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Perturbation Theory and Non-Renormalizable Scalar Fields

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Abstract. We study how to set up systematic summation rules that could permit us to interpret the divergent expressions arising in the perturbation theory of $:P(\varphi):_d$ when one does not allow any renormalization besides the usual coupling constants, mass and wave function renormalizations.

1. Introduction

Our main result is that it is possible to express the Schwinger functions (or the effective potentials) as formal power series of objects which we call "form factors" which, although divergent to all orders of perturbation theory if the cut-off N is removed, obey to all orders a formal equation which retains its meaning as $N \to \infty$.

We show that if the formal equation admits a solution verifying suitable bounds, then the formal power series for the Schwinger functions in terms of the form factors is bounded to all orders.

Hence there is the possibility of giving a meaning to perturbation theory of non-renromalizable interactions without introducing infinitely many new counterterms, but rather introducing infinitely many new constants, the form factors, which however are not independent but are related by an equation (which may or may not have some non-trivial solution).

We restrict ourselves to the case of renormalizable (but not superrenormalizable) or non-renormalizable polynomial interactions in integer dimension $d \geqq 3$. The superrenormalizable cases would require a separate treatment. It is conceivable that something like the results of this paper hold for some non-polynomial interactions (like sine-Gordon field in two dimensions): however the whole problem should be studied starting again from scratch.

While the ideas involved in this paper are partly already in the literature (see Parisi, 1973, 1975; Symanzik, 1973, and references therein) the bounds that we

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present here do not seem to have been studied so far, probably because of the difficulties related to the treatment of the "overlapping divergences" present in the usual approaches the renormalization theory.

2. Notations

Let $\varphi^{(\leqq N)}$ be the free field with cut-off at length γ^{-N}, where $\gamma>1$ is a fixed scale parameter,

$$\varphi_x^{(\leqq N)}=\sum_{j=0}^{N}\varphi_x^{(j)}, \tag{2.1}$$

where $\varphi^{(j)}$, $\varphi^{(j')}$ are free fields, independent for $j\neq j'$, with propagators

$$\mathscr{E}(\varphi_x^{(j)}\varphi_y^{(j)})=\frac{1}{(2\pi)^d}\int\gamma^{-2j}F_j(\gamma^{-2j}p^2)e^{ip(x-y)}dp, \tag{2.2}$$

and, if $\hat{F}_j$ is the Fourier transform of F_j:

$$\begin{aligned}&\sum_{j=0}^{\infty}\gamma^{-2j}F_j(\gamma^{-2j}p^2)=\frac{1}{1+p^2},\\ &|\partial^s\hat{F}_j(x)|\leqq K\exp-\kappa|x| \qquad \text{all}\, j,\ \text{all}\, s\leqq 4.\end{aligned} \tag{2.3}$$

The (2.3) imply that the free field samples $\varphi^{(j)}$ are very smooth and

$$\|\varphi^{(j)}\|_3=\sup_{x\in\Lambda}\sum_{p=0}^{3}|\partial^p\varphi_x^{(j)}|\gamma^{-\frac{d-2}{2}j}\gamma^{-pj}>B, \tag{2.4}$$

with probability $\exp-B^2\cdot\text{const}$, if Λ is fixed. Actually for simplicity we shall fix Λ to be a torus with side size L and periodize over Λ the propagators (2.2), for each j.

We define a $:P(\varphi):_d$ interaction on scale N to be any element of the t-dimensional space $\mathscr{I}_N$ spanned by:

$$\begin{aligned}I_N^{(\alpha)}(\varphi^{(\leqq N)})&\equiv\int_\Lambda :(\varphi_x^{(\leqq N)})^{2\alpha}:dx, \qquad \alpha=0,1,\dots,t-2,\\ I_N^{(t-1)}(\varphi^{(\leqq N)})&\equiv\int_\Lambda :(\partial\varphi_x^{(\leqq N)})^2:dx,\end{aligned} \tag{2.5}$$

where the : : always denote Wick ordering (with respect to the covariance of the random variable appearing in their argument). Note that

$$\int :\varphi_{x_1}^{(\leqq N)}\dots\varphi_{x_n}^{(\leqq N)}:P(d\varphi^{(N)})=\,:\varphi_{x_1}^{(\leqq N-1)}\dots\varphi_{x_n}^{(\leqq N-1)}:.$$

Given t constants $\lambda_N=(\lambda_N^{(0)},\dots,\lambda_N^{(t-1)})$ with $\lambda_N^{(t-1)}>0$ one defines the "effective potential $V^{(K)}$ on scale K" as:

$$e^{V^{(k)}(\varphi^{\leqq k})}=\int e^{V^{(N)}(\varphi^{(\leqq N)})}\prod_{j>k}^{N^4}P(d\varphi^{(j)}), \quad \text{if}$$

$$V(\varphi^{(\leqq N)})\equiv V^{(N)}=\sum_{\alpha=0}^{t-1}\lambda_N^{(\alpha)}I_N^{(\alpha)}(\varphi^{(\leqq N)}),$$

and $V^{(k)}$ admits a formal power series expansion in $\boldsymbol{\lambda}_N$

$$V^{(k)} = \sum_{\mathbf{m}=(m_0,\ldots,m_{t-1})} V_{\mathbf{m}}^{(k)}(\varphi^{(\leqq k)}; N)_{nr} \boldsymbol{\lambda}_N^{\mathbf{m}}, \tag{2.6}$$

where $V_{\mathbf{m}}$ are suitable functions of $\varphi^{(\leqq k)}$.

Simple is the rule to build $V_{\mathbf{m}}^{(k)}$. Given any family $x_1, \ldots, x_n$ of random variables and denoting $\mathscr{E}(\cdot)$ the integration on their distribution one defines the "truncated expectation" of $x_1, \ldots, x_n$ as

$$\mathscr{E}^T(x_1, \ldots, x_n) = \frac{\partial^n}{\partial\omega_1, \ldots, \partial\omega_n} \log \mathscr{E}\left(\exp \sum_{i=1}^{n} \omega_i x_i\right)\bigg|_{\boldsymbol{\omega}=\mathbf{0}}. \tag{2.7}$$

Then, if $\mathscr{E}_j(\cdot)$ denotes the integration over the distribution of $\varphi^{(j)}$,

$$V^{(N-1)}(\varphi^{(\leqq N-1)}) = \sum_{h=1}^{\infty} \frac{1}{h!} \mathscr{E}_N^T(V^{(N)}, \ldots, V^{(N)}) \tag{2.8}$$

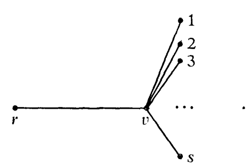
and

$$V^{(N-2)}(\varphi^{(\leqq N-2)}) = \sum_{k=1}^{\infty} \frac{1}{k!} \mathscr{E}_{N-1}^T\left(\sum_{h=1}^{\infty} \frac{1}{h!} \mathscr{E}_N^T(V^{(N)}, \ldots, V^{(N)}), \ldots, \sum_{h=1}^{\infty} \frac{1}{h!} \mathscr{E}_N^T(V^{(N)}, \ldots, V^{(N)})\right), \tag{2.9}$$

so that in general $V^{(k)}$ can be described in terms of "trees." A tree is an object built as follows:

1) Draw a horizontal segment rv: r is the "root" of the tree and rv its "trunk."

2) In v draw s segments, $s \geqq 0$, ending in $v_1, \ldots, v_s$, numbered from 1 to s from top to bottom



3) From each v_j draw $s_{v_j} \geqq 0$ segments ending in $v_{j_1}, \ldots, v_{j_{s_{v_j}}}$ numbered from 1 to s_v from top to bottom, etc., stopping after finitely many steps, having created a "tree" θ with n endpoints (the "degree" of θ is n).

To each vertex v of θ one can append a "frequency label" h_v compatibly with the order of the tree: $h_{v'} < h_v$ if $v' < v$ ($v' < v$ if v' is created first in building θ); h_r, if $r =$ root of θ, is the "root frequency" of the frequency assignment $\mathbf{h}$ to θ. The endpoints will conventionally be assigned index $N+1$.

Given a non-trivial tree θ, i.e. with one inner vertex at least, there are s_{v_0} trees $\theta_1, \ldots, \theta_{s_{v_0}}$, which have root $v_0 =$ first vertex of θ: clearly if θ is given a frequency assignment $\mathbf{h}$ then $\theta_1, \ldots, \theta_{s_{v_0}}$ inherit from θ frequency assignments $\mathbf{h}_1, \ldots, \mathbf{h}_{s_{v_0}}$ with root frequency h_{v_0}.

Then given a tree θ with frequency assignments $\mathbf{h}$, we define

$$V(\theta, \mathbf{h}) = \mathscr{E}_{h_r+1}, \ldots, \mathscr{E}_{h_{v_0}-1} \mathscr{E}_{h_{v_0}}^T(V(\theta_1, \mathbf{h}_1), \ldots, V(\theta_{s_{v_0}}, \mathbf{h}_{s_{v_0}})) \tag{2.9}$$

for θ non-trivial, while if $\theta = \theta_0 = \underset{r}{\bullet}\!\!\longrightarrow\!\!\underset{v_0}{\bullet}$

$$V(\theta_0, h_r) = \sum_{\alpha=0}^{t-1} \lambda_N^{(\alpha)} I_{h_r}^{(\alpha)}(\varphi^{(\leqq h_r)}). \tag{2.10}$$

The (2.9), (2.10) define inductively $V(\theta, \mathbf{h})$, and it is easy to check that

$$V^{(k)}(\varphi^{(\leqq k)}) = \sum_{\theta} \sum_{\mathbf{h}: h_r = k} \frac{V(\theta, \mathbf{h})}{n(\theta)}, \tag{2.11}$$

where the sum runs over all trees θ admitting labelings $\mathbf{h}$ with $h_r = k$, $h_v \leqq N$ (i.e. which do not have too many inner vertices), and $n(\theta)$ is a combinatorial factor

$$n(\theta) = \prod_{v \in \theta} s_v! \tag{2.12}$$

if s_v = number of lines into which θ bifuncates at v. By the multilinearity of the truncated expectations $V(\theta, \mathbf{h})$ is a part of the n^{th} order effective potential if degree $\theta = n$.

In general, fixed θ

$$\sum_{\mathbf{h}, h_r = k} V(\theta, \mathbf{h}) \text{ diverges as } N \to \infty \text{ for most } \varphi^{(\leqq k)}. \tag{2.13}$$

One therefore defines

$$\boldsymbol{\lambda}_N = \boldsymbol{\lambda} + \sum_{|\mathbf{m}| \geqq 2} \ell_{\mathbf{m}}(N) \boldsymbol{\lambda}^{\mathbf{m}}, \tag{2.14}$$

where $\boldsymbol{\lambda}$ are constants to be determined, called "renormalized constants", and one tries to define the coefficients $\ell_{\mathbf{m}}(N)$ so that $V^{(k)}$ re-expressed as a formal power series in $\boldsymbol{\lambda}$ via (2.14), (2.6),

$$V^{(k)}(\varphi^{(\leqq k)}) = \sum_{\mathbf{m}} V_{\mathbf{m}}^{(k)}(\varphi^{(\leqq k)}; N)_r \boldsymbol{\lambda}^{\mathbf{m}}, \tag{2.15}$$

is such that $V_{\mathbf{m}}^{(k)}(\varphi^{(\leqq k)}; N)_r$ is convergent as $N \to \infty$ for all $\varphi^{(\leqq k)}$ such that $\|\varphi^{(\leqq k)}\|_3 < \infty$ (see (2.4)), i.e. with proability 1.

Clearly, if there is one choice of $\ell_{\mathbf{m}}(N)$ for which this happens, any formal power series

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}' + \sum_{|\mathbf{m}| \geqq 2} L_{\mathbf{m}} \boldsymbol{\lambda}'^{\mathbf{m}} \tag{2.16}$$

permits, by substitution into (2.14), (2.15) and rearrangement, to define a new choice of $\ell_{\mathbf{m}}(N)$. But, of course, there is no reason that even one single choice of $\ell_{\mathbf{m}}(N)$ exists.

Theories for which "counterterms" $\ell_{\mathbf{m}}(N)$ can be found are called renormalizable while the others are called non-renormalizable.

The above models are renormalizable if $d=2$, for all t, if $d=3$ for $t=4$, 5 (": φ^4:$_3$; : φ^6:$_3$"), if $d=4$ for $t=4$ (": φ^4:$_4$"), only.

3. Expansion in Powers of the Form Factors

We now introduce scale dependent form factors (or running coupling constants) which describe the component of the effective potentials $V^{(k)}$ along the interaction space. Let $\mathscr{I}_k$ be the space of the interactions with cut-off k (see (2.5).. Assume that there is a family $\mathscr{L}_k$, $k = 0, 1, \ldots$ of projections on $\mathscr{I}_k$,

$$\mathscr{L}_k : M(P(d\varphi^{(\leqq k)})) \to \mathscr{I}_k, \tag{3.1}$$

such that

$$\mathscr{L}_k \mathscr{E}_{k+1} \ldots \mathscr{E}_p = \mathscr{E}_{k+1} \ldots \mathscr{E}_p \mathscr{L}_p, \quad \forall\, 0 \leqq k \leqq p. \tag{3.2}$$

If $F_0(p)>0$, see (2.3), as we shall assume for simplicity, a sequence of $\mathscr{L}_k$ can be constructed as follows: $\mathscr{L}_0$ is defined as a projection P on $\mathscr{I}_0$,

$$PV=\sum_{\alpha=0}^{t-1}\delta_0^{(\alpha)}(V)I_0^{(\alpha)}, \qquad V\in L_2(P(d\varphi^{\leqq 0})), \tag{3.3}$$

where $\delta^{(\alpha)}$ are suitable linear functionals. Then define, for $V\in L_2(P(d\varphi^{(\leqq k)}))$,

$$\mathscr{L}_k V=\sum_{\alpha=0}^{t-1}\delta_0^{(\alpha)}(\mathscr{E}_1\mathscr{E}_2\dots\mathscr{E}_k V)I_k^{(\alpha)}, \tag{3.4}$$

and $\mathscr{L}_k$, $k=0,1,\dots$, verify (3.1), (3.2). A simple choice of $\mathscr{L}_k$ is given by

$$\begin{aligned}
&\mathscr{L}_k : \varphi_{x_1}^{(\leqq k)}\dots\varphi_{x_q}^{(\leqq k)}: =0, \quad \text{if} \quad q>2(t-2),\\
&\mathscr{L}_k : \varphi_{x_1}^{(\leqq k)}\dots\varphi_{x_q}^{(\leqq k)}: =\frac{1}{|\Lambda|}\int_\Lambda :\varphi_x^{(\leqq k)q}: dx, \quad \text{if} \quad q=0,4,6,\dots,2(t-2),\\
&\mathscr{L}_k : \varphi_{x_1}^{(\leqq k)}\varphi_{x_2}^{(\leqq k)}: =\frac{1}{|\Lambda|}\int_\Lambda :\varphi_x^{(\leqq k)2}: dx-\frac{(x_1-x_2)_\pi^2}{2d|\Lambda|}\int_\Lambda :(\partial\varphi_x^{(\leqq k)})^2: dx,
\end{aligned} \tag{3.5}$$

where if $L=$ side size of the box Λ:

$$(x_1-x_2)_\pi^2=\sum_{i=1}^{d}\frac{1-\cos\frac{2\pi}{L}(x_i-y_i)}{2\left(\frac{2\pi}{L}\right)^2}\xrightarrow[L\to\infty]{}(x_1-x_2)^2. \tag{3.6}$$

See Appendix A for an interpretation of (3.5) as orthogonal projection on "$\mathscr{I}_{-1}$".

The effective potential on scale k can be decomposed into its component along $\mathscr{I}_k$ and the component transversal to $\mathscr{I}_k$. This gives a new recursion relation between the $V^{(k)}$'s:

$$\begin{aligned}
V^{(k)}=\mathscr{L}_k V^{(k)}+\mathscr{R}_k V^{(k)}&=\sum_{\alpha=0}^{t-1} r^{(\alpha)}(k)I_k^{(\alpha)}\\
&+\sum_{s=1}^{\infty}\frac{1}{s!}\mathscr{R}_k\mathscr{E}_{k+1}^T(V^{(k+1)},\dots,V^{(k+1)}),
\end{aligned} \tag{3.7}$$

where $\mathscr{R}_k=\mathbb{1}-\mathscr{L}_k$. The coefficients $r^{(\alpha)}(k)$, defined by (3.7) are called (dimensional) *form factors*. This recursion relation generates again a tree expansion, with the difference that each branching point of the trees bears an $\mathscr{R}_k$ operator, and the bare coupling constants are replaced by form factors

$$V^{(k)}=\sum_{\theta}\sum_{\mathbf{h}:h_r=k} V_R(\theta,\mathbf{h})/n(\theta), \tag{3.8}$$

with the recursive definition (to compare with (2.9), (2.10)),

$$\begin{aligned}
&V_R(\theta,\mathbf{h})=\mathscr{E}_{h_r+1}\dots\mathscr{E}_{h_{v_0}-1}\mathscr{R}_{h_{v_0}-1}\mathscr{E}_{h_{v_0}}^T\,(V_R(\theta_1,\mathbf{h}_1),\dots,V_R(\theta_s,\mathbf{h}_s)),\\
&V_R(\theta_0,h_r)=\sum_{\alpha=0}^{t-1} r^{(\alpha)}(h_r)I_{h_r}^{(\alpha)},
\end{aligned} \tag{3.9}$$

where θ_0 is the trivial tree. Equations (3.8), (3.9) provide an expansion of the effective potentials in powers of the form factors. We next derive a recursion relation between the form factors themselves. We write the recursive definition of the effective potentials as

$$V^{(k)}=\sum_{s=1}^{\infty}\frac{1}{s!}\mathscr{E}^T_{k+1}(V^{(k+1)},\ldots,V^{(k+1)})=\sum_{\alpha=0}^{t-1}r^{(\alpha)}(N)I^{(k)}_\alpha$$

$$+\sum_{h=k+1}^{N}\sum_{s=2}^{\infty}\frac{1}{s!}\mathscr{E}_{k+1}\ldots\mathscr{E}_{h-1}\mathscr{E}^T_h(V^{(h)},\ldots,V^{(h)})\,. \tag{3.10}$$

(Notice that $r^{(\alpha)}(N)=\lambda^{(\alpha)}_N$.) Acting on this equation with $\mathscr{L}_k$, and using (3.7) we get the relation

$$r^{(\alpha)}(k)=r^{(\alpha)}(N)+\sum_{\theta\,\text{non-trivial}}\sum_{\substack{\mathbf{h},h_r=k\\ h_v\leqq N}}\frac{r^{(\alpha)}(\theta,\mathbf{h})}{n(\theta)}, \tag{3.11}$$

where $r^{(\alpha)}(\theta,\mathbf{h})$ is given by

$$\sum_{\alpha=0}^{t-1}r^{(\alpha)}(\theta,\mathbf{h})I^{(\alpha)}_{h_r}=\mathscr{L}_{h_r}\mathscr{E}_{h_r+1}\ldots\mathscr{E}_{h_{v_0}-1}\mathscr{E}^T_{h_{v_0}}(V_R(\theta_1,\mathbf{h}_1),\ldots,V_R(\theta_s,\mathbf{h}_s))\,. \tag{3.12}$$

Subtracting (3.11) from the corresponding equation for $r^{(\alpha)}(k+1)$ gives

$$r^{(\alpha)}(k+1)-r^{(\alpha)}(k)=-\sum_{\theta\,\text{non-trivial}}\sum_{\substack{\mathbf{h},h_r=k\\ h_{v_0}=k+1\\ h_v\leqq N}}\frac{r^{(\alpha)}(\theta,\mathbf{h})}{n(\theta)}\,. \tag{3.13}$$

This relation is, via (3.9), a recursion between form factors on different scales. It is the analogue of the "flow equations" of the renormalization group.

4. Dimensionless Form Factors and Their "Natural Bounds"

In Sect. 3 we have reorganized perturbation theory in such a way that the effective potentials are expressed as power series in the "form factors," which obey a recursion relation (3.13).

Renormalizability of a theory implies that the effective potentials, and in particular the form factors, on some fixed scale k_0, can be expressed as power series in the renormalized coupling constants with finite coefficients. This means that, up to a finite renormalization (2.16), we can choose the renormalized coupling constant to be the form factors on some scale k_0, e.g. $k_0=0$, if the theory is renormalizable.

A way to proceed to "renormalize a non-renormalizable field theory" (as well as a renormalizable one) is simply to imagine that the form factors are the primary objects and that the power series for the Schwinger functions and the effective potentials have good convergence properties, at least at fixed order in the form factors. One can think that the non-renormalizability of a theory manifests itself only via the fact that the power series of the form factors in terms of the renormalized coupling constants has divergent coefficients. This is in fact very appealing: it is, indeed, clear that while the renormalized coupling constants have no direct physical meaning and can be thought of only as a device to classify divergences and organize their removal, the form factors are objects with direct physical meaning being in some sense the effective coupling constants on the various scales.

To insist on this viewpoint one has to check that the form factors should admit bounds good enough to ensure that $V^{(k)}$ can be written as a sum of terms of various orders in the form factors which are well-defined even if one lets $N \to \infty$.

Although this can only be answered in a satisfactory way after the theory of the model is completed, it is remarkable that it admits a "perturbative treatment."

To discuss this question we introduce the notion of "dimensionless form factors": the size of an interaction

$$I_N^{(\alpha)}(\varphi) = \int_\Delta : \varphi_x^{(\leqq N)2\alpha} : dx \tag{4.1}$$

is $\gamma^{\sigma(\alpha)N}$ with $\sigma(\alpha) = -d + 2\alpha \frac{d-2}{2}$. It measures the size of $\int_\Delta : \varphi_x^{(\leqq N)2\alpha} : dx$, when Δ is a box of size γ^{-N} and, therefore, measures the size of the interaction as it appears when we wish to integrate over the highest frequency component $\varphi^{(N)}$ of $\varphi^{(\leqq N)}$. This size is proportional to $\gamma^{\sigma(\alpha)N}$. Similarly the dimension of $\int : (\partial \varphi_x^{(\leqq N)})^2 : dx$ is $\sigma(t-1) = 0$, in the same sense. The $-d$ comes from $|\Delta| = \gamma^{-dN}$ and $\frac{d-2}{2}$ comes from $\varphi_x^{(\leqq N)} = O\left(\gamma^{\frac{d-2}{2}N}\right)$.

Therefore it is natural to think that the form factors $r^{(\alpha)}$ are written as

$$r^{(\alpha)}(k) = \gamma^{-\sigma(\alpha)k} \lambda^{(\alpha)}(k), \tag{4.2}$$

where the $(\lambda^{(\alpha)}(k))_{\substack{k=0 \\ \alpha=0,\ldots,t-1}}^{\infty}$ will be called the "dimensionless form factors".

If $\lambda^{(\alpha)}(k) \approx O(1)$ for all k, then the projection of $V^{(k)}$ on $\mathscr{I}_k$ gives an energy of $O(1)$ in each box of size γ^{-k}. We shall say that a sequence of form factors $\underline{\mathbf{r}}$ is "regular" if $|\lambda^{(\alpha)}(k)|$ is polynomially bounded as $k \to \infty$.

This means that the dimensional form factors corresponding to "relevant interactions", $\sigma(\alpha) < 0$, can grow exponentially fast with the frequency k still keeping finite energy per box of size γ^{-k}, while the dimensional form factors corresponding to "irrelevant interactions," $\sigma(\alpha) > 0$, have to vanish exponentially fast as $k \to \infty$ to keep finite the amount of energy that they contribute per box of side γ^{-k}.

We collect in the form of three theorems the results of our analytical work.

Theorem 1. *Assume that $r^{(\alpha)}(k)$ is a regular solution to* (3.13) *such that* $\sup_{\alpha,k} |\lambda^{(\alpha)}(k)| = \|\underline{\boldsymbol{\lambda}}\| < \infty$. *Then $V^{(k)}$ is a formal power series in the form factors and its n^{th} order term is a sum of terms like*

$$\int_{\Delta_1 \times \ldots \times \Delta_n} V^{(k)}(x_1, \ldots, x_n; \mathbf{P}) \mathbf{P}(x_1, \ldots, x_n) dx_1, \ldots, dx_n, \tag{4.3}$$

where $\mathbf{P}$ is a Wick monomial, of degree $\mathbf{p}$, in the fields $\varphi_x^{(\leqq k)}, \partial \varphi_x^{(\leqq k)}$. Then

$$\int_{\Delta_1 \times \ldots \times \Delta_n} |V^{(k)}(x_1, \ldots, x_n; \mathbf{P})| \, |\mathbf{P}| dx_1 \ldots dx_n \leqq \|\underline{\boldsymbol{\lambda}}\|^n C^{n-1} \mathscr{N}(\mathbf{p})$$
$$\|\varphi^{(\leqq k)}\|_3^{\mathbf{p}} \frac{(n(t-2))!}{n!} e^{-\kappa \gamma^k d(\Delta_1, \ldots, \Delta_n)}, \tag{4.4}$$

where $\Delta_1, \ldots, \Delta_n$ are n boxes of side γ^{-k}, κ and C are suitable positive constants, $\mathscr{N}(\mathbf{p})$ is a suitable function of the degree $\mathbf{p}$ of $\mathbf{P}$.

More generally one can get bounds of the same type if

$$\|\underline{\lambda}\|_j = \sup_{K,\alpha} (1+k)^{-j} |\lambda^{(\alpha)}(k)| < \infty \,, \tag{4.5}$$

but C^{n-1} is replaced by $(Ck^j)^{n-1}$, essentially.

This is a generalization of the $n!$-bounds in φ_4^4 of de Calan-Rivasseau, 1982, along the lines of Gallavotti-Nicolò, 1984; Gallavotti, 1984.

More interesting is the following theorem which puts bounds on the coefficients of (3.13).

First we rewrite (3.13) as an equation for the dimensionless form factors, $0 \leqq k \leqq N$:

$$\lambda^{(\alpha)}(k) = \gamma^{-\sigma(\alpha)} \lambda^{(\alpha)}(k+1) + \sum_{\theta \text{ non-trivial}} \sum_{\substack{\mathbf{h}, h_{v_0} = k+1 \\ h_r = k \\ h_v \leqq N}} \beta^{(\alpha)}(\theta, \mathbf{h}, \boldsymbol{\alpha}) \prod_i \lambda^{(\alpha_i)}(h_i) \,, \tag{4.6}$$

where the last product is over the endpoints of θ and h_i is the frequency attributed to the first inner vertex of θ to which the endpoint i is attached; v_0 is the first non-trivial vertex of θ.

Note that the β are N-independent (no N appears in the defining relation (3.13)). Then:

Theorem 2. *The coefficients* β *in* (4.6) *verify*

$$|\beta^{(\alpha)}(\theta, \mathbf{h}, \boldsymbol{\alpha})| \leqq \left[\frac{(n(t-2))!}{n!} \right] C^n \prod_{v \text{ inner}} \gamma^{-\bar{\varrho}(h_v - h_{v'})} \tag{4.7}$$

for some $\bar{\varrho} > 0$, $C > 0$ *and* $\forall N$, *if* n *is the number of endpoints of* θ.

The proof also gives that the coefficient in square brackets can be omitted if one considers the planar theory.

We call this series in (4.6) the "beta functional" because of its resemblance to the beta function of field theory: note that

$$(B_N \underline{\lambda})^{(\alpha)}(k) = \sum_{\theta \text{ non-trivial}} \sum_{\substack{\mathbf{h}: h_r = k \\ h_{v_0} = k+1 \\ h_v \leqq N}} \beta^{(\alpha)}(\theta, \mathbf{h}, \boldsymbol{\alpha}) \prod_i \lambda^{(\alpha_i)}(h_i) \tag{4.8}$$

is not a function of few variables (as the usual beta function is): in (4.8) $(B_N \underline{\lambda})^{(\alpha)}(N+1) \equiv 0$.

Since the β's are N-independent it is clear that the "beta functional" without cut-off,

$$(B \underline{\lambda})^{(\alpha)}(k) = \sum_{\theta \text{ non-trivial}} \sum_{\substack{\mathbf{h}: h_r = k \\ h_{v_0} = k+1}} \beta^{(\alpha)}(\theta, \mathbf{h}, \boldsymbol{\alpha}) \prod_i \lambda^{(\alpha_i)}(h_i) \,, \tag{4.9}$$

has coefficients verifying the same bounds (4.7).

The above theorem extends the analogous results for φ_4^4 of Gallavotti-Nicolò, 1984; Gallavotti, 1984.

It is tempting to try to define a perturbation theory for $P(\varphi)_d$ looking for a solution to

$$\lambda^{(\alpha)}(k) = \gamma^{-\sigma(\alpha)} \lambda^{(\alpha)}(k+1) + (B \underline{\lambda})^{(\alpha)}(k) \,, \tag{4.10}$$

in which $\lambda^{(\alpha)}(k)$ depend in a C^∞-way on $\mu^{(\alpha)} \equiv \lambda^{(\alpha)}(0)$.

One can find a formal solution to (4.10) as a power series in $\mu^{(\alpha)}$ if one writes (4.10) as

$$\begin{aligned}\lambda^{(\alpha)}(k+1) &= \gamma^{\sigma(\alpha)}(B\underline{\lambda})^{(\alpha)}(k)+\gamma^{2\sigma(\alpha)}(B\underline{\lambda})^{(\alpha)}(k-1)\\ &\quad +\ldots+\gamma^{\sigma(\alpha)(k+1)}(B\underline{\lambda})^{(\alpha)}(0)+\gamma^{\sigma(\alpha)(k+1)}\mu^{(\alpha)}\\ &\equiv(\mathscr{B}\underline{\lambda})^{(\alpha)}(k+1)+\gamma^{\sigma(\alpha)(k+1)}\mu^{(\alpha)}, \qquad k=0,1,2,\end{aligned} \tag{4.11}$$

and then setting

$$\begin{aligned}\lambda_0^{(\alpha)}(k+1) &= \gamma^{\sigma(\alpha)(k+1)}\mu^{(\alpha)}, \qquad k=0,1,2,\ldots,\\ \lambda_n^{(\alpha)}(k+1) &= \gamma^{\sigma(\alpha)(k+1)}\mu^{(\alpha)}+(\mathscr{B}\underline{\lambda}_{n-1})^{(\alpha)}(k+1).\end{aligned} \tag{4.12}$$

This recursive solution gives a power series finite to all order for renormalizable theories, i.e. if all $\sigma(\alpha)\leqq 0$. However, if for some α one has $\sigma(\alpha)>0$, as the dangerous factor $\gamma^{\sigma(\alpha)(k+1)}$ suggests, one expects divergences to appear. This is in fact the case for non-renormalizable theories. One is therefore lead to the idea of solving (4.10) with cut-off N finite, as a power series in $(\lambda^{(\alpha)}(0))_{\sigma(\alpha)\leqq 0}$, and $(\lambda^{(\alpha)}(N))_{\sigma(\alpha)>0}$. Instead of (4.11), we write

$$\begin{aligned}\lambda^{(\alpha)}(k) &= \gamma^{\sigma(\alpha)k}\mu^{(\alpha)}+(\mathscr{B}'_N\underline{\lambda})^{(\alpha)}(k), &&\text{for}\quad \sigma(\alpha)\leqq 0,\\ \lambda^{(\alpha)}(k) &= \gamma^{-\sigma(\alpha)(N-k)}\nu^{(\alpha)}+(\mathscr{B}'_N\underline{\lambda})^{(\alpha)}(k), &&\text{for}\quad \sigma(\alpha)>0,\end{aligned} \tag{4.13}$$

where $\mu^{(\alpha)}\equiv\lambda^{(\alpha)}(0)$, $\nu^{(\alpha)}\equiv\lambda^{(\alpha)}(N)$. This equation can be solved recursively as in (4.12).

The following theorem holds:

Theorem 3. (i) *If the theory is renormalizable, the recursive solution* (4.12) *of* (4.10) *yields*

$$\lambda^{(\alpha)}(k)=\gamma^{\sigma(\alpha)k}\mu^{(\alpha)}+\sum_{|\mathbf{m}|\geqq 2}\ell^{(\alpha)}(\mathbf{m};k)\boldsymbol{\mu}^{\mathbf{m}} \tag{4.14}$$

for some $\ell(\mathbf{m},k)<\infty$, *and "slowly" growing in* k:

$$|\ell(\mathbf{m},k)|\leqq C^{|\mathbf{m}|-1}(|\mathbf{m}|(t-3)-1)!\sum_{j=0}^{|\mathbf{m}|(t-3)-1}\frac{(\beta k)^j}{j!}. \tag{4.15}$$

(ii) *If the theory is not renormalizable, then the solution of the analogue of* (4.10) *with* B_N *replacing* B *obtained with the algorithm analogue to* (4.12) *yields* $\lambda^{(\alpha)}(k;N)$ *as a power series of the form* (4.14) *with coefficients* $\ell_N^{(\alpha)}(\mathbf{m};k)$, *which diverge as* $N\to\infty$, *in general.*

If $\sigma(\alpha)>0$, *then most of the coefficients* $\ell_N^{(\alpha)}(\mathbf{m};k)$ *with* $m_\alpha\neq 0$ *diverge.*

(iii) *The recursive solution of* (4.13) *is*

$$\begin{aligned}\lambda^{(\alpha)}(k) &= \gamma^{\sigma(\alpha)k}\mu^{(\alpha)}+\sum_{|\mathbf{m}|+|\mathbf{n}|\geqq 2}\ell_N^{(\alpha)}(\mathbf{m},\mathbf{n},k)\boldsymbol{\mu}^{\mathbf{m}}\mathbf{v}^{\mathbf{n}},\\ \lambda^{(\alpha)}(k) &= \gamma^{-\sigma(\alpha)(N-k)}\nu^{(\alpha)}+\sum_{|\mathbf{m}|+|\mathbf{n}|\geqq 2}\ell_N^{(\alpha)}(\mathbf{m},\mathbf{n},k)\boldsymbol{\mu}^{\mathbf{m}}\mathbf{v}^{\mathbf{n}},\end{aligned} \tag{4.16}$$

for some $\ell_N^{(\alpha)}(\mathbf{m},\mathbf{n},k)<\infty$, *with finite limit as* $N\to\infty$. *However,*

$$\lim_{N\to\infty}\ell_N^{(\alpha)}(\mathbf{m},\mathbf{n},k)=\begin{cases}0 & \text{if}\quad \mathbf{n}\neq 0\\ \ell^{(\alpha)}(\mathbf{m};k) & \text{if}\quad \sigma(\alpha)\leqq 0,\end{cases}$$

where $\ell^{(\alpha)}(\mathbf{m};k)$ *are the coefficients appearing in* (4.14).

The first part of the theorem is essentially contained in Gallavotti-Nicolò, 1984; Gallavotti, 1984. The second part is nothing but the rephrasing of the non-renormalizability property. The third part is a "perturbative triviality" result. It expresses the fact that if one includes non-renormalizable interactions and expands in powers of the bare dimensionless coupling constants for these interactions, one is back to the theory *without* these interactions in the limit $N \to \infty$. Nevertheless one can think of compensating the vanishing of these coefficients by the divergence of the full series. This possibility is expected to occur if there is a non-gaussian ultraviolet stable fixed point, and one can demonstrate this in the planar approximation. See Felder (to appear).

5. Proof of the Theorems

We follow here rather closely the ideas in Gallavotti-Nicolò, 1984; Gallavotti, 1984; however, technically the proof below contains a few new developments. In particular, the construction of the subtractions in terms of projection operators was only limited in Gallavotti, 1984b, while in the above papers the subtractions were discussed in a more empirical way.

The first task is to develop an algorithm to represent the truncated expectations appearing in (3.9).

We first develop the algorithm to represent the truncated expectations like (3.9), but modified by replacing $1-\mathscr{L}_{h_{v_0}-1}$ by 1 in the first of (3.9).

As the reader expects, the algorithm is based on Feynman graphs. Withouth entering into details we simply describe the result which can be easily proved by induction, using the Wick theorem, see Gallavotti, 1984a, for a suitable formulation of Wick theorem (Appendix C).

1) Draw n points in $\mathbb{R}^d$, $x_1, \ldots, x_n$, and to each of them associate an index $\alpha = 0, 1, \ldots, t-1$.

2) If $\alpha_i \leqq t-2$, draw $2\alpha_i$ distinct lines (called "half lines") emerging from the vertex x_i. If $\alpha = t-1$, draw two lines marked by a symbol ∂. These "half lines" are meant to be a symbolic representation of fields φ_{x_i} or $\partial\varphi_{x_i}$ (if they bear a ∂ symbol).

3) Connect some pairs of the above half lines to obtain a connected graph of "complete lines." The graph should be connected in such a way that if $v \in \theta$ is a vertex of the tree θ, being considered in (3.9), and if $a_1, \ldots, a_{n_v}$ are the endpoints that can be reached by climbing the tree starting from v, then the set of complete lines with endpoints in $x_{a_1}, \ldots, x_{a_v}$ ("cluster of vertices corresponding to $v \in \theta$") is connected.

In this way one obtains a graph G of some half lines and some complete lines and a family G_v, $v \in \theta$, of subgraphs of G with the property that G_v is connected by complete lines. The complete lines which connect vertices of the cluster corresponding to v ("cluster v") are said to be "internal to the cluster v."

4) Mark each complete line with an index h or s ("hard" or "soft") so that in each subgraph G_v, $v \in \theta$ the hard lines still connect the vertices of G_v, i.e. the points in the cluster v.

5) Suppose that G contains some half lines left as external lines, and suppose that they emerge from the vertices $x_{i_1}, \ldots, x_{i_q}, y_{j_1}, \ldots, y_{i_p}$: from the first q of them emerge lines not marked ∂ but from the last p emerge lines marked ∂. Then one

defines

$$\mathbf{P}_G \equiv :\prod_{r=1}^{q} \varphi_{x_{i_r}}^{(\le k)} \prod_{s=1}^{p} \partial\varphi_{y_{j_s}}^{(\le k)}:, \tag{5.1}$$

where k is the root frequency of θ in the frequency assignment $\mathbf{h}$.

Assuming that the degree n trees θ give to $V_R(\theta, \mathbf{h})$, defined as in (3.9) but with $1-\mathscr{L}$ replaced by 1, the contribution:

$$\sum_G \int V(\theta, \mathbf{h}, G)\mathbf{P}_G dx_1 \ldots dx_n, \tag{5.2}$$

where G is a decorated Feynman graph constructed in the above five steps ("F-graph compatible with θ"), then it is easy to prove inductively (using the above-mentioned Wick theorem) that

$$V(\theta, \mathbf{h}, G) = \prod_{\lambda\,\text{soft}} C_\lambda^{(<h_\lambda)} \prod_{\lambda\,\text{hard}} C_\lambda^{(h_\lambda)} \prod_{\text{endpoints}} r^{(\alpha_i)}(h_i), \tag{5.3}$$

where the products over λ run over the complete lines λ of soft or hard type. The meaning of $C_\lambda^{(<h_\lambda)}$ or $C_\lambda^{(h_\lambda)}$ is the following: if λ is a complete line it is an inner line for some subgraph G_v for some v and all its predecessors in θ but not for its followers; if h_v is the frequency of v in $(\theta, \mathbf{h})$ we say that λ has frequency $h_\lambda = h_v$. Then $C_\lambda^{(<h_\lambda)}$ or $C_\lambda^{(h_\lambda)}$ represent the covariances of the pair of fields symbolized by the two half lines forming λ, considered respectively as $\varphi^{(h_\lambda)}$, $\partial\varphi^{(h_\lambda)}$ or as $\varphi^{(<h_\lambda)}$, $\partial\varphi^{(<h_\lambda)}$ (as appropriate).

In general, one can associate a frequency to each half line: it is the "frequency h_v of the smallest graph G_v" which contains the vertex of the half line. A complete line has the frequency of the smallest frequency half line composing it.

Having understood how to represent graphically (3.9) with the simplification that $1-\mathscr{L}$ is replaced by 1, we can discuss (3.9) itself.

For this purpose it is useful to introduce auxiliary fields different from φ, $\partial\varphi$:

$$\begin{gathered} D_{xy} = \varphi_x - \varphi_y, \qquad S_{xy} = \varphi_x - \varphi_y - (x-y)_\pi \cdot \partial\varphi_y, \\ T_{xy} = \varphi_x - \varphi_y - (x-y)_\pi \cdot \partial\varphi_\eta - \tfrac{1}{2}(x-y)_\pi^2 \cdot \partial^2\varphi_y, \qquad D_{xy}^1 = \partial\varphi_x - \partial\varphi_y, \\ S_{xy}^1 = \partial\varphi_x - \partial\varphi_y - (x-y)_\pi \cdot \partial\partial\varphi_y, \qquad \mathscr{D}_{xyz}^1 = (x-y)_\pi \cdot D_{yz}^1, \end{gathered} \tag{5.4}$$

where $(x-y)_\pi$ is a "periodized version" of $(x-y)$, e.g.

$$\begin{aligned} ((x-y)_\pi)_i &\equiv \tfrac{1}{2}\frac{\partial}{\partial x_i}(x-y)_\pi^2 \equiv \tfrac{1}{2}\frac{\partial}{\partial x_i} 2\sum_{j=1}^{d} \frac{1-\cos\frac{2\pi}{L}(x_i - y_i)}{\left(\frac{2\pi}{L}\right)^2} \\ &\equiv \frac{L}{2\pi}\sin\frac{2\pi}{L}(x_i - y_i). \end{aligned} \tag{5.5}$$

The introduction of the above fields allows us to describe inductively the structure of V^R in terms of the form factors $\underline{\mathbf{r}}$.

The fields (5.4) have to be thought of as carrying a cut-off label (h) or $(\le h)$ as usual.

Assume that the $V^R(\theta, \mathbf{h})$ from a tree of degree n and frequency assignments $\mathbf{h}$ is described in terms of graphs G via a formula like (5.2) but using graphs G constructed via steps 1)–5) above, modifying 5) as follows:

5′) Add to the graph G extra labels β_v labeling each of the subgraphs G_v, $v \in \theta$. The labels β_v can take values $\beta = 0, 1, \ldots, \beta_0$ for some fixed β_0 (see below).

According to the value of β attached to G_v, the half lines emerging from G_v change meaning and can mean any of the fields (5.4) as well as φ, $\partial\varphi$, with position indices x, y belonging to the set of vertices of G_v. The precise rule to perform this change of meaning of the lines is specified below.

Starting from the innermost $v \in \theta$, one changes the meaning to the lines as prescribed by the β indices: a given line which is external to many G_v's can therefore change meaning several times.

One the above process of reinterpretation of the half lines is done we take G to represent a monomial P_G in the fields (5.4) given by the product of the fields associated with the external lines.

Assuming that V^R in (3.9) is given by (5.2) with the sum over G running on the graphs with the extra decorations described in 5′) above and a suitable $V^R(\theta, \mathbf{h}, G)$ replacing $V(\theta, \mathbf{h}, G)$, it is easy to prove by induction on n = degree of θ that

$$V^R(\theta, \mathbf{h}, G) = \prod_{\lambda\,\text{soft}} C_\lambda^{(<h_\lambda)} \prod_{\lambda\,\text{hard}} C_\lambda^{(h_\lambda)} \prod_{\text{endpoints}} r^{(\alpha_i)}(h_i), \tag{5.7}$$

where $C_\lambda^{(\cdot)}$ is the covariance of the two fields represented by the two half lines forming the complete line λ.

The induction argument is as follows.

Assume that the contribution $V^R(\theta, \mathbf{h})$ to the effective potential is given by

$$\sum_G \int V^R(\theta, \mathbf{h}, G) P_G dx_1, \ldots, dx_n \tag{5.8}$$

for all trees θ with degree $n \leqq n_0$ for some suitable rule to interpret the β_v-indices appended to each subgraph of G: we do not specify here this rule because it is going to be determined inductively as well.

Let θ be a non-trivial tree of degree $n_0 + 1$ which bifurcates at its first non-trivial vertex v_0 into s trees $\theta_1, \ldots, \theta_s$ at frequency $h_{v_0} = h$.

To proceed we must understand the action of the operator $\mathscr{L}_h$ on monomials in the fields φ $\partial\varphi$ and (5.4). $\mathscr{L}_h$ being defined by (3.5), it is clear that $\mathscr{L}_h \mathbf{P}$ vanishes unless $\mathbf{P}$ has degree $\leqq 2(t-2)$. If $\mathbf{P}$ has degree $p \leqq 2(t-2)$, the action of $\mathscr{L}$ vanishes if $\mathbf{p} \neq 2$ and $\mathbf{P}$ contains any of the fields (5.4) or any $\partial\varphi$ field (see the second of (3.5)). If $\mathbf{P} = :\varphi_{x_1}, \ldots, \varphi_{x_\mathbf{p}}:$, $2 \neq \mathbf{p} \leqq 2(t-2)$, the action of $\mathscr{L}_h$ is clear from the second of (3.5). Therefore the cases to be understood are those with $\mathbf{p} = 2$.

To understand, from the third of (3.5), the action of $\mathscr{L}_h$ on the second order monomials $\mathbf{P}$ in φ, $\partial\varphi$ and (5.4), it is convenient to write $\mathbf{P}$ as a sum of a "local term" involving only fields computed in the same point and of non-local terms which vanish to some order $\geqq \sigma(\mathbf{P})$ when the points appearing as indices in the fields of $\mathbf{P}$ coincide.

Denote x_1, x_2, ... simply by 1, 2, ... and $(x_2 - x_1)_\pi = \delta_{21}$, and let

$$\sigma(\mathbf{P}) = d - n\frac{d-2}{2} - n'\frac{d}{2} - v, \tag{5.9}$$

where n or n' are the numbers of fields of type zero in $\mathbf{P}$, i.e. of fields φ, D, S, T, of type 1 in $\mathbf{P}$, i.e. of fields $\partial\varphi$, D^1, S^1, $\mathscr{D}^1$, and v is the order of zero of $\mathbf{P}$, i.e. one unit per each D, D^1 field in $\mathbf{P}$, two units per each S, S^1, $\mathscr{D}^1$ field and three units per each T field.

Then we write each $\mathbf{P}$ of second order as follows:

$$
\begin{aligned}
&:\varphi_1\varphi_2: = :\varphi_1(\varphi_1+\delta_{21}\cdot\partial\varphi_1+\tfrac{1}{2}\delta_{21}^2\times\partial^2\varphi_1): + [:\varphi_1 T_{21}:],\\
&:\varphi_1 D_{23}: = :(T_{13}+\varphi_3+\delta_{13}\cdot\partial\varphi_3+\tfrac{1}{2}\delta_{13}^2\times\partial^2\varphi_3)(T_{23}+\delta_{23}\cdot\partial\varphi_3+\tfrac{1}{2}\delta_{23}^2\times\partial^2\varphi_3):\\
&\quad = :\varphi_3\delta_{23}\cdot\partial\varphi_3: +\tfrac{1}{2}:\varphi_3\delta_{23}^2\times\partial^2\varphi_3: + :\delta_{13}\cdot\partial\varphi_3\delta_{23}\cdot\partial\varphi_3:\\
&\qquad +[:T_{13}D_{23}: + :\varphi_3 T_{23}: +\delta_{13}:\partial\varphi_3 S_{23}:],\\
&:\varphi_1 S_{23}: = :(S_{13}+\varphi_3+\delta_{13}\cdot\partial\varphi_3)(T_{23}+\tfrac{1}{2}\delta_{23}^2\times\partial^2\varphi_3):\\
&\quad =\tfrac{1}{2}:\varphi_3\delta_{23}^2\times\partial^2\varphi_3:\\
&\qquad +[:S_{13}S_{23}:+:\varphi_3 T_{23}:+\delta_{13}:\partial\varphi_3 T_{23}:],\\
&:\varphi_1 T_{23}: = [:\varphi_1 T_{23}:],\\
&:\varphi_1\partial\varphi_2: = :\varphi_1(\partial\varphi_1+\delta_{21}\cdot\partial\partial\varphi_1): +[:\varphi_1 S_{21}^1:],\\
&:\varphi_1 D_{23}^1: = :(\varphi_3+\delta_{13}\cdot\partial\varphi_3+S_{13})(S_{23}^1+\delta_{23}\cdot\partial\partial\varphi_3):\\
&\quad = :\varphi_3\delta_{23}\cdot\partial\partial\varphi_3:+[:\varphi_3 S_{23}^1:+:S_{13}D_{23}^1:-:D_{13}D_{13}^1:],\\
&:\varphi_1 S_{23}^1: = [:\varphi_1 S_{23}^1:],\\
&:D_{12}D_{34}: = :\delta_{12}\cdot\partial\varphi_2\delta_{34}\cdot\partial\varphi_4:+[:S_{12}S_{14}:-:S_{12}D_{34}:-:S_{34}D_{12}:]\\
&\quad = :\delta_{12}\cdot\partial\varphi_2\delta_{34}\cdot\partial\varphi_2:\\
&\qquad +[:\delta_{12}\cdot\partial\varphi_2\delta_{34}D_{42}^1:+:S_{12}S_{34}:-:S_{12}D_{34}:-:D_{12}S_{34}:],\\
&:D_{12}S_{34}: = [:D_{12}S_{34}:], \qquad :D_{12}T_{34}: = [:D_{12}T_{34}:],\\
&:D_{12}\partial\varphi_3: = :(S_{12}+\delta_{12}\cdot\partial\varphi_2)(\partial\varphi_2+D_{32}^1):\\
&\quad = :\delta_{12}\cdot\partial\varphi_2\partial\varphi_2:+[:S_{12}\partial\varphi_3:+:\delta_{12}\partial\varphi_2 D_{32}^1:],\\
&:D_{12}D_{34}^1: = [:D_{12}D_{34}^1:], \qquad :D_{12}S_{34}^1: = [:D_{12}S_{34}^1:],\\
&:S_{12}S_{34}: = [:S_{12}S_{34}:], \qquad :S_{12}T_{34}: = [:S_{12}T_{34}:],\\
&:S_{12}\partial\varphi_3: = [:S_{12}\partial\varphi_3:], \qquad :S_{12}D_{34}^1: = [:S_{12}D_{34}^1:],\\
&:S_{12}S_{34}^1: = [:S_{12}S_{34}^1:], \qquad :T_{12}T_{34}: = [:T_{12}T_{34}:],\\
&:T_{12}\partial\varphi_3: = [:T_{12}\partial\varphi_3:], \qquad :T_{12}D_{34}^1: = [:T_{12}D_{34}^1:],\\
&:T_{12}S_{34}^1: = [:T_{12}S_{34}^1:], \qquad :\partial\varphi_1\partial\varphi_2: = :\partial\varphi_1\partial\varphi_1:+[:\partial\varphi_1 D_{21}^1:],\\
&:\partial\varphi_1 D_{23}^1: = [:\partial\varphi_1 D_{23}^1:], \qquad :\partial\varphi_1 S_{23}^1: = [:\partial\varphi_1 S_{23}^1:],\\
&:D_{12}^1 D_{34}^1: = [:D_{12}^1 D_{34}^1:], \qquad :D_{12}^1 S_{34}^1: = [:D_{12}^1 S_{34}^1:],\\
&:S_{12}^1 S_{34}^1: = [:S_{12}^1 S_{34}^1:], \qquad :(\text{any field})\mathscr{D}_{123}^1: = [:\text{same}:].
\end{aligned}
\tag{5.10}
$$

It is now easy to evaluate $\mathscr{L}$ in a useful form using (3.5) and its consequences:

$$
\begin{aligned}
&\mathscr{L}:\varphi_1\partial\varphi_1: = 0, \qquad \mathscr{L}:\varphi_1\partial_{ij}^2\varphi_1: = -\frac{\bar{\delta}_{ij}}{d|\Lambda|}I^{(t-1)},\\
&\mathscr{L}:\partial_i\varphi_1\partial_j\varphi_1: = \frac{\bar{\delta}_{ij}}{d|\Lambda|}I^{(t-1)}, \qquad \mathscr{L}:\varphi_1\varphi_1: = \frac{1}{|\Lambda|}I^{(1)},
\end{aligned}
\tag{5.11}
$$

where $\bar{\delta}_{ij}$ denotes the Kronecker delta symbol (to avoid confusion with $\delta_{ij}=(x_i-x_j)_\pi$).

One finds, denoting $[\Omega]$ the result of the action of $\mathscr{L}$ on the square bracket terms in (5.10) (one would have $[\Omega]\equiv 0$ if $(x_i-x_j)_\pi^2$ could be "replaced" by $(x_i-x_j)^2$: see comments after (5.15)), using (5.10), (5.11):

$$\begin{aligned}
&\mathscr{L}:\varphi_1\varphi_2:=\frac{1}{|\Lambda|}I^{(2)}-\frac{\delta_{21}^2}{2}\frac{1}{d|\Lambda|}I^{(t-1)}+[\Omega],\\
&\mathscr{L}:\varphi_1 D_{23}:=\frac{1}{d|\Lambda|}(-\tfrac{1}{2}\delta_{23}^2+\delta_{13}\cdot\delta_{23})I^{(t-1)}+[\Omega],\\
&\mathscr{L}:\varphi_1 S_{23}:=-\frac{1}{2d|\Lambda|}\delta_{23}^2 I^{(t-1)}+[\Omega],\\
&\mathscr{L}:\varphi_1\partial_j\varphi_2:=-\frac{(\delta_{21})_j}{d|\Lambda|}I^{(t-1)}+[\Omega],\\
&\mathscr{L}:\varphi_1(D_{23}^1)_j:=-\frac{(\delta_{23})_j}{d|\Lambda|}I^{(t-1)}+[\Omega],\\
&\mathscr{L}:D_{12}D_{34}:=\frac{1}{d|\Lambda|}\delta_{12}\cdot\delta_{34}I^{(t-1)}+[\Omega],\\
&\mathscr{L}:D_{12}\partial_j\varphi_3:=\frac{(\delta_{12})_j}{d|\Lambda|}I^{(t-1)}+[\Omega],\\
&\mathscr{L}:\partial_i\varphi_1\partial_j\varphi_2:=-\frac{\bar{\delta}_{ij}}{d|\Lambda|}I^{(t-1)}+[\Omega],\\
&\mathscr{L}:\mathbf{P}:=[\Omega]\quad\text{for the other }\mathbf{P}\text{'s of second degree,}\\
&\mathscr{L}:\varphi_1\ldots\varphi_q:=\frac{1}{|\Lambda|}I^{(q/2)}\quad\text{for}\quad q=4,6,\ldots,2(t-2),0.
\end{aligned}\tag{5.12}$$

We can now compute the result of the action of $(1-\mathscr{L}_{h-1})$ on the monomials of interest.

Using the inductive hypothesis

$$\begin{aligned}V(\theta,\mathbf{h})=&\sum_{G_1,\ldots,G_s}\mathscr{E}_{k+1}\ldots\mathscr{E}_{h-1}(1-\mathscr{L}_{h-1})\int\mathscr{E}_h^T(\mathbf{P}_{G_1},\ldots,\mathbf{P}_{G_s})\\
&\cdot V(\theta_1,\mathbf{h}_1,G_1)\ldots V(\theta_s,\mathbf{h}_s,G_s)dx_1\ldots dx_{n_0+1},\end{aligned}\tag{5.13}$$

and using the Wick theorem

$$\begin{aligned}V(\theta,\mathbf{h})=&\sum_G\mathscr{E}_{k+1}\ldots\mathscr{E}_{h-1}(1-\mathscr{L}_{h-1})\int\mathbf{P}_G\\
&\cdot V(\theta_1,\mathbf{h}_1,G_1)\ldots V(\theta_s,\mathbf{h}_s,G_s)dx_1\ldots dx_{n_0+1},\end{aligned}\tag{5.14}$$

where G is a decorated Feynman graph with the additional β_V indices on the v's following v_0 but not on v_0, because of our inductive assumption.

The (5.12) imply, for general kernels W which are assumed translation invariant on Λ and rotation covariant for rotations of multiples of $\frac{\pi}{2}$ in any direction (the only ones which make sense on Λ), that

$$(1-\mathscr{L})1=0,$$

$$\begin{aligned}
&(1-\mathscr{L})\int W(1,2):\varphi_1\varphi_2:d1d2\\
&\quad=\int W(1,2):\varphi_1\varphi_2:d1d2\\
&\qquad-\int W(1,2)d1:\varphi_2^2:d2+\tfrac{1}{2}\int W(1,2)\frac{\delta_{21}^2}{d}:(\partial\varphi_2)^2d1d2+\omega\\
&\quad=\int W(1,2):\varphi_1T_{21}:d1d2+\omega,\\
&(1-\mathscr{L})\int W(1,2,3):\varphi_1D_{23}:d1d2d3\\
&\quad=\int W(1,2,3):\varphi_1D_{23}:d1d2d3\\
&\qquad-\int W(1,2,3)(-\tfrac{1}{2}\delta_{23}^2+\delta_{13}\cdot\delta_{23})\frac{:(\partial\varphi_3)^2:}{d}d1d2d3+\omega\\
&\quad=\int W(1,2,3)\left(:\varphi_1D_{23}:+(\tfrac{1}{2}\delta_{23}^2-\delta_{13}\delta_{23})\frac{:\partial\varphi_3^2:}{d}\right)d1d2d3+\omega\\
&\quad=\int W(1,2,3)(:\varphi_1T_{23}:+:\varphi_3T_{13}:+:S_{13}D_{23}:-:T_{13}D_{23}:)d1d2d3+\omega,\\
&(1-\mathscr{L})\int W(1,2,3):\varphi_1S_{23}:d1d2d3\\
&\quad=\int W(1,2,3)\left(:\varphi_1S_{23}:+\frac{\delta_{23}^2}{2d}:(\partial\varphi_3)^2:\right)d1d2d3+\omega\\
&\quad=\int W(1,2,3)(:\varphi_1S_{23}:+\tfrac{1}{2}:(\delta_{23}\cdot\partial\varphi_3)^2:)d1d2d3+\omega\\
&\quad=\int W(1,2,3):\varphi_1T_{23}:d1d2d3+\omega,\\
&(1-\mathscr{L})\int\sum_j W_j(1,2):\varphi_1\partial_j\varphi_2:d1d2 \qquad (5.15)\\
&\quad=\int\sum_j W_j(1,2)\left(:\varphi_1\partial_j\varphi_2:+\frac{(\delta_{21})_j}{d}:(\partial\varphi_1)^2:\right)d1d2+\omega\\
&\quad=\int\sum_j W_j(1,2)(:\varphi_1\partial_j\varphi_2:-:\varphi_1(\delta_{21}\cdot\partial)\partial_j\varphi_1:)d1d2+\omega\\
&\quad=\int\sum_j W_j(1,2):\varphi_1(S_{21}^1)_j:d1d2+\omega,\\
&(1-\mathscr{L})\int\sum_j W_j(1,2,3):\varphi_1(D_{23}^1)_j:\\
&\quad=\int\sum_j W_j(1,2,3)(:D_{13}(D_{23}^1)_j:+:\varphi_3(S_{23}^1)_j:)d1d2d3+\omega,\\
&(1-\mathscr{L})\int W(1,2,3,4):D_{12}D_{34}:d1d2d3d4\\
&\quad=\int W(1,2,3,4)(:D_{12}S_{34}:+:S_{12}D_{34}:\\
&\qquad+:D_{12}\delta_{34}D_{42}^1:-:S_{12}\delta_{34}D_{42}^1:)d1d2d3d4+\omega,\\
&(1-\mathscr{L})\int\sum_j W_j(1,2,3):D_{12}\partial_j\varphi_3:\\
&\quad=\int\sum_j W_j(1,2,3)(:D_{12}(D_{32}^1)_j:+:S_{12}\partial_j\varphi_2:)+\omega,\\
&(1-\mathscr{L})\int W(1,2,\ldots,q):\varphi_1\ldots\varphi_q:d1\ldots dq\\
&\quad=\int W(1,2,\ldots,q)\sum_{p=1}^{q-1}:\varphi_1^pD_{p1}\varphi_{p+1}\ldots\varphi_q:d1\ldots dq,\ 2\neq q\leqq 2(t-2),\\
&(1-\mathscr{L})\int W\mathbf{P}=\omega\quad\text{for all other second order }\mathbf{P}\text{'s in (5.10)},
\end{aligned}$$

where ω denotes a term proportional to $I^{(t-1)} = \int_{\Lambda} :(\partial\varphi_x)^2: dx$, of course different in each expression.

The (5.15) follows from (5.11), (5.12) simply using the symmetry properties of the W's and integration by parts: one does *not* use the approximation $(x_1-x_2)_\pi^2 \approx (x_1-x_2)^2$. If such approximation was used, disregarding the consistency problems which arise from the fact that $(x_1-x_2)^2$ is not periodic on Λ and proceeding formally, one would have found $\Omega \equiv 0$, $\omega \equiv 0$ in the above formulae.

By inspection on (5.15), (5.10) one can summarize the above algebraic calculations by saying that $(1-\mathscr{L}_{h-1})\mathscr{E}_h^T$ is represented by

$$(1-\mathscr{L}_{h-1})\int W\mathbf{P}dx_1\dots dx_n = \int WR\mathbf{P}dx_1\dots dx_n,$$

where R has the form, if $\omega(\cdot)$ are suitable functions:

$$\begin{gathered}
R1=0, \qquad R:\varphi_1\varphi_2:=:\varphi_1 T_{21}:+\omega^{(1)}(12):(\partial\varphi_1)^2:,\\
R:\varphi_1 D_{23}:=:\varphi_3 T_{23}:+:\varphi_3 T_{12}:+:S_{13}D_{23}:-:T_{13}D_{23}:+\omega^{(2)}(123):(\partial\varphi_1)^2:,\\
R:\varphi_1 S_{23}:=:\varphi_1 T_{23}:+\omega^{(3)}(123):(\partial\varphi_1)^2:,\\
R:\varphi_1\partial\varphi_2:=:\varphi_1 S_{21}^1:+\omega^{(4)}(12):(\partial\varphi_1)^2:,\\
R:\varphi_1 D_{23}^1:=:D_{13}D_{23}^1:+:\varphi_3 S_{23}^1:+\omega^{(5)}(123):(\partial\varphi_1)^2:,\\
R:D_{12}D_{34}:=:D_{12}S_{34}:+:S_{12}D_{34}:-:S_{12}S_{34}:-:D_{12}\mathscr{D}_{342}^1:\\
+:S_{12}\mathscr{D}_{342}^1:+\omega^{(6)}(1234):(\partial\varphi_1)^2:,\\
R:D_{12}\partial_j\varphi_3:=:D_{12}(D_{32}^1)_j:+:S_{12}\partial_j\varphi_2:+\omega^{(7)}(123):(\partial\varphi_1)^2:,\\
R:\varphi_1\dots\varphi_q:=\sum_{p=1}^{q-1}:\varphi_1^p D_{p1}\varphi_{p+1}\dots\varphi_q:, \qquad 2\neq q\leqq 2(t-2),\\
R\mathbf{P}=\omega(\mathbf{P}):(\partial\varphi_1)^2: \quad \text{for the remaining second order polynomials},\\
R\mathbf{P}=\mathbf{P} \quad \text{for all other polynomials}.
\end{gathered} \tag{5.16}$$

The ω's are suitable functions which vanish to order $>\sigma(\mathbf{P})$ when all the arguments coincide and to the order of the zeros of $\mathbf{P}$ when the appropriate subsets of the arguments in which $\mathbf{P}$ has zeros degenerate into single points.

We now label by $\beta=0,1,\dots,\beta_0$ the various terms in each of (5.16), where β_0 is large enough, $\beta_0=\max(2(t-2)-1,6)$ (the 6 is to take into account the "worse in (5.16)," $RD_{12}D_{34}$ and $2(t-2)-1=\max_q(q-1)$).

If $\beta=0$, it means that $R\mathbf{P}=\mathbf{P}$, if $\beta=1,\dots,$ it means that $R\mathbf{P}\neq\mathbf{P}$ and we choose the β^{th} term in the expression (5.16) of $R\mathbf{P}$: is β is too large and no such β^{th} term exists we take this to mean that we choose 0. Also if $\beta=0$ but $R\mathbf{P}\neq\mathbf{P}$, we take this to mean that we choose 0.

Calling G^β the graph G with the label β associated with the vertex v_0, it is clear that we can write (5.14) as

$$V(\theta,\mathbf{h}) = \sum_{G,\beta}\int V(\theta_1,\mathbf{h}_1,G_1)\dots V(\theta_s,\mathbf{h}_s,G_s)\mathbf{P}_{G^\beta}dx_1\dots dx_{n_0+1}, \tag{5.17}$$

where $\mathbf{P}_{G^\beta}$ is obtained by changing the meaning of some external lines according to the value of $\beta_{v_0}=\beta$.

This proves the inductive hypothesis for n_0+1: since it is valid, obviously, for $n_0=1$ our proof of the evaluation rule by graphs of V^R is complete.

In (5.16) there are several terms (i.e. the ones with the ω's) proportional to $:(\partial\varphi_1)^2:$ which could be eliminated, thus reducing greatly the formal calculations necessary in any real computation: this could be achieved as follows:

1) By letting $L\to\infty$ at the end of the discussion because they give a vanishing contribution in this limit (in fact, their presence is due to the fact that $(x_1-x_2)^2_\pi \neq (x_1-x_2)^2$, but $(x_1-x_2)^2_\pi \xrightarrow[L\to\infty]{} (x_1-x_2)^2$);

2) or by modifying the definitions of $\mathscr{L}$ and R giving up linearity.

Then 2) has a strong disadvantage because if one eliminates the ω's in (5.15) and the Ω's in (5.12), $\mathscr{L}$ and R are no longer linear operators but just operations on functions, and one has to be careful to avoid, possible ambiguities (e.g. $\mathscr{L}\varphi_1\varphi_2 - \mathscr{L}\varphi_1\varphi_3 \neq \mathscr{L}\varphi_1 D_{23}$): see Gallavotti-Nicolò, 1984, Gallavotti, 1984, where this choice is adopted.

The 1) has been adopted in Felder (to appear). However, the considerations below could equally well be performed if one decided to pursue one of the two alternatives above. In particular, all the bounds would be qualitatively left unchanged.

Using (5.7) it is very easy to put bounds on $V^R(\theta,\mathbf{h},G)$.

Assume that

$$r^{(\alpha)}(h)=\gamma^{\sigma(\alpha)h}\lambda^{(\alpha)}(h)\,, \qquad \sup_{\alpha,h}|\lambda^{(\alpha)}(h)|=\|\underline{\lambda}\|<\infty\,. \tag{5.18}$$

Let $n^e=$ degree of $\mathbf{P}_G$, $\sigma=-d+n_0^e\frac{d-2}{2}+n_1^e\frac{d}{2}$, where n_0^e is the number of φ, D, S, T fields in P_G, n_1^e is the number of $\partial\varphi$, D^1, S^1 fields in $\mathbf{P}_G: n^e=n_0^e+n_1^e$.

Then the bound (2.3) on the covariances immediately gives us the bound

$$\begin{aligned}\int_{\Delta_1\times\ldots\times\Delta_n} &|V^R(\theta,\mathbf{h},G)|\,|P_G| \leqq C^{n^e} n^e!\,\|\varphi^{(\leqq k)}\|_3^{n^e}\,\|\underline{\lambda}\|^n \\ &\cdot\gamma^{\sigma k}\gamma^{dk}\prod_{\substack{1/2\text{ lines }\lambda\\ \text{of type }0}} K^{1/2}\gamma^{\frac{d-2}{2}h_\lambda}\prod_{\substack{1/2\text{ lines }\lambda\\ \text{of type }1}} K^{1/2}\gamma^{\frac{d}{2}h_\lambda}\prod_{i\text{ endpoints}}\gamma^{\sigma(\alpha_i)h_i} \\ &\cdot\int_{\Delta_1\times\ldots\times\Delta_n}\exp\Bigg(-\kappa\sum_{\substack{\lambda\text{ complete}\\ \text{hard}}}\gamma^{h_\lambda}|\lambda|\Bigg)\prod_{v\text{ inner}}\prod_{\substack{\lambda\text{ external}\\ \text{in }G_v}}(\gamma^{h_w}\,d(v))^{\zeta(\lambda,v)}dx_1\ldots dx_n\,,\end{aligned} \tag{5.19}$$

with $d(v)=$ graphical diameter of the vertices of G_v,[1] and where $\zeta(\lambda,v)$ is the variation of the order of zero carried by λ (D, D^1 have order of zero 1, S, S^1 have order of zero 2, T, $\mathscr{D}^1$ have order of zero 3) due to the fact that λ may change meaning as an external line of G_v because of the R-operation; $w_\lambda<v$ is the vertex of θ such that λ becomes internal to G_{w_λ} "for the first time" (i.e. G_{w_λ} is the smallest graph G_v, $v\in 0$ such that λ is internal to G_v); $w_\lambda\equiv r$ if λ is external.

In (5.19) K, C, $\kappa>0$ are constants and $C^m m!\,\sup|\varphi_i|^m$ is a bound on $:\varphi_1\varphi_2\ldots\varphi_m:$.

To bound (5.19) one uses the inequalities

$$\sum_{\substack{\lambda\text{ complete}\\ \text{hard}}}\gamma^{h_\lambda}|\lambda| \geqq (1-\gamma^{-1})\sum_v \gamma^{h_v}d(v)\,, \tag{5.20}$$

---

1 The graphical distance of n points is the length of the shortest set of lines connecting them

which are a consequence of the connectedness of each G_v and of $\gamma^h \geqq (1-\gamma^{-1})(\gamma^h + \gamma^{h-1} + \ldots + 1)$.

The last product in (5.19) is bounded by observing

$$\prod_{v\,\text{inner}} \prod_{\substack{\lambda\,\text{external}\\ \text{in}\,G_v}} (\gamma^{h_{w_\lambda}} \gamma^{-h_v})^{\zeta(\lambda,v)} \prod_v (\gamma^{h_v} d(v))^{\sum_\lambda \zeta(\lambda,v)}$$

$$= \prod_{v\,\text{inner}} \gamma^{-(h_v - h_{v'})\theta(v)} \prod_v (\gamma^{h_v} d(v))^{\sum_\lambda \zeta(\lambda,v)}, \tag{5.21}$$

where $\theta(v) = 1 - \left(-d + \frac{d-2}{2} n^e_{0,v} + \frac{d}{2} n^e_{1,v}\right)$ if $-d + \frac{d-2}{2} n^e_{0,v} + \frac{d}{2} n^e_{1,v} \leqq 0$ and $\theta(v) = 0$ otherwise; so that since $\sum_\lambda \zeta(\lambda, v) \leqq 3$, the last product in (5.19) is bounded for any $\varepsilon > 0$ by

$$\prod_{v\,\text{inner}} \gamma^{-(h_v - h_{v'})\theta(v)} \frac{3!}{\varepsilon^3} \exp \sum_v \varepsilon \gamma^{h_v} d(v), \tag{5.22}$$

which if combined with

$$\int e^{-\tilde{\kappa} \sum_v \gamma^{h_v} d(V)} dx_1 \ldots dx_n \leqq \tilde{C}^n \gamma^{-dk} \prod_{v\,\text{inner}} \gamma^{-d(s_v - 1)h_v} \tag{5.23}$$

which is easily proved for some $\tilde{C} > 0$, if $\tilde{\kappa} > 0$ (if s_v = number of lines emerging from v in θ) and some simple counting of the number of lines gives that the integral in (5.19) is bounded by

$$C^{n^e} n^e! \, \|\varphi^{(\leqq k)}\|_3^{n^e} e^{-\frac{\kappa}{2} d(\Delta_1, \ldots, \Delta_n)\gamma^k} D^n \prod_{v\,\text{inner}} \gamma^{-(\sigma(v) + \theta(v))(h_v - h_{v'})} \tag{5.24}$$

if D is a suitable constant and $d(\Delta_1, \ldots, \Delta_n)$ is the graphical distance between the cubes $\Delta_1, \ldots, \Delta_n$; clearly for some $\bar{\varrho} > 0$:

$$\sigma(v) + \theta(v) \geqq 2\bar{\varrho} n^e_v + \bar{\varrho}. \tag{5.25}$$

The exponential factor in (5.24) is obtained trivially by extracting $\exp - \frac{\kappa}{2} \sum_\lambda \gamma^{h_\lambda} |\lambda| \leqq \exp - \frac{\kappa}{2} d(\Delta_1, \ldots, \Delta_n)$ from the exponential in (5.19) before processing it, via (5.20), (5.23).

It is now easy to bound the total contribution to the effective potential due to the trees θ with n endpoints and due to the Wick monomials of degree n^e.

The basic facts to use are

1) The bound (5.24);

2) There are C_1^n different trees θ with n endpoints;

3) A given unlabeled Feynman graph can be labeled to be compatible with θ so that the subgraph G_v has n^e_v external lines in at most

$$n(\theta) C_\varepsilon^n e^{\varepsilon \sum_{v\,\text{inner}} n^e_v} \tag{5.26}$$

ways, for all $\varepsilon > 0$;

(4) There are at most $\frac{(n(t-2))!}{n!}$ unlabeled topologically non equivalent Feynman graphs.

Clearly the above four statements imply

$$\sum_{\theta, G}^{*} \int_{\Delta_1 \times \ldots \times \Delta_n} \frac{|V(\theta, \mathbf{h}, G)|}{n(\theta)} |\mathbf{P}_G| dx_1 \ldots dx_n \leqq \frac{(n(t-2))!}{n!} e^{-\kappa d(\Delta_1, \ldots, \Delta_n)\gamma^k}$$

$$\cdot C^{n^e} n^e! \, \|\varphi^{(\leqq k)}\|_3^{n^e} E^n \prod_{v \, \text{inner}} \gamma^{-(h_v - h_{v'})(\bar{\varrho} + n^e \bar{\varrho})}, \tag{5.27}$$

having taken ε small enough compared to $\bar{\varrho}$ and having used (5.25): in (5.27) the sum is extended to all trees θ with n endpoints and to all Feynman graphs compatible with θ and with n^e external lines.

This completes the proof of Theorem 1 because the sum over n_v^e can be performed in (5.27).

To prove Theorem 2 one starts from (3.13) which shows that the β's in (4.6) are

$$\beta^{(\alpha)}(\theta, \mathbf{h}, \boldsymbol{\alpha}) = \gamma^{-\sigma(\alpha)k} \sum_{\theta \, \text{non-trivial}} \frac{1}{n(\theta)}$$

$$\cdot \delta^{(\alpha)}(\mathscr{E}_1 \ldots \mathscr{E}_{k-1} \mathscr{E}_k^T (V^R(\theta_1, \mathbf{h}_1), \ldots, V^R(\theta_s, \mathbf{h}_s))), \tag{5.28}$$

with V^R evaluated using dimensionless form factors identically 1.

Of course the estimate of (5.28) is identical to the one in (5.27): it is true that no $(1-\mathscr{L}_k)$ acts after $\mathscr{E}_k^T$, but the purpose of $(1-\mathscr{L}_k)$ was to produce a zero of order $\sigma(\alpha)$ in the external fields which was subsequently bounded by $\gamma^{-\sigma(\alpha)k}$.

In (5.28) the factor $\gamma^{-\sigma(\alpha)k}$ is already in front of the sum to bound: therefore we can use the bound (5.27) with $n^e = 2\alpha$ if $\alpha = 0, \ldots, t-2$, and $n^e = 2$ if $\alpha = t-1$, and the result is Theorem 2.

Theorem 3 i) is proved in Gallavotti-Nicolò, 1984; Gallavotti, 1984, for φ_4^4: the same proof applies to the other renormalizable cases (it is in fact a simple corollary of Theorem 2); in the case of the non-renormalizable theories, Theorem 3 ii), it is essentially a rephrasing of the non-renormalizability property and it is left to the reader to check it by taking the lowest non-trivial divergent graph which requires, to be renormalized in the usual sense, terms which are not present in $\mathscr{I}_N$ $\left(\text{e.g. } \int_\Delta :(\Delta\varphi_x)^2: dx\right)$.

We now sketch the proof of iii). The solution of (4.13) can be represented as in Gallavotti-Nicolò, 1984; Gallavotti, 1984, in terms of "framed trees." In this representation one draws a frame around a subtree at each insertion of $\lambda^{(\alpha)}(k')$, $(\sigma(\alpha) \leqq 0)$, in the right-hand side of (4.13). The expansion in powers of $v^{(\alpha)}$ is the usual perturbation expansion in terms of the bare coupling constants, up to the dimensional factor $\gamma^{-\sigma(\alpha)(N-k)}$. This means that we have an expansion in trees which is identical with the one one would have in the theory without the non-renormalizable interactions, with the exception that some of the endpoints bear an α index with $\sigma(\alpha) > 0$. For each of these endpoints we have a factor $\gamma^{-\sigma(\alpha)(N-j)}$, where j is the frequency assigned to the vertex to which the endpoint is attached. From this factor, corresponding to one endpoint, togehter with the factors $\gamma^{-\bar{\varrho}(h_v - h_{v'})}$ on the way down from the endpoint to the root, we can extract a factor $\gamma^{-\delta(N-k)}$ for each tree containing at least one non-renormalizable endpoint, where $k = h_r$, $\delta > 0$. Then we simply delete all non-renormalizable endpoints, by estimating the remaining $\gamma^{-\sigma(\alpha)(N-j)}$ factors by 1 and we are left with tree contributions of the renormalizable theory times a factor $\gamma^{-\delta(N-k)} \to 0$, $N \to \infty$.

6. Concluding Remarks

The spirit of our method has some analogies with the paper by Polchinski, 1984. In the latter paper, which deals with the renormalizable case, however the problem of deriving the concrete bounds (4.4), (4.7), (4.15) is not tackled (essentially only cut-off-uniform finiteness is proved); also there is no separation between the flow of the coupling constants $\lambda(k)$ and that of all the other irrelevant local and non-local couplings of the effective interaction. In the case of a renormalizable theory this could perhaps be considered a matter of taste; however part (i) of Theorem 3 seems to show that this is an essential change in point of view in the non-renormalizable case.

Without introducing a subtraction procedure it is possible to introduce "form factors" describing the theory of scalar fields, whether interacting via a renormalizable or via a non-renormalizable polynomial interaction in any dimension $d>2$. Perturbation theory appears when one tries to solve by a power series the relations between the form factors.

Alternatively, the form factors could be introduced on the basis of a purely perturbative approach in which one introduces subtractions only for the operators in the interaction defining the model (see (2.5) for a $(t-2)$th order interaction). Proceeding as in Gallavotti-Nicolò, 1984; or Gallavotti, 1984 (see Sect. 9) one would eventually obtain the same results of Sect. 3 in this paper: the amount of work would be essentially equivalent and we preferred the derivation of Sect. 3 because it is clearly more intrinsic and one does not have to provide arguments for the analysis of the meaning of the formal renormalized coupling constants.

If the theory is not renormalizable one obtains a nice expansion of the effective potential in terms of form factors on various scales: they turn out to be functions of the renormalized constants given by formal power series whose coefficients diverge with the cut-off essentially in every order of perturbation theory, at least if they are non-trivial coefficients.

However the form factors verify, as long as the cut-off N is fixed, to all orders in perturbation theory an equation whose coefficients stay uniformly bounded in the ultraviolet cut-off.

One can decide to define the theory in terms of the form factors and one defines the form factors as solutions to the above-mentioned equations which have the property of remaining uniformly finite in the cut-off and in their frequency index k (the form factors are functions of a discrete index k, called their frequency, and for each k there is one form factor per coupling constant in the lagrangian).

In this way one can avoid the introduction of the renormalized coupling constants, provided of course one finds a solution to the form factor equations with the required boundedness properties. One also avoids the ambiguities of the arbitrariness of the subtraction constants because the equation obeyed by the form factors in *independent* on the arbitrary subtraction constants.

Optimistically, the impossibility of writing the form factors as formal power series of a t-parameter family of constants (see Theorem 3 i), ii)) indicates just that one should not expect to find t-parameter solutions of the form factor equations. On the other hand, it may well be that such equations do not admit any non-trivial solution (see Theorem 3 iii)). The triviality suspicion looms on the solutions of the form factors equation, at least if $\|\boldsymbol{\lambda}\|_\infty$ is small. The situation is not so clear if $\lambda^{(\alpha)}(N)\sim 1$, not even in dimension 3, for instance: this is because $\lambda^{(\alpha)}(N)\sim 1$ is outside the domain where constructive field theory applies (in constructive φ_3^4 one has $\lambda^{(2)}(N)\sim\gamma^{-N}$!).

Moreover, the appearance of non-Gaussian fixed points can drastically change the situation; see remarks after Theorem 3.

There is also another phenomenon which might happen: namely, the solution of the form factor equation in the presence of the cut-off

$$\lambda^{(\alpha)}(k)=\gamma^{-\sigma(\alpha)}\lambda^{(\alpha)}(k+1)+(B_N\underline{\lambda})^{(\alpha)}(k),\quad 0\leqq k\leqq N, \tag{6.1}$$

see (4.8), may not commute with the limit as $N\to\infty$. In this case it may be that

$$\lambda^{(\alpha)}(k)=\gamma^{-\sigma(\alpha)}\lambda^{(\alpha)}(k+1)+(B\underline{\lambda})^{(\alpha)}(k),\quad 0\leqq k<\infty, \tag{6.2}$$

see (4.10), may have a bounded solution although (6.1) does not have a uniformly bounded solution as N varies.

This apparently impossible event can actually happen only if the series defining $\mathbf{B}_N$ and $\mathbf{B}$ do not converge absolutely for $|\boldsymbol{\lambda}(k)|\leqq\|\underline{\boldsymbol{\lambda}}\|<\infty$, which of course has very little chance to be the case because of the bound in Theorem 2 which depends factorially on the order.

In other words it is even very unclear what could be the rigorous meaning to attach to expressions like $\mathbf{B}(\underline{\boldsymbol{\lambda}})$ or $\mathbf{B}_N(\underline{\boldsymbol{\lambda}})$ which will probably make sense only for special choices of $\underline{\boldsymbol{\lambda}}$, and it might be that the $\underline{\boldsymbol{\lambda}}$'s for which $\mathbf{B}(\underline{\boldsymbol{\lambda}})$ makes sense are not such that $\mathbf{B}_N(\underline{\boldsymbol{\lambda}})$ makes sense in that one cannot construct a solution of (6.1) from one of (6.2).

If the above phenomenon whereby the cut-off equations (6.1) do not admit a solution $\underline{\boldsymbol{\lambda}}_N$ such that $\|\underline{\boldsymbol{\lambda}}_N\|$ is uniformly bounded in N but the (6.2) do have a solution $\underline{\boldsymbol{\lambda}}$ with $\|\underline{\boldsymbol{\lambda}}\|<\infty$, then the field theory that one constructs from the solution of (6.2) could not be obtained in the usual way of introducing a cut-off and letting $N\to\infty$, an interesting possibility which unfortunately remains a speculative one for lack of examples.

The above properties of the solutions of (6.1), (6.2) are also strongly dependent on the regularization used. We briefly discuss the usual nearest neighbor lattice regularization because in this case it is known that $:\varphi^4:_d$, $d>4$ is trivial, Aizenman, 1982, Fröhlich, 1982.

Consider a lattice regularization in Λ: so Λ is replaced by a lattice with mesh a and periodic boundary conditions at distance fixed throughout the discussion.

To compare the two regularizations we can decompose into scaling fields, the lattice covariance thus representing the lattice field as a sum of an infinite sequence of lattice fields which live on scale γ^{-j}. Since there is the lattice cut-off a there is no need to introduce the cut-off fields $\varphi^{(\leqq N)}$. Nevertheless, it makes sense to investigate the effective potential $V^{(k)}$ on scale k defined, as in this paper, by integration of the exponential of the bare interaction over the high frequencies.

One can also define the beta functional $\mathbf{B}_{(a)}$. It should be clear that $\mathbf{B}_{(a)}$ and $\mathbf{B}_N$, B are strongly related because the $\mathbf{B}_{(a)}$-coefficients $\beta^{(\alpha)}(\theta,\mathbf{h},\boldsymbol{\alpha})$, although a-dependent will be almost identical to $\beta^{(\alpha)}(\theta,\mathbf{h},\boldsymbol{\alpha})$ if $\max\gamma^{h_i}\ll a^{-1}$.

However, for γ^h of the order of a^{-1} there will be major differences between the coefficients of $\mathbf{B}_{(a)}$ and of $\mathbf{B}$. This means that the two approaches cannot be really compared exactly for the same reasons for which one cannot establish a relation between (6.1), (6.2).

If one tries to compare $\mathbf{B}_N$ and $\mathbf{B}_{(a)}$ with $\gamma^{-N}\sim a$, one has to start to apply $\mathbf{B}_{(a)}$ to a bare interaction which is not of the same type as that to which one applies $\mathbf{B}_N$: in fact, the integration over the modes of the lattice field with frequencies larger than N should be rather trivial because such fields are essentially independent on the lattices sites (because the lattice spacing is a). However, it seems that the effective interaction that one gets on scale a after integrating the high frequency parts looks very gentle compared to the bare interaction: this is so because the exponential of the interaction in a given site appears as a convolution of the exponential of the bare interaction and a very flat gaussian (representing the high frequency components of the free field).

Therefore again it is unclear whether one can really compare the two approaches and it remains an open question to understand the deep implications of the triviality theorems on the solutions of (6.1), (6.2).

Finally, last but not least, even if the (6.2) did admit a bounded non-trivial solution, there would be a long way to go before really constructing a $P(\varphi)_d$ theory: one would have to understand the convergence properties and the summation rules to reconstruct the effective potentials from the form factors, check the necessary positivity properties for the effective potential necessary so that $\exp V^{(k)}$ can be interpreted as a probability density, and also one should check the basic euclidean invariance, cluster properties and Osterwalder-Schrader positivity (or equivalent properties) necessary to deduce that the objects constructed really represent a physically interesting quantum field theory.

Appendix A: Interpretation of (3.5)

Let us consider a field

$$\varphi=\varphi^{(-1)}+\varphi^{(0)}+\dots, \tag{A.1}$$

where $\varphi^{(-1)}$ has an arbitrary covariance C and is independent of $\varphi^{(0)}, \varphi^{(1)}, \dots$ which have the same meaning as in the rest of the paper.

Then, if $\mathscr{I}_{-1}\subset L_2(P(d\varphi^{(-1)})$ is defined as the linear span of the function in (2.5) with $k=-1$, one can easily compute the orthogonal projection $\mathscr{L}^{(c)}_{-1}$ on $\mathscr{I}_{-1}$.

Using the Graham-Schmidt orthogonalization method to compute the projection on the two-dimensional space spanned by the "quadratic interactions" ($\int:\varphi^2:$ and $\int:(\partial\varphi)^2:$) one finds:

$$\mathscr{L}^{(c)}_{-1}:\varphi^{(-1)}_{x_1}\dots\varphi^{(-1)}_{x_p}:=\frac{\int_\Lambda C_{xx_1}\dots C_{xx_p}dx}{\int_\Lambda C^p_{x0}dx}, \tag{A.2}$$

if $C_{xy}=\sum_p C(p)e^{ip(x-y)}$, $p=\frac{2\pi}{L}\nu$, $\nu\in\mathbb{Z}^\ell$,

$$\begin{aligned}\mathscr{L}^{(c)}_{-1}:\varphi^{(-1)}_{x_1}\varphi^{(-1)}_{x_2}:=&\frac{1}{|\Lambda|}\int:(\partial\varphi_x)^2:dx\frac{\sum_{pq}C(q)^2C(p)^2(p^2-q^2)\,(e^{ip(x_1-x_2)}-1)}{\sum_{pq}C(q)^2C(p)^2p^2(q^2-p^2)}\\&-\frac{1}{|\Lambda|}\int:\varphi^2_x:\frac{\sum_{pq}C(q)^2C(p)^2q^2(q^2-p^2)e^{ip(x_1-x_2)}}{\sum_{pq}C(q)^2C(p)^2p^2(q^2-p^2)}.\end{aligned} \tag{A.3}$$

If the covariance, so far arbitrary, C is taken the "simplest possible": i.e., in Fourier transform (on the torus Λ), as

$$C(p)=C^{(-1)}(p)=\varepsilon\left(1+\sum_{i=1}^{d}\varepsilon'\cos\frac{2\pi}{L}p_i\right), \quad |\varepsilon'|<\frac{1}{d}, \quad \varepsilon>0. \tag{A.4}$$

One finds that (A.2), (A.3) converge to the first two of (3.5) when one takes the limits $\lim_{\varepsilon\to 0}\lim_{\varepsilon'\to 0}$, at least formally, and concludes our heuristic motivation of (3.5).

The theory of the operator (3.4) with the choice (3.5) requires the (easy) analysis of the continuity properties of the operators $\delta_0^{(\alpha)}$ in $L_2(P(d\varphi^{(0)})$: the functionals $\delta^{(\alpha)0}$ are well-defined if $F_0(p)$, see (2.3), is strictly positive.

If F_0 does not have this property, the operators $\mathscr{L}_k$ can be defined, when $(x-y)_\pi^2$ is chosen as in (3.6), only for k large enough end one could carry through all our analysis along the same lines with a few minor modifications which we do not discuss.

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