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Finiteness of Ricci Flat Supersymmetric Non-linear σ-Models

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Abstract. Combining the constraints of Kähler differential geometry with the universality of the normal coordinate expansion in the background field method, we study the ultraviolet behavior of 2-dimensional supersymmetric non-linear σ-models with target space an arbitrary riemannian manifold M. We show that the constraint of $N = 2$ supersymmetry requires that all counterterms to the metric beyond one-loop order be cohomologically trivial. It follows that such supersymmetric non-linear σ-models defined on locally symmetric spaces are super-renormalizable and that $N = 4$ models are on-shell ultraviolet finite to all orders of perturbation theory.

1. Introduction

Non-linear σ-models are the quantum field theories of maps from spacetime into a riemannian manifold M. A few years ago, it was shown [1] that the ultraviolet properties of supersymmetric non-linear σ-models can be studied using the geometrical properties of Kähler manifolds. In particular, it was shown [1, 2] that supersymmetric σ-models on 2-dimensional spacetimes with target space M a Ricci flat manifold are ultraviolet finite up to three-loop order, and it was conjectured that such models were ultraviolet finite to all orders of perturbation theory. Specific examples of Ricci flat manifolds were shown in [3] to result in supersymmetric σ-models finite to all orders of perturbation theory, but these examples were not sufficiently general to imply the result for arbitrary Ricci flat manifolds. In this paper we will provide a proof that $N = 4$ supersymmetric σ-models are indeed finite to all orders of perturbation theory. We regard such theories as consequently likely to admit exact solutions. We also discuss extensions of our methods here to the case of arbitrary Ricci flat manifolds. Our intent is to illustrate as cleanly as possible the qualitative features underlying the finiteness of these models. On this basis we occasionally resort to compelling arguments rather than to truly rigorous proofs to establish certain intermediate results.

Part of the motivation for clarifying these issues is provided by recent progress [4] in string theory which suggests that certain 10-dimensional supersymmetric string theories provide a nearly unique framework in which quantum gravity may be consistently defined and unified with other known interactions. Phenomenological applications [5] require that the string theory be defined on a manifold of the form $M_4 \times M_6$, where M_4 is assumed to be a maximally symmetric 4-dimensional space time and M_6 is a compact 6-dimensional internal space (in the Kaluza-Klein sense). Whether string ground states of this form actually exist is intimately connected with the ultraviolet finiteness of 2-dimensional supersymmetric non-linear σ-models, where the 2 spacetime dimensions of the σ-model play the role of the internal coordinates on the string world sheet and the target space M plays the role of the compactified M_6. Ultraviolet finiteness implies that the theory is conformally invariant, and this in turn is required for a consistent formulation of the string theory on such backgrounds. The results of this paper suggest that only internal spaces M_6 which are Ricci flat should be considered, and these are in any event preferred for phenomenological reasons [5].

We shall begin in Sect. 2 with a review of known results concerning 2-dimensional supersymmetry, some geometrical preliminaries, and the basic features of the background field method in superspace which will enter into our subsequent analysis. In Sect. 3, we detail some important restrictions imposed on counterterms by the constraints of $N=2$ supersymmetry, and give various interpretations of the result. In Sect. 4, we combine the results of Sect. 3 with certain uniqueness properties of Ricci flat manifolds which act to constrain their geometry. Renormalization of the metric in perturbation theory is effectively prevented, proving that $N=4$ supersymmetric theories are finite to all orders of perturbation theory. We conclude with some comments concerning the application of our methods to the cases of $N=2$ and $N=1$ supersymmetric models.

2. Background

Let (M_n, g) be an n-dimensional riemannian manifold with metric g_{ij}. For 2-dimensional spacetime, an $N=1$ supersymmetric σ-model with scalar fields taking values on M_n is given by the lagrangian [6],

$$\begin{gathered}\mathscr{L}=\tfrac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j+\frac{i}{2}g_{ij}\bar{\psi}^i\gamma^\mu D_\mu\psi^j+\tfrac{1}{12}R_{ijkl}\bar{\psi}^i\psi^k\bar{\psi}^j\psi^l\\ D_\mu\psi^i=\partial_\mu\psi^i+\Gamma^i_{jk}\partial_\mu\phi^j\psi^k,\quad \gamma^0=\sigma_2,\quad \gamma^1=i\sigma_1,\end{gathered}\tag{2.1}$$

where Γ^i_{jk} is the Christoffel connection and R_{ijkl} is the Riemann curvature tensor. The ψ^i's are here and throughout two-component Majorana fermions. The supersymmetry transformation rules for (2.1) are given by

$$\delta\phi^i=\bar{\varepsilon}\psi^i,\quad \delta\psi^i=-i\partial\!\!\!/\phi^i\varepsilon-\Gamma^i_{jk}(\bar{\varepsilon}\psi^j)\psi^k.\tag{2.2}$$

The superspace form of the action (2.1) can be written in terms of a 2-dimensional real superfield $\Phi^i=\phi^i+\bar{\theta}\psi^i+\tfrac{1}{2}\bar{\theta}\theta F^i$, where θ is a real two-component constant Majorana spinor. The superspace form of (2.1) is just the naive extension of the purely bosonic σ-model

$$\mathscr{L} = \frac{1}{4i}\int d^2\theta g_{ij}(\Phi)\bar{D}\Phi^i D\Phi^j, \tag{2.3}$$

$$D_\alpha \Phi^i = \left(\frac{\partial}{\partial\bar{\theta}^\alpha} - i(\gamma^\mu\theta)_\alpha \frac{\partial}{\partial x^\mu}\right)\Phi^i,$$

with the superfield supersymmetry transformation rule given by

$$\delta\Phi^i = \bar{\varepsilon}_\alpha\left(\frac{\partial}{\partial\bar{\theta}_\alpha} + i(\partial\!\!\!/\theta)_\alpha\right)\Phi^i. \tag{2.4}$$

Note that for 2-dimensional spacetimes, it is always possible to write an $N=1$ supersymmetric σ-model for an arbitrary manifold M as long as it possesses a riemannian structure (this is as opposed to the case of 4-dimensions [7], where the chirality constraint in $N=1$ supersymmetry naturally requires in addition a Kähler structure for M; such a model dimensionally reduces to the $N=2$ model in 2-dimensions, to be discussed shortly). Furthermore, the supersymmetry transformation rules (2.2), though apparently non-covariant, commute with coordinate reparametrizations of M: under an infinitesimal coordinate reparametrization $\delta_c\phi^i = \xi^i(\phi)$, $\delta_c\psi^i = (\partial_j\xi^i)\psi^j$, it is easy to check that $[\delta_c, \delta] = 0$ acting on ϕ^i and ψ^i.

The action (2.1), (2.3) admits a second supersymmetry [7, 3] if and only if M is a Kähler manifold. (M_n, g) is said to be Kähler if there exists a tensor $f^i{}_j$ (the complex structure) which satisfies

$$f^i{}_k f^k{}_j = -\delta^i{}_j, \tag{2.5a}$$

$$g_{ij} f^i{}_k f^j{}_l = g_{kl}, \tag{2.5b}$$

$$\nabla_i f^j{}_k = 0. \tag{2.5c}$$

Equations (2.5*a*–*c*) first of all imply that M is a complex manifold, i.e. that M can be covered smoothly with complex coordinate charts $(z^\alpha, \bar{z}^\beta)$ such that the transition functions in overlapping coordinate patches are holomorphic. In these adapted complex coordinates, the complex structure $f^i{}_j$ becomes simply multiplication by $i(-i)$ on holomorphic (antiholomorphic) vectors (i.e. $f^\alpha{}_\beta = i\delta^\alpha{}_\beta$, $f^{\bar\alpha}{}_{\bar\beta} = -i\delta^{\bar\alpha}{}_{\bar\beta}$; $f^\alpha{}_{\bar\beta} = f^{\bar\alpha}{}_\beta = 0$). The metric components in these coordinates are written $g_{\alpha\bar\beta}$, so that

$$ds^2 = g_{ij}dx^i \otimes dx^j = 2g_{\alpha\bar\beta}dz^\alpha \otimes d\bar{z}^\beta. \tag{2.6}$$

The conditions (2.5*a*–*c*), giving $\nabla_l(g_{kj}f^k{}_i) = 0$, then moreover require that the Kähler-form J, defined as

$$J = \tfrac{1}{2}g_{kj}f^k{}_i dx^i \wedge dx^j = ig_{\alpha\bar\beta}dz^\alpha \wedge d\bar{z}^\beta, \tag{2.7}$$

be closed: $dJ = 0$. This is equivalent to the curl-free conditions

$$\partial_\gamma g_{\alpha\bar\beta} = \partial_\alpha g_{\gamma\bar\beta}, \quad \partial_{\bar\gamma} g_{\alpha\bar\beta} = \partial_{\bar\beta} g_{\alpha\bar\gamma}, \tag{2.8}$$

which tell us that the metric components $g_{\alpha\bar\beta}$ may be written locally as

$$g_{\alpha\bar\beta} = \frac{\partial^2}{\partial z^\alpha \partial\bar{z}^\beta}K(z, \bar{z}), \tag{2.9}$$

where $K(z,\bar{z})$ is known as the Kähler potential. K is unique up to Kähler transformations $K(z,\bar{z}) \to K(z,\bar{z}) + f(z) + \bar{f}(\bar{z})$, where $f(z)$ is a holomorphic function.

On a Kähler manifold the standard formulae of riemannian geometry simplify dramatically. For example, the only non-vanishing components of Γ^i_{jk} are $\Gamma^\alpha_{\beta\gamma} = g^{\alpha\bar{\rho}}\partial_\beta g_{\gamma\bar{\rho}}$ and $\Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} = (\Gamma^\alpha_{\beta\gamma})^*$ (which are symmetric in β, γ and $\bar{\beta}, \bar{\gamma}$, respectively, by (2.8)). The curvature tensor becomes

$$R^\alpha{}_{\beta\bar{\rho}\gamma} = \partial_{\bar{\rho}}\Gamma^\alpha_{\beta\gamma}, \tag{2.10}$$

and the cyclic and Bianchi identities reduce to

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\gamma\bar{\beta}\alpha\bar{\delta}}, \quad R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\alpha\bar{\delta}\gamma\bar{\beta}}, \tag{2.11a}$$

$$\nabla_\lambda R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \nabla_\alpha R_{\lambda\bar{\beta}\gamma\bar{\delta}}, \quad \nabla_{\bar{\lambda}} R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \nabla_{\bar{\beta}} R_{\alpha\bar{\lambda}\gamma\bar{\delta}}. \tag{2.11b}$$

It is useful to notice that the only non-vanishing components $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ of the curvature tensor on a Kähler manifold have two holomorphic and two antiholomorphic indices, and otherwise satisfy the usual symmetries of the Riemann tensor. Finally, the Ricci tensor on a Kähler manifold turns out to take the simple form

$$R_{\alpha\bar{\beta}} = g^{\mu\bar{\lambda}} R_{\bar{\lambda}\mu\alpha\bar{\beta}} = -\partial_\alpha\partial_{\bar{\beta}} \ln\det(g). \tag{2.12}$$

The condition (2.5b) means that the metric is a hermitian tensor with respect to the complex structure $f^i{}_j$. Generally, an arbitrary second rank tensor T_{ij} is hermitian if

$$T_{ij} f^i{}_k f^j{}_l = T_{kl}. \tag{2.13}$$

In complex coordinates, this means that $T_{\alpha\beta} = T_{\bar{\alpha}\bar{\beta}} = 0$, i.e. only the mixed components $T_{\alpha\bar{\beta}}$ and $T_{\bar{\alpha}\beta}$ may be non-zero. A symmetric hermitian tensor T_{ij} is called a Kähler tensor if it satisfies as well the Kähler condition

$$\partial_\lambda T_{\alpha\bar{\beta}} = \partial_\alpha T_{\lambda\bar{\beta}}, \quad \partial_{\bar{\lambda}} T_{\alpha\bar{\beta}} = \partial_{\bar{\beta}} T_{\alpha\bar{\lambda}}. \tag{2.14}$$

On a Kähler manifold, we can consider forms which are p-times holomorphic (contain p dz's) and q times antiholomorphic (q $d\bar{z}$'s), so that the space of r-forms Λ^r splits naturally into $\Lambda^r = \Lambda^{(r,0)} \oplus \Lambda^{(r-1,1)} \oplus \cdots \oplus \Lambda^{(0,r)}$. Similarly, the exterior derivative d and its adjoint $\delta = d^*$ can be split as $d = \partial + \bar{\partial}$, $\delta = \partial^* + \bar{\partial}^*$ ($\partial^* = -*\bar{\partial}*$, $\bar{\partial}^* = -*\partial*$), so that ∂: $\Lambda^{p,q} \to \Lambda^{p+1,q}$, $\bar{\partial}$: $\Lambda^{p,q} \to \Lambda^{p,q+1}$, ∂^*:$\Lambda^{(p,q)} \to \Lambda^{(p-1,q)}$, and $\bar{\partial}^*$:$\Lambda^{(p,q)} \to \Lambda^{(p,q-1)}$. It also follows (see, for example, [8]) from the covariant constancy of the complex structure that the laplacian on forms satisfies $\Box \equiv d\delta + \delta d = 2(\partial\partial^* + \partial^*\partial) = 2(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})$, and that $\partial\bar{\partial}^* + \bar{\partial}^*\partial = \bar{\partial}\partial^* + \partial^*\bar{\partial} = 0$. Since the generators of the cohomology groups are the harmonic forms, we have $H^r = H^{(r,0)} \oplus H^{(r-1,1)} \oplus \cdots \oplus H^{(0,r)}$, where $H^{(p,q)}$ is generated by harmonic forms of type (p,q). Equation (2.14) says that if T_{ij} is a symmetric hermitian tensor, then the type (1, 1) 2-form[1]

$$\tau = \tfrac{1}{2} T_{kj} f^k{}_i dx^i \wedge dx^j = i T_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta, \tag{2.15}$$

is closed, $d\tau = 0$, if and only if T_{ij} is Kähler. A Kähler tensor thus represents a cohomology class in the Hodge-DeRham group $H^{1,1}(M)$. In particular, for a

1 Note that $T_{kj} f^k{}_i$ is automatically antisymmetric in i and j due to the hermiticity condition (2.13) and symmetry of T_{kj}

compact manifold the Kähler form J of (2.7) always generates a non-trivial element of $H^{1,1}(M)$. This is because for a compact manifold M of complex dimension n, the volume form is expressable as $J^n/n!$, and were this exact, the manifold would have zero volume. Another hermitian tensor is the Ricci tensor $R_{\alpha\bar{\beta}}$, and from contracting the Bianchi identity (2.11b) or directly from (2.12), it is evident that it also satisfies the Kähler condition (2.14). Its associated 2-form $\Sigma = iR_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta$, known as the Ricci form, can be shown [9] to represent the first Chern class of the manifold. This characterization of the first Chern class in terms of the Ricci form plays a central role in the Calabi conjecture [9] proven by Yau [10], to be discussed in Sect. 4.

Finally, it can be shown [3] that each supersymmetry beyond $N=2$ requires an independent Kähler structure $f^{(a)i}{}_j$ satisfying

$$f^{(a)i}{}_k f^{(b)k}{}_j + f^{(b)i}{}_k f^{(a)k}{}_j = -2\delta^i{}_j \delta^{ab}. \tag{2.16}$$

It should be clear that $N=3$ supersymmetry automatically implies $N=4$, because if $f^{(1)}$ and $f^{(2)}$ are two Kähler structures which satisfy (2.5a–c) and (2.16), so also does $f^{(3)i}{}_j = f^{(1)i}{}_k f^{(2)k}{}_j$. If M is an irreducible manifold (i.e. does not split into a product of lower dimensional manifolds), then $N=4$ is the largest number of supersymmetries that one can have. The reason is that any covariantly constant tensor is invariant under the action of the holonomy group [11]. If the manifold is irreducible, the tangent space provides a real irreducible representation of the holonomy group. Since each of the $f^{(a)}$'s is covariantly constant, any matrix $H^i{}_j$ in the holonomy group satisfies $H^i{}_k f^{(a)k}{}_j = f^{(a)i}{}_k H^k{}_j$. But by Schur's lemma [12], a representation is real irreducible only if all the matrices which commute with the representation form a division algebra over the real numbers. The only non-trivial possibilities are the complex numbers (in which case the single complex structure plays the role of the imaginary unit) or the quaternions (in which case the $f^{(a)}$'s, $a = 1, 2, 3$, represent the three imaginary units). The first case corresponds to $N=2$ supersymmetry in which the holonomy group of a manifold of $2n$ real dimensions is reduced from SO$(2n)$ to U(n), and the manifold is Kähler. The second case is that of $N=4$ supersymmetry in which the holonomy group of a manifold of $4n$ real dimensions is further reduced to the Sp(n) subgroup of U$(2n)$, and the manifold is then called hyperkähler.

We now briefly recall the methodology for analyzing the ultraviolet divergences induced by quantum corrections to the σ-model (2.3). Study of the ultraviolet structure of (2.3) is greatly facilitated by use of the superspace background field method (see [13] for details) which involves expanding Φ around a generic solution Φ_0 to equations of motion derived from (2.3). These classical field equations are the superspace geodesic equations,

$$\bar{D}D\Phi_0^i + \Gamma^i_{jk}(\Phi_0)\bar{D}\Phi_0^j D\Phi_0^k = 0. \tag{2.17}$$

Since the S-matrix of the theory is expected to be reparametrization invariant, rather than making the naive splitting of the field Φ^i in terms of a classical solution plus a quantum field: $\Phi^i = \Phi_0^i + u^i$, it is preferable to parametrize Φ^i in terms of normal coordinates centred upon Φ_0^i. Φ^i is thereby parametrized in terms of the tangent vector ξ^i at Φ_0^i tangent to the unit speed geodesic joining Φ_0^i to Φ^i:

$$\Phi^i = \Phi_0^i + \xi^i - \tfrac{1}{2}\Gamma^i_{jk}(\Phi_0)\xi^j\xi^k + \cdots. \tag{2.18}$$

The advantage of this procedure is that the functional Taylor expansion of the action around Φ_0 has manifestly covariant coefficients which can be written exclusively in terms of the curvature tensor and its covariant derivatives. For example, expanding to second order in ξ gives [13]

$$\mathscr{L} = \frac{1}{4i}\int d^2\theta [g_{ij}(\Phi_0)\bar{D}\Phi_0^i D\Phi_0^j + g_{ij}(\Phi_0)\bar{D}\xi^i D\xi^j + R_{iklj}\xi^k\xi^l \bar{D}\Phi_0^i D\Phi_0^j + O(\xi^3)]. \tag{2.19}$$

It is easy then to verify that the one-loop divergences are proportional to $R_{ij}\bar{D}\Phi_0^i D\Phi_0^j$, and are thus generated by the Ricci tensor.

Since the higher order effects simply duplicate the structure (2.3), the renormalization of the original lagrangian can always be expressed in terms of changes in the metric tensor. The renormalization group equations [14] are written in the form

$$\mu\frac{d}{d\mu}g_{ij} = -\beta_{ij}(g), \quad \beta_{ij}(g) = \alpha R_{ij} + \cdots,$$

where $\alpha = 1/2\pi$ for the supersymmetric model (2.1). It is important to note that in the normal coordinate expansion (2.19), the vertices of the theory contain explicitly neither the metric nor the Ricci tensor; they involve only the Riemann tensor and its covariant derivatives. Since loop computations involve only such vertices and their contractions with the metric g^{ij} (the propagator of the quantum field ξ^i is proportional to $g^{ij}(\Phi_0)$), the counterterms which can be generated in perturbation theory are somewhat constrained. There is no way, for example, to generate explicitly a counterterm of the form $g_{ij}S$, where S is a scalar function constructed in terms of curvatures and covariant derivatives (see [13] and references therein for more details). It is possible to determine whether a given tensor may appear at a given order in the loop expansion by probing with a constant conformal rescaling of the metric: $g_{ij} \to \lambda^{-1}g_{ij}$. Since this would be equivalent to a rescaling of $\hbar$ (had it appeared) in the original action, it follows that λ can be used as a loop counting parameter. Thus if a tensor T_{ij} appears at l-loop order, it must behave as $T_{ij} \to \lambda^{l-1}T_{ij}$ under the conformal rescaling. For example, at one-loop order, only tensor counterterms with zero conformal weight ($l = 0$) are allowed, and there are only two such tensors: R_{ij} and $g_{ij}R$, where R is the Ricci scalar. But we have already argued that $g_{ij}R$ cannot appear in perturbation theory so we quickly conclude that the only possible tensor counterterm at the one-loop level is the Ricci tensor itself.

A crucial property of the background field expansion (2.19) is that the coefficients of the divergences are universal, in the sense that they are independent of the local or global properties of the manifold M on which the model is defined. This means that geometrical constraints imposed on M may affect only the values of the curvature polynomials themselves and not their numerical coefficients in the background field expansion. For example, if the manifold is Ricci flat, then this universality together with the discussion of the preceding paragraph suffice to show that the theory is one-loop finite.

As mentioned, the on-shell renormalization of the model (2.3) is represented by changes of the original metric g_{ij}. If the theory is regulated with supersymmetric dimensional regularization, the bare metric g^B is given in terms of the renormalized metric g^R

$$g_{ij}^B = \mu^\varepsilon \left[g_{ij}^R + \sum_{\nu=1}^{\infty} \frac{T_{ij}^{(\nu)}(g^R)}{\varepsilon^\nu} \right]. \tag{2.20}$$

The tensors $T_{ij}^{(\nu)}(g^R)$ are polynomials in the curvature tensor and its covariant derivatives. Since poles of order ν will appear first at ν-loop order, $T_{ij}^{(\nu)}(g^R)$ can be expressed in perturbation theory as a series

$$T_{ij}^{(\nu)} = \sum_{l=\nu-1}^{\infty} T_{ij}^{(\nu,l)}, \tag{2.21}$$

where $T_{ij}^{(\nu,l)}$ is a tensor with conformal weight l with respect to rescalings of the metric, generated in $(l+1)$-loop order. The one- and two-loop results are [13]

$$\begin{aligned} T_{ij}^{(1,0)} &= \frac{1}{2\pi} R_{ij}, \\ T_{ij}^{(1,1)} &= 0, \\ T_{ij}^{(2,1)} &= -\frac{1}{4(2\pi)^2} (\nabla^k \nabla_k R_{ij} + [\nabla_i, \nabla^k] R_{kj} + [\nabla_j, \nabla^k] R_{ik}). \end{aligned} \tag{2.22}$$

The generalized renormalization group equations [14] are obtained exactly as in ordinary field theory, here by requiring that a change in the mass scale μ in (2.20) be compensated by a change in g_{ij}^R. The result is

$$\begin{aligned} \mu \frac{\partial}{\partial \mu} g_{ij}^R &= -\beta_{ij}(g^R), \\ \beta_{ij}(g^R) &= \varepsilon g_{ij}^R + \left(1 + \lambda \frac{\partial}{\partial \lambda}\right) T_{ij}^{(1)}(\lambda^{-1} g^R) \Big|_{\lambda=1}, \end{aligned} \tag{2.23}$$

together with the generalized pole equations

$$\begin{aligned} &\lim_{\lambda \to 1} \left(1 + \lambda \frac{\partial}{\partial \lambda}\right) T^{(\nu+1)}(\lambda^{-1} g^R) \\ &= \lim_{\substack{\eta \to 0 \\ \lambda \to 1}} \eta^{-1} \left[T^{(\nu)}\left(g^R + \eta \left(1 + \lambda \frac{\partial}{\partial \lambda}\right) T^{(1)}(\lambda^{-1} g^R) \right) - T^{(\nu)}(g^R) \right]. \end{aligned} \tag{2.24}$$

As in the standard 't Hooft renormalization group pole equations [15], the pole terms $1/\varepsilon^\nu$ with $\nu \geqq 2$ are completely determined by the $1/\varepsilon$ poles. This is quite useful in providing a consistency check on multiloop computations. For example, the tensor $T_{ij}^{(2,1)}$ in (2.22) is completely determined by the one-loop counterterm $T_{ij}^{(1,0)}$ and the pole equations (2.24). Up to two-loop order, we see from the vanishing of $T_{ij}^{(1,1)}$ in (2.22) that the β-function defined in (2.23) is porportional to the Ricci tensor alone. It thus follows that theories on Ricci flat manifolds are two-loop on-shell finite. Further explicit computations [2] show that theories with $R_{ij} = 0$ are as well finite at the three-loop level.

Before closing this section, we wish to comment on issues concerning the regularization of (2.3). The most efficient way to compute Feynman diagrams containing loops is to use the supersymmetric dimensional regularization scheme known as dimensional reduction (for a review of supersymmetric regulators, see [16]). No ambiguities arise up to three-loop order for either 2- or 4-dimensional theories. The rules of dimensional reduction as they currently stand, however, are not obviously consistent to higher order and this procedure may need some modifications for complete reliability. In any event, a simple power counting argument shows that the divergences expected in (2.3) are logarithmic, so a single Pauli-Villars superfield should suffice as a regulator for 2-dimensional theories. We shall thus assume (and all of our results to follow depend on this assumption) that (2.3) may be regulated in a manner consistent with supersymmetry.

With this assumption, we still need to ask whether there is a regularization which preserves extended supersymmetry for 2-dimensional models. For $N=2$, we know that (M,g) must be a Kähler manifold. The theory (2.1) then admits a global, non-chiral SO(2) symmetry,

$$\delta\psi^i = \varepsilon f^i{}_j \psi^j, \tag{2.25}$$

generated by the complex structure. In terms of coordinates adapted to the complex structure, ψ has components ψ^α and $\psi^{\bar\alpha}$, and invariance under (2.25) simply corresponds to infinitesimal invariance under the U(1) rotation $\psi^\alpha \to e^{i\varepsilon}\psi^\alpha$, $\psi^{\bar\alpha} \to e^{-i\varepsilon}\psi^{\bar\alpha}$. Similarly, for the case of $N=4$ supersymmetry, the hyperkähler structure gives (2.1) a full SU(2) symmetry,

$$\delta\psi^i = (\varepsilon_1 f^{(1)i}{}_j + \varepsilon_2 f^{(2)i}{}_j + \varepsilon_3 f^{(3)i}{}_j)\psi^j, \tag{2.26}$$

generated by the three complex structures. This corresponds to a global SU(2) rotation in the Sp(1) indices of the tangent space (see Sect. 4 for more details). Hence the preservation of $N=2$ or $N=4$ supersymmetry depends on having a regulator which preserves the $N=1$ supersymmetry and the SO(2) or SU(2) global symmetries which exchange the different supercharges. This would be satisfied by dimensional reduction, so it is likely that any consistent $N=1$ regulator can be extended to the higher cases as well.

We point out finally that there is a refinement of the normal coordinate expansion which makes the SO(2) symmetry manifest. In the standard normal coordinate expansion, the freedom to make coordinate reparametrizations is completely fixed by the choice of normal coordinates. For a Kähler manifold, some extra care is necessary since arbitrary diffeomorphisms will in general mix holomorphic and antiholomorphic coordinates. This necessitates finding a method of fixing the coordinate gauge which involves only holomorphic coordinate reparametrizations. The simplest way of doing this is to impose at the point ϕ_0, $\bar\phi_0$ (the origin of the normal coordinate system), the conditions

$$\begin{aligned} \partial_{\alpha_1}\cdots\partial_{\alpha_n} g_{\mu\bar\lambda}(\phi_0,\bar\phi_0) &= 0, \\ \partial_{\bar\alpha_1}\cdots\partial_{\bar\alpha_n} g_{\mu\bar\lambda}(\phi_0,\bar\phi_0) &= 0, \end{aligned} \tag{2.27}$$

to all orders in partial derivatives. Using the special properties of Kähler geometry, it is then easy to show that the background field expansion is manifestly covariant, and moreover that $\phi^\alpha = \phi_0^\alpha + u^\alpha(\xi^\beta)$, where u is a holomorphic function of the geodesic generating tangent vector ξ. Since the fermion fields transform like vectors, the new quantum fermion field is obtained from ψ^α according to $\psi^\alpha \to (\partial u^\alpha / \partial \xi^\beta)\psi^\beta$, which has a well-defined transformation rule under SO(2). These properties of the normal coordinate expansion for $N=2$ supersymmetry are useful because they mean that the only on-shell counterterms which will be generated in perturbation theory must be hermitian tensors with respect to the complex structure, and this will be seen to simplify the classification of possible divergences.

3. Constraints on Counterterms

The renormalization counterterms generated in perturbation theory are given by symmetric tensors constructed from the curvature tensor and its covariant derivatives, with no further reference to the underlying geometry of the σ-model manifold M. If M were taken to be a Kähler manifold, then preservation of $N=2$ supersymmetry would imply that the only possible on-shell counterterms are Kähler tensors, i.e. tensors which satisfy the hermiticity condition (2.13) and curl condition (2.14). Universality of the coefficients of the background field expansion then tells us that the only tensors which may be generated on an *arbitrary* riemannian manifold M are those which when restricted to Kähler manifolds satisfy these same conditions. Kähler tensors come in two types: those which are generators of non-trivial classes of $H^{1,1}$, and those which belong to the trivial class. Any tensor of the first type can be written locally in terms of a Kähler potential $T_{\alpha\bar\beta} = \partial_\alpha \partial_{\bar\beta} \Lambda(z, \bar z)$, where Λ is not globally defined. Λ can instead only be defined patchwise, and changes from patch to patch by a Kähler gauge transformation, $\Lambda \to \Lambda + f(z) + \bar f(\bar z)$, where $f(z)$ depends on the holomorphic transition functions between coordinates on the different patches. Kähler tensors in the second class may be written similarly as $T_{\alpha\beta} = \partial_\alpha \partial_{\bar\beta} S(z, \bar z) = \nabla_\alpha \nabla_{\bar\beta} S(z, \bar z)$, but where S is now a globally defined scalar function, invariant from patch to patch.

We must now discuss when tensors of each of these types may appear in perturbation theory. Counterterms generated in $(l+1)$-loop order on a Kähler manifold can always be written using the Bianchi identities as $T_{\alpha\bar\beta} = \nabla_\alpha \nabla_{\bar\beta} S$, where S and $T_{\alpha\bar\beta}$ both have conformal weight l and S is a polynomial constructed from curvatures and their covariant derivatives. If $T_{\alpha\bar\beta}$ generates a non-trivial cohomology class, then S must change from patch to patch according to $S \to S + f(z) + \bar f(\bar z)$. It is not easy to see how a polynomial in the curvatures with all of its coordinate indices covariantly contracted can be non-invariant under a change of coordinates. In fact, the only way to construct such an S with well-defined conformal weight using only the local tensor constructions available in perturbation theory is in the form $S = \log \det M_{\alpha\bar\beta}$ (products of logarithms, for example, would not have well-defined conformal weight). Independent of the conformal weight of the tensor $M_{\alpha\bar\beta}$, however, $T_{\alpha\bar\beta} = \nabla_\alpha \nabla_{\bar\beta} S$ then necessarily has zero conformal weight, and hence non-trivial generators of cohomology can only appear as tensor counterterms in one-loop order. But we already know that the only counterterm generated in one-loop order is the Ricci tensor R_{ij}, so it follows that the only two-index tensor of the first type (i.e. generating non-trivial $H^{1,1}$ cohomology) which appears in σ-model perturbation theory to any order is the Ricci form

$$\Sigma = iR_{\alpha\bar{\beta}} dz^{\alpha} \wedge d\bar{z}^{\beta}. \tag{3.1}$$

All counterterms generated in σ-model perturbation theory beyond one-loop order are of the second type, i.e. cohomologically trivial.

(Although not necessary to our purposes in what follows, we mention that this result is probably related to a much stronger mathematical result, namely that the only generators of cohomology below the top dimension constructible from polynomials in the curvature and its covariant derivatives are identically the generators of the Pontryagin classes in the general riemannian case, and the Chern classes in the Kähler case. To prove this requires a strengthening of Gilkey's lemma, which plays an important role in the heat kernel derivation of the index theorem (for details see [17] which also includes references to the earlier literature). The heat kernel method allows the computation of the index density for an elliptic operator on a manifold as a local function of its curvature, and in principle also its covariant derivatives. Integrating this index density over the manifold then gives an index of the operator in question, universal in the sense that it depends only on topological properties of the manifold. This universality motivates, both from conceptual and practical points of view (to simplify the heat kernel expansions), a characterization of the universal cohomology classes generated by polynomials in the curvature and its covariant derivatives. In [17], it is reported that the only generators of non-trivial cohomology among such polynomials of conformal weight $l \leqq 0$ are the Pontryagin and Chern classes (all having zero conformal weight). This result is easily strengthened to encompass polynomials of arbitrary conformal weight [18], at least for cohomology up to half the dimension of the manifold.)

The result may be made particularly transparent for superfield afficionados by means of a manifestly $N=2$ invariant superspace formulation [7] of the σ-model (2.1). In this formulation, a general superfield is a function $\Phi(x, \theta, \bar{\theta})$, where the spacetime coordinates x^{μ} ($\mu = 1, 2$) and the constant complex 2-component spinors θ and $\bar{\theta}$ parametrize superspace. Taylor expanding in θ and $\bar{\theta}$, however, shows Φ to contain too many degrees of freedom to describe a single scalar multiplet, so instead Φ is required to be a chiral superfield by imposing the constraint $\bar{D}_{\alpha}\Phi = 0$, where $\bar{D}_{\alpha}$ is the spinor derivative defined in (2.3) but with θ and $\bar{\theta}$ now taken as independent variables (see [16] for more details). In terms of chiral superfields, the lagrangian can be written

$$\mathscr{L} = \int d^2\theta d^2\bar{\theta} K(\Phi^i, \bar{\Phi}^j), \tag{3.2}$$

i.e., entirely in terms of the Kähler potential. The counterterms in this formalism then appear as corrections to the Kähler potential of the original metric. In the one-loop approximation, the effective action is given as the superdeterminant of the quadratic approximation to (3.2), and its divergent part turns out simply proportional to $\log\det \partial_{\alpha}\partial_{\bar{\beta}}K$, the Kähler potential for the Ricci tensor. Corrections to K generated in higher loop orders, on the other hand, are proportional only to globally defined scalar curvature polynomials S having non-zero conformal weight with respect to the transformation $K \to \lambda^{-1}K$. This is because, as argued above, $S = \log\det M_{\alpha\bar{\beta}}$ would have zero conformal weight, and an $S = K \cdot P(R)$, with $P(R)$ a curvature polynomial, cannot be generated because this would induce a counterterm of the form $g_{ij}P(R)$, earlier argued to be excluded in riemannian perturbation theory. New non-trivial log det's can thus never occur and the induced higher order counterterms $T_{\alpha\bar{\beta}}$ to the metric $g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K$ are then automatically of the form $\partial_\alpha \partial_{\bar{\beta}} S$, with S globally defined. The only difficulty in carrying out this procedure explicitly is that it is not known at present how to formulate the normal coordinate expansion around a non-trivial background superfield Φ_0^i in a way which preserves the chirality constraint. Using instead a naive background field expansion, the vertices derived from (3.2) are not necessarily manifestly covariant, although the on-shell counterterms so calculated will of course emerge in covariant form. Modulo this purely technical difficulty of simultaneously manifesting $N=2$ supersymmetry and coordinate covariance (which in any event is probably surmountable), the $N=2$ superspace formulation thus provides a straightforward and rather intuitive means of understanding the trivial topological structure of counterterms generated beyond one-loop.

In closing this section, we point out that $N=2$ σ-models on locally symmetric spaces (for which $\nabla_m R_{ijkl} = 0$) are super-renormalizable. This is because the Kählerity together with the cohomological triviality of the counterterms beyond one-loop forces them to take the form $\nabla_\alpha \nabla_{\bar{\beta}} S$ with S a scalar curvature polynomial. Expanded out, all such terms vanish because they necessarily contain covariant derivatives acting on the curvature tensor, and the only non-vanishing counterterm is hence the one-loop contribution from the Ricci tensor itself. This result could be extended by universality to cover $N=1$ models defined on locally symmetric spaces as well by showing that there are no riemannian polynomials constructed purely from the curvature tensor (i.e. without using covariant derivatives) which generically vanish upon restriction to Kähler manifolds.

4. Finiteness of $N=4$ Models

In this section, we will show how the uniqueness of the Ricci flat metric satisfying the conditions of the Calabi conjecture constraints the geometry sufficiently to prevent any counterterms from appearing in perturbation theory.

Let us begin with a manifold M of real dimension $4n$. The curvature 2-form

$$\Omega^i{}_j = \tfrac{1}{2} R^i{}_{jkl} dx^k \wedge dx^l \tag{4.1}$$

is valued in the Lie algebra of the holonomy group, typically SO($4n$) for an arbitrary riemannian manifold. If the manifold is Kähler, then Ω is instead valued in the Lie algebra of U($2n$), and (4.1) becomes

$$\Omega^\alpha{}_\beta = R^\alpha{}_{\beta\delta\bar{\gamma}} dz^\delta \wedge d\bar{z}^\gamma. \tag{4.2}$$

The trace of the curvature 2-form picks out the U(1) part of the holonomy group and, by (2.12) and (3.1), is related to the Ricci form Σ by

$$\operatorname{tr} \Omega = \Omega^\alpha{}_\alpha = R^\alpha{}_{\alpha\delta\bar{\gamma}} dz^\delta \wedge d\bar{z}^\gamma = R_{\delta\bar{\gamma}} dz^\delta \wedge d\bar{z}^\gamma = -i\Sigma. \tag{4.3}$$

It follows that a Kähler manifold has no U(1) holonomy if and only if it is Ricci flat. Now if M is hyperkähler, Ω is valued in the Lie algebra of Sp(n). Since Sp(n) lies entirely within the SU(2n) subgroup of U(2n), its generators are all traceless and we learn that hyperkähler manifolds are automatically Ricci flat. The case of $N=4$ supersymmetric σ-models requires a manifold M which is hyperkähler, and moreover the preservation of $N = 4$ supersymmetry in perturbation theory requires that the metric plus induced counterterm T_{ij} preserve the consequent Ricci flatness. Thus, for hyperkähler manifolds it is guaranteed that

$$R_{ij}(g + T) = 0. \tag{4.4}$$

In the previous section, we argued that Kähler tensors of the first kind, i.e. non-trivial generators of cohomology classes, cannot appear as counterterms in perturbation theory, and thus that $T_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} S$, where S is a globally defined scalar curvature polynomial. In order to show how this argument works together with the Calabi conjecture to imply the finiteness of Ricci flat σ-models, we now digress momentarily to describe the conjecture and its implications.

Suppose M is a compact complex manifold admitting an infinitely differentiable Kähler metric with Kähler form J and Ricci form Σ. Calabi conjectured [9] that if Σ' is any closed form of type (1, 1) cohomologous to Σ, then there exists a unique Kähler metric with associated Kähler form J' cohomologous to J and Ricci form equal to Σ'. Assuming the existence of such a metric, Calabi went on to prove its uniqueness; Yau [10] later provided the proof of its existence. The work of Calabi and Yau in particular establishes that if we have a manifold M whose first Chern class vanishes (so that its Ricci form Σ is automatically cohomologous to zero), then for a given cohomology class of the Kähler form there is a unique metric for which the Ricci tensor vanishes identically. The interpretation of this result is straightforward: the vanishing of the first Chern class represents a condition on the integral of the Ricci form over arbitrary closed 2-surfaces in M and hence the absence of a global (integrated) obstruction to the existence of a Ricci flat metric. Even in the absence of such a *global* obstruction to removing the U(1) part of the holonomy, there might in principle still be a problem smoothing it away *locally*. Yau's work assures us that this is not the case, i.e. that the absence of a global obstruction insures that there is no local obstruction to removing the U(1) holonomy.

This uniqueness property of the metric for Ricci flat manifolds can now be used to give an immediate proof of finiteness for hyperkähler manifolds.[2] Substituting $T_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} S$ in (4.4) implies that $R_{ij}(g + \partial\bar{\partial}S) = 0$, and then the uniqueness of the Ricci flat metric within a given topological class requires that $\partial_\alpha \partial_{\bar{\beta}} S = 0$, since the original metric g is already Ricci flat. Finiteness thus follows simply from the fact that counterterms from perturbation theory cannot change the topological class of the metric, and hence uniqueness of the Ricci flat metric allows no counterterms at all. The $N=4$ condition is essential to this argument because otherwise the counterterms would not necessarily have to satisfy (4.4), i.e. in principle there might occur counterterms which alter the metric structure to have a riemannian connection with non-vanishing U(1) holonomy and uniqueness arguments would no longer apply.

Since we need here only the much more easily proven uniqueness result rather than the full existence result, the essential features of this argument can be made

2 A different proof of the finiteness of $N=4$ supersymmetric σ-models has been given by Hull [19]

more apparent by considering the linearized form of (4.4),

$$-\tfrac{1}{2}(\nabla^k\nabla_k T_{ij} - \nabla^k\nabla_i T_{kj} - \nabla^k\nabla_j T_{ik} + \nabla_i\nabla_j T^k{}_k) = 0. \qquad (4.4)'$$

We first rewrite (4.4)′ in the form

$$-(\nabla^k\nabla_k T_{ij} + [\nabla_i, \nabla^k]T_{kj} + [\nabla_j, \nabla^k]T_{ik}) \equiv \Delta_L T_{ij} = 0, \qquad (4.5)$$

having made use of the fact that T_{ij} is Kähler, and hence satisfies $\nabla^k T_{ki} = \frac{1}{2}\nabla_i T^k{}_k$. Since a hyperkähler manifold is in particular Kähler, (4.5) requires that T_{ij} be a zero mode of the Lichnerowicz laplacian of type (1, 1), i.e. $\Delta_L T_{\alpha\bar\beta} = 0$. Equivalently, this means that the 2-form $\tau = \frac{1}{2}T_{kj}f^k{}_i dx^i \wedge dx^j = iT_{\alpha\bar\beta}dz^\alpha \wedge d\bar z^\beta$ is harmonic of type (1, 1): $\Box\tau = 0$ (this is because $\Box\tau \equiv (d\delta + \delta d)(\tau_{ij}dx^i \wedge dx^j) = (\Delta_L \tau_{ij})dx^i \wedge dx^j$). But now we have a contradiction because we have already established that counterterms generated beyond one-loop order are associated with forms which are cohomologically trivial, i.e. exact. By the Hodge decomposition, the only way a form can be both harmonic and exact is by vanishing identically. Thus the conflict between the exactness of any counterterm $T_{\alpha\bar\beta}$ generated beyond one-loop and the linearized condition (4.5) requiring that it be also harmonic renders supersymmetric σ-models on arbitrary hyperkähler manifolds finite. We note that this argument excludes counterterms from appearing in any order of perturbation theory on any Ricci flat manifold in which a symmetry acts to maintain the Ricci flatness of the metric plus counterterms. Although our argument has been formulated for compact hyperkähler manifolds, we may now appeal to the universality of the coefficient expansion to extend the result to arbitrary hyperkähler manifolds: any counterterm generated on a non-compact hyperkähler manifold is excluded by virtue of having a non-vanishing analog on some compact hyperkähler manifold.

To further elucidate our results, we introduce a formalism better suited to taking advantage of the geometrical features of hyperkähler manifolds (see, for example, [20]). Since the holonomy group of a hyperkähler manifold is $\mathrm{Sp}(n) \subset \mathrm{SU}(2n) \subset \mathrm{SO}(4n)$, we can covariantly split the tangent space as the tensor product of the 2-dimensional representation of Sp(1) and the fundamental $2n$-dimensional representation of Sp(n). Each tangent space index i can then be represented in terms of an Sp(1) index $\alpha = 1, 2$ and an Sp(n) index $A = 1, 2n$. The interpolation between indices i and (α, A) is then provided by covariantly constant matrices $\sigma^i_{\alpha A}$ (this construction is the natural generalization of the familiar decomposition of $\mathrm{SO}(4) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \approx \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ in which case the σ's are the usual $\sigma^\mu_{\alpha\beta}$ matrices). Taking $\varepsilon_{\alpha\beta}$ and ε_{AB} to be the fundamental antisymmetric invariant tensors of Sp(1) and Sp(n) respectively, it follows that the metric g_{ij} adapted to these $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ frames is simply $g_{ij}\sigma^i_{\alpha A}\sigma^j_{\beta B} = \varepsilon_{\alpha\beta}\varepsilon_{AB}$. Because of the vanishing Sp(1) holonomy, the connection 1-form becomes $\omega_{\alpha A,\beta B} = \varepsilon_{\alpha\beta}\omega_{AB}$, where antisymmetry of ω_{ij} implies the symmetry $\omega_{AB} = \omega_{BA}$. Similarly, the curvature referred to these frames is

$$R_{\alpha A,\beta B,\gamma C,\delta D} = R_{ijkl}\sigma^i_{\alpha A}\sigma^j_{\beta B}\sigma^k_{\gamma C}\sigma^l_{\delta D} = \varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\Omega_{ABCD}, \qquad (4.6)$$

where the usual symmetries of R_{ijkl} imply that Ω_{ABCD} is totally symmetric in A, B, C, and D. The three Kähler structures $f^{(a)i}{}_j$ are given in this language by $f^{(a)k}{}_j g_{ki}\sigma^i_{\alpha A}\sigma^j_{\beta B} = \{\delta_{\alpha\beta}\varepsilon_{AB}, (\tau_1)_{\alpha\beta}\varepsilon_{AB}, (\tau_3)_{\alpha\beta}\varepsilon_{AB}\}$ (where τ_1 and τ_3 are the usual 2×2 Pauli matrices). These tensors are manifestly antisymmetric in αA, βB and covariantly constant (hence satisfy (2.5*a*–*c*)), and automatically satisfy the Clifford algebra condition (2.16).

To see how our results fit into this framework, consider the symmetric tensor counterterm T_{ij} generated in perturbation theory, now written as $T_{\alpha A,\beta B} = T_{\beta B,\alpha A}$. If the $\mathrm{SU}(2) \approx \mathrm{Sp}(1)$ global symmetry is preserved, preserving the hyperkähler condition, then we must have $T_{\alpha A,\beta B} = \varepsilon_{\alpha\beta} T_{AB}$ with $T_{AB} = -T_{BA}$. In these Sp(n) frames, (4.5) thus becomes

$$\nabla^k \nabla_k T_{AB} - 2\Omega_{CABD} T^{CD} = 0. \tag{4.7}$$

Antisymmetry of T_{CD} together with the symmetry of Ω_{ABCD} then reduces (4.7) to $\nabla^k \nabla_k T_{AB} = 0$. Multiplying by T_{AB} and integrating by parts then gives $\nabla_k T_{AB} = 0$, as usual by the positivity of the metric. But if the manifold is irreducible, then the only covariantly constant tensor is ε_{AB}, and thus we have that $T_{\alpha A,\beta B} = (\mathrm{const}) \cdot \varepsilon_{\alpha\beta} \varepsilon_{AB}$, reproducing our result that T_{ij} must be proportional to the metric tensor and thus cannot be generated in perturbation theory (the metric itself, of course, cannot be generated in higher loop perturbation theory because it has conformal weight -1; no polynomial in the curvatures accidentally proportional to the metric could be generated because it would then generate non-trivial cohomology). The integration by parts argument above requires working on either a compact hyperkähler manifold or on a non-compact one with appropriate decay of the curvature at infinity. There are actually known examples of both types. Two infinite classes of irreducible compact hyperkähler manifolds of any real dimension $4n$ are exhibited in [21], and an infinite class of asymptotically locally euclidean non-compact manifolds, comprising the Calabi series of hyperkähler metrics for the cotangent bundle of CP^N (the lowest dimensional example of which corresponds to the Eguchi-Hanson gravitational instanton), is given in [22] (see also [23]). An appeal to universality then establishes the result, as before, for all hyperkähler manifolds.[3]

Our finiteness arguments following (4.4) depended on a symmetry, in that case $N = 4$ supersymmetry, to maintain the Ricci flatness of the hyperkähler metric plus counterterms, whereupon uniqueness of the Ricci flat metric excludes any counterterms. It turns out [24] that Ricci flat $N = 2$ models are distinguished among $N = 2$ models in having as well a symmetry, in this case a fermionic axial symmetry, which acts to maintain the Ricci flatness of the Kähler metric in perturbation theory. The methodology developed here then extends directly to a proof of finiteness to all orders of perturbation theory for $N = 2$ models (see [24]). Since we have no such corresponding uniqueness properties for general riemannian metrices, a proof of finiteness for $N = 1$ models must proceed on a different basis. As mentioned in Sect. 2, universality places rather severe restrictions on the tensor counterterms generated

3 Unless, as pointed out to us by C. Hull, there is some obscure reason for which *all* compact and asymptotically locally euclidean hyperkähler manifolds are so special that some combination of curvature tensors happens to vanish automatically and hence wouldn't necessarily be excluded by the above arguments from appearing as a counterterm on some more general non-compact manifold. We consider this unlikely since among the infinite classes of hyperkähler manifolds mentioned above are some which are irreducible and have no isometries, and should thus be sufficiently generic to eliminate any such accidental degenerate relations among curvature invariants

in perturbation theory: they must reduce to Kähler tensors upon imposing the Kähler condition on the metric. Those which reduce to non-vanishing Kähler tensors are excluded by a proof of finiteness for $N=2$ models. The only potential difficulty, then, would come from riemannian tensor counterterms which do not vanish on Ricci flat manifolds but would vanish upon imposing the Kähler condition. Few, if any, such terms can appear, however, since the complex structure $f^i{}_j$ is not generated explicitly in perturbation theory, forbidding the construction of any obvious projection operators. (It should also be clear that no polynomial in the curvature and covariant derivatives $P^i{}_j$ generated in perturbation theory can be accidentally equal to the complex structure. This is because the form $P_{ij}dx^i \wedge dx^j$ would then be proportional to the Kähler form J, contradicting the result that no non-trivial generators of cohomology can be generated in perturbation theory beyond one-loop order.) We consequently expect that the proof of finiteness for $N=2$ models can be extended to the $N=1$ case by showing that no potential counterterms are lost in going from arbitrary Ricci flat riemannian to Kähler manifolds and then making use of the universal dimension independence of the counterterms.

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