Commun. Math. Phys. 104, 353–385 (1986)

Communications in Mathematical Physics

Critical Scaling for Monodromy Fields

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Abstract. The large scale asymptotics of the correlations for a family of two dimensional lattice field theories is calculated at the critical "temperature".

Introduction

This paper is devoted to an examination of the large scale asymptotics of the correlation functions for monodromy fields at the critical point. Monodromy fields on the lattice were introduced in [25] based on a natural generalization of the two dimensional Ising field. They are lattice versions of the continuum fields studied by Sato, Miwa and Jimbo in [33] and have also appeared in work on the Federbush and massless Thirring model again in a continuum setting [29, 30, 32]. In [25] the asymptotics of the correlations were examined in the limit that sends the lattice spacing to zero and the "temperature" to the critical point so that the correlation length remains fixed, (massive scaling regime). In this paper we examine the large scale asymptotics of the correlations at the critical point (massless regime) [18]. The original inspiration for this work was the calculation of the critical asymptotics for the two dimensional Ising model. The Ising field is a special case of a monodromy field in the following sense. For each $p \times p$ matrix M we define a field operator $\sigma_m(M)$ $(m \in \mathbb{Z}^2)$. The terminology "monodromy field" used in connection with $\sigma_m(M)$ is motivated by the fact that it is possible to "create" monodromy (M) located at $m \in \mathbb{Z}^2$ in the solution to a certain linear difference equation on the lattice through a formula involving $\sigma_m(M)$ (see 4.1 in [25]). When M is the scalar -1 (1×1 matrix) one has the Ising model in the sense that the vacuum expectation of a product $\sigma_{m_1}(-1) \cdots \sigma_{m_n}(-1)$ gives the *square* of an Ising correlation (see [20] and [21]).

The critical scaling limit of the Ising model is of interest for a variety of reasons related to the renormalization group analysis of critical phenomena [18]. Attached to a critical point in a statistical mechanical system are various critical exponents which measure how different physical quantities behave as the critical temperature is approached [13]. It is observed experimentally that wide varieties of physical systems have the same critical exponents [13]. To "explain" this universality one posits a connection between the critical exponents which measure the behaviour of the system near the critical point and the large scale asymptotics of the correlations *at* the critical point (scaling hypothesis). The large scale asymptotics of the correlations at the critical point are governed by scale invariant random fields [18]. The possibilities for such random fields are presumably circumscribed in much the same way that the central limit theorem limits the possibilities for the distribution of fluctuations of sums of independent random variables (they are Gaussian in the finite variance case). The difference between the assignment of critical exponents and the assignment of scale invariant random fields to a critical point may be likened to the difference between the assignment of Betti numbers and the assignment of cohomology groups to a topological space. In the first case it is hoped and in the second case it is known that one has achieved a deeper level of understanding.

The Ising field is one of the few models in more than one space dimension in which an explicit calculation of the critical scaling limit appears feasible with current technology. Indeed much work has already been done on this problem. In [36] T. T. Wu rigorously calculated the large scale asymptotics of the two point function on the diagonal, and made a formidable asymptotic analysis of the horizontal correlations at $T = T_c$. Kadanoff calculated the collinear $2n$-point asymptotics based on some ideas about operator product expansions. Kadanoff's formula does not have an obvious extension to non-collinear points. H. Au Yang calculated the 4 point asymptotics for two pairs of points, each lying on a diagonal [37]. In [10] Luther and Peschel conjecture a general formula, and in [1] Bander and Richardson demonstrate the non-obvious fact that the Luther–Peschel conjecture agrees with the Kadanoff prescription for collinear points. There are many other references [9, 19, 2] where similar results are described, but I think it is fair to say that the "proofs" of these results lack conviction. A description of the Luther and Peschel analysis should suffice to indicate the difficulties to be found in much of this work. Luther and Peschel examine the Baxter model at the critical temperature in its incarnation as pair of Ising models coupled by a four spin interaction. Almost immediately they introduce a continuum field which is intended to capture the large scale behavior of the correlations at the critical point. This continuum field is defined via a continuous analogue of the Jordan–Wigner transformation. They make assumptions about the dispersion relation for the spin transfer matrix (an elusive object in the Baxter analysis which deals directly with the arrow transfer matrix) and they make an arbitrary choice in the definition of the continuum Jordan–Wigner transformation which directly effects the outcome of the calculations. Cutoffs are introduced to make sense of the continuum fields and an analytic continuation is made to take advantage of formally simpler calculations in the Minkowski regime. When rotational invariance is lost in the final answer it is put back in again by hand. The bold assumptions made in this analysis should be compared with the fact that the asymptotics of the two point function for Ising spins separated along a coordinate axis at the critical temperature involve the application of Szego's theorem to a delicate borderline case which is not yet rigorously justified in spite of recent progress on the singular case [3].

These reservations not withstanding the Luther–Peschel conjecture is an attractive one. In a sense that will be explained later it corresponds to approximating the Ising field $\sigma(-1)$ on the lattice by the symmetrical choice $(\sigma(\lambda) + \sigma(\lambda)^*)/$

$2(\lambda \to -1)$. It is not clear that the fields $(\sigma(\lambda) + \sigma(\lambda)^*)/2$ are random fields on the lattice, but there are strong indications that the critical scaling limits for these fields can be identified with chiral fields in two dimensions (formally the cosine of the non-existent mass zero Gaussian [7]). Thus the scaling limits ought to possess the positivity properties which would be inherited by a limit of random fields. Such speculations will be addressed at greater length in the final section of this paper.

We are now ready to discuss the results of the present paper. To a certain extent these results show that the Luther–Peschel analysis can be made rigorous for monodromy fields (we do not know about the application to the critical Baxter model).

The first section is devoted to showing that the monodromy fields $\sigma(M)$ have correlations which are continuous functions of "temperature" up to and including the critical point. These results also show that the monodromy correlations are continuous functions of the matrices M at the critical point. This implies that the Ising correlations are limits of correlations for the monodromy fields $\sigma(\lambda)$ as $\lambda \to -1$ at the critical point. The first section ends with a formula for "paired" monodromy correlations in which the two point functions $\langle \sigma_a(M)\sigma_b(M)^{-1} \rangle$ are factored out and the remaining portion is expressed as an explicit perturbation determinant. This formula is the basis for the scaling calculations in Sects. 2 and 3 and it is here we must make the additional assumption that none of the monodromy matrices has spectrum on the non-positive axis. The Ising case is thus excluded from explicit consideration in Sects. 2, 3 and 4.

In Sect. 2, the two point functions $\langle \sigma_a(M)\sigma_b(M)^{-1} \rangle$ are expressed as block Toeplitz determinants with a singular symbol. We use a matrix generalization of some results of Basor and Widom [3] to calculate the leading order contribution to the asymptotics. The principal result, Theorem 2.1, is stated at the end of this section.

The third section analyzes the scaling behavior of the perturbation determinant. As in [21, 25] the entries in the determinant are isometrically embedded in a "limiting" Hilbert space and then shown to converge in Schmidt norm. The isometric embedding is carefully chosen to straighten out the Q matrix of the Ising model. This makes possible the identification of the limiting $Q_{\pm}$ decomposition of $L^2(\mathbb{R}, \mathbb{C})$ with a familiar Hardy space splitting. This is used in the next section.

In the fourth section the structure of the limiting perturbation determinant is exploited to evaluate it explicitly in the case in which all the monodromy fields commute with one another. It is here that the arguments most closely resemble those of Luther and Peschel. The structure of the determinant suggests that it is the vacuum expectation of a product in a current group. Unfortunately, the elements in this current group are singular and so a cutoff must be introduced to make sense of the vacuum expectation of a product. Inside the current group there is a cocycle formula which permits the explicit calculation of the vacuum expectation. The vacuum expectation of the cutoff product can also be calculated using determinants. The idea is then to control the convergence of the two results as the cutoff is removed. In order to do this in the determinant formula one must have non-zero transfer. This upsets the current group cocycle calculation but may be circumvented by introducing pure imaginary transfer, then doing an analytic continuation. When this is done and the cutoff is removed the perturbation determinant converges to the result of Sect. 3 and the cocycle formula converges to an explicit product of homogeneous functions. The identity of these two expressions is then the principal result of this section. The results of Sect. 2 and this section may then be combined in the following theorem:

Theorem 4.1. *Suppose* $M_j \in \mathrm{Gl}(p, \mathbb{C})$ $(j=1,\ldots,n)$ *and that no matrix* M_j *has an eigenvalue which is 0 or negative. Suppose* $m_j=(p_j,r_j)$, *and* $n_j=(q_j,r_j)$ *with* $p_j < q_j (j=1,\ldots,n)$ *and* $r_1 < r_2 \cdots < r_n$. *Let* $G_j(N)=\sigma_{Nm_j}(M_j)\sigma_{Nn_j}(M_j)^{-1}$, *where* $\sigma_m(M)$ *has the "unitary" normalization* (1.9). *Let*:

$$\tau_N(m,n)=\prod_{i=1}^{n} N^{-2\mathrm{Tr}(L_i^2)}\langle G_1(N)\cdots G_n(N)\rangle_{T=T_c},$$

where:

$$2\pi L_j=\frac{1}{2\pi 1}\int_C \frac{\log z}{(z-M_j)}dz,$$

and the contour C is a counterclockwise oriented simple closed curve which surrounds the spectrum of M_j *and does not intersect the non-positive real axis. Then* $\lim_{N\to\infty} \tau_N(m,n)=\tau_\infty(m,n)$ *exists, and in the event that all the* M_j *commute amongst themselves the limit is given by*:

$$\tau_\infty(m,n)=c\prod_{i=1}^{n}|m_i-n_i|^{2\mathrm{Tr}(L_i^2)}\prod_{i<j}\left[\frac{|m_i-n_j|\,|n_i-m_j|}{|m_i-m_j|\,|n_i-n_j|}\right]^{2\mathrm{Tr}(L_iL_j)},$$

where c is a constant that depends on $M_1,\ldots,M_n$.

One observation we would like to make about this result is that the choice of logarithms for the M_j is fixed. The way in which the lattice theories "pick out" a choice of logarithms is a somewhat subtle point in the choice of approximations which is discussed further in Sect. 4.

The final section of this paper concerns speculations about the application of the results of this paper to the Ising case. The idea is that an indirect approach looks more promising than a frontal assault. Also discussed is a rather natural conjecture for the scaling limit in the general case. It should agree with the tau-function introduced for the Riemann–Hilbert problem by Sato, Miwa, and Jimbo [33]. Recently, the existence of the tau-function was clarified in a beautiful geometric analysis of Schlesinger's equations by Malgrange [11]. There are, however, some obstacles to overcome in forging a link between these continuum results and the lattice theories examined here.

In conclusion I would like to point out than A. Carey, S. Ruijsenaars, and J. Wright have a recent preprint which clarifies the status of the Klaiber n-point functions for the massless Thirring model (Wightman positivity is proved) [38]. In another preprint A. Carey and S. Ruijsenaars develop results similar to those used in Sect. 4 for current algebras [32].

Section 1

In this section we introduce the induced rotation for the transfer matrix of the Ising model and examine certain aspects of its behavior at the critical point which will be of use to us later on. We then recall the definition of monodromy fields from [25] and we investigate the behavior of the correlations for these fields as the critical point is approached (Proposition 1.0). For those familiar with the Ising model the results in the first part of this section are analogues of the Montroll, Potts, and Ward formulas for the Ising correlations as finite dimensional Pfaffians. Finally we give a formula (1.14) for "paired" correlations in which the two point functions are factored out, using Theorem 3.0 in [24]. This formula will be the basis for the scaling calculations which are carried out in Sects. 2 and 3.

We begin by recalling some notation from [25]. Let $H = L^2(S^1, \mathbb{C}^2)$ and define:

$$T(\theta) = \begin{bmatrix} c^2/s - \cos\theta & s\sin\theta - i(c/s - c\cos\theta) \\ s\sin\theta + i(c/s - c\cos\theta) & c^2/s - \cos\theta \end{bmatrix}, \tag{1.0}$$

where $c,\ s > 0$ and $c^2 - s^2 = 1$. The matrix valued multiplication operator on $L^2(S^1, \mathbb{C}^2)$ associated with $T(\theta)$ is the induced rotation for the transfer matrix of the Ising model [21]. The parameter s is a function of the temperature in the Ising model. For our purposes it is enough to know that $s < 1$ corresponds to $T < T_c$, $s = 1$ corresponds to $T = T_c$, and $s > 1$ corresponds to $T > T_c$. In this paper we are mainly interested in $T = T_c$ or $s = 1$. However, in this section we will also consider correlations in the limit $s \uparrow 1$ as results for this limit are important if one is to make the connection with critical Ising correlations. We now make the preliminary study of the transfer matrix as $s \uparrow 1$ which will be used to prove Proposition 1.0. For $0 < s < 1$ define functions $\gamma(\theta) > 0$ and $\alpha(\theta)$ by

$$\begin{aligned} \cosh\gamma(\theta) &= c^2/s - \cos\theta \\ \sinh\gamma(\theta)e^{i\alpha(\theta)} &= (c/s - c\cos\theta) + is\sin\theta. \end{aligned} \tag{1.1}$$

We normalize $\alpha(\theta)$ so that $\alpha(\pi) = 0$. The importance of these functions for us is the formula:

$$T(\theta) = \exp[-\gamma(\theta)Q(\theta)] = e^{-\gamma(\theta)}Q_+(\theta) + e^{\gamma(\theta)}Q_-(\theta), \tag{1.2}$$

where

$$Q(\theta) = \begin{bmatrix} 0 & ie^{i\alpha(\theta)} \\ -ie^{-i\alpha(\theta)} & 0 \end{bmatrix} \quad \text{and} \quad Q_\pm(\theta) = \tfrac{1}{2}[1 \pm Q(\theta)].$$

The projections $Q_\pm$ will determine the spin representation in which the monodromy fields live. It is not hard to see that when $0 < s < 1$ the functions $\gamma(\theta)$ and $\alpha(\theta)$ are smooth functions. Indeed solving (1.1) for $\gamma(\theta)$ one finds:

$$\gamma(\theta) = \log(A(\theta) + [A(\theta)^2 - 1]^{1/2}), \tag{1.3}$$

where $A(\theta) = c^2/s - \cos\theta$. When $s < 1$ the quantity $c^2/s = s + s^{-1} > 2$, so that $A(\theta) > 1$. It follows from this and (1.3) that $\gamma(\theta)$ is a smooth function of θ. The pointwise limit in (1.3) as $s\uparrow 1$ is elementary and one finds:

$$\gamma_c(\theta)=\log(2-\cos\theta+[(2-\cos\theta)^2-1]^{1/2}), \tag{1.4}$$

where $\gamma_c(\theta)=\lim_{s\uparrow 1}\gamma(\theta)$. The smoothness in θ is lost in the limit and $\gamma_c(\theta)$ has a cusp at $\theta=0$.

Now take the second equation in (1.1) and divide it by its own complex conjugate. One finds

$$e^{2i\alpha(\theta)}=\frac{c/s-c\cdot\cos\theta+is\cdot\sin\theta}{c/s-c\cdot\cos\theta-is\cdot\sin\theta}.$$

Next differentiate this relation in θ to obtain:

$$\frac{d\alpha}{d\theta}=\frac{c(\cos\theta-s)}{(\cos\theta-c^2/s-1)(\cos\theta-c^2/s+1)}. \tag{1.5}$$

For θ in intervals $[a,b]$ which do not contain 0 the derivative $d\alpha/d\theta$ converges uniformly to $\sqrt{2}\,(\cos\theta-3)$ as $s\uparrow 1$. Thus by dominated convergence:

$$\lim_{s\uparrow 1}(\alpha(b)-\alpha(a))=\int_a^b\frac{\sqrt{2}d\theta}{\cos\theta-3},\quad 0\notin[a,b]. \tag{1.6}$$

But the normalization point $\alpha(\pi)=0$ shows that $\lim_{s\uparrow 1}\alpha(\pi)=0$. Together with (1.6) this shows that the limit of $\alpha(\theta)$ as $s\uparrow 1$, $\alpha_c(\theta)$, exists for $\theta\neq 0$ and we have:

$$\alpha_c(\theta)=\int_\pi^\theta\frac{\sqrt{2}d\phi}{\cos\phi-3}\quad -\pi<\theta<\pi.$$

A simple calculation shows that $\lim_{\theta\to 0\pm}\alpha_c(\theta)=\pm\pi/2$. It will be convenient to make use of this to rewrite the result for α_c as follows:

$$\alpha_c(\theta)=\frac{\pi\varepsilon(\theta)}{2}+\int_0^\theta\frac{\sqrt{2}d\phi}{\cos\phi-3}, \tag{1.7}$$

where $\varepsilon(\theta)=\begin{cases}1 & \theta>0\\ -1 & \theta<0\end{cases}$. The jump discontinuity in $\alpha_c(\theta)$ at $\theta=0$ revealed in (1.7) will be the source of all the special difficulty we have in studying the critical correlations. It is the reason that the single site monodromy fields do not make sense at $s=1$ and is thus the reason we are forced to consider only paired correlations. We will write Q_c and $Q_c^\pm$ for Q and $Q_\pm$ at the critical temperature (i.e., with $\alpha(\theta)$ replaced by $\alpha_c(\theta)$).

We now recall the definition of the monodromy fields in [25]. Let p denote a positive integer and write $H^p=L^2(S^1,\mathbb{C}^2)\otimes\mathbb{C}^p$ (which we sometimes identify with $L^2(S^1,\mathbb{C}^{2p})$. Suppose $M\in\mathrm{Gl}(p,\mathbb{C})$ and define an action of M on H^p by:

$$f\to I\otimes Mf.$$

For $f(\theta)\in L^2(S^1,\mathbb{C}^{2p})$ define:

$$\hat{f}(k)=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}f(\theta)d\theta, \tag{1.8}$$

where $k\in\mathbb{Z}+\frac{1}{2}=\mathbb{Z}_{1/2}$. These Fourier coefficients establish a well known isomorphism between $L^2(S^1,\mathbb{C}^{2p})$ and $l^2(\mathbb{Z}_{1/2},\mathbb{C}^{2p})$, which we often make use of without mentioning the transformation (1.8) explicitly. Let ε act on H^p by multiplying the half integer Fourier coefficients by $\varepsilon(k)=\begin{cases}1 & k>0\\ -1 & k<0\end{cases}$. Define $s(M)=(1-\varepsilon)/2+((1+\varepsilon)2)M$. In [25] it was shown that $s(M)\in \mathrm{Gl}_Q(H^p)$ (we write T, Q and $Q_\pm$ for $T\otimes I_p$, $Q\otimes I_p$, and $Q_\pm\otimes I_p$ acting on H^p. On the other hand the discontinuity in $\alpha_c(\theta)$ is directly responsible for the fact that $s(M)\notin \mathrm{Gl}_{Q_c}(H^p)$ (see [24] for a definition of $\mathrm{Gl}_Q(H)$). We now make the assumption that none of the eigenvalues for M are negative real numbers or 0. In this case the operator $Q_-s(M)Q_-$ is invertible for all values of the parameters $0<s\leqq 1$. This follows from the results of Sect. 1 of [25], but it is instructive to notice that it can be understood in a completely elementary fashion as follows. The matrix M is similar to its own Jordan normal form J_M with eigenvalues $\lambda_1,\ldots,\lambda_p$ on the diagonal. Thus $Q_-s(M)Q_-$ is similar to $Q_-s(J_M)Q_-$ which has entries on the diagonal given by $(1/2)Q_-((\lambda_k+1)+(\lambda_k-1)\varepsilon)Q_-$. Evidently $Q_-s(J_M)Q_-$ will be invertible precisely when each operator $Q_-((\lambda_k+1)+(\lambda_k-1)\varepsilon)Q_-$ is invertible. The operator $Q_-\varepsilon Q_-$ is self-adjoint with spectrum contained in the interval $[-1,1]$. Thus $(\lambda_k+1)+(\lambda_k-1)Q_-\varepsilon Q_-$ is invertible provided λ_k is not a negative real number or 0. (The precise spectrum of $Q_-s(M)Q_-$ is calculated in [25].) In order to represent $s(M)$ in $\hat{\mathrm{Gl}}_Q(H^p)$ we choose the factorization of $s(M)$ described in Sects. 1 and 2 of [25]. We write:

$$s(M)=\underset{\sim}{s}(M)D(M),$$

where $D(M)=I_+\oplus(P_+\otimes I_p+P_-\otimes M)_-$ is defined after (2.5) in [25].

The monodromy field at site $(0,0)$ in $\mathbb{Z}^2$ is now defined to be:

$$\sigma(M)=\Gamma_Q(\underset{\sim}{s}(M))\Gamma(D(M)). \tag{1.9}$$

The homomorphisms Γ_Q and Γ are defined in Sect. 3 of [24].

Next we translate these fields around on the lattice. Let z denote multiplication by $e^{i\theta}$ on $H^p=L^2(S^1,\mathbb{C}^{2p})$. Write $V_1=\Gamma(z)$ and $V_2=\Gamma(T)$ in the Q spin representation of $\hat{\mathrm{Gl}}_Q(H^p)$. We define:

$$s_a(M)=T^{a_2}z^{a_1}s(M)z^{-a_1}T^{-a_2},\quad \sigma_a(M)=V_2^{a_2}V_1^{a_1}\sigma(M)V_1^{-a_1}V_2^{-a_2}$$

for points $a\in Z^2$. The monodromy fields $\sigma_a(M)$ act on a common dense invariant domain in the alternating tensor algebra $A(H_+^p\oplus\overline{H_-^p})$, where $H_\pm^p=Q_\pm H^p$ and $\bar{H}$ is the conjugate of H (see [24] for details). We wish to study the behavior of the correlations $\langle\sigma_{a_1}(M_1)\cdots\sigma_{a_n}(M_n)\rangle$ (vacuum expectations) as $s\uparrow 1$. At first we consider the case in which the monodromy fields occur in pairs $\sigma_m(M)\sigma_n(M)^{-1}$. In a sense that will be made precise in a moment one can imagine the "monodromy" associated with such a pair emerging at "m" moving along a path joining "m" to "n" and then disappearing at "n". We conjecture that this "confinement" of monodromy is important if one hopes to get non-trivial limits for the correlations as $s\uparrow 1$. More specifically we will show that:

$$\lim_{s\uparrow 1} \langle \sigma_{m_1}(M_1)\cdots\sigma_{m_n}(M_n)\rangle$$

exists and is finite if $M_1\cdots M_n = I$. If $M_1\cdots M_n \neq I$, then we conjecture this limit is 0 or ∞.

The method we have for dealing with "paired" correlations $\langle \sigma_{m_1}(M_1)\sigma_{n_1}(M_1)^{-1}\cdots\sigma_{m_r}(M_r)\sigma_{n_r}(M_r)^{-1}\rangle$ depends on the results (2.9), (2.11), (2.12) and (2.13) in [25] which we summarize here for the convenience of the reader. Let $e_1=(1,0)$ and $e_2=(0,1)$. Then:

$$\begin{aligned}\sigma_0(M)\sigma_{e_1}(M)^{-1} &= (\det M)^{-1}\Gamma_Q(s_0(M)s_{e_1}(M)^{-1}),\\ \sigma_0(M)\sigma_{e_2}(M)^{-1} &= \Gamma_Q(s_0(M)s_{e_2}(M)^{-1}).\end{aligned} \tag{1.10}$$

This is (2.9) and (2.11a) in [25]. We also have:

$$\begin{aligned} s(M)zs(M)^{-1} &= z + (M-1)P_{1/2}z,\\ s(M)T(z)s(M)^{-1} &= T(z) + T_+(M-1)P_{1/2}z + T_-(M^{-1}-1)P_{-1/2}z^{-1},\end{aligned} \tag{1.11}$$

where $T(z)=T_+z+T_0+T_-z^{-1}$ (see 2.1 in [25]) and P_k is the projection in $l^2(\mathbb{Z}_{1/2},\mathbb{C}^{2p})$ on those functions supported at $k\in\mathbb{Z}_{1/2}$. These are Eqs. (2.12) and (2.13) in [25]. We are now prepared to state one of the principal results of this section.

Proposition 1.0. *Let $M_j\in \mathrm{Gl}(p,\mathbb{C})$ $(j=1,\ldots,r)$. Suppose that none of the eigenvalues for M_j is a negative real number or 0. Then:*

$$\begin{aligned}\lim_{s\uparrow 1}&\langle \sigma_{m_1}(M_1)\sigma_{n_1}(M_1)^{-1}\cdots\sigma_{m_r}(M_r)\sigma_{n_r}(M_r)^{-1}\rangle\\ &=\left(\prod_{j=1}^{r}(\det M_j)^{\pi_1(m_j-n_j)}\right)\langle \Gamma_{Q_c}(G_1)\cdots\Gamma_{Q_c}(G_r)\rangle,\end{aligned} \tag{1.12}$$

where $G_j = s_{m_j}(M_j)s_{n_j}(M_j)^{-1}$.

Proof. Let m and n denote points in $\mathbb{Z}^2$ and let M denote an element of $\mathrm{Gl}(p,\mathbb{C})$ with no negative or 0 eigenvalues. Join m to n with a sequence of horizontal, $(c,c\pm e_1)$, and vertical, $(c,c\pm e_2)$, bonds. Write $\sigma_m(M)\sigma_n(M)^{-1}$ as the appropriately path ordered product of horizontal, $\sigma_c(M)\sigma_{c\pm e_1}(M)^{-1}$, and vertical, $\sigma_c(M)\sigma_{c\pm e_2}(M)^{-1}$, factors. One finds:

$$\sigma_m(M)\sigma_n(M)^{-1} = \prod_{(c,c')\in \mathrm{Path}(m,n)} \sigma_c(M)\sigma_{c'}(M)^{-1}, \tag{1.13}$$

where c and c' are nearest neighbors and the product is path ordered along a path, $\mathrm{Path}(m,n)$, joining m to n on the lattice $\mathbb{Z}^2$. Equation (1.10) shows that $\sigma_c(M)\sigma_{c'}(M)^{-1}$ is an s independent multiple of $\Gamma_Q(G)$, where $G=s_c(M)s_{c'}(M)^{-1}$. Equations (1.11) show that in each case G is a finite rank perturbation of the identity. Choose a basis e_k for H^p so that only finitely many of the vectors $(G-1)e_k$ are non-zero. Formula (1.1) of [24] applies and we may write:

$$\Gamma_Q(G) = F_Q(\theta \exp \textstyle\sum (G-1)e_k \wedge \bar{e}_k),$$

where F_Q is the Q-Fock representation, and θ is the j ordering map from the Grassmann algebra $A(H^p \oplus \overline{H^p})$ to the Clifford algebra $(H^p \oplus \overline{H^p})$ (see Sect. 1 of [20]). Consulting (1.11) one sees that the canonical basis of $H^p = l^2(\mathbb{Z}_{1/2}, \mathbb{C}^2) \otimes \mathbb{C}^p$ has the property that $G - 1$ vanishes on all but finitely many elements of this basis. We may write the elements of this basis in the following manner:

$$e_i^\alpha(k) = \delta(\cdot - k) \begin{bmatrix} \delta(\alpha, +) \\ \delta(\alpha, -) \end{bmatrix} \otimes e_i,$$

where e_i is the standard basis of $\mathbb{C}^p$, α takes values in $\{+, -\}$, $i = 1, \ldots, p$, and $k \in \mathbb{Z}_{1/2}$. The power series expansion of the exponential in the Grassmann algebra has only finitely many non-zero terms and it follows that $\sigma_c(M)\sigma_{c'}(M)^{-1}$ is an element of the Clifford algebra of finite sums of finite products of elements $F_Q(e_i^\alpha(k))$ and $F_Q(\overline{e_i^\alpha(k)}) = F_Q(e_i^\alpha(k))^*$. Furthermore, since the matrix elements of $G - 1$ in this canonical basis depend on s only through linear combinations of $s^{\pm 1}$, $c = (s^2 + 1)^{1/2}$, and c/s, it follows that the coefficients in this Clifford algebra representation are continuous function of s at $s = 1$. What has just been established for $\sigma_c(M)\sigma_{c'}(M)^{-1}$ obviously remains true for $\sigma_m(M)\sigma_n(M)^{-1}$ because of the product formula (1.13). Taking products again we may say that $\sigma_{m_1}(M_1)\sigma_{n_1}(M_1)^{-1} \cdots \sigma_{m_r}(M_r)\sigma_{n_r}(M_r)^{-1}$ is a finite sum of finite products in the Clifford algebra generated by $F_Q(e_i^\alpha(k))$ and $F_Q(e_i^\alpha(k))^*$. The coefficients in this representation are continuous functions of s at $s = 1$. Thus to establish the existence of limit as $s \uparrow 1$ for a "paired" correlation we need only investigate the vacuum expectations of products with factors selected from $F_Q(e_i^\alpha(k))$ and its adjoint. The Fock state on the Clifford algebra has the property that all such vacuum expectations can be expressed as determinants whose entries are two point functions $\langle F_Q(e_j^\beta(l)^* F_Q(e_i^\alpha(k)) \rangle$. The two point functions of this sort which do not vanish identically depend on $\alpha(\theta)$ only through linear combinations of the Fourier coefficients $1/2\pi \int_{-\pi}^{\pi} e^{\pm i\alpha(\theta)} e^{\pm i(k-l)\theta} d\theta$. But $\lim_{s \uparrow 1} e^{i\alpha(\theta)} = e^{i\alpha_c(\theta)}$ $(\theta \neq 0)$, so dominated convergence implies the continuity of these Fourier coefficients at $s = 1$.

Making use of (1.10) and the product representation (1.13) one finds:

$$\sigma_m(M)\sigma_n(M)^{-1} = (\det M)^{\pi_1(m-n)} \Gamma_Q(s_m(M)s_n(M)^{-1}).$$

Together with the preceding discussion this suffices to establish (1.12).

Q.E.D.

Suppose that $M_1 \cdots M_n = I$, we will show that $\sigma_{m_1}(M_1) \cdots \sigma_{m_n}(M_n)$ can be expressed as a product of "paired" monodromy fields. The fact that the map $M \to \sigma_m(M)$ is a homomorphism for each fixed m (see Sect. 2 of [25]) makes this easy:

$$\sigma_{m_1}(M_1) \cdots \sigma_{m_n}(M_n) = \prod_{j=1}^{n-1} \sigma_{m_j}(M_1 \cdots M_j)\sigma_{m_{j+1}}(M_1 \cdots M_j)^{-1},$$

where the product on the right is ordered from left to right with increasing j. We made use of:

$$\sigma_{m_j}(M_1 \cdots M_{j-1})^{-1}\sigma_{m_j}(M_1 \cdots M_j) = \sigma_{m_j}(M_j) \quad \text{and} \quad \sigma_{m_n}(M_1 \cdots M_{n-1})^{-1} = \sigma_{m_n}(M_n),$$

since $M_1 \cdots M_n = I$. Thus Proposition (1.0) implies that $\lim_{s \uparrow 1} \langle \sigma_{m_1}(M_1) \cdots \sigma_{m_n}(M_n) \rangle$ exists and is finite if $M_1 \cdots M_n = I$.

In the remainder of this section we will specialize our considerations to the case in which $\pi_2(m_i) = \pi_2(n_i)$ $(i = 1, \ldots, r)$. This will permit us to verify in an elementary fashion that Theorem (3.2) of [24] applies to the correlation $\langle \Gamma_Q(G_1) \cdots \Gamma_Q(G_r) \rangle$, and in the next section it will allow us to reduce the two point asymptotics to a Szego limit result for a singular block Toeplitz determinant.

The hypothesis of Theorem (3.2) of [24] which must be checked in order to apply this theorem to evaluate $\langle \Gamma_Q(G_1) \cdots \Gamma_Q(G_r) \rangle$ is that $\langle \Gamma_Q(G_j) \rangle \neq 0$ $j = 1, \ldots, r$. This is a consequence of the following lemma:

Lemma 1.1. *Let* $G = s_m(M) s_n(M)^{-1}$ *with* $\pi_2(m - n) = 0$ *and* M *an element of* $\mathrm{Gl}(p, \mathbb{C})$ *with no eigenvalues that are negative or* 0. *Let* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be the matrix of* G *relative to the decomposition of* $H^p = H^p_+ \oplus H^p_-$. *Then* d *is invertible on* H^p_- *and the inverse is bounded in norm by a constant which depends on* M *but does not depend on the first coordinates* $\pi_1(m)$, $\pi_1(n)$.

Proof. Write $m_j = \pi_j(m)$ and $n_j = \pi_k(n)$ $(j = 1, 2)$. It is convenient to suppose $m_1 < n_1$ (the case $n_1 < m_1$ may be dealt with in a precisely analogous fashion observing only that if M satisfies the hypothesis of the lemma then so does M^{-1}). Recalling the definition of $s_m(M)$ and the hypothesis $m_2 = n_2$, one finds:

$$s_m(M) s_n(M)^{-1} = T^{m_2}(I + (M - I) P[m_1, n_1]) T^{-m_2},$$

where $P[m_1, n_1]$ is the orthogonal projection in $l^2(\mathbb{Z}_{1/2}, \mathbb{C}^{2p})$ on the space of functions $f(k)$ which are non-vanishing only for those half-integer values of k for which $m_1 < k < n_1$. The maps $T^{\pm m_2}$ commute with Q_- and do not effect the uniformity in the variables m_1, n_1 which we wish to establish, so we will drop them from further consideration. Let $A = Q_- P[m_1, n_1] Q_-$. Then A is clearly a non-negative self-adjoint operator whose spectrum (only a finite number of non-zero eigenvalues) lies in the interval $[0, 1]$. The operator d whose inverse we wish to examine is thus the restriction to H^p_- of $I + (M - I)A$. But $(M - I)$ commutes with A, and it is thus clear from a consideration of the eigenvector expansion for A that to show $I + (M - I)A$ is invertible it is enough to demonstrate that $I + (M - I)a$ is invertible for all real numbers $a \in [0, 1]$.

Furthermore:

$$\|d^{-1}\| \leqq \sup_{0 \leqq a \leqq 1} \|[I + (M - I)a]^{-1}\|. \tag{1.14}$$

It is an elementary exercise to show that our assumption on the spectrum of M suffices to render the right-hand side of (1.14) finite. By making a similarity transformation it is enough to consider matrices M which are in Jordan normal form. Indeed it suffices to consider a single elementary Jordan block $\begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$ for M. We want to estimate the norm of the inverse of $\begin{bmatrix} \lambda_a & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_a \end{bmatrix}$, where $\lambda_a = 1 + (\lambda - 1)a$ and $0 \leqq a \leqq 1$. Define a function $N(\lambda)$ in the complex plane $\lambda =$

$\alpha + i\beta$ by:

$$N(\lambda) = \begin{cases} 1 & \alpha \geqslant 1 \\ [\alpha^2 + \beta^2]^{-1/2} & \text{for} \quad (\alpha - \frac{1}{2})^2 + \beta^2 \leqslant (\frac{1}{2})^2 \\ [((1-\alpha)^2 + \beta^2)/\beta^2]^{1/2} & \text{all other } \lambda = \alpha + i\beta \end{cases}$$

Then it is easy to see that $N(\lambda)$ is finite if λ is not negative or 0 and by minimizing a quadratic form in "a" on the interval $0 \leqq a \leqq 1$ one sees that: $|\lambda_a|^{-1} \leqq N(\lambda)$. The inverse of the Jordan block with diagonal λ_a is found by factoring out the diagonal and expanding the remaining piece in a finite geometric series. Without difficulty the norm of the inverse is seen to be bounded by $N(\lambda) + N(\lambda)^2 + \cdots + N(\lambda)^q$, where q is the dimension of the Jordan block. This finishes the proof of Lemma 1.1.

Q.E.D.

We are now prepared to apply Theorem (3.2) to the paired correlation $\langle \Gamma_Q(G_1) \cdots \Gamma_Q(G_r) \rangle$.

Theorem 1.2. *Let $M_j \in \mathrm{Gl}(p, \mathbb{C})$ $(j = 1, \ldots, r)$. Suppose that none of the eigenvalues for M_j is a negative real number or 0. Suppose $\pi_2(m_j) = \pi_2(n_j)$ $(j = 1, \ldots, r)$, and write $G_j = \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix}$ for the matrix of G_j relative to the $H^p_+ \oplus H^p_-$ decomposition of H^p. Then:*

$$\langle \Gamma_Q(G_1) \cdots \Gamma_Q(G_r) \rangle = \prod_{j=1}^{r} \langle \Gamma_Q(G_j) \rangle \det_2(I + LR), \tag{1.14}$$

where L is the $r \times r$ block matrix with entries:

$(i < k)$
$$l_{ik} = \begin{cases} -Q_+ & k = i+1 \\ -a_{i+1} Q_+ & k = i+2 \\ -a_{i+1} \cdots a_{k-1} Q_+ & k > i+2 \end{cases}$$

$(i > k)$
$$l_{ik} = \begin{cases} Q_- & i = k+1 \\ d_{k+1}^{-1} Q_- & i = k+2 \\ d_{i-1}^{-1} \cdots d_{k+1}^{-1} Q_- & i > k+2 \end{cases}$$

$(i = k)$
$$l_{ii} = 0,$$

and R is the $r \times r$ block diagonal matrix $R_1 \oplus \cdots \oplus R_r$ with entries R_k on the diagonal given by:

$$R_k = \begin{bmatrix} -b_k d_k^{-1} c_k & b_k d_k^{-1} \\ d_k^{-1} c_k & 0 \end{bmatrix},$$

in the $H_+ \oplus H_-$ decomposition of H.

Proof. The preceding lemma shows that each d_j is invertible and hence that Theorem 3.2 of [24] applies. The rest is just a transcription of this theorem which we include here for the reader's convenience. Q.E.D.

Section 2

In this section we will examine the asymptotics of the two point function at the critical point for monodromy fields paired horizontally. The principal result, Theorem 2.0, appears at the end of this section.

We begin the two point analysis. Because of translation invariance it is enough to consider correlations of the form $\langle \sigma_0(M)\sigma_m(M)^{-1}\rangle$, where $m\in\mathbb{Z}$, and for simplicity we have written $\sigma_m(M)$ for $\sigma_{(m,0)}(M)$. We shall reduce the calculation of this two point function to a block Toeplitz determinant and then use a matrix generalization [26] of a result of Basor and Widom [3] to calculate the asymptotics. Proposition 1.0 implies that:

$$\begin{aligned}\langle \sigma_0(M)\sigma_m(M)^{-1}\rangle &= (\det M)^{-m}\langle \Gamma_Q(s_0(M)s_m(M)^{-1})\rangle\\ &= (\det M)^{-m}\det[Q_- s_0(M)s_m(M)^{-1}Q_-],\end{aligned}$$

(see Lemma 3.1 of [24]). But $s_0(M)s_m(M)^{-1} = I + (M-I)P_m$, where P_m is the orthogonal projection in $l^2(\mathbb{Z}_{1/2}, \mathbb{C}^{2p})$ on those functions $f(k)$ which vanish outside the interval $0<k<m(k\in\mathbb{Z}_{1/2})$. The two point function we are interested in is thus:

$$\begin{aligned}\det Q_-(I+(M-I)P_m)Q_- &= \det[Q_+ + Q_-(I+(M-I)P_m)Q_-]\\ &= \det[I+Q_-(M-I)P_mQ_-] = \det[I+Q_-(M-I)P_m],\end{aligned}$$

where we made repeated use of $\det\begin{bmatrix}1 & b\\ 0 & d\end{bmatrix} = \det d$. This last identity and the fact that P_m is a projection imply that $\det(I+XP_m) = \det[(P_m+XP_m)+(I-P_m)] = \det P_m(I+X)P_m$. Therefore $\det[I+Q_-(M-I)P_m] = \det\ P_m(I+Q_-(M-I))P_m = \det P_m[M_+ + M_-Q]P_m$, where $M_\pm = (I\pm M)/2$. Thus we have shown that:

$$\langle \sigma_0(M)\sigma_m(M)^{-1}\rangle = (\det M)^{-m}\det P_m[M_+ + M_-Q]P_m. \tag{2.0}$$

Since M acts on $\mathbb{C}^{2p}\simeq\mathbb{C}^2\otimes\mathbb{C}^p$ as $I\otimes M$ and Q acts on $l^2(\mathbb{Z}_{1/2}, C^2)\otimes\mathbb{C}^p$ as convolution with $\hat{Q}(\cdot)\otimes I_p$, it follows that we may put M in Jordan normal form with a similarity transformation that commutes with Q. This similarity does not depend on the argument $\mathbb{Z}_{1/2}$ so that it clearly commutes with the projection P_m as well. Therefore in calculating the determinant (2.0) we may as well suppose that M is already in Jordan normal form. In fact, only the diagonal terms contribute to the Toeplitz determinant det $P_m[M_+ + M_-Q]P_m$ as we shall now prove. Let $e_j^\pm = e_\pm\otimes e_j$ where $e_j (j=1,\dots,p)$ is the standard basis for $\mathbb{C}^p$ and $e_+ = \begin{bmatrix}1\\0\end{bmatrix}$, $e_- = \begin{bmatrix}0\\1\end{bmatrix}$. We suppose that M is in Jordan normal form with respect to the basis $e_1,\dots,e_p$ with entries $\lambda_1,\dots,\lambda_p$ on the diagonal and 1's or 0's on the superdiagonal living in elementary Jordan blocks. Let $e_j^\pm(k) = e^{ik\theta}e_j^\pm$ $(k\in\mathbb{Z}_{1/2})$. Next consider the matrix for $P_m(M_+ + M_-Q)P_m$ with respect to the basis $e_j^\pm(k)$ $(j=1,\dots,p,\ 0<k<m)$ for the range of P_m ordered in the following manner:

$$\begin{gathered}e_1^+(1/2), e_1^-(1/2),\dots,e_1^+(m-1/2), e_1^-(m-1/2),\\ e_2^+(1/2), e_2^-(1/2)\ \dots\ e_2^+(m-1/2),\ e_2^-(m-1/2)\cdots\\ \cdots\cdots\\ e_p^+(1/2),\ e_p^-(1/2)\ \dots\ e_p^+(m-1/2),\ e_p^-(m-1/2).\end{gathered}$$

This matrix is block upper triangular with $m\times m$ block Toeplitz matrices on the diagonal given by $P_m(\lambda_j^+ + \lambda_j^- Q)P_m$, where $\lambda_j^\pm = (1\pm\lambda_j)/2$, and we have written P_m

for the projections on the span of $\{e_j^{\pm}(k) | j \text{ fixed}, 0 < k < m\}$. Thus:

$$\det P_m(M_+ + M_- Q)P_m = \prod_{j=1}^{p} \det P_m(\lambda_j^+ + \lambda_j^- Q)P_m. \tag{2.1}$$

Suppose $\lambda \in \mathbb{C}$ and write $\lambda_\pm = (1 \pm \lambda)/2$. Then since $Q = \begin{bmatrix} 0 & ie^{i\alpha(\theta)} \\ -ie^{-i\alpha(\theta)} & 0 \end{bmatrix}$ one sees that $P_m(\lambda_+ + \lambda Q)P_m$ is an $m \times m$ block Toeplitz matrix with generating function:

$$\begin{bmatrix} \lambda_+ & i\lambda_- e^{i\alpha(\theta)} \\ -i\lambda_- e^{-i\alpha(\theta)} & \lambda_+ \end{bmatrix}.$$

It is convenient to introduce $\beta(\theta) = \alpha(\theta) + \pi/2$. We may then write the generating function as:

$$\begin{bmatrix} \lambda_+ & \lambda_- e^{i\beta} \\ \lambda_- e^{-i\beta} & \lambda_+ \end{bmatrix}. \tag{2.2}$$

Next we make a similarity transformation of (2.2) to put the terms which are singular at the critical point on the diagonal. Transform (2.2) by conjugation with $(1/\sqrt{2})$ $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. The result is:

$$G(\theta) = \begin{bmatrix} \lambda_+ + \lambda_- \cos\beta(\theta) & i\lambda_- \sin\beta(\theta) \\ -i\lambda_- \sin\beta(\theta) & \lambda_+ - \lambda_- \cos\beta(\theta) \end{bmatrix}. \tag{2.3}$$

At the "critical temperature" $\beta(\theta) = \beta_c(\theta)$ has a discontinuity only at $\theta = 0$. From (1.7) it follows that $\lim_{\theta \to 0+} \beta_c(\theta) = \pi$ and $\lim_{\theta \to 0-} \beta_c(\theta) = 0$. But this means that $\sin\beta_c(\theta)$ is continuous at $\theta = 0$; indeed $\lim_{\theta \to 0} \sin\beta_c(\theta) = 0$. Next we define

$$\psi_1(\theta) = \begin{bmatrix} (\lambda_+ - \lambda_- \cos\beta_c(\theta))^{-1} & 0 \\ 0 & 1 \end{bmatrix} \psi_3(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & (\lambda_+ - \lambda_- \cos\beta_c(\theta)) \end{bmatrix} \tag{2.4}$$

$$\psi_2(\theta) = \psi_1(\theta)^{-1} G_c(\theta) \psi_3(\theta)^{-1},$$

where $G_c(\theta)$ is $G(\theta)$ evaluated at $\beta(\theta) = \beta_c(\theta)$. Observe that $G_c(\theta) = \psi_1(\theta)\psi_2(\theta)\psi_3(\theta)$.

Before we proceed it is convenient to introduce some notation that will be in force throughout the proof of Theorem (2.0). Let P_+ (P_-) denote the projection in $l^2(\mathbb{Z}_{1/2}, \mathbb{C}^2)$ on the subspace of functions with support on the positive (negative) elements in $\mathbb{Z}_{1/2}$. If ϕ is an operator on $l^2(\mathbb{Z}_{1/2}, \mathbb{C}^2)$ let $\begin{bmatrix} a(\phi) & b(\phi) \\ c(\phi) & d(\phi) \end{bmatrix}$ denote the matrix of ϕ relative to the decomposition $P_+ l^2 \oplus P_- l^2$ of l^2. We also write $\begin{bmatrix} a_m(\phi) & b_m(\phi) \\ c_m(\phi) & d_m(\phi) \end{bmatrix}$ for matrix of ϕ relative to the decomposition $P_m l^2 \oplus (I - P_m) l^2$.

Our analysis of the asymptotics of $\det a_m(G_c)$ will proceed in two stages. First we will show that the singularities in the factorization of $\psi_1\psi_2\psi_3$ of G_c are "non-overlapping" in a sense that they will permit the application of Theorem 2.0 below to show that

$$\det a_m(G_c) \sim \prod_{j=1}^{3} \det a_m(\psi_j), \quad m \to \infty, \tag{2.5}$$

where here and in what follows we use $A_m \sim B_m$ to signify that $\lim_{m\to\infty} A_m/B_m$ exists and is $\neq 0$.

We will then apply the analysis of Basor and Widom [3] to the asymptotics for the singular scalar Toeplitz determinants det $a_m(\psi_j)$ $(j=1, 3)$ and the results of Widom [35] for the asymptotics of regular block Toeplitz determinant det $a_m(\psi_2)$ (observe that since sin $\beta_c(\theta)$ and $\cos^2 \beta_c(\theta)$ are continuous at 0 it follows that $\psi_2(\theta)$ is piecewise smooth and continuous).

Before we state the result needed to prove (2.5) we introduce some terminology. Suppose $\psi: S^1 \to \mathrm{Gl}(p, \mathbb{C})$. We will say that ψ is piecewise C^2 if ψ is twice continuously differentiable except at a finite number of points where it has right and left hand limits in $\mathrm{Gl}(p, \mathbb{C})$. If ψ and ϕ are two maps from S^1 to the $p \times p$ complex matrices then we say that the ordered pair (ψ, ϕ) has separated singularities if there exists a smooth matrix valued partition of unity $(g, (I-g))$ such that ψg and $(I-g)\phi$ are continuous. The n tuple $(\psi_1, \ldots, \psi_n)$ will be said to have non overlapping singularities if for each increasing chain $1 \leqq i < \cdots < k+1 \leqq n$ with $i \leqq k$ the ordered pair $(\psi_i \cdots \psi_k, \psi_{k+1})$ has separated singularities. The result we require to prove (2.5) is the following slight generalization of a result of Basor and Widom:

Theorem 2.0. *For $j = 1, \ldots n$, suppose ψ_j is a piecewise C^2 map from S^1 to* $\mathrm{Gl}(p, C)$. *Suppose $a(\psi_j)$ and $d(\psi_j)$ are invertible for $j = 1, \ldots, n$. If $a(\psi_1 \cdots \psi_n)$ is invertible and the n-tuple $(\psi_1, \ldots, \psi_n)$ has non-overlapping singularities then:*

$$\lim_{m\to\infty} \frac{\det a_m(\psi_1 \cdots \psi_n)}{\prod_{j=1}^{n} \det a_m(\psi_j)} = AD,$$

where

$$A = \det[a(\psi_1 \cdots \psi_n) a(\psi_n)^{-1} \cdots a(\psi_1)^{-1}]$$

and

$$D = \det[d(\psi_1 \cdots \psi_n) d(\psi_n)^{-1} \cdots d(\psi_1)^{-1}].$$

Remark. It is part of the theorem that the determinants defining A and D are absolutely convergent infinite determinants of the form $\det(I + \text{trace class})$. When $a(\psi_1 \cdots \psi_n)$ and $d(\psi_1 \cdots \psi_n)$ are both invertible then A and D are both non-zero.

Proof. The proof is a straightforward generalization of the proof in Basor and Widom [3] combined with results for the strong convergence of $a_m(\psi_j)^{-1}$ and $d_m(\psi_j)^{-1}$ to $a(\psi_j)^{-1}$ and $d(\psi_j)^{-1}$ as $m \to \infty$ that can be found in Silberman [39].

As a first step in proving (2.5) we will show that the Toeplitz operators $a(\psi_j)$ and $d(\psi_j)$, $(j = 1, 2, 3)$ are invertible if λ is not 0 or a negative number. This is one of the hypotheses of Theorem 2.0, and our success in proving this for ψ_2 motivated our choice of three factors for $G_c(\theta)$. There are a number of ways to factor $G_c(\theta)$ into a single singular diagonal factor and a continuous loop in $\mathrm{Gl}(2, \mathbb{C})$. However, we were unable to give a simple proof of the invertibility for the Toeplitz part of the continuous loop for any such factorization. Had we been able to use a two part factorization, the technique of Basor and Widom [3] (see their "main lemma") would have sufficed for our purposes. For a piecewise continuous scalar symbol ϕ with only a finite number of discontinuities and right and left limits at each discontinuity, the invertibility of $a(\phi)$ is determined as follows (see [40]). Complete the broken curve described by $\phi(\theta)$ by joining the right and left limits at points of discontinuity by straight line segments. If the resulting unbroken curve does not pass through 0 and has winding number 0 then $a(\phi)$ is invertible. The line segments needed to complete the broken curve $\theta \to (\lambda_+ - \lambda_- \cos \beta_c(\theta))^{\pm 1}$ join 1 to $\lambda^{\pm 1}$. Thus they will not pass through 0 if λ is not 0 or a negative real number. The imaginary part of $\lambda_+ - \lambda_- \cos \beta_c(\theta)$ is $(1 + \cos \beta_c(\theta)) \operatorname{Im}(\lambda)/2$, so the curves of interest stay in the upper or lower half plane and thus have winding number 0. This shows that $a(\psi_1)$ and $a(\psi_3)$ are invertible. Next we consider $a(\psi_2)$. It is a Fredholm operator [6]. To show that it is invertible it is thus enough to show that $a(\psi_2)$ and $a(\psi_2)^*$ have trivial null spaces. Suppose $\begin{bmatrix} x \\ y \end{bmatrix} \in P_+ l^2$ is in the null space of $a(\psi_2)$. Then one finds:

$$[\lambda_+^2 - \lambda_-^2 a(\cos^2 \beta_c)]x + i\lambda_- a(\sin \beta_c) y = 0, \quad -i\lambda_- a(\sin \beta_c)x + y = 0.$$

Since $\lambda_+^2 - \lambda_-^2 a(\cos^2 \beta_c) = \lambda + \lambda_-^2 a(\sin^2 \beta_c)$ and $a(\sin^2 \beta_c) - a(\sin \beta_c) a(\sin \beta_c) = b(\sin \beta_c) c(\sin \beta_c) = c^*(\sin \beta_c) c(\sin \beta_c)$, it follows that:

$$\lambda x + \lambda_-^2 c^*(\sin \beta_c) c(\sin \beta_c) x = 0. \tag{2.6}$$

If $\lambda = 1$ this equation becomes $x = 0$ for which there are clearly no non-trivial solutions. If $\lambda \neq 1$ then (2.6) is equivalent to:

$$c^* c x = -[(\lambda^{1/2} - \lambda^{-1/2})/2]^{-2} x. \tag{2.7}$$

But c^*c clearly has non-negative spectrum so that (2.7) will not have any solutions unless $\lambda^{1/2} - \lambda^{-1/2}$ is pure imaginary. This happens only for $\lambda = 0$ or λ negative and so we have proved the null space of $a(\psi_2)$ is trivial. The proof that the null space of $a(\psi_2)^*$ is $\{0\}$ is precisely analogous. The arguments for the invertibility of $d(\psi_j)$ $(j = 1, 2, 3)$ are similar. The operators $a(\psi_1 \psi_2 \psi_3) = a(G_c)$ and $d(\psi_1 \psi_2 \psi_3) = d(G_c)$ are invertible as a consequence if the same elementary spectral theory considerations that are used in Lemma 1.1. Thus $A \neq 0$ and $D \neq 0$ in Theorem 2.0.

Finally, we leave to the reader the simple construction of partitions of unity for the chains encountered in verifying that the factorization $\psi_1 \psi_2 \psi_3$ satisfies the hypothesis of Theorem 2.0. We note only that it is helpful to observe that $\theta \to (\lambda_+ - \lambda_- \cos \beta_c(\theta))^{-1} \sin \beta_c(\theta)$ is continuous on the circle. We have established (2.5) and we turn to the consideration of the asymptotics for $\det a_m(\psi_j)$ $j = 1, 2, 3$.

Let $\log(\cdot)$ denote the branch of the logarithm cut along the negative axis and normalized so that $\log 1 = 0$. Let $e_\lambda(\theta) = \exp\{-[(\pi - \theta)\log \lambda]/2\pi\}$ and define $\phi(\theta) = e_\lambda(\theta)(\lambda_+ - \lambda_- \cos \beta_c(\theta))^{-1}$ for $0 \leqq \theta < 2\pi$. By comparing jumps at $0+$ and $2\pi-$ one sees that $\phi(\theta)$ is continuous at 0; it is also smooth elsewhere. The technique in the main lemma of Basor and Widom [3] applies to the factorization $(\lambda_+ - \lambda_- \cos \beta_c(\theta))^{-1} = e_\lambda^{-1}(\theta)\phi(\theta)$ of the symbol of ψ_1 and one finds the asymptotics of

det $a_m(\psi_1)$ is up to a constant the same as the asymptotics of the product det $a_m(e_\lambda^{-1})$ det $a_m(\phi)$. The asymptotics of det $a_m(e_\lambda^{-1})$ can be found in Fisher and Hartwig [4].
It is

$$\det a_m(e_\lambda^{-1}) \sim m^{[(\log\lambda)/2\pi]^2} \quad (m\to\infty). \tag{2.8}$$

The asymptotics of the regular Toeplitz determinant det $a_m(\phi)$ can be found using results of Widom [35].
It is:

$$\det a_m(\phi) \sim \exp\left[\frac{m}{2\pi}\int_0^{2\pi} \log\det\phi(\theta)d\theta\right] \quad (m\to\infty). \tag{2.9}$$

The symbol $(\lambda_+ - \lambda_- \cos\beta_c(\theta))$ of ψ_3 may be factored as $e_\lambda(\theta)\phi(\theta)^{-1}$. The asymptotics of det $a_m(e_\lambda)$ is the same as that for det $a_m(e_\lambda^{-1})$ in (2.8). The asymptotics of det $a_m(\phi^{-1})$ is easily seen to involve the reciprocal of the exponential in (2.9). Thus

$$\det a_m(\psi_1)\cdot\det a_m(\psi_3) \sim m^{2[(\log\lambda)/2\pi]^2} \quad (m\to\infty). \tag{2.10}$$

Finally another application of Widom's result [35] for the block Toeplitz case shows that:

$$\det a_m(\psi_2) \sim \exp\left[\frac{m}{2\pi}\int_0^{2\pi} \log\det\psi_2(\theta)d\theta\right] = \exp(m\log\lambda) = \lambda^m \tag{2.11}$$

as $m\to\infty$. If we combine (2.10) and (2.11) with (2.5) and compare the result with (2.1) we find:

$$\det P_m(M_+ + M_-Q)P_m \sim m^{\Sigma_j 2[(\log\lambda_j)/2\pi]^2}(\det M)^m.$$

Consulting (2.0) we have the main result of this section:

Theorem 2.1. *Suppose no eigenvalue for M is negative or 0. Then $\langle\sigma_0(M)\sigma_m(M)^{-1}\rangle$ is asymptotic to $m^{\Sigma_j 2[(\log\lambda_j)/2\pi]^2}$ as $n\to+\infty$. The λ_j are eigenvalues of M and the logarithm has its branch cut on the negative real axis.*

Section 3

In this section we will investigate the asymptotics of the perturbation determinant in Theorem 1.2 for large separation of the spin sites. As in previous investigations of this sort [21, 25], it is convenient to rewrite (1.14) in a manner that takes advantage of the smoothing properties of the transfer matrix. To be able to do this we must now make the additional assumption that the second coordinates occur in increasing order $\pi_2(n_1) < \pi_2(n_2) < \cdots < \pi_2(n_r)$ (recall that $\pi_2(m_j) = \pi_2(n_j)$, $j = 1, 2, \ldots, r$). This assumption will be in force throughout this section. To avoid writing $\pi_1(m_j)$, $\pi_2(m_j)$, etc., we introduce the notation $m_j = (p_j, r_j)$, $n_j = (q_j, r_j)$. Next define

$$\tilde{G}_j = T^{(r_j - r_{j-1})/2} s_{p_j}(M_j) s_{q_j}(M_j)^{-1} T^{(r_{j+1}-r_j)/2},$$

where n_{r+1} and n_0 are arbitrary except for $\pi_2(n_0) < \pi_2(n_1)$ and $\pi_2(n_r) < \pi_2(n_{r+1})$. The reader may easily check that $\langle\Gamma_Q(G_1)\cdots\Gamma_Q(G_r)\rangle = \langle\Gamma_Q(\tilde{G}_1)\cdots\Gamma_Q(\tilde{G}_r)\rangle$ by making use of the intertwining property $\Gamma(h)\Gamma_Q(g)\Gamma(h)^{-1} = \Gamma_Q(hgh^{-1})$ and $\Gamma(h)1 = 1$. For precisely the same reason $\langle \Gamma_Q(\tilde{G}_j)\rangle = \langle \Gamma_Q(G_j)\rangle$. Thus the only alteration in (1.14) is that the operators a_j, b_j, c_j and d_j that occur in the description of L and R which follows (1.14) ought now to be regarded as the matrix elements of $\tilde{G}_j$ rather than G_j. We do this without introducing new notation and write:

$$\tilde{G}_j = \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix} \text{ relative to } H = H_+ \oplus H_-. \tag{3.1}$$

Now let $N > 0$ denote a positive integer and write $\tilde{G}_j(N) = \begin{bmatrix} a_j(N) & b_j(N) \\ c_j(N) & d_j(N) \end{bmatrix}$ for $\tilde{G}_j$ with m_j and n_j replaced by Nm_j and Nn_j. Let $L(N)$ and $R(N)$ denote the operators L and R of Theorem 1.2 with $\tilde{G}_j$ replaced by $\tilde{G}_j(N)$. We are interested in $\lim_{N\to\infty} \det_2$ $(I + L(N)R(N))$ as a function of $(m_1, n_1, \ldots, m_r, n_r)$. Observe that if N tended to ∞ through powers of 2 and m_j and n_j had components which were dyadic rationals, then Nm_j and Nn_j would eventually end up on the integer lattice and stay there for all sufficiently large N. Thus we would still be studying the asymptotics of the lattice correlations in such circumstances. We will not therefore continue to suppose that m_j and n_j are on the integer lattice. We will, however, maintain that Nm_j and Nn_j stay in $\mathbb{Z}^2$ as this will simplify an estimate at the end of this section.

The scheme of the convergence proof follows that in [21] and [25]. We will introduce a family of isometric embeddings i_N mapping $L^2(S^1, \mathbb{C}^{2p})$ into $L^2(\mathbb{R}, \mathbb{C}^{2p})$ such that $i_N R(N) i_N^*$ converges in Schmidt norm and $i_N L(N) i_N^*$ converges strongly to a limit in $L^2(\mathbb{R})$. The continuity of $\det_2(1 + A)$ in the Schmidt norm for A then finishes the proof. We will now present the details of the argument for the case in which each M_j is a scalar λ_j; The matrix case is identical except for more involved notation.

Suppose $f(\theta) \in L^2(S^1, \mathbb{C}^2)$, we define $i_N f \in L^2(\mathbb{R}, \mathbb{C}^2)$ as follows (note: $L^2(S^1, \mathbb{C}^2)$ is identified with $L^2([-\pi, \pi], \mathbb{C}^2)$):

$$i_N f(k) = \chi_N(k) I_N(k) f(k/N), \tag{3.2}$$

where $\chi_N(k) = \begin{cases} 1 & |k| \leqq N\pi \\ 0 & |k| > N\pi \end{cases}$ and

$$I_N(k) = \left(\frac{N}{2}\right)^{1/2} \begin{bmatrix} e^{-i\delta/2} & e^{i\delta/2} \\ -e^{-i\delta/2} & e^{i\delta/2} \end{bmatrix}, \quad \delta = \delta(k/N), \tag{3.3}$$

and

$$\delta(\theta) \overset{\text{def}}{=} \alpha(\theta) - \frac{\varepsilon(\theta)\pi}{2} = \int_0^\theta \frac{\sqrt{2}d\phi}{\cos\phi - 3} \quad \text{(see 1.7).}$$

It is not difficult to check that i_N is an isometric embedding from $L^2(S^1, (d\theta/2\pi))$ to $L^2(\mathbb{R}, (dk/d\pi))$. There are two observations concerning this embedding we would like to make. The first is that $I_N(k)$ is actually a smooth function of k for $|k| < \pi N$. The second observation is:

$$i_N Q_c(\theta) i_N^* = \begin{bmatrix} -\varepsilon(k) & 0 \\ 0 & \varepsilon(k) \end{bmatrix}, \tag{3.4}$$

where $Q_c(\theta)$ is the matrix valued multiplication operator $\begin{bmatrix} 0 & ie^{i\alpha_c(\theta)} \\ -ie^{-i\alpha_c(\theta)} & 0 \end{bmatrix}$ on $L^2(S^1, \mathbb{C}^2)$. These two properties essentially decided the choice of I_N for us. The second property (3.4) will allow us to identify the scaling results obtained here with similar results obtained for a current group representation associated with the usual Hardy space decomposition of $L^2(\mathbb{R}, \mathbb{C}^2)$. This will be done in Sect. 4 and it will provide an explicit evaluation of the scaled perturbation determinant in certain cases.

We now isometrically embed the Hilbert space on which L and R act into a direct sum of copies of $L^2(\mathbb{R}, \mathbb{C}^2)$ using the direct sum of copies of i_N which we continue to denote by i_N. The invariance of $\det_2$ under unitary transformations is easily seen to imply:

$$\det_2(I + L(N)R(N)) = \det_2(I + i_N L(N)R(N)i_N^*).$$

It is well known that the product of a strongly convergent sequence of operators with a sequence that converges in Schmidt norm will converge in Schmidt norm to the appropriate product of the limits. A glance at the structure of the operators L and R shows that it will suffice to prove strong convergence for:

$$i_N a_j(N) i_N^*, \quad i_N d_j^{-1}(N) i_N^* \quad (j = 1, \ldots, r) \tag{3.5}$$

and convergence in Schmidt norm for:

$$i_N b_j(N) d_j^{-1}(N) c_j(N) i_N^*, \quad i_N b_j(N) d_j^{-1}(N) i_N^* \tag{3.6}$$

and

$$i_N d_j^{-1}(N) c_j(N) i_N^* \quad (j = 1, \ldots, r).$$

There is no need to carry around the subscripts j and so we shall prove (3.5) and (3.6) for the matrix elements of an operator $\tilde{G}$ of the special form

$$\tilde{G} = T^{n_1} s_{-m}(\lambda) s_m(\lambda)^{-1} T^{n_2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{3.7}$$

where n_1, n_2 and m are all > 0. We made use of the "translation" invariance of i_N ($e^{im\theta} \to e^{iNmk/N} = e^{imk}$) to locate the first coordinates in $\tilde{G}$ symmetrically about the origin (this is not essential but simplifies the appearance of the integral kernels which arise). Write:

$$\tilde{G}(N) = T^{Nn_1} s_{-Nm}(\lambda) s_{Nm}(\lambda)^{-1} T^{Nn_2},$$

and recall the running assumption that $Nm \in \mathbb{Z}$. Then $s_{-Nm}(\lambda) s_{Nm}(\lambda)^{-1}$ is given by $I + (\lambda - I) P_{Nm}$, where P_{Nm} is an integral operator on $L^2(S^1, \mathbb{C}^2)$ with kernel:

$$P_{Nm}(\theta - \theta') = \frac{\sin mN(\theta - \theta')}{2\pi \sin[(\theta - \theta')/2]}.$$

Our first result is:

Lemma 3.0. *The operator $i_N P_{Nm} i_N^*$ converges as $N \to \infty$ in the strong operator topology to an operator $P(m)$ on $L^2(S^1, \mathbb{C}^2)$ given by:*

$$P(m)f(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin m(k-k')}{k-k'} f(k')dk'. \tag{3.8}$$

Proof. The operator $i_N P_{Nm} i_n^*$ is an integral operator with matrix valued kernel given by:

$$\frac{\chi_N(k)\chi_N(k')\sin m(k-k')}{2\pi N \sin[(k-k')/2N]} \begin{bmatrix} \cos(\delta-\delta')/2 & i\sin(\delta-\delta')/2 \\ i\sin(\delta-\delta')/2 & \cos(\delta-\delta')/2 \end{bmatrix}, \tag{3.9}$$

where $\delta = \delta(k/N)$ and $\delta' = \delta(k'/N)$. The operator associated with this kernel is uniformly bounded since P_{Nm} is. Thus to prove strong convergence as $N \to \infty$ it will suffice to prove convergence on a dense set. We take this dense set to be the set of integrable function in $L^2(\mathbb{R})$ with compact support. Suppose then that $f(k)$ is an L^1 function contained in $L^2(\mathbb{R})$ whose support is contained in the interval $[-L, L]$. We examine the diagonal part of (3.9) first. Thus we wish to show that the function $g_N(k)$ given by

$$g_N(k) = \chi_N(k) \int_{-N\pi}^{N\pi} \frac{\sin m(k-k')}{2\pi N \sin[(k-k')/2N} \cos[(\delta-\delta')/2] f(k')dk',$$

converges in L^2 as $N \to \infty$, where $\delta = \delta(k/N)$ and $\delta' = \delta(k'/N)$. Choose $\varepsilon > 0$ so that $\varepsilon < \pi/2$. For sufficiently large N we have $L/2N < \varepsilon$. For such an N the relevant values of $(k-k')/2N$ in the integral defining $g_N(k)$ lie between $-(\pi/2)-\varepsilon$ and $(\pi/2)+\varepsilon$. For $\varepsilon < (\pi/2)$ it is easy to see that there exists a constant c_ε such that $|\sin\theta| \geqq c_\varepsilon |\theta|$ for all $\theta \in [-(\pi/2)-\varepsilon, (\pi/2)+\varepsilon]$. Thus $|N \sin[(k-k')/2N]|^{-1} \leqq 2c_\varepsilon^{-1} |k-k'|^{-1}$ for $(k-k')/2N \in [-(\pi/2)-\varepsilon, (\pi/2)+\varepsilon]$. This last estimate, the fact that $\lim_{N\to\infty} \delta(k/N) = 0$, and dominated convergence together imply that:

$$\lim_{N\to\infty} g_N(k) = g(k) \stackrel{\text{def.}}{=} \int_{-\infty}^{\infty} \frac{\sin m(k-k')}{\pi(k-k')} f(k')dk',$$

in the sense of pointwise convergence. If we can prove that $|g_N(k)|^2$ is bounded by a fixed integrable function, then dominated convergence shows that g_N actually converges to g in L^2. Make the same estimate in the definition of $g_N(k)$ that was used in the proof of pointwise convergence and use the Cauchy–Schwartz inequality in an obvious fashion to obtain:

$$|g_N(k)|^2 \leqq (\text{const}) \| f \|_1 \int_{-\infty}^{\infty} \left| \frac{\sin m(k-k')}{k-k'} \right|^2 |f(k')| dk', \tag{3.10}$$

where $\| f \|_1$ is the L^1 norm of f. The convolution involves two L^1 functions and hence is in L^1. This last inequality is thus sufficient for the application of dominated convergence to prove $g_N \to g$ in L^2.

Next we consider the off diagonal operators in (3.9). Let f be given as above.

Then we define:

$$h_N(k)=\chi_N(k)\int_{-N\pi}^{N\pi}\frac{i\sin m(k-k')}{2\pi N\sin[(k-k')/2N]}\sin[(\delta-\delta')/2]f(k')dk',$$

where $\delta=\delta(k/N)$ and $\delta'=\delta(k'/N)$. If we employ the same estimate for $(N\sin[k-k')/2N])^{-1}$ used above, then one application of dominated convergence implies $\lim_{N\to\infty} h_N(k)=0$ pointwise and a second application using an analogue of (3.10) shows that $h_N\to 0$ in L^2. Q.E.D.

Introduce the notation $S_N(m)=i_N s_{-m}(\lambda)s_m(\lambda)^{-1}i_N^*$ and $S(m)=I+(\lambda-I)P(m)$. Then Lemma (3.0) implies that $s\text{-}\lim_{N\to\infty} S_N(Nm)=S(m)$. Furthermore since $Q_\pm=\begin{bmatrix}\theta(\mp k) & 0\\ 0 & \theta(\pm k)\end{bmatrix}$ $\left(\text{where } \theta(k)=\begin{cases}1 & k>0\\ 0 & k<0\end{cases}\right)$ in the $L^2(\mathbb{R},\mathbb{C}^2)$ representation, it is clear that:

$$s\text{-}\lim_{N\to\infty} a(S_N(Nm))=a(S(m)),$$

$$s\text{-}\lim_{N\to\infty} d(S_N(Nm))=d(S(m)),\tag{3.11}$$

where $S=\begin{bmatrix}a(S) & b(S)\\ c(S) & d(S)\end{bmatrix}$ is the matrix of S relative to the splitting $Q_+L^2\oplus Q_-L^2$. When λ is not 0 or negative Lemma 1.1 implies $d(S_N(Nm))$ is invertible on $i_NL^2(S^1,\mathbb{C}^2)$ with an inverse that is uniformly bounded in N. Since $P(m)$ is well known to be the orthogonal projection on those functions $f(p)$ whose Fourier transforms $\hat{f}(x)$ have support contained in $[-m,m]$ the same spectral theory arguments used to prove Lemma 1.1 also imply that $S(m)$ is invertible on $L^2(\mathbb{R},\mathbb{C}^2)$. Because of the uniform bound (3.11) implies:

$$s\text{-}\lim_{N\to\infty} d(S_N(Nm))^{-1}=d(S(m))^{-1},\tag{3.12}$$

(for this result we set $d(S_N)^{-1}f=0$ on those functions f in $L^2(\mathbb{R},\mathbb{C}^2)$ whose support lies outside $[-N\pi,N\pi]$). We will now use (3.11) and (3.12) to obtain the strong convergence results needed in (3.5). The reader should have no difficulty in checking that:

$$i_Na(N)i_N^*=e^{-Nn_1\gamma}a(S_N(Nm))e^{-Nn_2\gamma},$$
$$i_Nd(N)^{-1}i_N^*=e^{-Nn_2\gamma}d(S_N(Nm))^{-1}e^{-Nn_1\gamma},\tag{3.13}$$

where for brevity we have written $e^{-Nn\gamma}$ for the multiplication operator $e^{-Nn\gamma(k/N)}\chi_N(k)$ on $L^2(\mathbb{R},\mathbb{C}^2)$. We will now show that $e^{-Nn\gamma}$ converges uniformly to a limit as a multiplication operator on $L^2(\mathbb{R},\mathbb{C}^2)$ as $N\to\infty$ (provided $n>0$). The function $\gamma(\theta)$ is not differentiable at 0 but it does have the one sided derivatives

$$\lim_{\theta\to 0\pm}\frac{\gamma(\theta)-\gamma(0)}{\theta}=\pm 1\quad(\text{note } \gamma(0)=0).$$

Thus $\lim_{N\to\infty} N\gamma(k/N)=|k|$. Furthermore the function $\gamma(\theta)$ is concave down between

$-\pi$ and 0 and between 0 and π. Thus there exists a constant $c > 0$ (which one could take to be $\gamma(\pi)$) so that $\gamma(\theta) \geqq c|\theta|$, $|\theta| \leqq \pi$. It follows that $e^{-Nn\gamma(k/N)} \leqq e^{-cn|k|}$ for $|k| \leqq N\pi$. This bound implies $\lim_{N\to\infty} e^{-Nn\gamma(k/N)}\chi_N(k) = e^{-n|k|}$ in the uniform norm (i.e., in L^∞). Combining (3.11), (3.12), and (3.13) we have shown the strong convergence of the operators appearing in (3.5).

We turn now to the problem of showing Schmidt norm convergence for those operators appearing in (3.6). It will suffice to illustrate the method for $i_N d(N)^{-1} c(N) i_N^*$. A straightforward calculation shows that:

$$i_N d(N)^{-1} c(N) i_N^* = e^{-Nn_2\gamma} d(S_N(Nm))^{-1} c(S_N(Nm)) e^{-Nn_1\gamma}.$$

The strong convergence results for the last two factors on the right-hand side of this equation show that it is enough to prove convergence in Schmidt norm for $c(S_N(Nm))e^{-Nn_1\gamma}$. We will actually prove a little more; namely that $S_N(Nm)e^{-Nn\gamma}$ converges in Schmidt norm as $N \to \infty$ to the operator on $L^2(\mathbb{R}^2, \mathbb{C}^2)$ with kernel $(\sin m(k-k')/\pi(k-k'))e^{-n|k'|}$ $(n > 0)$. Let:

$$M(k,k') = \begin{bmatrix} \cos(\delta-\delta')/2 & i\sin(\delta-\delta')/2 \\ i\sin(\delta-\delta')/2 & \cos(\delta-\delta')/2 \end{bmatrix},$$

where $\delta = \delta(k/N)$ and $\delta' = \delta(k'/N)$.

The difference of the kernel for $S_N(Nm)e^{-Nn\gamma}$ and $(\sin m(k-k')/\pi(k-k'))e^{-n|k'|}$ may be written as the sum of three terms:

$$\begin{aligned} &\left[\frac{\sin m(k-k')}{2\pi N \sin[(k-k')/2N]} - \frac{\sin m(k-k')}{\pi(k-k')}\right] M(k,k') e^{-Nn\gamma(k'/N)}\chi_N(k)\chi_N(k') \\ &\quad + \frac{\sin m(k-k')}{\pi(k-k')}(M(k,k')-I)e^{-Nn\gamma(k'/N)}\chi_N(k)\chi_N(k') \\ &\quad + \frac{\sin m(k-k')}{\pi(k-k')}(e^{-Nn\gamma(k'/N)}\chi_N(k)\chi_N(k') - e^{-n|k'|}). \end{aligned} \tag{3.14}$$

We wish to show that each of the three terms in this sum converges to 0 in $L^2(\mathbb{R}, dkdk')$. The last two terms are easy to deal with. The uniform estimate $e^{-N\gamma(k'/N)}\chi_N(k') \leqq e^{-c|k'|}$ shows that each of these two kernels is bounded in absolute value by a constant times the $L^2(\mathbb{R}^2)$ function: $\left|\frac{\sin m(k-k')}{k-k'}\right| e^{-cn|k'|}$. Thus dominated convergence implies the L^2 norm of the last two terms does tend to zero as $N \to \infty$ (we use $M(k,k') \to I$ as $N \to \infty$). Since $M(k,k')$ is uniformly bounded and $e^{-Nn\gamma(k'/N)}\chi_N(k') \leqq e^{-cn|k'|}$ to show that the first term in (3.14) tends to zero in $L^2(\mathbb{R}^2)$ it is enough to prove:

$$\lim_{N\to\infty} \int_{-N\pi}^{N\pi} dk \int_{-N\pi}^{N\pi} dk' \left[\frac{\sin m(k-k')}{2\pi N \sin[(k-k')/N]} - \frac{\sin m(k-k')}{\pi(k-k')}\right]^2 e^{-2cn|k'|} = 0.$$

In order to prove this split the range of the dk' integration into the pieces $|k'| \leqq \varepsilon N$ and $\varepsilon N \leqq |k'| \leqq \pi N$, where $0 < \varepsilon < \pi$. On the first interval $(-\pi-\varepsilon)N \leqq k-k' \leqq (\pi+\varepsilon)N$ or $|(k-k')/2N| \leqq (\pi+\varepsilon)/2$, and as above we have the estimate $|2N \sin$

$[(k-k')/2N]|^{-1} \leqq \text{const}\,|k-k'|^{-1}$ for $|k| \leqq \pi N$ and $|k'| \leqq \varepsilon N$. This is sufficient to apply dominated convergence and we conclude that this part of the L^2 norm vanishes in the limit $N \to \infty$. In the remaining integral over the domain $\varepsilon N \leqq |k'| \leqq \pi N$ make the change of variables $\theta = k/N$ and $\theta' = k'/N$ to obtain:

$$\int_{-\pi}^{\pi} d\theta \int_{\varepsilon \leqq |\theta'| \leqq \pi} d\theta' \left[\frac{\sin Nm(\theta-\theta')}{N \sin[(\theta-\theta')/2]} - \frac{2 \sin Nm(\theta-\theta')}{N(\theta-\theta')} \right]^2 e^{-2cN|\theta'|} N^2 .$$

Since Nm is an integer, $(\sin Nm(\theta-\theta'))/\sin[(\theta-\theta')/2]$ is the sum of $2Nm$ terms in a geometric series, each term having absolute value 1. Dividing by N one gets a uniform bound. We also have $|(\sin Nm(\theta-\theta'))/Nm(\theta-\theta')| \leqq \sup_u |(\sin u)/u| \leqq \infty$. But $|\theta'|$ is bounded away from 0, so $e^{-2cN|\theta'|}N^2$ goes uniformly to zero. Thus the whole integral tends to zero. We have finished the proof that $i_N d(N)^{-1} c(N) i_N^*$ converges in Schmidt norm.

In order to state the principal result of this section it is useful to introduce some notational conventions for the continuum situation which arises. In order to emphasize the similarity in the structure of the continuum and the lattice determinants we will adopt the lattice notations for the continuum objects without burdening the notation with distinctions. As this abuse of notation will be confined to the remainder of this section and Sect. 4 it is hoped that the reader will find it suggestive rather than confusing. Let $H = L^2(\mathbb{R}, \mathbb{C}^2)$, and let Q denote the matrix valued multiplication operator $\begin{bmatrix} -\varepsilon(k) & 0 \\ 0 & \varepsilon(k) \end{bmatrix}$, where $\varepsilon(k) = 1$ for $k > 0$ and $\varepsilon(k) = -1$ for $k < 0$. Let $H^p = H \otimes \mathbb{C}^p$ and write $Q = Q \otimes I_p$ for the direct sum of p-copies of Q acting on H^p. If $X: H^p \to H^p$ is a linear map we write $X = \begin{bmatrix} a(X) & b(X) \\ c(X) & d(X) \end{bmatrix}$ for the matrix of X relative to the $Q_+ H^p \oplus Q_- H^p$ decomposition of H^p ($Q_\pm = \frac{1}{2}(I \pm Q)$). If $M \in \mathrm{Gl}(p, \mathbb{C})$, then we let M act on H^p by:

$$f \to I \otimes Mf.$$

For $m < n$ let $P(m,n)$ denote the orthogonal projection in H^p on those functions whose Fourier transforms are supported in the interval $[m,n]$. Thus

$$P(m,n)f(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i(k'-k)n} - e^{i(k'-k)m}}{k'-k} f(k')dk'.$$

For $m < n$ and $M \in \mathrm{Gl}(p, C)$ we also write:

$$S_{m,n}(M) = I + (M-I)P(m,n). \tag{3.15}$$

We now state the principal result of this section:

Theorem 3.1. *Suppose $M_j \in \mathrm{Gl}(p, C)$ $(j = 1, \ldots, n)$ and that no matrix M_j has an eigenvalue that is 0 or negative. Suppose $m_j = (p_j, r_j)$, $n_j = (q_j, r_j)$ are in $\mathbb{Q} \times \mathbb{Q}$ ($\mathbb{Q}$ = rationals) and $p_j < q_j$ $(j = 1, \ldots, n)$ and $r_1 < r_2 < \cdots < r_n$. Recalling the operators $L(N)$ and $R(N)$ introduced at the beginning of this section we have:*

$$\lim_{N \to \infty} \det{}_2(I + L(N)R(N)) = \det{}_2(I + L_\infty R_\infty) \quad (Nm_j, Nn_j \in \mathbb{Z}^2),$$

where L_∞ *is the* $n \times n$ *block matrix with entries*:

$$(i<k) \qquad l_{ik} = \begin{cases} -Q_+ & k=i+1 \\ -a_{i+1}Q_+ & k=i+2 \\ -a_{i+1}\cdots a_{k-1}Q_+ & k>i+2 \end{cases}$$

$$(i>k) \qquad l_{ik} = \begin{cases} Q_- & i=k+1 \\ d_{i-1}Q_- & i=k+2 \\ d_{i-1}^{-1}\cdots d_{k+1}^{-1}Q_- & i>k+2 \end{cases}$$

$$(i=k) \qquad l_{ii}=0.$$

R_∞ *is the block diagonal matrix* $R_1 \oplus \cdots \oplus R_n$ *with entries on the diagonal*:

$$R_k = \begin{bmatrix} -b_k d_k^{-1} c_k & b_k d_k^{-1} \\ d_k^{-1} c_k & 0 \end{bmatrix}.$$

The operators a_k, b_k, c_k *and* d_k *are the matrix elements of* $\tilde{G}_k$ *relative to the* $Q_+H^p \oplus Q_-H^p$ *splitting of* H^p, *where*:

$$\tilde{G}_k = T^{(r_k - r_{k-1})/2} S_{p_k,q_k}(M_k) T^{(r_{k+1}-r_k)/2},$$

and T *is the operator of multiplication by* $\exp(-|k|Q(k))$ *on* H^p. *Observe that although* T *is unbounded on* H^p *the contributions which* T *makes to the operators* L_∞ *and* R_∞ *are all bounded operators.*

Proof. This is simply a transcription to the matrix situation of the scaling result whose proof is detailed in the first part of this section.

In the next section we will make use of the fact that $\det_2(I + L_\infty R_\infty)$ looks like the vacuum expectation of a product in a spin representation to evaluate this determinant in the abelian case. Some technical problems are caused by the fact that $S_{n,m}(M) \notin \mathrm{Gl}_Q(H^p)$.

Section 4

In this section we will use some results for the representation of current groups (known as loop groups when S^1 is involved instead of $\mathbb{R}$) to evaluate the perturbation determinant which appears in Theorem 3.1 in the abelian case (i.e., when all the monodromy matrices commute).

We begin with a description of the representation of interest. Let $H = L^2(\mathbb{R}, \mathbb{C}^2)$, and let Q denote the matrix valued multiplication operator $\begin{bmatrix} -\varepsilon(k) & 0 \\ 0 & \varepsilon(k) \end{bmatrix}$, where $\varepsilon(k)=1$ for $k>0$ and $\varepsilon(k)=-1$ for $k<0$. Let $H^p = H \otimes \mathbb{C}^p$. We continue to write $Q = Q \otimes I_p$ acting on H^p. Recall that associated with H^p and Q there is the restricted general linear group $\mathrm{Gl}_Q(H^p)$ determined by the property that the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of a group element relative to the $Q_+H^p \oplus Q_-H^p$ splitting of H^p has diagonal elements "a" and "d" which are Fredholm operators of index 0 and off diagonal elements "b" and "c" which are Schmidt class operators. We also have the spin representation of the central extension $\hat{Gl}_Q(H^p)$ of $Gl_Q(H^p)$ described in [24] (see also [34]) which will be the object of interest for us here. Suppose $g \in Gl_Q(H^p)$ and write $\begin{bmatrix} a(g) & b(g) \\ c(g) & d(g) \end{bmatrix}$ for the matrix of g relative to the $Q_+H^p \oplus Q_-H^p$ splitting of H^p. Consider the collection of elements g in $Gl_Q(H^p)$ with $d(g)$ invertible. This is not a subgroup of $Gl_Q(H^p)$ but it has a cross section in $\hat{Gl}_Q(H^p)$ which we write as:

$$\pi(g) = \Gamma_Q(gd(g)^{-1})\Gamma(d(g)), \tag{4.1}$$

where for brevity we have written $d(g)$ for $\begin{bmatrix} 1 & 0 \\ 0 & d(g) \end{bmatrix}$ acting on H^p. See [24] for definitions of $\Gamma_Q(\cdot)$ and $\Gamma(\cdot)$. Suppose g_1 and g_2 are in $Gl_Q(H^p)$ and $d(g_1)$, $d(g_2)$, and $d(g_1g_2)$ are all invertible. We may then write down a simple determinant formula for the cocycle associated with the cross section π in (4.1):

$$\pi(g_1)\pi(g_2) = c(g_1, g_2)\pi(g_1g_2), \quad c(g_1, g_2) = \det(d(g_1g_2)d(g_2)^{-1}d(g_1)^{-1}). \tag{4.2}$$

To see this we first make use of the fact that $\pi(g)$ is characterized by having vacuum expectation 1. This is a straightforward calculation $\langle \Gamma_Q(gd(g)^{-1})\Gamma(d(g))\rangle = \langle \Gamma_Q(gd(g)^{-1})\rangle = \det d(gd(g)^{-1}) = \det I = 1$ (see [24]). It follows from this that $c(g_1, g_2) = \langle \pi(g_1)\pi(g_2)\rangle$. We now make use of $\Gamma(h)\Gamma_Q(g)\Gamma(h^{-1}) = \Gamma_Q(hgh^{-1})$ and $\Gamma_Q(g_1)\Gamma_Q(g_2) = \Gamma_Q(g_1g_2)$ to write:

$$\begin{aligned} \langle \pi(g_1)\pi(g_2)\rangle &= \langle \Gamma_Q(g_1g_2d(g_2)^{-1}d(g_1)^{-1})\rangle = \det d(g_1g_2d(g_2)^{-1}d(g_1)^{-1}) \\ &= \det(d(g_1g_2)d(g_2)^{-1}d(g_1)^{-1}) \end{aligned}$$

(see Lemma 3.1 in [24]).

Next we wish to describe the current representation. Split H^p into two copies of $L^2(S^1, \mathbb{C}^p)$, so that on the first copy Q acts as $-\varepsilon$ and on the second copy Q acts as ε. We write $H^p = H_1^p \oplus H_2^p$ for this splitting of H^p. Let $\hat{f}(x)$ denote the (inverse) Fourier transform of $f(k) \in L^2(\mathbb{R}, \mathbb{C}^p)$:

$$\hat{f}(x) = (2\pi)^{-1/2} \int f(k)e^{ikx}dk. \tag{4.3a}$$

Let $x \to M(x) \in Gl(p, \mathbb{C})$ denote a matrix valued function of x. Suppose $M(x) - I = \int_{-\infty}^{\infty} F(k)e^{ikx}dk$, where $F(\cdot) \in L^1(\mathbb{R}, \mathbb{C}^{p^2})$ and $\int_{-\infty}^{\infty} |k| \mathrm{Tr}(F^*(k)F(k))dk < \infty$. For each such matrix $M(x)$ associate an operator M on H^p defined by:

$$M\hat{f}(x) = M(x)\hat{f}_1(x) \oplus M(x)\hat{f}_2(x), \tag{4.3}$$

where $f = f_1 \oplus f_2 \in H_1^p \oplus H_2^p$. Such an operator M is in $Gl_Q(H^p)$. To see this, first observe that the commutator of M and Q (or equivalently the commutator $M - I$ and Q) has finite Schmidt norm as a consequence of $\int_{-\infty}^{\infty} |k| \mathrm{Tr}(F^*(k)F(k))dk < \infty$. Thus $b(M)$ and $c(M)$ are in the Schmidt class. It remains to check that $d(M)$ has index 0. Evidently $d_Q(M) = d_{-\varepsilon}(M) \oplus d_\varepsilon(M) = a_\varepsilon(M) \oplus d_\varepsilon(M)$, where we use the subscripts Q, ε and $-\varepsilon$ to distinguish the operators which produce the relevant splittings of the spaces involved. But then:

$$\mathrm{Ind}(d_Q(M)) = \mathrm{Ind}(d_\varepsilon(M)) + \mathrm{Ind}(a_\varepsilon(M)).$$

However, this last sum is 0 since $\begin{bmatrix} a_\varepsilon & 0 \\ 0 & d_\varepsilon \end{bmatrix}$ is a compact perturbation of the invertible operator $M = \begin{bmatrix} a_\varepsilon & b_\varepsilon \\ c_\varepsilon & d_\varepsilon \end{bmatrix}$ which has index 0. It is not hard to see that the family of matrix multiplication operators just described is a group. Suppose $M_1(x) - I$ and $M_2(x) - I$ are each the Fourier transforms of L^1 functions; then $M_1 M_2(x) - I = (M_1(x) - I)(M_2(x) - I) - (M_1(x) - I) - (M_2(x) - I)$ is also the Fourier transform of an L^1 function since the convolution of two L^1 functions is again in L^1. If $M_j(x) - I = \int_{-\infty}^{\infty} F_j(k) e^{ikx} dk$ $(j = 1, 2, 3)$ with $F_j \in L^1$ and $M_3(x) = M_1(x) M_2(x)$, then $\int_{-\infty}^{\infty} |k| \operatorname{Tr}(F_j^*(k) F_j(k)) dk < \infty$ $(j = 1, 2)$ implies $\int_{-\infty}^{\infty} |k| \operatorname{Tr}(F_3^*(k) F_3(k)) dk < \infty$. The easiest way to see this is to note that $\int_{-\infty}^{\infty} |k| \operatorname{Tr}(F_j^*(k) F_j(k)) dk < \infty$ is equivalent to the commutator $[\varepsilon, M_j - I]$ being a Schmidt class operator. Thus the expression for $M_3 - I$ in terms of $(M_1 - I)$ and $(M_2 - I)$ given above, the derivation property for $[\varepsilon, \cdot]$, and the fact that the Schmidt class operators constitute an ideal in the bounded operators altogether finish the demonstration that finite $H_{1/2}$ norms for F_1 and F_2 imply a finite $H_{1/2}$ norm for F_3.

Definition. Let $M(\mathbb{R}, \mathrm{Gl}(p))$ denote the group of $\mathrm{Gl}(p, \mathbb{C})$ valued functions of x whose elements $M(x)$ have the property that $M(x)^{\pm 1} - I = \int_{-\infty}^{\infty} F_{\pm}(k) e^{ikx} dk$, where $F_{\pm}(\cdot)$ is an L^1 function and $\int_{-\infty}^{\infty} |k| \operatorname{Tr}(F_{\pm}^*(k) F_{\pm}(k)) dk < \infty$.

Evidently we may regard $M(\mathbb{R}, \mathrm{Gl}(p))$ as a subgroup of $\mathrm{Gl}_Q(H^p)$ via the action (4.3). The group $\hat{M}(\mathbb{R}, \mathrm{Gl}(p))$ which covers $\mathrm{Gl}_Q(H^p)$ in $\hat{\mathrm{Gl}}_Q(H^p)$ is an example of what is sometimes called a current group.

We are now prepared to specialize the cocycle formula (4.2) to the "abelian" case.

Lemma 4.1. *Let $M(x)$ and $N(x)$ denote elements of $M(\mathbb{R}, \mathrm{Gl}(p))$. Suppose $M(x) = e^{m(x)}$ and $N(x) = e^{n(x)}$ and that $m(x)$ and $n(x)$ are the Fourier transforms of functions in $L^1 \cap H_{1/2}$. Let $m_{\pm}(x) = \varepsilon_{\pm} m(x)$ and $n_{\pm}(x) = \varepsilon_{\pm} n(x)$, where $\varepsilon_{\pm} = (1 \pm \varepsilon)/2$. Finally suppose that the matrix valued function $m_{\pm}(x), n_{\pm}(x)$ commute among themselves. Then the cocycle in (4.2) is given by:*

$$\det(d_Q(MN) d_Q(N)^{-1} d_Q(M)^{-1}) = e^{\operatorname{Tr}(b(m_+)c(n_-) + c(m_-)b(n_+))};$$

we have abbreviated $b_\varepsilon(\cdot)$ and $c_\varepsilon(\cdot)$ as $b(\cdot)$ and $c(\cdot)$.

Proof. Since $d_Q(M) = a_\varepsilon(M) \oplus d_\varepsilon(M)$, the formula (4.2) for the cocycle becomes:

$$\det[a(MN)a(N)^{-1}a(M)^{-1}] \det[d(MN)d(N)^{-1}d(M)^{-1}],$$

where again we write $a_\varepsilon(\cdot) = (\cdot)$, and $d_\varepsilon(\cdot) = d(\cdot)$. Let $M_{\pm} = e^{m_{\pm}}$ and $N_{\pm} = e^{n_{\pm}}$. Then we make use of $a(M_- M_+) = a(M_-)a(M_+)$, $a(M_{\pm} N_{\pm}) = a(M_{\pm})a(N_{\pm})$ and other similar identities together with the commutativity of $m_{\pm}$ and $n_{\pm}$ to see that

$$\begin{aligned}
&\det[a(MN)a(N)^{-1}a(M)^{-1}]\\
&\quad=\det[a(M_-)a(N_-)a(M_+)a(N_+)a(N_+)^{-1}a(N_-)^{-1}a(M_+)^{-1}a(M_-)^{-1}]\\
&\quad=\det[a(N_-)a(M_+)a(N_-)^{-1}a(M_+)^{-1}]=\det[e^{a(n_-)}e^{a(m_+)}e^{-a(n_-)}e^{-a(m_+)}]\\
&\quad=\det[e^{[a(n_-),a(m_+)]}]=\det e^{b(m_+)c(n_-)}=e^{\mathrm{Tr}(b(m_+)c(n_-))}.
\end{aligned}$$

The commutator $[a(n_-), a(m_+)] = b(m_+)c(n_-)$, since n_- and m_+ commute. Write $M=M_+M_-$ and $N=N_+N_-$ to take advantage of $d(M_+M_-)=d(M_+)d(M_-)$. A calculation similar to the one above shows that $\det(d(MN)d(N)^{-1}d(M)^{-1})=e^{\mathrm{Tr}(c(m_-)b(n_+))}$. This finishes the proof of Lemma 4.1. Q.E.D.

Suppose now that $M_j\in\mathrm{Gl}(p,\mathbb{C})$ and that none of the matrices M_j has spectrum which intersects the non-positive real axis ($j=1,\ldots,r$). Let C denote a simple closed curve which surrounds the eigenvalues of all the matrices M_j, does not intersect the non-positive real axis, and is counterclockwise oriented. Define:

$$2\pi L_j=\frac{1}{2\pi i}\int_C\frac{\log z}{(z-M_j)}dz, \tag{4.4}$$

where $\log z$ is normalized so that $\log 1 = 0$ and has its branch cut on the negative real axis. We now suppose that the M_j all commute amongst themselves. It follows from (4.4) that the L_j all commute amongst themselves as well. For $m<n$ define

$$S_{m,n}(M_j)=I+(M_j-I)P(m,n)=\exp 2\pi L_jP(m,n).$$

We would like to form a product of operators $\pi(S_{m,n}(M_j))$, and evaluate the vacuum expectation of this product in two different ways, the first way using the determinant formulas in [24], the second way using the cocycle formula in Lemma 4.2. Unfortunately $S_{n,m}(M)\notin\mathrm{Gl}_Q(H^p)$ in general, so we must be more devious. First we introduce a "smooth" version of $P_\delta(m,n)$ of $P(m,n)$ so that for $\delta>0$ we have:

$$S^\delta_{m,n}(M)=(I+(M-I)P_\delta(m,n))\in\mathrm{Gl}_Q(H^p),$$

and

$$\lim_{\delta\to 0}S^\delta_{m,n}(M)=S_{m,n}(M).$$

We then use the cocycle and determinant formulas to evaluate the expectation of the product of smoothed operators $\pi(S^\delta_{m,n}(M_j))$. In order to let $\delta\to 0$ in the determinant formulas we must introduce non-zero transfer. The same result is obtained by analytic continuation in the cocycle formulas. Finally we pass to a limit $\delta\to 0$. The cocycle calculation yields an explicit product of homogeneous functions and the determinant formula may be matched with the results of Theorem 3.1. The results of Theorem (2.0) and Theorem (3.1) are then combined in a more satisfactory account of the asymptotics in the abelian case (Theorem 4.1).

We now describe our choice for $P_\delta(m,n)$.

$$P^\delta_{m,n}(x)=\begin{cases}0 & x<m \text{ or } x>n\\ (x-m)/\delta & m\leqq x\leqq m+\delta\\ 1 & m+\delta\leqq x\leqq n-\delta\\ (n-x)/\delta & n-\delta\leqq x\leqq n.\end{cases}$$

Let $P_\delta(m,n)$ denote the operator of multiplication by $P^\delta_{m,n}(x)$ in the Fourier transform variables (4.3a).

Next we introduce the transfer matrix with pure imaginary argument, T^{ir}, given by multiplication $\begin{bmatrix} e^{irk} & 0 \\ 0 & e^{-irk} \end{bmatrix}$ in the $H^p_1 \oplus H^p_2$ decomposition of H^p. This will allow us to introduce analytic continuation variables into the cocycle calculations. Suppose $m_j=(p_j,r_j)$ and $n_j=(q_j,r_j)$ with $p_j<q_j$ $(j=1,\dots,r)$. Let $S^\delta_{p_j,q_j}(M_j)=I+(M_j-I)P_\delta(p_j,q_j)$. Then one may easily check that:

$$T^{ir_j}S^\delta_{p_j,q_j}(M_j)T^{-ir_i}=S^\delta_{p_j-r_j,q_j-r_j}(M_j)\oplus S^\delta_{p_j+r_j,q_j+r_j}(M_j), \tag{4.5}$$

where the direct sum occurs in the $H^p_1 \oplus H^p_2$ decomposition of H^p.

Let $G_j=T^{ir_j}S^\delta_{p_j,q_j}(M_j)T^{-ir_j}$; we wish to calculate $\langle \pi(G_1)\cdots\pi(G_r)\rangle$ using the cocycle formula in Lemma 4.1. Observe, however, that Lemma 4.1 does not directly apply since G_j acts differently on H^p_1 and H^p_2. The modification needed is simple and will now be described. Write $G_j=e^{g_j}=e^{g_{1j}\oplus g_{2j}}$, where g_{kj} acts on H^p_k $(k=1,2)$. Then:

$$\pi(e^{g_j})\pi(e^{g_k})=e^{\alpha(g_j,g_k)+\beta(g_j,g_k)}\pi(e^{g_j+g_k}),$$

where

$$\alpha(g_j,g_k)=\mathrm{Tr}(b(g^+_{1j})c(g^-_{1k})) \tag{4.6}$$

and

$$\beta(g_j,g_k)=\mathrm{Tr}(c(g^-_{2j})b(g^+_{2k})).$$

The proof follows that of Lemma 4.1 without essential change. We now state the principal result of this section:

Theorem 4.0. *Suppose $M_j\in\mathrm{Gl}(p,\mathbb{C})$ $(j=1,\dots,r)$ and that no matrix M_j has an eigenvalue that is 0 or negative. Let L_j be defined by (4.4) and suppose all the M_j (and hence the L_j) commute amongst themselves $(j=1,\dots,n)$. Suppose $m_j=(p_j,r_j)$ and $n_j=(q_j,r_j)$ are in $\mathbb{Q}\times\mathbb{Q}$ ($\mathbb{Q}$ = rationals), $p_j<q_j$ $(j=1,\dots,n)$, and $r_1<r_2\cdots<r_n$. Recall $L(N)$ and $R(N)$ introduced at the beginning of Sect. 3. Then:*

$$\lim_{N\to\infty}\det{}_2(I+L(N)R(N))=\prod_{i<j}\left[\frac{|m_i-n_j|\,|n_i-m_j|}{|m_i-m_j|\,|n_i-n_j|}\right]^{2\mathrm{Tr}(L_iL_j)}$$

where N tends to ∞ so that Nm_j and Nn_j are eventually in $\mathbb{Z}^2$.

Proof. We wish to apply the determinant formula in Theorem 3.0 of [24] to the evaluation of $\langle\pi(G_1)\cdots\pi(G_n)\rangle$. It is convenient to introduce $\tilde{G}_j=T^{i(r_j-r_{j-1})/2}S^\delta_{p_j,q_j}(M_j)T^{i(r_{j+1}-r_j)/2}$ to facilitate comparison of the result with Theorem 3.1. Without difficulty one sees that $\langle\pi(G_1)\cdots\pi(G_n)\rangle=\langle\pi(\tilde{G}_1)\cdots\pi(\tilde{G}_n)\rangle$. In the determinant formula for this last expectation, the exponential factors $\begin{bmatrix} e^{\pm i(r_{j+1}-r_j)k/2} & 0 \\ 0 & e^{\mp i(r_{j+1}-r_j)k/2} \end{bmatrix}$ always occur in conjunction with the corresponding projections $Q_\pm$. Thus these exponentials have analytic continuations into the lower half plane in the difference variables $r_{j+1}-r_j$ which are strongly continuous up to the real axis. It follows that the determinants also possess analytic continuations in the difference variables which are continuous up to the real axis, since the product of a strongly continuous map and a Schmidt class map is continuous in the Schmidt norm. We now effect such an analytic continuation by the substitution $(r_{j+1}-r_j)\to -i(r_{j+1}-r_j)$. It is now important that $r_1<r_2\cdots<r_n$, so that the analytic continuation takes place in the appropriate half planes. At this point we wish to let $\delta\to 0$, and compare the result with Theorem 3.0. This may be done along the lines of the proof of Theorem 3.0 but is even simpler. The strong convergence of $S^{\delta}_{m,n}(M)$ to $S_{m,n}(M)$ as $\delta\to 0$ is a trivial consequence of dominated convergence and implies the strong convergence of $a(S^{\delta})$ and $d(S^{\delta})$ to $a(S)$ and $d(S)$ as $\delta\to 0$. The operators $b(S^{\delta})$ and $c(S^{\delta})$ always occur in conjunction with decaying exponential factors $\theta(\pm k)\exp\pm(r_{j+1}-r_j)k/2$, and it is trivial to supply the estimates for convergence in Schmidt norm. The only ingredient worth discussing in more detail is the strong convergence of $d(S^{\delta})^{-1}$. Since $S^{\delta}_{m,n}(M)=I+(M-I)P_{\delta}(m,n)$ and $P_{\delta}(m,n)$ is a self-adjoint operator with spectrum $[0,1]$ the arguments in Lemma (1.1) apply without change to show that $d(S^{\delta}_{m,n}(M))$ has a uniformly (in δ) bounded inverse provided M has no eigenvalues which are 0 or negative. Thus the strong convergence of $d(S^{\delta}_{m,n}(M))$ implies the strong convergence of the inverse $d(S^{\delta}_{m,n}(M))^{-1}$ as $\delta\to 0$. The simplicity of this argument is one reason we choose to work with the cutoff in the form S^{δ} rather than $\exp 2\pi LP_{\delta}$. This second form is considerably simpler to use in the cocycle calculation we are about to do but has the disadvantage that a uniform bound for $d(\exp 2\pi LP_{\delta})^{-1}$ is hard to come by. Indeed, for the wrong choice of $2\pi L=\log M$ such uniform bounds fail and the cocycle calculation leads to results which depend on the choice of logarithms for M. The reader should have no difficulty assembling the various convergence results described above into a proof that analytic continuation in the difference variables $r_{j+1}-r_j$ followed by passage to the limit $\delta\to 0$, transforms $\langle\pi(\tilde{G}_1)\cdots\pi(\tilde{G}_n)\rangle$ into $\det_2(I+L_\infty R_\infty)$, defined in Theorem 3.1.

We now consider the same two step sequence for the cocycle calculation. It is clear that (4.6) may be iterated to obtain:

$$\pi(G_1)\cdots\pi(G_n)=\prod_{i<j}\exp(\alpha_{ij}+\beta_{ij})\pi(G_1\cdots G_n),$$

where $\alpha_{ij}=\alpha(g_i,g_j)$ and $\beta_{ij}=\beta(g_i,g_j)$. Since $\langle\pi(G_1\cdots G_n)\rangle=1$ we have:

$$\langle\pi(G_1)\cdots\pi(G_n)\rangle=\prod_{i<j}\exp(\alpha_{ij}+\beta_{ij}). \tag{4.8}$$

A little calculation shows that:

$$\alpha(g_i,g_j)=\frac{1}{(2\pi)^2}\int_0^\infty k\,\mathrm{Tr}(\hat{g}_{1i}(k)\hat{g}_{1j}(-k))dk,$$
$$\beta(g_i,g_j)=\frac{1}{(2\pi)^2}\int_0^\infty k\,\mathrm{Tr}(\hat{g}_{2i}(-k)\hat{g}_{2j}(k))dk, \tag{4.9}$$

where

$$\hat{g}(k)=\int_{-\infty}^{\infty}g(x)e^{-ikx}dx,$$

and

$$g_{1j}(x)=\log(S^{\delta}_{p_j-r_j,q_j-r_j}(M_j)[x]),\quad g_{2j}(x)=\log(S^{\delta}_{p_j+r_j,q_j+r_j}(M_j)[x]). \tag{4.10}$$

The spectrum of $S^{\delta}_{m,n}(M)[x]=I+(M-I)P^{\delta}_{m,n}(x)$ never intersects the non-positive real axis and we choose the logarithm in (4.10) as is done in (4.4). To proceed with the calculation of $\alpha(g_i,g_j)$ and $\beta(g_i,g_j)$, it is useful to make some simplifications. Let M denote one of the matrices M_j and write:

$$f(x)=\log(I+(M-I)x),\quad 0\leqq x\leqq 1,$$

where again the logarithm is defined as in (4.4). The Fourier transforms of the functions which occur in (4.10) are of the form ($m=p_j\pm r_j$, $n=q_j\pm r_j$):

$$\int_m^{m+\delta} f\left(\frac{x-m}{\delta}\right)e^{-ikx}dx+\int_{m+\delta}^{n-\delta} f(1)e^{-ikx}dx+\int_{n-\delta}^{n} f\left(\frac{n-x}{\delta}\right)e^{-ikx}dx.$$

In this last formula we may integrate by parts once, cancelling all the boundary terms to obtain:

$$(ik)^{-1}\int_m^{m+\delta} f'\left(\frac{x-m}{\delta}\right)e^{-ikx}\delta^{-1}dx-(ik)^{-1}\int_{n-\delta}^{n} f'\left(\frac{n-x}{\delta}\right)e^{-ikx}\delta^{-1}dx.$$

Now make the change of variables $(x-m)/\delta\leftarrow x$ in the first integral and $(x-n)/\delta\leftarrow x$ in the second integral. One finds:

$$(ik)^{-1}e^{-ikm}\int_0^1 f'(x)e^{-i\delta kx}dx-(ik)^{-1}e^{-ikn}\int_{-1}^{0} f'(-x)e^{-i\delta kx}dx. \tag{4.11}$$

Next observe that $\int_0^1 f'(x)dx=\int_{-1}^0 f'(-x)dx=\log M$, then add and subtract $(ik)^{-1}(e^{-ikm}-e^{-ikn})e^{-\delta|k|}\log M$ in (4.11). This result may be written:

$$\begin{aligned}&e^{-ikm}\int_0^1 f'(x)(ik)^{-1}(e^{-i\delta kx}-e^{-\delta|k|})dx-e^{-ikn}\int_{-1}^0 f'(-x)(ik)^{-1}(e^{-i\delta kx}-e^{-\delta|k|})dx\\&\quad+e^{-\delta|k|}(ik)^{-1}(e^{-ikm}-e^{-ikn})\log M.\end{aligned} \tag{4.12}$$

We claim that when this last sum is substituted into (4.9) and an analytic continuation is made in the difference variables $(r_{j+1}-r_j)$, as described earlier, the first two terms in (4.12) do not make a contribution in the limit $\delta\to 0$. The reason for this is easy to see; as a function of k the integral $\int_0^1 f'(x)(ik)^{-1}(e^{-i\delta kx}-e^{-\delta|k|})dx$ is dominated by a constant for $|k|\leqq 1$ and by (constant/$|k|$) for $|k|\geqq 1$. The same can be said for $\int_1^0 f'(-x)(-ik)^{-1}(e^{-i\delta kx}-e^{-\delta|k|})dx$. Each of the integrals just described tends to zero pointwise in k as $\delta\to 0$, and the effect of analytic continuation is to introduce exponential factors $e^{-(r_{j+1}-r_j)k}$ into the integrals in (4.9). Thus dominated convergence applies and all the terms involving these two integrals vanish in the limit $\delta\to 0$. We may thus make the replacements:

$$\hat{g}_{1j}(k)\leftarrow e^{-\delta|k|}(ik)^{-1}(e^{-ik(p_j-r_j)}-e^{-ik(q_j-r_j)})\log M_j,$$

$$\hat{g}_{2j}(k)\leftarrow e^{-\delta|k|}(ik)^{-1}(e^{-ik(p_j+r_j)}-e^{-ik(q_j+r_j)})\log M_j,$$

in the calculation of the integrals in (4.9). One finds:

$$\alpha(g_i, g_j) = \mathrm{Tr}(L_iL_j)A_{ij}, \quad \beta(g_i, g_j) = \mathrm{Tr}(L_iL_j)B_{ij},$$

where:

$$\begin{aligned} A_{ij} &= \log(q_j - p_i + r_i - r_j + 2\delta i) - \log(p_j - p_i + r_i - r_j + 2\delta i) \\ &\quad - \log(q_j - q_i + r_i - r_j + 2\delta i) + \log(p_j - q_i + r_i - r_j + 2\delta i), \qquad (4.13) \\ B_{ij} &= \log(q_j - p_i + r_j - r_i - 2i\delta) - \log(p_j - p_i + r_j - r_i - 2i\delta) \\ &\quad - \log(q_j - q_i + r_j - r_i - 2i\delta) + \log(p_j - q_i + r_j - r_i - 2i\delta). \end{aligned}$$

The logarithm which appears in both expressions is normalized so that $\log 1 = 0$ and has its branch cut on the negative real axis. It is evident that we may analytically continue $(r_j - r_i)$ $(i < j)$ to negative imaginary values (i.e., replace $(r_j - r_i)$ with $-i(r_j - r_i)$), where we now require $r_1 < r_2 \cdots < r_n$. Do this in (4.13) and then take the limit $\delta \to 0$. One finds after some elementary algebra that $A_{ij} + B_{ij}$ becomes:

$$2\log\frac{|m_i - n_j|\,|n_i - m_j|}{|m_i - m_j|\,|n_i - n_j|}.$$

Together with (4.8) this finishes the proof of Theorem 4.0.

It is natural at this point to combine the results of Theorem (2.0) and Theorem (4.0).

Theorem 4.1. *Suppose $M_j \in \mathrm{Gl}(p, \mathbb{C})$ $(j = 1, \ldots, n)$ and that no matrix M_j has an eigenvalue that is 0 or negative. Suppose $m_j = (p_j, r_j)$ and $n_j = (q_j, r_j)$ with $p_j < q_j$ $(j = 1, \ldots, n)$ and $r_1 < \cdots < r_n$. Let $G_j(N) = \sigma_{Nm_j}(M_j)\sigma_{Nn_j}(M_j)^{-1}$, where $\sigma_m(M)$ is given the "unitary normalization" described in Sect. 1 (1.9). Let*

$$\tau_N(m, n) = \prod_{i=1}^{n} N^{-2\mathrm{Tr}(L_i^2)} \langle G_1(N) \cdots G_n(N) \rangle_{T = T_c},$$

where

$$2\pi L_i = \frac{1}{2\pi i} \int_C \frac{\log z}{(z - M_j)} dz,$$

and the countour C is a counterclockwise oriented simple closed curve which surrounds the spectrum of M_j and does not intersect the non-positive real axis. Then $\lim_{N \to \infty} \tau_N(m, n)$ exists, where N tends to ∞ so that Nm and Nn are eventually in Z^2. In the event that M_j commute amongst themselves the limit, $\tau_\infty(m, n)$, is given by:

$$\tau_\infty(m, n) = c \prod_{i=1}^{n} |m_i - n_i|^{2\mathrm{Tr}(L_i^2)} \prod_{i<j} \left[\frac{|m_i - n_j|\,|n_i - m_j|}{|m_i - m_j|\,|n_i - n_j|}\right]^{2\mathrm{Tr}(L_iL_j)} \qquad (4.14)$$

where c is a constant that depends on $M_1, \ldots, M_n$.

Section 5

In this section we present some speculations concerning the generalization of Theorem 4.1 to arbitrary configurations. We begin with the abelian case. The reader will have no trouble checking that if we obliterate the distinction between the points m_j and n_j labeling all the points m_j with associated monodromy $M_j = \exp(2\pi L_j)$, then (4.14) can be rewritten:

$$c \prod_{i \neq j} |m_i - m_j|^{-\operatorname{Tr}(L_i L_j)}. \tag{5.0}$$

This is not quite a good conjecture for the general configuration in the abelian case. One may see this by examining what happens when the points m_i and n_i in (4.7) are separated by a long distance. Suppose $|m_i - n_i| = R$ for each $i = 1, \ldots, n$ so that the m_i configuration and the n_i configuration are congruent and separated by a distance R. Substitute this into (4.14) and use $|m_i - n_j| = |m_i - n_i + n_i - n_j| = R(1 + 0(R^{-1}))$ and $|n_i - m_j| = |n_i - m_i + m_i - m_j| = R(1 + O(R^{-1}))$. Then one encounters $R^{2\operatorname{Tr}(\Sigma L_i)^2}$, and another factor which becomes the product of two copies of (5.0) as $R \to \infty$. Now when $\operatorname{Re}\operatorname{Tr}(\sum L_i)^2 < 0$, the limit of $R^{2\operatorname{Tr}(\Sigma L_i)^2}$ is 0 as $R \to \infty$, and when $\operatorname{Re}\operatorname{Tr}(\sum L_i)^2 > 0$, the limit of $R^{2\operatorname{Tr}(\Sigma L_i)^2}$ is ∞ as $R \to \infty$. Thus if one believes that the limiting correlations cluster, then these results suggests that (5.0) is good only when $\sum L_i = 0$ or $M_1 \cdots M_n = I$. It is interesting to specialize further to the case when the L_j are pure imaginary scalars il_j with $-1/2 \leqq l_j \leqq 1/2$. The square of the critical Ising correlations is then given by:

$$\langle \sigma_{m_1}(e^{2\pi i l_1}) \cdots \sigma_{m_n}(e^{2\pi i l_n}) \rangle_{T = T_c, l_j = \pm 1/2}. \tag{5.1}$$

Without difficulty one may check that (5.0) does not give a sensible conjecture for the Ising scaling limit in the sense that the result depends on which choices $l_j = \pm 1/2$ are made in (5.1). The Luther–Peschel conjecture amounts to approximating $\sigma_m(-1)$ by $[\sigma_m(e^{2\pi i l}) + \sigma_m(e^{-2\pi i l})]/2$ $(l \to 1/2)$. Observe that in this case the expansion of product $\prod_j (\sigma_{m_j}(e^{2\pi i l_j}) + \sigma_{m_j}(e^{-2\pi l_j}))$ yields a sum of products of monodromy fields in each of which $\operatorname{Re}(\sum L_i^2)^2 \leqq 0$, and using the conjectured result (5.0) one finds:

$$\sum_{l_i, l_j = \pm 1/2} \prod_{i \neq j} |m_i - m_j|^{l_i l_j} \delta(l_1 + \cdots + l_n).$$

The term $\delta(l_1 + \cdots + l_n)$ restricts the sum of those choices of $l_1, \ldots, l_n$ which sum to 0. This is the Luther–Peschel conjecture. It also resembles what one finds if the Ising case is represented as a sum of Fourier series in the monodromy variables and various limits are freely interchanged.

We next briefly consider the non-abelian case. The obvious conjecture for the scaling limit in this case is the τ-function of Sato, Miwa and Jimbo [33] whose existence has recently been established by Malgrange [11] in the regular singular case. Of course, some modification is necessary since we have a splitting determined by $\begin{bmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}$ rather than by ε, but this is a small matter. A more serious problem is that Malgrange establishes the existence of τ (or $d \log \tau$) by first proving a generalized Painlevé property for the Schlesinger equations. He then uses a formula due to Sato, Miwa, and Jimbo which express $d(\log \tau)$ in terms of the solution to the Schlesinger equations. It seems hard to make analogous constructions on the lattice, but the McCoy, Wu, Perk "deformation" equations for the Ising correlations [16] certainly suggest that the structure is there. We hope to pursue this investigation at a later date.

Acknowledgements. The author would like to thank Doug Pickrell and Professor E. Basor for a patient critique of the arguments in Sect. 4. Doug Pickrell is also responsible for the simple proof of Lemma 4.1.

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Communicated by K. Osterwalder

Received December 3, 1984; in revised form August 6, 1985