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Bounds for the Limiting Variance of the "Heavy Particle" in R^1

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Abstract. We consider a one-dimensional system consisting of a tagged particle of mass M surrounded by a gas of unit-mass hard-point particles in thermal equilibrium. Denoting by Q_t the displacement of the tagged particle, we give lower and upper bounds – independent of M – for $\overline{\underline{\lim}}\, E\frac{Q_t^2}{t}$. It results from the proof that the correct nontrivial norming of Q_t – if any – is $\sqrt{t}$.

1. Introduction

Consider the following one-dimensional system of point-like particles: a particle of mass M (the "heavy particle") is surrounded by particles of mass 1 ("light particles") distributed on the line according to a Poisson distribution with density $\varrho = 1$. The velocities of the particles are distributed independently according to Gaussian laws with mean zero: that of the heavy particle with variance $M^{-1/2}$, those of the light ones with variances 1 (Maxwellian distributions with inverse temperature $\beta = 1$). No interaction among the light particles exists and the heavy particle interacts with the light ones through a hard-core potential of radius 0. That is: they collide elastically. It is well known that the dynamics of this system is well defined with probability one and the measure as seen from the heavy particle is stationary.

Let us denote by V_t the velocity and by $Q_t = \int_0^t V_s ds$ the displacement of the heavy particle. It is widely believed that the suitably scaled trajectory of the heavy particle converges to a Brownian motion, but, at present, there is only partial progress in this direction. Before describing it, we make two simple remarks.

Remarks. 1. As far as the behaviour of the heavy particle is concerned only, this system is equivalent with the system of the same particles also assuming elastic collisions between the light particles.

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2. Other equilibrium velocity distributions are also imaginable (e.g. with a.s. bounded velocities). Our proofs – *mutatis mutandis* – work in those cases, too, replacing the m_k-s (to be defined in the next section) with the k^{th} moments of the light particle velocity distribution.

In his Turán Memorial Lecture held in Budapest in January 1985 Ya. G. Sinai presented results concerning the problem stated above, obtained jointly with M. R. Soloveichik (see [3]). He announced two theorems: the first one states that $t^{-(1/2+\varepsilon)}Q_t$ converges in probability to zero, the second one gives an asymptotic representation of $t^{-1/2}Q_t$ as the difference of two random variables for both of which the CLT holds with variance $\sigma^2 = \frac{m_1}{2}$ (though nothing is known about their dependence, this decomposition implies the stochastic boundedness of $t^{-1/2}Q_t$). A corollary of the first theorem was also presented, asserting that with probability one each light particle collides finitely many times with the heavy one.

The present note can be considered as a comment on Sinai's Turán Memorial Lecture (and, consequently, the first parts of [3]). In fact, by a – simple but quite useful – new idea, we can give bounds for the limiting variance of $t^{-1/2}Q(t)$ (see the theorem of Sect. 5). To do this we should also improve the aforementioned first and second results in order to obtain the remainder terms in L_2 rather than in convergence in measure. We emphasize throughout that very little information about the real dynamics of the system is used. The proofs mainly rely on fluctuation theorems concerning Poisson point processes (some elementary geometry and probability theory).

Our upper bound is sharp: it is exactly the value of the limiting variance for the solved case $M=1$ (see [1, 4]). We guess that on the contrary, the lower bound is not sharp, and in fact the limiting variance doesn't depend on M (see the conjecture of Sect. 5).

Using completely different methods, a third theorem was obtained by Sinai and Soloveichik (the last theorem of [3]) from which the same lower bound as ours follows for the limiting variance of $t^{-1/2}Q_t$.

2. General Formalism and Notation

The phase-space of our system is:

$$\mathfrak{X} = \mathbb{R} \times \Omega = \{\mathfrak{x} = (V, \omega) : V \in \mathbb{R}, \omega = (q_i, v_i)_{i \in I} \in \Omega\}, \tag{2.1}$$

where I is a countable infinite index set, Ω the set of infinite, but locally finite point systems in $\mathbb{R} \times \mathbb{R}$. Interpretation: V is the velocity of the heavy particle, $(q_i, v_i)_{i \in I}$ are the coordinates (relative to the position of the heavy particle) and velocities (absolute) of the light particles. We say that ω is the environment seen by the heavy particle. Ω is a Polish space, endowed with the natural σ-algebra $\mathscr{F}_0$ generated by counting functions on compact sets. The σ-algebra on $\mathfrak{X}$ is $\mathscr{F} = \mathscr{B} \times \mathscr{F}_0$, $\mathscr{B}$ being the Borel-algebra on $\mathbb{R}$. The equilibrium measure described in the introduction is:

$$d\mu = dF_M \cdot d\mu_0, \tag{2.2}$$

with μ_0 being the Poisson measure on $(\Omega, \mathscr{F}_0)$ with intensity $dx \cdot dF_1(v)$;

$$dF_M(V) = \sqrt{\frac{M}{2\pi}} \exp\left(-\frac{MV^2}{2}\right) dV, \tag{2.3}$$

$$dF_1(v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right) dv. \tag{2.4}$$

We shall use (in Sect. 5) the notation:

$$m_k = E|v|^k = \int_{-\infty}^{\infty} |v|^k dF_1(v), \tag{2.5}$$

$\left(m_1 = \sqrt{\frac{2}{\pi}}, m_2 = 1, \text{and } m_3 = 2\sqrt{\frac{2}{\pi}} \text{ will appear}\right)$. $S_t : \mathfrak{X} \to \mathfrak{X}$, $t \in \mathbb{R}$ is the dynamics of the system determined by the laws of classical mechanics assuming no interaction among the light particles and with elastic collisions between the light particles and the heavy one.

The following two facts are assumed to be known:

a) on a set $\mathfrak{X}' \subset \mathfrak{X}$ of μ-measure 1 the maps S_t are well defined for any $t \in \mathbb{R}$ and $S_{t+s} = S_t \circ S_s$ (existence of dynamics);

b) for any $t \in \mathbb{R}$ S_t is μ-measure preserving (stationarity).

Beside this one-parameter group, the following map will have an important role:

$$U : \mathfrak{X} \to \mathfrak{X}; \quad U(V, (q_i, v_i)_{i \in I}) = (-V, (q_i, -v_i)_{i \in I}), \tag{2.6}$$

which simply inverts the velocity of each particle. U is measure-preserving (due to the symmetry of the velocity distributions) and for any $t \in \mathbb{R}$

$$U \circ S_t = S_{-t} \circ U. \tag{2.7}$$

(Let $S_s(\mathfrak{x})$, $s \in [0, t]$ be a path of time-length t of the system, $S_s(U \circ S_t \mathfrak{x})$, $s \in [0, t]$ is the same path observed backwards!)

We shall use the notations

$$V(\mathfrak{x}) = V \quad \text{and} \quad \omega(\mathfrak{x}) = \omega \quad \text{iff} \quad \mathfrak{x} = (V, \omega), \tag{2.8}$$

$$V_t(\mathfrak{x}) = V(S_t \mathfrak{x}), \tag{2.9}$$

$$Q_t(\mathfrak{x}) = \int_0^t V_s(\mathfrak{x}) ds. \tag{2.10}$$

(Usually, if no confusion may arise we omit the notation of dependence on $\mathfrak{x}$.)

The main task is to deduce limit theorems for the random variable Q_t defined on the probability space $(\mathfrak{X}, \mathscr{F}, \mu)$ as $t \to \infty$. The special case $M = 1$ was solved in the classical papers of Harris [1] and Spitzer [4], and it was found that

$$\text{for} \quad M = 1 \quad \frac{Q_{\alpha t}}{\sqrt{\alpha}} \Rightarrow w_\sigma \quad \text{as} \quad \alpha \to \infty,$$

where $\Rightarrow$ stands for convergence in distribution on $C[0, \infty)$ and w_σ is a Wiener-process of dispersion $\sigma^2 = m_1$. We expect the same to hold for $M \neq 1$, too, but we are still far from proving it.

The technical parts of our proofs rely on the following "large deviation estimates" of elementary probability theory. Let ν_λ be a random variable with Poisson-distribution of parameter λ. There are positive constants A, α, and β not depending on λ such that

$$\mathrm{Prob}(\nu_\lambda > c\cdot\lambda) < A\exp(-\alpha c^\beta), \tag{2.11}$$

$$\mathrm{Prob}(|\nu_\lambda - \lambda| > c\cdot\sqrt{\lambda}) < A\exp(-\alpha\cdot c^\beta). \tag{2.12}$$

The rest of notation and formalism will be introduced where motivation arises.

3. "Large Deviation Estimate" for the Trajectory of the Heavy Particle

In the present section we prove the following:

Lemma 1. *For any $\varepsilon>0$ there exist positive constants A, α, and β depending on ε, such that, for $c>0$,*

$$\mu\left(\left\{\mathfrak{x}\,\middle|\,\sup_{t>1}\frac{|Q_t(\mathfrak{x})|}{t^{1/2+\varepsilon}}>c\right\}\right)<A\exp(-\alpha\cdot c^\beta). \tag{3.1}$$

Proof. Let us define the following functions on $\mathfrak{X}$ ($a>0$, $s>0$):

$$M_a^\pm(\mathfrak{x}) = \#\{(q,v)\in\omega\,|\,0\leqq q\leqq a \text{ and } v>0 (v<0)\}, \tag{3.2}$$

$$M_a(\mathfrak{x}) = M_a^+(\mathfrak{x}) + M_a^-(\mathfrak{x}), \tag{3.3}$$

$$K_{a,s}(\mathfrak{x}) = \#\{(q,v)\in\omega\,|\,a\leqq q \text{ and } q+vs\leqq a\}, \tag{3.4}$$

$$C_s(\mathfrak{x}) = \#\{(q,v)\in\omega\,|\,0\leqq q \text{ and } \exists s'\in[0,s]: q+vs' = Q_{s'}(\mathfrak{x})\}. \tag{3.5}$$

$M_a^\pm$ is the number of light particles situated at $t=0$ in the interval $[0,a]$ and having positive (negative) velocities. $K_{a,s}$ is the number of light particles which, in the free dynamics, would have passed from the right to the left through the point of coordinate a in the time interval $[0,s]$. C_s is the number of light particles coming from the right which interact with the heavy particle in the time interval $[0,s]$. Observe that the M's and K's are defined in terms of the free dynamics of the light particles, and so they are easy to handle. Further, we have the following simple but very useful relations among the random variables just introduced:

a) For any $\tau>0$ and $a>0$, if $Q_\tau(\mathfrak{x})\geqq a$ and $s'\leqq\tau\leqq s$, then

$$C_s(\mathfrak{x})\geqq K_{a,s'}(\mathfrak{x}) + M_a^-(\mathfrak{x}).$$

b) For any $s>0$ and $a>0$, if $\max\limits_{0\leqq s'\leqq s} Q_{s'}(\mathfrak{x})\leqq a$, then

$$C_s(\mathfrak{x})\leqq K_{a,s}(\mathfrak{x}) + M_a(\mathfrak{x}).$$

c) For any $s>0$ and $\mathfrak{x}\in\mathfrak{X}$

$$C_s(\mathfrak{x}) = C_s(U\circ S_s\mathfrak{x}).$$

One can convince himself of the truth of these relations by simple inspection.

Let τ_c denote the first hitting time of the "parabola" $y(s)=(c\cdot s)^{1/2+\varepsilon}$:

$$\tau_c(\mathfrak{x}) = \min\{s>0: Q_s(\mathfrak{x}) = (c\cdot s)^{1/2+\varepsilon}\}, \tag{3.6}$$

and put

$$\mathscr{B}_{c,n} = \{\mathfrak{x} \in \mathfrak{X} : n < \tau_c(\mathfrak{x}) \leqq n+1\}, \qquad n = 0, 1, \ldots. \tag{3.7}$$

We have

$$\mathscr{B}_{c,0} \subset \left\{\mathfrak{x} : \max_{0 \leqq s \leqq 1} V_s \geqq c^{1/2+\varepsilon}\right\}, \tag{3.8}$$

and, for $n \geqq 1$,

$$\begin{aligned}
\mathscr{B}_{c,n} \subset & \Big\{\mathfrak{x} : (n < \tau_c \leqq n+1) \text{ and } (C_{n+1} \geqq K_{(cn)^{1/2+\varepsilon},n} + M^-_{(cn)^{1/2+\varepsilon}}) \text{ and} \\
& \Big(\min_{n \leqq s \leqq n+1} V_s > -(cn)^{\varepsilon}\Big)\Big\} \cup \left\{\mathfrak{x} : \min_{n \leqq s \leqq n+1} V_s \leqq -(cn)^{\varepsilon}\right\} \\
& \subset \left[\{\mathfrak{x} : C_{n+1} \geqq K_{(cn)^{1/2+\varepsilon},n} + M^-_{(cn)^{1/2+\varepsilon}}\} \cap \left\{\mathfrak{x} : \max_{0 \leqq s \leqq n+1} (Q_{n+1-s} - Q_{n+1}) \leqq (cn)^{\varepsilon}\right\}\right] \\
& \cup \left\{\mathfrak{x} : \min_{n \leqq s \leqq n+1} V_s \leqq -(cn)^{\varepsilon}\right\} \\
& \stackrel{\text{def}}{=} [\mathscr{E}_{c,n} \cap \mathscr{F}_{c,n}] \cup \mathscr{D}_{c,n}.
\end{aligned} \tag{3.9}$$

In the first inclusion we have used property a), in the second one the fact that for $0 \leqq s \leqq \tau_c$, $Q_s - Q_{\tau_c} \leqq 0$ and for $\tau_c \leqq s \leqq n+1$, $Q_s - Q_{n+1} \leqq -\min_{n \leqq s \leqq n+1} V_s$. But, by using properties b) and c) consecutively, we find that

$$\begin{aligned}
\mathscr{F}_{c,n} &= U \circ S_{n+1} \left\{\mathfrak{x} : \max_{0 \leqq s \leqq n+1} Q_s \leqq (cn)^{\varepsilon}\right\} \\
&\subset U \circ S_{n+1} \{\mathfrak{x} : C_{n+1} \leqq K_{(cn)^{\varepsilon},n+1} + M_{(cn)^{\varepsilon}}\} \\
&= \{\mathfrak{x} : C_{n+1}(\mathfrak{x}) \leqq K_{(cn)^{\varepsilon},n+1}(U \circ S_{n+1}\mathfrak{x}) + M_{(cn)^{\varepsilon}}(U \circ S_{n+1}\mathfrak{x})\}.
\end{aligned} \tag{3.10}$$

Let us define

$$\xi_{n,c}(\mathfrak{x}) = K_{(cn)^{1/2+\varepsilon},n}(\mathfrak{x}) + M^-_{(cn)^{1/2+\varepsilon}}(\mathfrak{x}), \tag{3.11}$$

$$\eta_{n,c}(\mathfrak{x}) = K_{(cn)^{\varepsilon},n+1}(U \circ S_{n+1}\mathfrak{x}) + M_{(cn)^{\varepsilon}}(U \circ S_{n+1}\mathfrak{x}). \tag{3.12}$$

Combining (3.9) and (3.10),

$$\mathscr{B}_{c,n} \subset \{\mathfrak{x} : \xi_{c,n} \leqq \eta_{c,n}\} \cup \mathscr{D}_{c,n} \stackrel{\text{def}}{=} \mathscr{G}_{c,n} \cup \mathscr{D}_{c,n}. \tag{3.13}$$

By a very simple argument one can show that

$$\mu\left(\left\{\mathfrak{x} \,\middle|\, \max_{0 \leqq s \leqq 1} |V_s| > c\right\}\right) < A_1 \exp(-\alpha_1 c^{\beta_1}) \tag{3.14}$$

for some positive A, α_1, and β_1. (It relies on the fact that $\max_{0 \leqq s \leqq 1} |V_s|$ is large iff the initial velocity of the heavy particle was large or it met a light one with large velocity: the probability of both events is exponentially small.) Hence

$$\mu(\mathscr{B}_{c,0}) < A_1 \exp(-\alpha_1 c^{\beta_1(1/2+\varepsilon)}), \tag{3.15}$$

and, using stationarity,

$$\mu(\mathscr{D}_{c,n}) < A_1 \exp(-\alpha_1 (cn)^{\beta_1 \cdot \varepsilon}). \tag{3.16}$$

$\xi_{c,n}$ and $\eta_{c,n}$ are Poissonian random variables (strongly dependent!) with means $\frac{n}{\sqrt{2\pi}} + \frac{(cn)^{1/2+\varepsilon}}{2}$ and $\frac{n+1}{\sqrt{2\pi}} + (cn)^{\varepsilon}$ respectively. Applying standard large deviation estimates (2.12) to them, we find

$$\mu(\mathscr{G}_{c,n}) < A_2 \exp(-\alpha_2 \cdot c^{\beta_2} \cdot n^{\gamma_2}) \tag{3.17}$$

with some positive constants A_2, α_2, β_2, and γ_2. Further, by the last two estimates and (3.13)

$$\mu(\mathscr{B}_{c,n}) < A_3 \exp(-\alpha_3 \cdot c^{\beta_3} \cdot n^{\gamma_3}). \tag{3.18}$$

From (3.15), (3.18) and similar arguments applied to $-Q$, the lemma follows.

Remarks. 1. The arguments of the proof are inspired by the proof of Theorem 1 in [3], suitably adjusted for our different purposes.

2. Note that the only information used about the dynamics were: the relations a)–c) and large deviation estimates concerning the equilibrium distribution. Consequently we think that a similar proof can be carried over for systems with more general interactions, too.

4. The Main Lemma

This section is devoted to the proof of the Main Lemma of our note. It is an improvement of an argument from the proof of Theorem 2 [3]. Beside the fact that we improve it suitably to be used in the next section, we also consider that the exposition is more economic (and hence more transparent).

Let us define four notable collections of light particles:

$$\mathscr{N}_t^{\pm}(\mathfrak{x}) = \{(q,v) \in \omega \mid q \leqq 0 (q \geqq 0) \text{ and } q + vt \geqq 0 (q + vt \leqq 0)\}, \tag{4.1}$$

$$\ell_t^{\pm}(\mathfrak{x}) = \{(q,v) \in \omega \mid q \leqq 0 (q \geqq 0) \text{ and } \exists s \in [0,t] : q + vs = Q_s(\mathfrak{x})\}. \tag{4.2}$$

For $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying

$$E\varphi^2(v) = \int_{-\infty}^{\infty} \varphi^2(v) dF_1(v) < \infty, \tag{4.3}$$

let

$$\begin{aligned} a_\varphi^+ &= E\varphi(v)\chi_{\{v>0\}} = \int_0^{\infty} \varphi(v) dF_1(v), \\ a_\varphi^- &= E\varphi(v)\chi_{\{v<0\}} = \int_{-\infty}^{0} \varphi(v) dF_1(v), \end{aligned} \tag{4.4}$$

and define the random processes

$$N_{\varphi,t}^{\pm} = \sum_{(q,v) \in \mathscr{N}_t^{\pm}} \varphi(v), \tag{4.5}$$

$$C_{\varphi,t}^{\pm} = \sum_{(q,v) \in \ell_t^{\pm}} \varphi(v). \tag{4.6}$$

The N's are relatively simply defined by the Poisson field $(\Omega, \mathscr{F}_0)$, so we can determine anything we need to know about them.

Let

$$\delta^{\pm}_{\varphi,t} = C^{\pm}_{\varphi,t} \pm a^{\pm}_{\varphi} Q_t - N^{\pm}_{\varphi,t}. \tag{4.7}$$

Main Lemma. *For any* φ *satisfying* (4.3) *and* $\varepsilon > 0$,

$$\frac{\delta^{\pm}_{\varphi,t}}{t^{1/4+\varepsilon}} \xrightarrow{L_2(\mathfrak{X},\mu)} 0. \tag{4.8}$$

The proof goes through combining Lemma 1 with a statement (Lemma 2 below) concerning the fluctuations of the Poisson point-process ω. No new information about the dynamics of the system is used!

We are going now to formulate Lemma 2, the proof of which is postponed to the Appendix.

Let Γ_t be the set of continuous functions $y:[0,t]\to\mathbb{R}$ with $y(o)=0$.

Let

$$\Gamma_t^{(c,\varepsilon)} = \{y \in \Gamma_t \mid \text{ for } s \in [0,t] : |y(s)| < c \cdot s^{1/2+\varepsilon};\ |y(t-s) - y(t)| < c \cdot s^{1/2+\varepsilon}\}.$$

For $y \in \Gamma_t$, let

$$\mathscr{D}^{\pm}_{y,t}(\mathfrak{x}) = \{(q,v) \in \omega \mid q \leqq 0 (q \geqq 0) \text{ and } \exists s \in [0,t] : q + vs = y(s)\}, \tag{4.9}$$

$$D^{\pm}_{\varphi,y,t} = \sum_{(q,v)\in\mathscr{D}^{\pm}_{y,t}} \varphi(v), \tag{4.10}$$

$$\delta^{\pm}_{\varphi,y,t} = D^{\pm}_{\varphi,y,t} \pm a^{\pm}_{\varphi} \cdot y(t) - N^{\pm}_{\varphi,t}. \tag{4.11}$$

Lemma 2. *For any* $\varepsilon > 0$ *sufficiently small and* φ *as above, there are positive constants* B, η, ϱ, σ, *and* θ *such that*

$$\mu\left(\left\{\mathfrak{x} \,\middle|\, \sup_{y\in\Gamma_t^{(c,\varepsilon)}} |\delta^{\pm}_{\varphi,y,t}| > d \cdot t^{1/4+\varepsilon}\right\}\right) < B \cdot c \cdot \exp\left(-\eta \frac{d^{\varrho} \cdot t^{\sigma}}{c^{\theta}}\right). \tag{4.12}$$

Proof of the Main Lemma. We shall prove that

$$\mu(\{\mathfrak{x} \mid |\delta^{\pm}_{\varphi,t}| > d \cdot t^{1/4+\varepsilon}\}) < A \cdot \exp(-\alpha d^{\beta} \cdot t^{\gamma}), \tag{4.13}$$

from which both the convergence in probability to zero and the uniform integrability follow.

Let ε and φ be fixed,

$$\mathscr{A}_c = \left\{\mathfrak{x} \,\middle|\, \sup_{t>0} \frac{|Q_t|}{t^{1/2+\varepsilon}} < c\right\}, \tag{4.14}$$

$$\mathscr{B}_{c,t} = \mathscr{A}_c \cap (U \circ S_t \mathscr{A}_c); \qquad \bar{\mathscr{B}}_{c,t} = \mathfrak{X} \backslash \mathscr{B}_{c,t}, \tag{4.15}$$

$$\mathscr{F}_{d,t} = \{\mathfrak{x} \mid |\delta_{\varphi,t}| > d \cdot t^{1/4+\varepsilon}\}, \tag{4.16}$$

$$\mathscr{E}_{d,c,t} = \left\{\mathfrak{x} \,\middle|\, \sup_{y\in\Gamma_t(c,\varepsilon)} |\delta_{\varphi,y,t}| > d \cdot t^{1/4+\varepsilon}\right\}. \tag{4.17}$$

By definition $\mathscr{F}_{d,t} \cap \mathscr{B}_{c,t} \subset \mathscr{E}_{d,c,t}$. Consequently

$$\mu(\mathscr{F}_{d,t}) \leqq \mu(\mathscr{E}_{d,c,t}) + \mu(\bar{\mathscr{B}}_{c,t}) < B \cdot c \cdot \exp\left(-\eta \frac{d^{\varrho} \cdot t^{\sigma}}{c^{\theta}}\right) + 2A \exp(-\alpha c^{\beta}) \tag{4.18}$$

by Lemmas 1 and 2. Choosing a suitable c we gain exactly (4.13).

5. Bounds on Correlations and Variances

We introduce special notations for two particular choices of φ:

a) for $\varphi(v)=1$,

$$N_{1,t}^{\pm}=N_t^{\pm}, \quad C_{1,t}^{\pm}=C_t^{\pm}, \quad N_t^{+}-N_t^{-}=N_t, \quad C_t^{+}-C_t^{-}=C_t, \tag{5.1}$$

b) for $\varphi(v)=v$,

$$N_{v,t}^{\pm}=R_t^{\pm}, \quad C_{v,t}^{\pm}=P_t^{\pm}, \quad R_t^{+}+R_t^{-}=R_t, \quad P_t^{+}+P_t^{-}=P_t. \tag{5.2}$$

The Main Lemma applied to these choices of φ reads

$$C_t^{\pm}\pm\tfrac{1}{2}Q_t=N_t^{\pm}+\delta_t^{\pm}, \tag{5.3}$$

$$P_t^{\pm}+\frac{m_1}{2}Q_t=R_t^{\pm}+\eta_t^{\pm}, \tag{5.4}$$

where $\delta_t^{\pm}/\sqrt{t}$ and $\eta_t^{\pm}/\sqrt{t}$ tend to zero in L_2 as $t\to\infty$. Further:

$$C_t+Q_t=N_t+\delta_t, \tag{5.5}$$

$$P_t+m_1Q_t=R_t+\eta_t. \tag{5.6}$$

Let X_t stand for any process defined on the path-space of the system. Denote by $\bar{X}_t$ the process

$$\bar{X}_t(\mathfrak{x})=X_t(U\circ S_t\mathfrak{x}). \tag{5.7}$$

(Vaguely speaking: the same observable taken on the backward path.) We already know from the third section that

$$\bar{C}_t^{\pm}=C_t^{\pm}, \tag{5.8}$$

because the number of light particles met by the heavy one on the forward and reversed trajectories is the same. We also have

$$\bar{Q}_t=-Q_t \tag{5.9}$$

from simple geometry.

Conservation of momentum implies

$$P_t+\bar{P}_t=(V_t-V_0)M. \tag{5.10}$$

Equations (5.3), (5.8), and (5.9) imply Theorem 2 of [3]. We go in another direction and obtain

Theorem. *For* $0<M<\infty$,

$$\frac{m_2^2}{m_3}\leqq \varliminf_{t\to\infty} E\frac{Q_t^2}{t}\leqq \varlimsup_{t\to\infty} E\frac{Q_t^2}{t}\leqq m_1. \tag{5.11}$$

Proof. $U\circ S_t$ is measure-preserving, so from (5.8), (5.9),

$$E(Q_t\cdot C_t)=0. \tag{5.12}$$

Hence, by (5.5),

$$E\frac{C_t^2}{t}+E\frac{Q_t^2}{t}=E\frac{N_t^2}{t}+o(1)=m_1+o(1). \tag{5.13}$$

From (5.8), (5.10) by using uniform L_2-boundedness of V_t and $C_t/\sqrt{t}$,

$$E(C_tP_t)=\mathcal{O}(\sqrt{t})\,. \tag{5.14}$$

Hence, by (5.5) and (5.6),

$$E\frac{N_tR_t}{t}-E\frac{R_tQ_t}{t}=o(1)\,. \tag{5.15}$$

(We have used $E(N_tQ_t)=EQ_t^2+o(t)$, which is a consequence of (5.12).) Hence, by Schwartz's inequality

$$E\frac{Q_t^2}{t}\geqq\left(E\frac{N_tR_t}{t}\right)^2\Big/E\frac{R_t^2}{t}+o(1)=\frac{m_2^2}{m_3}+o(1)\,. \tag{5.16}$$

The theorem is proved.

Having these bounds we also automatically have

$$0\leqq\underline{\lim}\,E\frac{C_t^2}{t}\leqq\overline{\lim}\,E\frac{C_t^2}{t}\leqq\frac{m_1m_3-m_2^2}{m_3}, \tag{5.17}$$

$$\frac{(m_3-m_1m_2)^2}{m_3}\leqq\underline{\lim}\,E\frac{P_t^2}{t}\leqq\overline{\lim}\,E\frac{P_t^2}{t}\leqq m_3-2m_1m_2+m_1^3\,. \tag{5.18}$$

It is interesting to compare these bounds with the values found in the solvable cases $M=1$ (see Harris [1] and Spitzer [4]) and $M=\infty$ (trivial):

	$M=1$	$M=\infty$	$0<M<\infty$ lower bound	$0<M<\infty$ upper bound
$E\frac{Q_t^2}{t}$	m_1	0	$\frac{m_2^2}{m_3}$	m_1
$E\frac{C_t^2}{t}$	0	m_1	0	$\frac{m_1m_3-m_2^2}{m_3}$
$E\frac{P_t^2}{t}$	$m_3-2m_1m_2+m_1^3$	m_3	$\frac{(m_3-m_1m_2)^2}{m_3}$	$m_3-2m_1m_2+m_1^3$

Observe that the interval of admitted values for $E\frac{P_t^2}{t}\,(0<M<\infty)$ intersects the interval determined by the values taken in the two solvable cases in a single point: the value taken in the case $M=1$. Expecting monotonicity in M of the limiting variances and correlations, we

Conjecture.

$$\lim_{t\to\infty}E\frac{Q_t^2}{t}=m_1\quad\text{independently of } M. \tag{5.19}$$

From (5.19)

$$\frac{Q_t-N_t}{\sqrt{t}}\xrightarrow{L_2}0$$

would follow, which would be a strong result. It would mean that the asymptotical displacement of the heavy particle, normed by $\sqrt{t}$, does not depend on its mass. As a consequence the CLT with variance m_1 would hold. One could prove (5.19), for instance, by showing that

$$\varepsilon E C_t^2 < E(C_t P_t) \quad \text{for some} \quad \varepsilon > 0,$$

which also seems to be plausible. Physical arguments also seem to support our conjecture.

Appendix

For the sake of definiteness we will prove Lemma 2 for $\delta^+_{\varphi,y,t}$ with $\varphi(v)=1$. Thus, for simplicity of notation, we are allowed to drop the upper index $+$ and the lower index φ inside the Appendix. For other choices of φ one proves the lemma in a similar way, regarding separately the positive and negative parts of φ.

Let

$$N_t(\mathfrak{x}) = \#\{(q,v)\in\omega \mid q\leqq 0 \text{ and } q+vt\geqq 0\}, \tag{A.1}$$

$$D_{y,t}(\mathfrak{x}) = \#\{(q,v)\in\omega \mid q\leqq 0 \text{ and } \exists s\in[0,t]: q+vs=y(s)\}, \qquad y\in\Gamma_t, \tag{A.2}$$

$$\hat{D}_{x,t}(\mathfrak{x}) = \#\{(q,v)\in\omega \mid q\leqq 0 \text{ and } q+vt\geqq x\}, \qquad x\in\mathbb{R}. \tag{A.3}$$

We have

$$\begin{aligned}\delta_{y,t} &= D_{y,t}+\tfrac{1}{2}y(t)-N_t = [D_{y,t}-\hat{D}_{y(t),t}]-[N_t-\hat{D}_{y(t),t}-\tfrac{1}{2}y(t)]\\ &\overset{\text{def}}{=} \delta^1_{y,t}-\delta^2_{y,t}.\end{aligned} \tag{A.4}$$

But

$$\sup_{y\in\Gamma_t^{(c,\varepsilon)}} |\delta^2_{y,t}| = \sup_{|x|<ct^{1/2+\varepsilon}} \left| N_t-\hat{D}_{x,t}-\frac{x}{2}\right| \tag{A.5}$$

and, for any $t>0$, $(N_t-\hat{D}_{x,t})_{x\in\mathbb{R}}$ is a Poisson process with expectation $\frac{x}{2}+o(x)$. Consequently, from standard estimates on Poisson processes we have the desired bound of the form (4.12) for δ^2.

Now, let us look at δ^1. For $k\in\mathbb{Z}$, let

$$y_k^c:[0,t]\to\mathbb{R}, \qquad y_k^c(s)=\max(-cs^{1/2+\varepsilon}, k-c(t-s)^{1/2+\varepsilon}). \tag{A.6}$$

For $y\in\Gamma_t^{(c,\varepsilon)}$, if $y(t)\in[k-1,k]$, the following inequalities hold:

$$D_{y,t}\leqq D_{y^c_{k-1},t}\overset{\text{def}}{=}D^c_{k-1,t}; \qquad \hat{D}_{y(t),t}\geqq\hat{D}_{k,t}. \tag{A.7}$$

(The positivity of the function φ is used. Hence the warning that in case of arbitrary φ one has to take the positive and negative parts separately.)

From (A.7),

$$\begin{aligned}\mu\left(\left\{\mathfrak{x}\,\middle|\, \sup_{y\in\Gamma_t^{(c,\varepsilon)}}\delta^1_{y,t}>d\cdot t^{1/4+\varepsilon}\right\}\right) &\leqq 2c\cdot t^{1/2+\varepsilon}\max_{|k|\leqq ct^{1/2+\varepsilon}}\\ &\cdot\mu(\{\mathfrak{x}\mid D^c_{k-1,t}-\hat{D}_{k,t}>dt^{1/4+\varepsilon}\}).\end{aligned} \tag{A.8}$$

But $D^c_{k-1,t}-\hat{D}_{k,t}$ are Poissonian random variables with expectation less than $A\cdot c^2t^{2\varepsilon}$, for A sufficiently large. (Geometry and absolute continuity of the light particle velocity distribution is used.) Applying (2.11) we gain the desired bound also for δ^1. Lemma 2 is proved.

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Notes added

1. Recent simulations done by Dürr, Omerti, and Ronchetti (manuscript) strongly suggest dependence on the mass of the limiting variance in contrast to our conjecture.

2. R. Holley [Z. Wahrscheinlichkeitstheor. Verw. Geb. **17**, 181 (1971)] had considered a similar mechanical model for Brownian motion with the essential difference that in his case the mass of the tagged particle was normed in the same way as the time scale. That is: he considered the asymptotics, as $\varepsilon \to 0$, of the random processes $\varepsilon^{-1/2} V_{\varepsilon^{-1}t}^{(\varepsilon^{-1}\mu)}$ and $\varepsilon^{1/2} Q_{\varepsilon^{-1}t}^{(\varepsilon^{-1}\mu)}$, with the superscript denoting now the mass of the tagged particle. He obtained the famous result that – under our conditions: $\varrho = \beta = m = 1$–these processes converge in distribution to the Ornstein-Uhlenbeck velocity and position processes respectively given by the equations

$$dV_t = -\gamma V_t dt + \sqrt{D}\, dw_t, \qquad dQ_t = V_t dt,$$

with $\gamma = \frac{4}{\mu}\sqrt{\frac{2}{\pi}}$, $D = \frac{8}{\mu^2}\sqrt{\frac{2}{\pi}}$. Now, performing Holley's limit first and then letting $\mu \to 0$, the position process converges in distribution to a Wiener process with variance exactly equal to our *lower* bound. We consider this fact a heavy argument against our conjecture.

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