we have

$$\frac{d}{dt} \mu_t(X)|_{t=0} =: (\mathscr{L}_N \mu_0)(X) \in L^2(\Gamma_N). \quad \text{(A.14)}$$

Then the restriction of μ_0 to $\tilde{\Gamma}$ belongs to $\mathscr{D}(\tilde{\mathscr{L}})$. For a bounded function $u_k \in \mathscr{D}(\mathscr{L}_k)$

which depends only on the first k particles, we have

$$\frac{d}{dt}\int_{\Gamma} u_k\mu_t dX = \frac{d}{dt}\int_{\mathscr{F}^+} d\sigma^+(Y)\int_0^{\varphi(Y)} ds\, u_k(\psi^{-1}(Y,s))\mu_t(\psi^{-1}(Y,s))$$

$$= -\int_{\mathscr{F}^+} d\sigma^+(Y)\int_0^{\varphi(Y)} ds\, (\mathscr{L}_k u_k)(\psi^{-1}(Y,s))\mu_t(\psi^{-1}(Y,s))$$

$$+ \int_{\mathscr{F}^+} d\sigma^+(Y)\{u_k(\psi^{-1}(Y,\varphi(Y)))\mu_t(\psi^{-1}(Y,\varphi(Y)))$$

$$- u_k(\psi^{-1}(Y,0))\mu_t(\psi^{-1}(Y,0))\}, \tag{A.15}$$

since $u_k\mu_t|_\Gamma \in \mathscr{D}(\bar{\mathscr{L}})$ and the integration by parts is possible by Lemma A.

With $F = u_k \cdot \mu_t$, we have by Liouville's theorem,

$$\int_{\mathscr{F}^+} d\sigma^+(Y)F(Y,\varphi(Y)) = -\int_{\mathscr{F}^-} d\sigma^-(Y)F(Y,0), \tag{A.16}$$

and so the last integral in the right-hand side of (A.15) reduces to

$$\int dX^k u(X^k) \sum_{j=k+1}^{N} dx_{k+1}\cdots dx_{j-1}dx_{j+1}\cdots dx_N dy_j n_{ij}(v_j - v_i)$$

$$\cdot dv_{k+1}\cdots dv_N \mu_t^N(x_1\cdots x_{j-1}y_j x_{j+1}\cdots x_N, v_1\cdots v_N). \tag{A.17}$$

We notice that by the continuity property of μ_0 along the trajectories, μ_t is $d\sigma^+$ a.e. defined in $\mathscr{F}^+$. Therefore,

$$f_t^{k+1}(x_1,\ldots,x_k,x_k + nd, v_1,\ldots,v_{k+1})$$

$$= \int dX^{N-k+1}\mu_t(x_1\cdots x_k, x_k + nd, v_1\cdots v_{k+1}; X^{N-k+1}) \tag{A.18}$$

is a.e. defined. (Eqn. A.17) can be written in the form

$$\int dX^k u_k(X^k)C_{k,k+1}f_t^{k+1}(X^k). \tag{A.19}$$

Finally

$$\frac{d}{dt}\int dX^k u_k(X^k)f_t^k(X^k) = \frac{d}{dt}\int_{\Gamma^\infty} dX u_k(X)\mu_t(X) + \frac{d}{dt}\int_\Gamma dX u_k(X)\mu_t(X)$$

$$= -\int_{\Gamma^\infty} dX(\mathscr{L}_k u_k)(X)\mu_t(X) + \text{right-hand side of (A.15)} \tag{A.20}$$

$$= -\int dX^k(\mathscr{L}_k u_k)(X^k)f_t^k(X^k) + \int dX^k u_k(X^k)C_{k,k+1}f_t^{k+1}(X^k),$$

where $\mathscr{L}_k$ is the generator associated with the flow ϕ_t^k. The series expansion $(2.4)_2$ satisfies the same differential equation for all sufficiently smooth u_k. Thus the joint distributions f_t^k defined by μ_t must coincide a.e. with the corresponding expressions given by the series expansion.

Proof of Lemma A. It is easy to verify that D restricted to the functions of type (A.12) and satisfying (A.13) (let us denote this set by $\mathscr{D}_0$) is antisymmetric. Moreover, this domain is invariant for the flow ϕ_t^N. It remains to show that if $w \in \mathscr{D}(D^*)$, then $w \in \mathscr{D}_0$.

For all $v \in \mathscr{D}_0$,

$$\int_{\mathscr{F}^+} d\sigma^+ \int_0^{\varphi(Y)} dt\, w(Y,t)(Dv)(Y,t) = -\int_{\mathscr{F}^+} d\sigma^+ \int_0^{\varphi(Y)} dt(D^*w)(Y,t)v(Y,t)$$

$$= -\int_{\mathscr{F}^+} d\sigma^+ \int_0^{\varphi(Y)} dt(D^*w)(Y,t)v(Y,0) - \int_{\mathscr{F}^+} d\sigma^+ \int_0^{\varphi(Y)} dt(D^*w)(Y,t)\int_0^t ds(Dv)(Y,s)$$

$$= -\int_{\mathscr{F}^+} d\sigma^+ \int_0^{\varphi(Y)} dt(D^*w)(Y,t)v(Y,0) - \int_{\mathscr{F}^+} d\sigma^+ \int_0^{\varphi(Y)} ds(Dv)(Y,s)\int_s^{\varphi(Y)} dt(D^*w)(Y,t).$$

$$(A.21)$$

The above identity is valid for all v such that $v(Y,0) = 0$. Therefore $w(Y,t) + \int_t^{\varphi(Y)} ds$ $(D^*w)(Y,s) \in L^2(\psi(\tilde{\Gamma}), d\sigma^+ dt)$ is only dependent on Y, i.e. there is a function $w_0 \in L^2(\mathscr{F}^+, d\sigma^+/\varphi)$ such that

$$w(Y,t) = w_0(Y, \varphi(Y)) - \int_t^{\varphi(Y)} ds(D^*w)(Y,s). \qquad (A.22)$$

Inserting this expression in (A.21), we are led to the identity

$$\int_{\mathscr{F}^+} d\sigma^+(Y)w(Y,\varphi(Y))[v(Y,\varphi(Y)) - v(Y,0)]$$

$$= \int_{\mathscr{F}^+} d\sigma^+(Y)v(Y,0)[w(Y,0) - w(Y,\varphi(Y))], \qquad (A.23)$$

which is equivalent to

$$\int_{\mathscr{F}^+} d\sigma^+(Y)[w(T^{-1}Y, \varphi(T^{-1}Y)) - w(Y,0)]v(Y,0) = 0. \qquad (A.24)$$

Since $v(\cdot, 0)$ is arbitrary in $L^2(\mathscr{F}^+, d\sigma^+/\varphi)$, the proof is complete.

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