Commun. Math. Phys. 102, 175–206 (1985)

Communications in
Mathematical
Physics

Monodromy Fields on $\mathbb{Z}^2$

John Palmer
Department of Mathematics, University of Arizona, Tucson, *AZ* 85721, USA

Abstract. Lattice monodromy fields are defined and the massive scaling regime is controlled.

Introduction

In this paper we introduce a family of lattice fields in two dimensions which are related to and include the two dimensional Ising field. The introduction of these fields was inspired by work of Sato, Miwa, and Jimbo (SMJ) on the Riemann–Hilbert problem [17, II] and the analysis of the Euclidean Dirac equation in [17, III] and [17, IV]. In a sense explained in Sect. 2, the fields introduced here are lattice analogues of the continuum fields used by SMJ in [17, IV].

I believe these lattice fields are interesting for a number of reasons. First, by working on the lattice and controlling the scaling limit one may make mathematically precise sense of the Euclidean wave functions used to such good effect in SMJ [17, IV]. Much of the present paper is devoted to laying the foundation for an analysis of the scaled correlations carried out along the lines of the analysis of the Ising correlations in [9]. Second, the numerous analogues of continuum structures on the lattice suggests the possibility of a discrete "SMJ" analysis of the lattice correlations. Work of McCoy, Wu and Perk [3–5, 11], which demonstrates that the Ising model correlations may be expressed in terms of the solutions to non-linear partial difference equations leaves one with little doubt for the future success of such a program for the more general models considered here. The second section of this paper is devoted to a cursory look at the lattice wave functions, the finite difference equation they satisfy and the characterization of a finite dimensional family of solutions with prescribed branch points and monodromy (Theorem 2.0). The difference between the lattice case and the continuum case is instructive and I believe the role of the τ-function is considerably clarified in the lattice formulation (see the proof of Theorem 2.0). A third reason I am interested in these fields concerns critical scaling limits. By such a scaling limit we mean the large scale asymptotics for the correlations at the critical temperature (zero mass limit). It is something of a scandal that despite the enormous amount of work that has been done on the two dimensional Ising correlations there is still no definitive account of the large scale asymptotics for the higher n-point functions at T_c (by definitive I do not mean mathematically rigorous, I mean an account which informed physicists agree settles the matter). I believe the reason is a somewhat hidden mathematical subtlety in the Ising model and that the matter will not be resolved short of a rigorous demonstration. In a sequel to this paper we will rigorously calculate the long range critical asymptotics for a subclass of the monodromy fields considered here. It will be seen that the Ising model sits on a branch cut for these asymptotics as a function of the monodromy parameter. In this paper, the special role of the Ising monodromy parameter $\lambda = -1$ can also be seen in the third section on the massive scaling limit. Finally the fields introduced here may be used to write down "solvable" lattice approximations to the Euclidean versions of the Federbush and massless Thirring models. Recently, S. Ruijsenaars has established the Wightman axioms for the Federbush model [13]. Ruijsenaars works directly in the Minkowski regime and interprets his fields as densely defined quadratic forms. In his work relativistic covariance is easy but locality is hard. This situation is reversed in the Euclidean lattice approach (see [9]). In other work Ruijsenaars has established scattering theory results for the Federbush and massless Thirring fields [15] and has determined the short distance singularities of the two point functions for these fields [16]. The results of the present paper might be used to make a Euclidean attack on these problems with additional results expressing the correlations in terms of solutions to nonlinear differential equations.

This paper depends heavily on the formalism developed in [10] which in turn was inspired by Segal and Wilson's paper [18]. The reader is referred to these works and [7] for the material on spin representations necessary to understand what is going on here. We are now prepared to describe the results in this paper.

Section 1 contains a detailed analysis of the "spin operator" ε and its relation to the transfer and translation operators. The joint spectrum of transfer and translation is shown to be an elliptic curve $M^{\mathbb{C}}$ following [17, V]. There is a splitting of the Hilbert space on which the induced rotation for the transfer matrix acts which determines the representation in which the monodromy fields act. This splitting can be realized as $L^2(M_+, \mathbb{C}) \oplus L^2(M_-, \mathbb{C})$, where $M_\pm$ are two distinguished cycles on $M^{\mathbb{C}}$. When the matrix elements of ε relative to this splitting are analysed it turns out that they are convolution operators with respect to the parametrization of the cycles given by the integral of the abelian differential. The kernels of these convolution operators are elliptic functions (see 1.12) and this permits a diagonalization which is carried out at the end of Sect. 1 and which is used extensively in the 3rd section on massive scaling limits. In part this is an elaboration of the early work of Yang [19] on the calculation of the spontaneous magnetization in the Ising model (and was also suggested by results of Ruijsenaars [14] in a continuum limit). We have tried to keep track of what is happening on the full complex curve as this will facilitate the residue calculations of integrals such as those encountered in the analysis of local expansions in [9]. We also wished to expose the geometry of the situation as clearly as possible in the hope that a better understanding would clarify the relation with elliptic substitutions that produce difference kernels in more complicated situations (see Baxter [1] p. 272). This common feature of solvable lattice models does not seem to be well understood. We calculate the one point functions for the monodromy fields as a ratio of theta functions (1.14a) and their asymptotics as the critical temperature is approached (1.14b). At the end of this first section we also indicate what happens "at the critical temperature" where $M^{\mathbb{C}}$ degenerates to a singular curve.

In Sect. 2 we introduce the monodromy fields $\sigma(M)$ and attempt to motivate this terminology. The correlations for these fields are lattice analogues of the τ-functions introduced by Sato, Miwa and Jimbo [17, IV] for monodromy preserving deformations, and by Miwa, Kashiwara, Jimbo and Date [2] for "spectrum" preserving deformations.

We construct a family of solutions to a lattice analogue of the Euclidean Dirac equation with prescribed branch points $a_1, \ldots, a_n$ and monodromy $M_1, \ldots, M_n$. We show that there are finite dimensional families of such solutions which can be characterized in much the same way that SMJ characterized a distinguished family of solutions to the Euclidean Dirac equation [17, III] (see Theorem 2.0). It is interesting that "most of the time" ($\tau(a, M) \neq 0$) there are *no* non-trivial solutions to the homogeneous finite difference equations with prescribed monodromy which vanish at ∞. There are however families of solutions with localized inhomogeneities. The solutions are uniquely determined by the inhomogeneous term precisely when the τ-function ($\tau(a, M)$) does not vanish. The formula (2.8) for the wave functions with localized inhomogeneity is then a group theoretic Fredholm formula for the solution of a linear equation. It is interesting to see what happens when the lattice solutions are scaled to their continuum limits. The inhomogeneity can be "hidden" at the branch points (this is done for the Ising case in [9]) or it can be left "out in the open" to become a normalization point (as is done for the Riemann–Hilbert problem [17, II], see also the recent work of Malgrange [6] for a beautiful geometric analysis of the deformation problem). Section 2 concludes with some results for monodromy fields $\sigma_a(M)\sigma_b(M^{-1})$. Such products can be expressed as path ordered exponentials of "currents" for paths joining a to b on the lattice. The monodromy M can be thought of as emerging at M and then disappearing at b. This "containment" of monodromy seems to be important for results concerning limits of correlations as the critical temperature is approached. In a sequel we will study the behavior of the critical correlations for products of pairs $\sigma_{a_i}(M_i)\sigma_{b_i}(M_i^{-1})(i = 1, \ldots n)$. We also show that the critical correlations $\langle \sigma_{a_i}(M_1) \ldots \sigma_{a_n}(M_n) \rangle$ exist when $M_1 \ldots M_n = 1$, by making use of the similar result for such paired correlations. Finally (2.23) shows that conjugation by the induced rotations for the monodromy fields preserves the finite difference band structure of the transfer matrix. This is reminiscent of the iso-spectral deformation of Toda type and may be a "reason" for the surprisingly intimate relationship between $s(M)$ and T revealed in Sect. 1.

In the third section we prove the convergence of the τ-functions in the massive scaling regime. We also obtain explicit formulas for these functions as determinants, but they are not written out in the general case; the general formulas being somewhat unenlightening. We don't expect that these "mixed" τ-functions will be the Schwinger functions of a quantum field theory except when all the monodromy matrices are taken to be equal. In this case we do expect to be able to prove the Osterwalder–Schrader axioms along the lines developed in [9] for the scaling limit of the Ising model. In that analysis symmetry, positivity and clustering are easy. The hard axioms is rotational invariance. Local rotational invariance (in FKN sectors) turns out to be a consequence of the association of the τ-functions with solutions to non-linear differential equations that have a rotational symmetry and global rotational invariance follows from invariance of the correlations under rotation by $\pi/2$ radians (known in the Ising case). This last invariance property is not anticipated for the monodromy fields considered here, but there are intriguing relations between rotation by $\pi/2$, a shift of the cycles $M_{\pm}$ on $M^{\mathbb{C}}$, and the duality transformation for the Ising model which remain to be investigated (see Sect. 2).

Section 1

In the analysis of the 2-dimensional Ising model which is presented in [17, V, 8, 9] there are two principal ingredients, the transfer matrix and the spin operator. In this section we will introduce the analogous operators for the scalar monodromy fields that we consider first. As in the Ising case both operators belong to a group representation which lives in the spin representation of a Clifford algebra. We begin with a description of the induced rotation for the transfer matrix. It is essentially a direct sum of two copies of the Ising model version, but because we use the formalism of [10] rather than [7] it will not at first appear to be so. We use the Ising transfer matrix since we wish to make applications of these results to critical scaling calculations later on. Let $H = L^2(S^1, C^2)$ and define the matrix valued multiplication operator T by:

$$Tf(\theta) = \begin{bmatrix} c^2/s - \cos\theta & s\sin\theta - i(c/s - c\cos\theta) \\ s\sin\theta + i(c/s - c\cos\theta) & c^2/s - \cos\theta \end{bmatrix} f(\theta),$$

where $c > 0$, $s > 0$ and $c^2 - s^2 = 1$. In the notation of [8], $c = \cosh 2K^*$. When $s < 1$ this is the Ising model below the critical temperature, at $s = 1$ it is the critical Ising model, and $s > 1$ corresponds to the Ising model above the critical temperature. In the present section we will restrict our attention to the noncritical values $s < 1$. As in [8] we introduce functions $\gamma(\theta) > 0$ and $\alpha(\theta)$ defined by:

$$\cosh\gamma(\theta) = c^2/s - \cos\theta,$$
$$\sinh\gamma(\theta)e^{i\alpha(\theta)} = (c/s - c\cos\theta) + is\sin\theta.$$

For $s \neq 1$ the functions $\gamma(\theta)$ and $\alpha(\theta)$ may be chosen so that they are smooth on the circle. The curve $\theta \to e^{i\alpha(\theta)}$ has winding number 0 when $s < 1$ and winding number -1 when $s > 1$. In each case we fix a determination of $\alpha(\theta)$ by requiring $\alpha(0) = 0$. It is easy to check that $T(\theta) = \exp[-\gamma(\theta)Q(\theta)]$, where

$$Q(\theta) = \begin{bmatrix} 0 & ie^{i\alpha(\theta)} \\ -ie^{-i\alpha(\theta)} & 0 \end{bmatrix}.$$

Multiplication by $Q(\theta)$ is a self-adjoint idempotent which we denote by Q. Let $Q_{\pm} = \frac{1}{2}(1 \pm Q)$. Let $\bar{H}$ denote the Hilbert space conjugate to H. It is the same real Hilbert space as H except that complex multiplication is given by $(-i)$ instead of i. If X is a linear map on H it induces a linear map $\bar{X}$ on $\bar{H}$ (given by $\bar{X} = X$ with the obvious abuse of notation). Let $W = H \oplus \bar{H}$, and write P for the conjugation on W

given by $\overline{x \oplus y} = y \oplus x$. Then $Q_W \stackrel{\text{def}}{=} Q \oplus (-\bar{Q})$ is a self-adjoint idempotent on W which anti-commutes with P. We are interested in the Q_W Fock state on the Clifford algebra $C(W, P)$ whose associated representation lives on the alternating tensor algebra $A(W_+)$, where $W_\pm = Q_W^\pm W$ and $Q_W^\pm = \frac{1}{2}(1 \pm Q_W)$ [10]. In this representation the complex orthogonal $T \oplus \bar{T}^{*-1}$ (complex orthogonal in the sense that it preserves the complex bilinear form $\langle \cdot, \bar{\cdot} \rangle$) induces an automorphism of $C(W, P)$ which is implemented by

$$V_2 = I \oplus \tilde{T} \oplus (\tilde{T} \otimes \tilde{T}) \oplus \cdots,$$

where $\tilde{T} = (Q_+ T) \oplus (Q_- \bar{T}^{*-1}) = (Q_+ T) \oplus (Q_- \bar{T}^{-1}) \sim$ "multiplication by $e^{-\gamma(\theta)} \oplus e^{-\gamma(\theta)}$" as may be easily checked. The operator V_2 is a self adjoint contraction on $A(W_+)$, which we refer to as the transfer matrix (it is isomorphic to the tensor product of two copies of the Ising transfer matrix).

Next we introduce the induced rotations for the scalar monodromy fields. Let

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} d\theta, \quad k \in Z + \tfrac{1}{2} = Z_{1/2}$$

denote the half integer Fourier coefficients of f. Define a map ε on H by $\widehat{\varepsilon f}(k) = \operatorname{sgn}(k) \hat{f}(k)$, and write $s(\lambda) = \frac{1}{2}((1 - \varepsilon) + (1 + \varepsilon)\lambda)$ for $\lambda \in \mathbb{C}$ (the Ising field is related to the case $\lambda = -1$). In the terminology of [10] we wish to show that $s(\lambda) \in \mathrm{Gl}_Q(H)$. If $H_+ \oplus H_-$ is the decomposition of H into the $+1$ and -1 eigenspaces for Q, we must show that the matrix $\begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix}$ of $s(\lambda)$ relative to this decomposition consists of transformations $a(\lambda)$ and $d(\lambda)$ which are Fredholm of index 0 and transformations $b(\lambda)$ and $c(\lambda)$ which are in the Schmidt class [10]. It is easy to see that the off diagonal parts $b(\lambda)$ and $c(\lambda)$ will be in the Schmidt class provided the commutator of Q and $s(\lambda)$ is in the Schmidt class. Thus it is enough to prove this for $[Q, \varepsilon]$. It is not hard to show that the Schmidt norm of $[Q, \varepsilon]$ is proportional to $\sum_{n \in \mathbb{Z}^+} n \operatorname{Tr}(\hat{Q}(n)\hat{Q}(-n))$, where $\mathbb{Z}^+$ = set of positive integers. This will be finite when $s \neq 1$ since $\alpha(\theta)$ may be chosen to be smooth on S^1 in this event. In a moment we will consider the problem of diagonalizing $d(\lambda)$. This will prove possible for $s < 1$ and will show at the same time that $d(\lambda)$ is invertible (except for an explicit sequence of negative values for λ) and hence of index 0. Once this is proved we know from [10] that there is an element $\sigma(\lambda) \in \hat{Gl}_Q(H)$ whose induced rotation is $s(\lambda)$ and whose vacuum expectation $\langle \sigma(\lambda) \rangle$ does not vanish. We postpone describing precisely the normalization we choose for $\sigma(\lambda)$ until after we have diagonalized $d(\lambda)$, but note here that the normalization will make $\sigma(\lambda)$ unitary when $|\lambda| = 1$. Now let V_1 denote the element of $\hat{\mathrm{Gl}}_Q(H)$ whose induced rotation on H is $e^{i\theta}$ normalized so that $V_1 1 = 1$ (V_1 is spacial translation by one unit). For $a \in \mathbb{Z}^2$ let $V(a) = V_1^{a_1} V_2^{a_2}$, and write $\sigma_a(\lambda) = V(a)\sigma(\lambda)V(a)^{-1}$. Then $\sigma_a(\lambda) \in \hat{\mathrm{Gl}}_Q(H)$, and we shall be interested in correlations of the form $\left\langle \vec{T} \prod_{j=1}^{n} \sigma_{a_j}(\lambda_j) \right\rangle$, where $\vec{T}$ is "time" ordering: the elements in the product are ordered so that the second coordinates of the a_j are increasing as one moves from left to right in the product. We will see later that if a and b have the same second coordinate, then $\sigma_a(\lambda)$ and $\sigma_b(\mu)$ commute so that there is no ambiguity in this definition. When $\lambda = -1$

these are squares of Ising correlations when the fields $\sigma(\lambda)$ are normalized as described below.

Next we present an analysis of the operators $d(\lambda)$. For this purpose it is convenient to first analyse the spectral decomposition of the transfer matrix in more detail. Let $z=e^{i\theta}$, following [17, V] we consider the characteristic equation $\det(T(z)-wI)=0$. After a little calculation one finds this is equivalent to:

$$\frac{z+z^{-1}}{2}+\frac{w+w^{-1}}{2}=c^2/s.$$

For $s\neq\pm 1,0$ this determines a non-singular elliptic curve $M^{\mathbb{C}}$. For the spectral analysis of T the relevant part of this complex curve consists of the two cycles $M_+=\{(z,w)\in M|(z,w)=(e^{i\theta},e^{-\gamma(\theta)})\}$ and $M_-=\{(z,w)\in M|(z,w)=(e^{-i\theta},e^{\gamma(\theta)})\}$. (The orientation on M_- is reversed to make the holomorphic differential, introduced later, positive on each cycle $M_\pm$). Curiously, some features of the diagonalization of $d(\lambda)$ depend on constructions which are natural on the complex curve $M^{\mathbb{C}}$, in particular the holomorphic differential on $M^{\mathbb{C}}$ and the associated uniformization parameter. Again following [17, V] we introduce homogeneous coordinates $(\zeta_0,\zeta_1,\zeta_2)$ so that $z=\zeta_1/\zeta_0$ and $w=\zeta_2/\zeta_0$. The branch points in the two sheeted covering projection $M^{\mathbb{C}}\ni(\zeta_0,\zeta_1,\zeta_2)\to(\zeta_0,\zeta_1)\in P^1(\mathbb{C})$ (i.e., the projection on z) are easily seen to be the zeros of the discriminant:

$$\frac{c^2}{s}\zeta_0\zeta_1-\frac{\zeta_1^2+\zeta_0^2}{2}-\zeta_1^2\zeta_0^2=0$$

or in terms of z (since $\zeta_0=0$ does not produce a root):

$$\left(\frac{c^2}{s}z-\frac{z^2+1}{2}\right)^2-z^2=0,$$

which has the 4 roots:

$$\alpha_1=\frac{(c-s)(c-1)}{s},\quad \alpha_1^{-1}=\frac{(c+s)(c+1)}{s},\quad \alpha_2=\frac{(c+s)(c-1)}{s},$$
$$\alpha_2^{-1}=\frac{(c-s)(c+1)}{s}.$$

A holomorphic differential on M is thus given by $idz/\pi\sqrt{(z-\alpha_1)(z-\alpha_1^{-1})\cdot(z-\alpha_2)(z-\alpha_2^{-1})}$, which after some calculation is seen to be $\omega=idz/\pi z(w-w^{-1})$ (making an arbitrary choice of sign which is fixed hereafter). On $M_\pm$ this differential is thus $d\theta/2\pi\sinh\gamma(\theta)$.

Living on the curve M is a line bundle whose fiber over (z,w) is the null space of $T(z)-wI$. It is natural to realize the direct integral decomposition of $L^2(S^1,\mathbb{C}^2)$ which diagonalizes $T(\theta)$ as a direct sum of L^2 sections of this bundle lying over the cycles $M_\pm$. We will make the transformation to this direct integral decomposition explicit by using the abelian differential $d\theta/2\pi\sinh(\theta)$ as a measure on the cycles $M_\pm$ and by choosing trivializations of the line bundle over $M_\pm$. To motivate this observe that

$$T(\theta)=e^{-\gamma(\theta)}Q_+(\theta)+e^{\gamma(\theta)}Q_-(\theta),$$

where $Q_\pm(\theta) = \frac{1}{2}\begin{bmatrix} 1 & \pm ie^{i\alpha(\theta)} \\ \mp ie^{-i\alpha(\theta)} & 1 \end{bmatrix}$. It is clear from this that $Q_\pm\ (\pm\theta)$ projects onto the fiber in the bundle over the point $(e^{\pm i\theta}, e^{\mp\gamma(\theta)})$ in $M_\pm$. If $f(\theta) \in L^2(S^1, \mathbb{C}^2)$, then $(\sinh\gamma(\theta))^{1/2}\, Q_+(\theta) f(\theta)$ is the part of f lying over M_+ (square integrable with respect to $d\theta/2\pi \sinh\gamma(\theta)$) and $(\sinh\gamma(\theta))^{1/2} Q_-(-\theta) f(-\theta)$ is the analogous part of f lying over M_-. Observe that

$$\begin{aligned}
(\sinh\gamma(\theta))^{1/2} Q_+(\theta) f(\theta) &= (\sinh\gamma(\theta)/2)^{1/2}(e^{-i\alpha(\theta)/2} f_1(\theta) + ie^{i\alpha(\theta)/2} f_2(\theta)) e_+(\theta), \\
(\sinh\gamma(\theta))^{1/2} Q_-(\theta) f(\theta) &= (\sinh\gamma(\theta)/2)^{1/2}(e^{-i\alpha(\theta)/2} f_1(\theta) - ie^{i\alpha(\theta)/2} f_2(\theta)) e_-(\theta), \qquad (1.1)
\end{aligned}$$

where $e_\pm(\theta) = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\alpha(\theta)/2} \\ \mp ie^{i\alpha(\theta)/2} \end{bmatrix}$. The reason for choosing the trivializations $e_\pm(\theta)$ has to do with the fact that the coefficients $(\sinh\gamma e^{\pm i\alpha})^{1/2}$ are somewhat easier to work with than $(\sinh\gamma)^{1/2}$ or $e^{i\alpha/2}$ separately.

Based on (1.1) we introduce:

$$\begin{aligned}
\sqrt{2} f_+(\theta) &= (\sinh\gamma(\theta)^{1/2}(e^{-i\alpha(\theta)/2} f_1(\theta) + ie^{i\alpha(\theta)/2} f_2(\theta)), \qquad -\pi \leqq \theta \leqq \pi, \\
\sqrt{2} f_-(\theta) &= (\sinh\gamma(\theta)^{1/2}(e^{-i\alpha(\theta)/2} \bar{f}_1(-\theta) + ie^{i\alpha(\theta)/2} \bar{f}_2(-\theta)), \quad -\pi \leqq \theta \leqq \pi.
\end{aligned} \qquad (1.2)$$

The functions $f_\pm$ may be regarded as functions on $M_\pm$. We have changed θ to $-\theta$ in f_- to match the orientation on M_-, and we have also introduced complex conjugation in f_- so that the real orthogonal map $(f_1, f_2) \xrightarrow{D} (f_-, f_+)$ from $L^2(S^1, \mathbb{C}^2)$ to $L^2(M_-) \oplus L^2(M_+)$ will take the complex structure $\Lambda = iQ$ into i on $L^2(M_-) \oplus L^2(M_+)$. This has the useful consequence that the decomposition of $s(\lambda)$ into $\begin{pmatrix} a(\lambda) & 0 \\ 0 & d(\lambda) \end{pmatrix}$ and $\begin{pmatrix} 0 & b(\lambda) \\ c(\lambda) & 0 \end{pmatrix}$ corresponds to the linear-conjugate linear decomposition of $Ds(\lambda)D^*$. We are now prepared to consider the diagonalization of $d(\lambda)$. It is evident from $s(\lambda) = (1+\varepsilon)/2 + \lambda(1-\varepsilon)/2$, that it will suffice to diagonalize $Q_-\varepsilon Q_-$. Let $s(\theta) = 2e^{-i\theta/2}/(1 - e^{-i\theta})$; then we have the principal value representation:

$$\varepsilf(\theta) = \tfrac{1}{2}\int_0^{2\pi} [s(\theta - \theta' + i0) + s(\theta - \theta' - i0)] f(\theta') \frac{d\theta'}{2\pi}.$$

We will use (1.2) to transform this integral operator to $L^2(M_-) \oplus L^2(M_+)$. We will then introduce a uniformization parameter u such that $du = d\theta/2\pi \sinh\gamma(\theta)$ and verify that the operators $Q_\pm\varepsilon Q_\pm$ become convolution operators in the u variable. At this point it is convenient to restrict our attention to the case $s < 1$. In this event $e^{i\alpha(\theta)}\sinh\gamma(\theta)$ has winding number 0 and we may choose a unique continuous square root on the circle by requiring that this square root is positive for $\theta = 0$. This choice will be implicit in (1.2) for the rest of this section. Before proceeding further it will also be useful to make a fractional linear transformation of $z = e^{i\theta}$, which puts the branch points $\alpha_1^{\pm 1}$, $\alpha_2^{\pm 1}$ into canonical positions 0, ∞, k^2, 1 where $k = s^2$. This will allow us to use the standard Jacobi elliptic functions. A fractional linear transformation that does this is:

$$x = k\frac{z-\alpha_1}{1-\alpha_1 z}, \quad z = \frac{x+\alpha_1 k}{k+\alpha_1 x}. \tag{1.3}$$

As z winds counterclockwise around the circle $|z|=1$, the variable x winds counterclockwise around the circle $|x|=k$. Let $z_1 = e^{i\theta}$, $z_2 = e^{i\theta'}$, then the kernel for ε is the principal value determination of

$$\frac{2\sqrt{z_2^{-1}}\sqrt{z_1}}{1-z_2^{-1}z_1},$$

where the square roots are taken with $\arg(z_j)\in[0,2\pi)$. We transform this kernel into the x variables, making use of:

$$\begin{aligned} \frac{dz}{2\pi iz} &= \frac{k(1-\alpha_1^2)}{(1+\alpha_1 kx^{-1})(k+\alpha_1 x)}\frac{dx}{2\pi ix}, \\ \sqrt{z} &= \frac{(1+\alpha_1 kx^{-1})^{1/2}}{(k+\alpha_1 x)^{1/2}}\sqrt{x}, \end{aligned} \tag{1.4}$$

where $1+\alpha_1 kx^{-1}$ and $k+\alpha_1 x$ have winding number 0 on $|x|=k$ and square roots normalized to $+(1+\alpha_1)^{1/2}$ and $+(k(1+\alpha_1))^{1/2}$ at $x=k$. The square root of x which appears is fixed by $\arg x\in[0,2\pi)$.

The map:

$$f(z)\to f(z(x))\frac{(k(1-\alpha_1^2))^{1/2}}{(1+\alpha_1 kx^{-1})^{1/2}(k+\alpha_1 x)^{1/2}} \tag{1.5}$$

is a unitary transformation from $L^2(S^1, dz/2\pi iz)$ to $L^2(S_k^1, dx/2\pi ix)$, where S_k^1 = circle of radius k. Under this transformation the kernel for ε becomes the principal value determination of $s(x_2,x_1) \overset{\mathrm{def}}{=} 2\sqrt{x_2^{-1}}\sqrt{x_1}/(1-x_2^{-1}x_1)$. The abelian differential in the x variables is (substitute $z=z(x)$ in dz and $(z-\alpha_i^{\pm 1})$ in the formula for ω):

$$\omega = \frac{k^{1/2}}{[(1-k^2x^{-1})(1-x)]^{1/2}}\frac{dx}{2\pi ix}, \tag{1.6}$$

where the sign ambiguity in the square root on the right-hand side cannot, of course, be resolved in the x coordinates. In order to use (1.1) we wish to transform $(e^{\pm i\alpha(\theta)}\sinh\gamma(\theta))^{1/2}$ into the x variables. Since $e^{i\alpha(\theta)}\sinh\gamma(\theta) = ((c+1)/2s)(1-\alpha_1 z)\cdot(1-\alpha_2 z^{-1})$, one finds

$$e^{i\alpha(\theta)}\sinh\gamma(\theta) = \frac{2(c-1)}{s}\frac{1-k^2x^{-1}}{(k+\alpha_1 x)(1+\alpha_1 kx^{-1})},$$

so that

$$(e^{i\alpha(\theta)}\sinh\gamma(\theta))^{1/2} = \left(\frac{2(c-1)}{s}\right)^{1/2}\frac{(1-k^2x^{-1})^{1/2}}{(k+\alpha_1 x)^{1/2}(1+\alpha_1 kx^{-1})^{1/2}}. \tag{1.7}$$

The square roots are all normalized so that they are positive at $x=k$. Since $z\to\bar{z}$ induces the transformation $x\to\bar{x}$, we may rewrite (1.2) in the x variables taking into account (1.7) and composition with (1.5) as follows:

$$\begin{aligned}\sqrt{2}f_+(x) &= \overline{q(x)}f_1(x) + iq(x)f_2(x),\\ \sqrt{2}f_-(x) &= \overline{q(x)}\bar{f}_1(\bar{x}) + iq(x)\bar{f}_2(\bar{x}),\end{aligned} \tag{1.7}$$

where $q(x) = [sc(s+1)(c-s)]^{-1/2}(1-k^2x^{-1})^{1/2}$. The inverse map is

$$\begin{aligned}\sqrt{2}f_1(x) &= \overline{q(x)^{-1}}(f_+(x) + \bar{f}_-(\bar{x})),\\ i\sqrt{2}f_2(x) &= q(x)^{-1}(f_+(x) - \bar{f}_-(\bar{x})).\end{aligned} \tag{1.8}$$

The transformation of ε using (1.6) and (1.7) yields:

$$\begin{aligned}(\varepsilon f)_+(x_2) &= \oint_{S_k^1} s(x_2,x_1)\,\mathrm{Re}\,(q(x_2)q(x_1)^{-1})f_+(x_1)\frac{dx_1}{2\pi i x_1}\\ &\quad - i\oint_{S_k^1} s(x_2,x_1)\,\mathrm{Im}\,(q(x_2)q(x_1)^{-1})\bar{f}_-(\bar{x}_1)\frac{dx_1}{2\pi i x_1},\\ (\varepsilon f)_-(x_2) &= i\oint_{S_k^1} \bar{s}(\bar{x}_2,x_1)\,\mathrm{Im}\overline{(q(x_2)}q(x_1)^{-1})\bar{f}_+(x_1)\frac{dx_1}{2\pi i x_1}\\ &\quad + \oint_{S_k^1} \bar{s}(\bar{x}_2,x_1)\,\mathrm{Re}\overline{(q(x_2)}q(x_1)^{-1})f_-(\bar{x}_1)\frac{dx_1}{2\pi i x_1}.\end{aligned} \tag{1.9}$$

To give this a more symmetrical look we may change the integration variable x_1 to $\bar{x}_1$ in the equation for $(\varepsilon f)_-(x_2)$, make use of $\bar{s}(\bar{x}_2,\bar{x}_1) = s(x_2,x_1)$, $q(\bar{x}_1) = \overline{q(x_1)}$ and take care with the orientation of the integrals to obtain:

$$\begin{aligned}(\varepsilon f)_-(x_2) &= -i\oint_{S_k^1} s(x_2,x_1)\,\mathrm{Im}\,(q(x_2)q(x_1)^{-1})\bar{f}_+(\bar{x}_1)\frac{dx_1}{2\pi i x_1}\\ &\quad + \oint_{S_k^1} s(x_2,x_1)\,\mathrm{Re}\,(q(x_2)q(x_1)^{-1})f_-(x_1)\frac{dx_1}{2\pi i x_1}.\end{aligned} \tag{1.10}$$

In (1.9) and (1.10) we now replace the measure $dx_1/2\pi i x_1$ with the abelian differential (1.6). Those integrals in (1.9) and (1.10) which involve f_+ may be thought of as integrals on M_+. On M_+ we have

$$\omega = \frac{k^{1/2}}{[(1-k^2x^{-1})(1-x)]^{1/2}}\frac{dx}{2\pi i x},$$

where the square root is taken to be positive (note $(1-k^2x^{-1})(1-x) > 0$ for $|x| = k$). Those integrals in (1.9) and (1.10) which involve f_- may be thought of as integrals on M_-. On M_- we have

$$\omega = -\frac{k^{1/2}}{[(1-k^2x^{-1})(1-x)]^{1/2}}\frac{dx}{2\pi i x},$$

where the square root is again taken to be positive. We may thus rewrite (1.9) and (1.10):

$$(\varepsilon f)_+(x_2)=\frac{1}{\sqrt{k}}\int_{M_+} s(x_2,x_1)\mathrm{Re}((1-k^2x_2^{-1})^{1/2}(1-x_1)^{1/2})f_+(x_1)\omega(x_1)$$
$$-\frac{i}{\sqrt{k}}\int_{M_-} s(x_2,x_1)\,\mathrm{Im}\,((1-k^2x_2^{-1})^{1/2}(1-x_1)^{1/2})\bar{f}_-(\bar{x}_1)\omega(x_1),$$
$$(\varepsilon f)_-(x_2)=\frac{1}{\sqrt{k}}\int_{M_-} s(x_2,x_1)\,\mathrm{Re}((1-k^2x_2^{-1})^{1/2}(1-x_1)^{1/2})f_-(x_1)\omega(x_1)$$
$$-\frac{i}{\sqrt{k}}\int_{M_+} s(x_2,x_1)\mathrm{Im}((1-k^2x_2^{-1})^{1/2}(1-x_1)^{1/2})\bar{f}_+(\bar{x}_1)\omega(x_1), \quad (1.11)$$

where we made use of the fact that $[(1-k^2x_1^{-1})(1-x_1)]^{1/2}$ is real and all the square roots in (1.11) are normalized to be positive when $x=k$. We also integrate over M_- with the orientation given by $\theta\to(e^{-i\theta},e^{\gamma(\theta)})$. The integrals in (1.11) are ripe for an elliptic substitution. Since we don't want to lose contact with the curve $M^{\mathbb{C}}$ we will not do this in the most direct fashion. Instead we now introduce a convenient uniformization of the entire curve $M^{\mathbb{C}}$. Let $p_0=(-1,e^{-\gamma(\pi)})$ (z variables) $=(-k,e^{-\gamma(\pi)})$(x variables). Then $p_0\in M_+$. Let iK' denote the imaginary quarter period associated with the elliptic modulus k. Define:

$$u=\frac{ik'}{2}+\frac{\pi}{\sqrt{k}}\int_{p_0}^{p}\omega.$$

The abelian differential is $\omega=\sqrt{k}[(1-k^2x^{-1})(1-x)]^{-1/2}\,dx/2\pi ix$. The sign of the square root is determined on M_+ so that ω is positive on counterclockwise oriented tangent vectors. Now let $x=sn^{-2}(v_++iK'/2)$, where $-K\leqq v_+\leqq K$. Then $v_+\to sn^{-2}(v_++iK'/2)$ runs counterclockwise around $|x|=k$ starting at $x=k$ as v_+ goes from $-K$ to K. For $p\in M_+$ we have:

$$\frac{\pi}{k}\int_{p_0}^{p}\omega=\frac{\pi}{k}\int_{-k}^{x(p)}\frac{\sqrt{k}}{[(1-k^2x^{-1})(1-x)]^{1/2}}\frac{dx}{2\pi ix}=\int_0^{v_+}dv_+=v_+,$$

where

$$[(1-k^2x^{-1})(1-x)]^{1/2}=\frac{idn\left(v_++\frac{iK'}{2}\right)cn\left(v_++\frac{iK'}{2}\right)}{sn\left(v_++\frac{iK'}{2}\right)}$$

on M_+, and we used $cn^2(u)+sn^2(u)=1$, $dn^2(u)+k^2sn^2(u)=1$ and $d(sn(u))=cn(u)dn(u)du$. Thus on M_+ we have $u=v_++iK'/2$ and $x=sn^{-2}(u)$. We next wish to locate M_- in the period parallelograms of u:

$$-K\leqq \mathrm{Re}(u)\leqq K,\quad -K'\leqq \mathrm{Im}\,(u)\leqq K'.$$

M_- is determined by $|x|=k$ and $(w-w^{-1})>0$. From (1.6),

$$w-w^{-1}=\frac{2((1-k^2x^{-1})(1-x))^{1/2}}{(1+\alpha_1kx^{-1})(k+\alpha_1x)}=\frac{-2idn(u)cn(u)sn^{-1}(u)}{(1+\alpha_1kx^{-1})(k+\alpha_1x)}\ \text{on}\ M_+.$$

Both the left- and right-hand sides are meromorphic on all of M, and since

$(1+\alpha_1 kx^{-1})(k+\alpha_1 x)>0$ for $|x|=k$ the condition that $(x,w)\in M_-$ becomes

$$|sn^{-2}(u)|=k, \quad -idn(u)cn(u)sn^{-1}(u)>0.$$

These conditions are satisfied for $u=v_- - iK'/2$ with $-K\leqq v_-\leqq K$. The curve $v_-\to sn^{-2}(v_- - iK'/2)$ winds clockwise around $|x|=k$ starting at $x=k$ as v_- goes from $-K$ to K. This is the appropriate orientation for M_- and we find $\omega=(k/\pi)dv_-$ on M_-.

Finally we make the substitutions $x_j=sn^{-2}(v_j^+ + iK'/2)$ when $x_j\in M_+$ and $x_j=sn^{-2}(v_j^- - iK'/2)$ when $x_j\in M_-$ in (1.11) (deciding $x_j\in M_\pm$ if it is the argument of a function $g_\pm$). It is useful to observe that since $\bar{x}_j=k^2/x_j$ we have:

$$\begin{aligned}2\mathrm{Re}((1-k^2x_2^{-1})^{1/2}(1-x_1)^{1/2}) &= (1-k^2x_2^{-1})^{1/2}(1-x_1)^{1/2}\\ &\quad +(1-x_2)^{1/2}(1-k^2x_1^{-1})^{1/2},\\ 2i\mathrm{Im}((1-k^2x_2^{-1})^{1/2}(1-x_1)^{1/2}) &= (1-k^2x_2^{-1})^{1/2}(1-x_1)^{1/2}\\ &\quad -(1-x_2)^{1/2}(1-k^2x_1^{-1})^{1/2}.\end{aligned}$$

The result is:

$$\begin{aligned}(\varepsilon f)_+(v_2^+) &= \frac{i}{\pi}\int_{-K}^{K} ds(v_2^+ - v_1^+)f_+(v_1^+)dv_1^+ - \frac{k}{\pi}\int_{-K}^{K} cn(v_2^+ - v_1^-)\bar{f}_-(-v_1^-)dv_1^-,\\ (\varepsilon f)_-(v_2^-) &= -\frac{i}{\pi}\int_{-K}^{K} ds(v_2^- - v_1^-)f_-(v_1^-)dv_1^- - \frac{k}{\pi}\int_{-K}^{K} cn(v_2^- - v_1^+)\bar{f}_+(-v_1^+)dv_1^+,\end{aligned} \tag{1.12}$$

where we made use of $ds(u_2-u_1)=(s_1d_1c_2+s_2d_2c_1)/(s_1^2-s_2^2)$ with $s_j=sn(u_j)$, $d_j=dn(u_j)$ and $c_j=cn(u_j)$ and $ds(u+iK')=-ikcn(u)$. (Note $ds(u)=dn(u)/sn(u)$.) The integrals involving $ds(\cdot)$ should be interpreted in the principal value sense. We also made use of the fact that since $\bar{x}=k^2/x$ we have $\overline{ns^2(v+(iK'/2))}=k^2sn^2(v+(iK'/2))$. But $k^2sn^2(v+(iK'/2))=ns^2(v-(iK'/2))$ (see Table 16.8 in [20]) and $ns^2(v-(iK'/2))=ns^2(-v+(iK'/2))$. Thus $x\to\bar{x}$ is equivalent to $v\to -v$.

If $\varepsilon=\begin{bmatrix} a & b\\ c & d\end{bmatrix}$ in the $H_+\oplus H_-$ decomposition of H, then as mentioned earlier, Eq. (1.12) gives a particularly simple form for the kernels of a,b,c,d. The subspaces H_+ and H_- are identified with $L^2(M_+,\omega)$ and $L^2(M_-,\omega)$ respectively. Thus for $f\in L^2(M_-,\omega)$ we have:

$$\begin{aligned}df(x) &= -\frac{i}{\pi}\int_{-K}^{K} ds(x-y)f(y)dy \,(\text{principal value}),\\ bf(x) &= -\frac{k}{\pi}\int_{-K}^{K} cn(x-y)\bar{f}(-y)dy.\end{aligned}$$

We are now prepared to diagonalize d. Let $q=e^{-\pi K'/K}$, then

$$ds(u)=\frac{\pi}{2K}\csc\frac{\pi u}{2K}+\frac{i\pi}{K}\sum_{l\in Z_{1/2}}\mathrm{sgn}(l)\frac{q^{2|l|}}{1+q^{2|l|}}e^{il\pi u/K} \tag{1.13}$$

(see 16.23.11 in [20]), where $Z_{1/2}=Z+1/2$ and $\varepsilon(l)=\mathrm{sgn}(l)$. Replace $\csc(\pi u/2K)$ with $(1/2)(\csc\pi(u+i\varepsilon)/2K+\csc\pi(u-i\varepsilon)/2K)$ in the last formula and call the result $ds_\varepsilon(u)$

(taking principal values of the "regular" part is unnecessary). Then d is given by the limit $\varepsilon \to 0$ of convolution with $-(i/\pi)d_\varepsilon$ on the interval $[-K, K]$. Suppose $l \in Z_{1/2}$, then

$$\frac{i}{\pi}\int_{-K}^{K} ds_\varepsilon(x-y)e^{(il\pi/K)y}dy = \frac{i}{\pi}\int_{-K-x}^{K-x} ds_\varepsilon(-y)e^{(il\pi/K)(y+x)}dy$$

$$= \left[\frac{i}{\pi}\int_{-K-x}^{K-x} ds_\varepsilon(-y)e^{(il\pi/K)}dy\right]e^{(il\pi/K)x} = \left[\frac{i}{\pi}\int_{-K}^{K} ds_\varepsilon(y)e^{-(il\pi/K)y}dy\right]e^{il\pi x/K}.$$

This last equality follows from the fact that $d_\varepsilon(-y)\,e^{(il\pi/K)y}$ is a $2K$ periodic function of y, although this is not true for the factors separately. Thus the functions $e^{il\pi x/K}$ are eigenfunctions for convolution with ds_ε, with eigenvalues $\int_{-K}^{K} ds_\varepsilon(y)\,e^{(-il\pi/K)y}\,dy$. A simple residue calculations shows that:

$$\frac{1}{2K}\int_{-K}^{K}\csc\frac{\pi(y+i\varepsilon)}{2K}e^{-il\pi y/K}dy = \begin{cases} -2ie^{-l\varepsilon} & l>0 \\ 0 & l<0\text{'} \end{cases}$$

$$\frac{1}{2K}\int_{-K}^{K}\csc\frac{\pi(y-i\varepsilon)}{2K}e^{-il\pi y/K}dy = \begin{cases} 0 & l>0 \\ 2ie^{l\varepsilon} & l<0\text{'} \end{cases}$$

where $l \in Z_{1/2}$. It follows from this and (1.13) that the eigenvalues for convolution with $-(i/\pi)ds_\varepsilon$ are $-\operatorname{sgn}(l)(e^{-|l|\varepsilon} - 2q^{2|l|}/(1+q^{2|l|}))$ with eigenfunctions $e^{(il\pi/K)x}$. The limit $\varepsilon \to 0$ is now obvious and reveals eigenvalues $-\operatorname{sgn}(l)(1-q^{2|l|})/(1+q^{2|l|}) = (-1+q^{2l})/(1+q^{2l})$ for d. Since $s(\lambda) = (\lambda+1)/2 + ((\lambda-1)/2)\varepsilon$ it follows that $d(\lambda) = (\lambda+1)/2 + ((\lambda-1)/2)d$. Thus $d(\lambda)$ is Fredholm for $\lambda \in \mathbb{C}$, and as is easily seen it is invertible for $\lambda \neq -q^{2l}$, $l \in Z_{1/2}$. (Note that when λ is not a negative number, $d(\lambda)$ is invertible for the completely trivial reason that d is self-adjoint with spectrum between -1 and 1.)

We are now prepared to discuss the normalization of $\sigma(\lambda)$. Recall from [10] that $\hat{\mathrm{Gl}}_Q(H)$ is a semi-direct product of the subgroup of $\mathrm{Gl}_Q(H)$ with the "d slot" = identity + trace class, and the subgroup of elements of the form $\begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}$. A choice of normalization for $\sigma(\lambda)$ is equivalent to a choice of factorization $d(\lambda) = \underline{d}(\lambda)D(\lambda)$ where $(\underline{d}(\lambda) - 1)$ is in the trace class, and $D(\lambda)$ is invertible. The diagonalization of $d(\lambda)$ suggests we choose $D(\lambda)$ to be multiplication by $(\lambda+1)/2 - ((\lambda-1)/2)\operatorname{sgn}(l)$ on the eigenfunctions $e^{i\pi l y/K}$ for $d(\lambda)$. It is easy to see that $\underline{d}(\lambda) = d(\lambda)D(\lambda)^{-1} = 1 +$ trace class. If we define $\sigma(\lambda) = \Gamma_Q\begin{bmatrix} a(\lambda) & \underline{b}(\lambda) \\ c(\lambda) & \underline{d}(\lambda) \end{bmatrix}\Gamma\begin{bmatrix} 1 & 0 \\ 0 & D(\lambda) \end{bmatrix}$ with $b(\lambda) = \underline{b}(\lambda)D(\lambda)$, this normalization is equivalent to:

$$\langle \sigma(\lambda) \rangle = \det \underline{d}(\lambda) = \prod_{l>0}\left[\frac{1+\lambda^{-1}q^{2l}}{1+q^{2l}}\frac{1+\lambda q^{2l}}{1+q^{2l}}\right], \tag{1.14a}$$

where $l \in Z_{1/2}$. It is also not hard to check that when $|\lambda| = 1$, $D(\lambda)$ is unitary. This implies $\sigma(\lambda)$ is unitary as well [10]. In the case $\lambda = -1$ the quantity $\langle \sigma(-1) \rangle$ is the square of the spontaneous magnetization for the Ising model. The behavior of this quantity as the temperature approaches the critical temperature is a famous result of

Onsager first proved by Yang in 1952 [19]. Yang's proof suggested that an elliptic substitution would be fruitful in the present work. The formula (1.14) for $\langle\sigma(\lambda)\rangle$ as an infinite product may be reexpressed as ratio of θ functions. Namely

$$\langle\sigma(\lambda)\rangle = \frac{\vartheta_3\left(-\dfrac{i\log\lambda}{2}, q\right)}{\vartheta_3(0,q)}.$$

See Henrici [21, 8.26] and formula (16.27.3) in [20]. Presumably information about ϑ functions could be used to extract the critical behavior for $\langle\sigma(\lambda)\rangle(k\uparrow 1)$. Here we observe that taking the logarithmic derivative of $\langle\sigma(\lambda)\rangle$ permits an elementary analysis. A simple calculation using (1.14) shows that:

$$\frac{d}{d\lambda}\log\langle\sigma(\lambda)\rangle = (\lambda-\lambda^{-1})\sum_{l>0}(\lambda+q^{2l})^{-1}(\lambda+q^{-2l})^{-1}.$$

Suppose λ is not negative or 0 and let C_λ denote a contour which joins 1 to λ without crossing the negative axis or going through 0. Then:

$$\log\langle\sigma(\lambda)\rangle = \int_{C_\lambda} dz(z-z^{-1})\sum_{l>0}(z+q^{2l})^{-1}(z+q^{-2l})^{-1}.$$

Now multiply both sides of this last equation by π/K and introduce $x = l\pi/K$. One finds:

$$\log\langle\sigma(\lambda)\rangle^{\pi/K} = \int_{C_\lambda} dz(z-z^{-1})\sum_{x>0}(z+e^{-2K'x})^{-1}(z+e^{2K'x})^{-1}\Delta x.$$

Next observe that $K'\to\pi/2$ as $k\uparrow 1$, so dominated convergence implies:

$$\begin{aligned}\lim_{k\uparrow 1}\log\langle\sigma(\lambda)\rangle^{\pi/K} &= \int_{C_\lambda} dz(z-z^{-1})\int_0^\infty (z+e^{-\pi x})^{-1}(z+e^{\pi x})^{-1}dx\\ &= \frac{1}{\pi}\int_{C_\lambda}\frac{\log z}{z}dz = \frac{(\log\lambda)^2}{2\pi},\end{aligned}$$

where the branch cut of $\log z$ is taken along the negative axis and $\log 1 = 0$. It is know that $\lim_{k\to 1}[K-\frac{1}{2}\log(16/(1-k^2))]=0$ (see (17.3.24) [20]), so this last limit may also be written:

$$\lim_{k\to 1}\frac{\ln\langle\sigma(\lambda)\rangle}{\ln(1-k)} = \frac{(\log\lambda)^2}{4\pi^2}. \tag{1.14b}$$

Next we calculate the R matrix associated with $s(\lambda)$ in the terminology of [10]. Here we require $\lambda \neq -q^{2l}$ $(l\in Z_{1/2})$; we refer to such λ as nonexceptional. The matrix R is:

$$R = \begin{bmatrix} -b(\lambda)d(\lambda)^{-1}c(\lambda) & b(\lambda)d(\lambda)^{-1}\\ d(\lambda)^{-1}c(\lambda) & 0\end{bmatrix}.$$

It will be enough to illustrate the calculation for $b(\lambda)d(\lambda)^{-1}$. First define a conjugation J on $L^2(M_\pm)$ by $Jf(v_\pm)=\bar{f}(-v_\pm)$. Using $ds(-u) = -ds(u)$ and the fact that $ds(u)$ is real when u is real, it is easy to see that $Jd = dJ$. Using $s(\lambda) = (\lambda+1)/2 + ((\lambda-1)/2)\varepsilon$, it is easy to see that $b(\lambda) = ((\lambda-1)/2)b$. Thus $b(\lambda)d(\lambda)^{-1} = b(\lambda)Jd(\lambda)^{-1}J$. But $b(\lambda)J$ is convolution with $((1-\lambda)k/2\pi)cn(u)$, and since

$$cn(u) = \frac{\pi}{kK} \sum_{l \in Z_{1/2}} \frac{q^l}{1+q^{2l}} e^{(i\pi l/K)u}$$

(see (16.23.2) in [20]) it follows that $b(\lambda)J$ has eigenvectors $e^{(i\pi l/K)u}$ $(l \in Z_{1/2})$ with eigenvalues $(1-\lambda)q^l/(1+q^{2l})$. Thus $b(\lambda)Jd(\lambda)^{-1}$ is multiplication by $(1-\lambda)q^l/(1+q^{2l})$ $((\lambda + q^{-2l})/(1+q^{-2l}))^{-1} = (1-\lambda)q^l/(1+\lambda q^{2l})$ on the eigenfunction $e^{(i\pi l/K)u}$. Similar calculations for $d(\lambda)^{-1}c(\lambda)$ and $-b(\lambda)d(\lambda)^{-1}c(\lambda)$ suggest the following definitions:

$$R_a^\lambda(u) = -\frac{(\lambda-1)^2}{2K} \sum_{l \in Z_{1/2}} (1+q^{2l})^{-1}(\lambda + q^{-2l})^{-1} e^{i\pi l u/K},$$

$$R_b^\lambda(u) = \frac{(\lambda-1)i}{2K} \sum_{l \in Z_{1/2}} q^l(1+\lambda q^{2l})^{-1} e^{i\pi l u/K}. \tag{1.15}$$

With these definitions:

$$\begin{aligned} -b(\lambda)d(\lambda)^{-1}c(\lambda)f(u) &= \int_{-K}^{K} R_a^\lambda(u-v)f(v)dv, \\ b(\lambda)d(\lambda)^{-1}f(u) &= \int_{-K}^{K} R_b^\lambda(u-v)\bar{f}(-v)dv, \\ d(\lambda)^{-1}c(\lambda)f(v) &= \int_{-K}^{K} R_b^\lambda(u-v)\bar{f}(-v)dv \end{aligned} \tag{1.16}$$

(note that in the case $\lambda = -1$ the function R_b^λ is simply related to $sn(u)$, see (16.23.1) in [20]).

We conclude this section with a remark which will be useful when we turn to the consideration of scaling limits. The substitution we used to diagonalize $a(\lambda)$ and $d(\lambda)$ was normalized so that $z=1$ or $x=k$ corresponds to $v_\pm = \pm K$. This is important if we want the substitution to remain sensible in the limit $k \uparrow 1$. In this limit the curve $M^{\mathbb{C}}$ becomes singular with the real quarter period K tending to ∞. In this limit the elliptic substitution goes over to the hyperbolic one:

$$x = \tanh^{-2}(v_\pm \pm i\pi/4) \quad \left(K' = \frac{\pi}{2}\right),$$

and the operator ε becomes:

$$\begin{aligned} (\varepsilon f)_+(v_2^+) &= \frac{i}{\pi}\int_{-\infty}^{\infty} \operatorname{csch}(v_2^+ - v_1^+)f_+(v_1^+)dv_1^+ \text{ (principal value)} \\ &\quad -\frac{1}{\pi}\int_{-\infty}^{\infty} \operatorname{sech}(v_2^+ - v_1^-)\bar{f}_-(-v_1^-)dv_1^-, \\ (\varepsilon f)_-(v_2^-) &= -\frac{i}{\pi}\int_{-\infty}^{\infty} \operatorname{csch}(v_2^- - v_1^-)f_-(v_1^-)dv_1^- \text{ (principal value)} \\ &\quad -\frac{1}{\pi}\int_{-\infty}^{\infty} \operatorname{sech}(v_2^- - v_1^+)\bar{f}_+(-v_1^+)dv_1^+, \end{aligned} \tag{1.17}$$

where $\mathrm{csch}(x) = 1/\sinh(x)$ and $\mathrm{sech}(x) = 1/\cosh(x)$. The monodromy fields $\sigma(\lambda)$ no longer exist (in $\hat{\mathrm{Gl}}_Q(H)$) since the off diagonal pieces of $s(\lambda)$ are not Schmidt class operators. One could perhaps still use (1.17) to study the critical model using quadratic form ideas like those used by Ruijsenaars in his study of the massless Thirring model [15]. This will not be attempted here, however. In this paper we will mainly consider the massive scaling limit, as in [8, 9]. This scaling limit emphasizes large scale behaviour of the correlations coming from the vicinity of $z = 1$. It is inconvenient to have $z = 1$ correspond to $\pm K$ in the $v_\pm$ variables. A simple way to remedy this is to change the substitution $x = sn^{-2}(v_\pm \pm iK'/2)$ to:

$$x = sn^{-2}\left(v_\pm + K \pm \frac{iK'}{2}\right), \quad -K \leqq v_\pm \leqq K. \tag{1.18}$$

Then $x \to \bar{x}$ still corresponds to $v_\pm \to -v_\pm$ and (1.12), (1.15) and (1.16) remain unaltered. It is this substitution we will use when we consider the massive scaling limit. Observe that $z = 1$ corresponds to $v_\pm = 0$.

Section 2

In this section we introduce a broader class of fields than the scalar monodromy fields considered in Sect. 1. Let p denote a positive integer and write $H^p = L^2(S^1, C^2) \otimes C^p = H \otimes C^p$. Let T denote the induced rotation for the transfer matrix on H defined in Sect. 1. Let Q denote the difference of spectral projections for T defined in Sect. 1. Let z denote the operator of multiplication by $e^{i\theta}$ on $H = L^2(S^1, C^2)$. We extend these transformations to H^p in the following manner: $T = T \otimes I_p$, $Q = Q \otimes I_p$, and $z = z \otimes I_p$, making use of the abuse of notation implicit in these definitions whenever it proves convenient.

Write $W^p = H^p \oplus \bar{H}^p$ and define the conjugation P on W^p by $P(x \oplus y) = \overline{x \oplus y} = y \oplus x$. Then $Q_W \overset{\text{def}}{=} Q \oplus (-Q)$ is a self-adjoint idempotent on W^p which anti-commutes with P. We are interested in the Q_W Fock state of the Clifford algebra $C(W^p, P)$ whose associated representation lives in the alternating tensor algebra $A(W_+^p)$, where $W_\pm^p = Q_W^\pm W^p$ and $Q_W^\pm = (1 \pm Q_W)/2$.

Suppose $w \in W^p$. Then the generators of Q_W Fock representation are given by:

$$F(w) = a^*(Q_W^+ w) + a(\overline{Q_W^- w}),$$

where the operators $a^*(\cdot)$, $a(\cdot)$ are the usual creation and annihilation operators on $A(W_+^p)$ [7]. Let $u \in H^p$; it will be useful to introduce $\psi(u) = F(u \oplus 0)$. Observe that $\psi^*(u) \overset{\text{def}}{=} \psi(u)^* = F(0 \oplus u)$. Let $\{e_j\}$ denote the standard basis for C^p. For $u \in H$ we write $\psi_j(u) = \psi(u \otimes e_j)$ and $\psi_j^*(u) = \psi^*(u \otimes e_j)$.

We next define the restricted general linear group $\mathrm{Gl}_Q(H^p)$. This group consists of bounded invertible linear maps on H^p with bounded inverses whose matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ relative to the $H_+^p \oplus H_-^p$ splitting of H^p have elements b and c which are Schmidt class operators and elements a and d which are Fredholm maps of index 0. Let $\mathrm{Gl}_Q^0(H^p)$ denote the subgroup of elements g of $\mathrm{Gl}_Q(H^p)$ defined by the condition that $d(g)$ is a trace class perturbation of the identity on H^p_-. Suppose D is a linear space, then we write $L(D)$ for the collection of linear maps from D into D.

One of the principal results of [10] is that there exists a dense linear domain $D \subseteq A(W^p_+)$ and a group homomorphism $\Gamma_Q: \mathrm{Gl}^0_Q(H^p) \to L(D)$ with the following property:

$$\Gamma_Q(g)F(w) = F(g \oplus g^{*-1}w)\Gamma_Q(g), \quad g \in \mathrm{Gl}^0_Q(H^p), \tag{2.1}$$

which is understood as an equality on D.

Let $\mathrm{Gl}^1_Q(H^p)$ denote the subgroup of elements g of $\mathrm{Gl}_Q(H^p)$ such that $b(g) = c(g) = 0$. Then there exists a homomorphism $\Gamma: \mathrm{Gl}^1_Q(H^p) \to L(D)$ so that:

$$\Gamma(g)F(w) = F(g \oplus g^{*-1}w)\Gamma(g), \quad g \in \mathrm{Gl}^1_Q(H^p) \tag{2.2}$$

(see [10]).

It is also proved in [10] that Γ acts on Γ_Q:

$$\Gamma(h)\Gamma_Q(g)\Gamma(h)^{-1} = \Gamma_Q(hgh^{-1}). \tag{2.3}$$

If $\mathrm{Gl}^0_Q(H^p) \times \mathrm{Gl}^1_Q(H^p)$ is the semi-direct product with composition rule $g_1 \times h_1 \cdot g_2 \times h_2 = g_1 h_1 g_2 h_1^{-1} \times h_1 h_2$, then (2.1), (2.2) and (2.3) may be summarized by saying that $g \times h \to \Gamma_Q(g)\Gamma(h)$ is a homomorphism. The kernel K of this homomorphism is $\{g \times h \mid gh = 1 \text{ and } \det d(g) = 1\}$. The group $\widehat{\mathrm{Gl}}_Q(H^p) = \mathrm{Gl}^0_Q(H^p) \times \mathrm{Gl}^1_Q(H^p)/K$ is an interesting central extension of $\mathrm{Gl}_Q(H^p)$ [18]. One may check that $(g \times h)K \to gh$ is a well defined homomorphism $T: \widehat{\mathrm{Gl}}_Q(H^p) \to \mathrm{Gl}_Q(H^p)$ with kernel C^*. From now on we identify $\widehat{\mathrm{Gl}}_Q(H^p)$ with its image in $L(D)$. If $g \in \widehat{\mathrm{Gl}}_Q(H^p)$ then (2.1) and (2.2) imply:

$$gF(w) = F(T(g) \oplus T(g)^{*-1}w)g, \quad w \in W^p. \tag{2.4}$$

Equality is understood on D. One may translate (2.4) into:

$$\begin{aligned} g\psi(u) &= \psi(T(g)u)g, && u \in H^p \\ g\psi^*(u) &= \psi^*(T(g)^{*-1}u)g, && u \in H^p. \end{aligned} \tag{2.5}$$

We are now prepared to introduce the monodromy fields $\sigma(M)$. Let $M \in GL(p, C)$. Then M acts on $H^p \simeq H \otimes C^p$ by $I \otimes M$. Let ε denote the operator on H studied in Sect. 1, and write $\varepsilon_\pm = (1 \pm \varepsilon)/2$. Define:

$$s(M) = \varepsilon_- \otimes I_p + \varepsilon_+ \otimes M.$$

The results of Sect. 1 imply that $s(M) \in \mathrm{Gl}_Q(H^p)$. We will define $\sigma(M) \in \widehat{\mathrm{Gl}}_Q(H^p)$ so that $T(\sigma(M)) = s(M)$. As in Sect. 1 what is needed is an appropriate factorization of $s(M)$. We now describe the factorization we will use. Let P_+ (P_-) denote the orthogonal projection on the subspace in $L^2([-K, K], C)$ whose elements have Fourier expansions in $\exp(i\pi l x/K)$ with no l negative (positive) terms. In Sect. 1 we introduced $D(\lambda) = I_+ \oplus (P_+ + \lambda P_-)_-$, where I_+ is the identity on H_+ and $(P_+ + \lambda P_-)_-$ acts on $H_- \simeq L^2([-K, K], C)$. In the matrix case we define $D(M) = I_+ \oplus (P_+ \otimes I_p + P_- \otimes M)_-$, where I_+ is the identity on H^p_+ and $(P_+ \otimes I_p + P_- \otimes M)_-$ acts on $H^p_- \simeq L^2([-K, K]), C) \otimes C^p$. Using the results of Sect. 1 it is not hard to see that $\underline{s}(M) \stackrel{\text{def}}{=} s(M)D(M)^{-1} \in \mathrm{Gl}^0_Q(H^p)$. Thus we may define:

$$\sigma(M) = \Gamma_Q(\underline{s}(M))\Gamma(D(M)).$$

One very useful feature of this choice of normalization for $\sigma(M)$ is that $M \to \sigma(M)$ is a homomorphism. This follows from the easily verified fact that $M \to s(M)$ and $M \to D(M)$ are homomorphisms and the calculation:

$$\begin{aligned}\sigma(M_1)\sigma(M_2) &= \Gamma_Q(\underline{s}(M_1))\Gamma(D(M_1))\Gamma_Q(\underline{s}(M_2))\Gamma(D(M_2))\\ &= \Gamma_Q(s(M_1)s(M_2)D(M_2)D(M_2)^{-1}D(M_1)^{-1})\Gamma(D(M_1))\Gamma(D(M_2))\\ &= \Gamma_Q(s(M_1M_2)D(M_1M_2)^{-1})\Gamma(D(M_1M_2)) = \sigma(M_1M_2),\end{aligned}$$

where we used the fact that $\Gamma_Q(\cdot)$ and $\Gamma(\cdot)$ are homomorphisms and $\Gamma(D)\Gamma_Q(\underline{s})\Gamma(D)^{-1} = \Gamma_Q(D\underline{s}D^{-1})$.

For $a \in Z^2$ we write $V(a) = \Gamma(T)^{a_2}\Gamma(z)^{a_1}$ which acts on $D \subseteq A(W_+^p)$, and we define $\sigma_a(M) = V(a)\sigma(M)V(a)^{-1}$. We next introduce some notation that will allow us to formulate the principal result of this section: a relation between monodromy fields and the solution to finite difference equations on Z^2 with prescribed monodromy and exponential decay at ∞. The Fourier coefficients:

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}d\theta$$

identify $L^2(S^1, C^2)$ with $l^2(Z_{1/2}, C^2)$. On l^2 the transfer matrix T is a finite difference operator which may be written

$$Tf(k) = T_-z^{-1}f(k) + T_0f(k) + T_+zf(k),$$

where $z^{\pm 1}f(k) = f(k \mp 1)$ and

$$\begin{aligned}T_- &= -\frac{1}{2}\begin{bmatrix} 1 & -i(c+s) \\ i(c-s) & 1 \end{bmatrix},\\ T_0 &= \frac{c}{s}\begin{bmatrix} c & -i \\ i & c \end{bmatrix},\\ T_+ &= -\frac{1}{2}\begin{bmatrix} 1 & -i(c-s) \\ i(c+s) & 1 \end{bmatrix}.\end{aligned} \tag{2.6}$$

Now let $e_+(k) = \delta(k - \cdot)\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $e_-(k) = \delta(k - \cdot)\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ denote the standard basis for $l^2(Z_{1/2}, C^2)$. For $k \in Z_{1/2} \times Z$ define:

$$\begin{aligned}\psi_j(k, \pm) &= \Gamma(T)^{k_2}\psi_j(e_\pm(k_1))\Gamma(T)^{-k_2} = \psi_j(T^{k_2}e_\pm(k_1)),\\ \psi_j^*(k, \pm) &= \Gamma(T)^{k_2}\psi_j^*(e_\pm(k_1))\Gamma(T)^{-k_2} = \psi_j^*(T^{-k_2}e_\pm(k_1)).\end{aligned}$$

Observe that $\psi_j(k_1, k_2, \pm)^* = \psi_j^*(k_1, -k_2, \pm)$. Let

$$\psi_j^*(k) = \begin{bmatrix} \psi_j^*(k, +) \\ \psi_j^*(k, -) \end{bmatrix}.$$

Then $\psi_j^*(k)$ satisfies the following difference equation:

$$\psi_j^*(k_1, k_2 - 1) = T(z)\psi_j^*(k_1, k_2). \tag{2.7}$$

To see this observe that if A is a 2×2 complex matrix, then

$$A\psi_j^*(k) = A\begin{bmatrix} \psi_j^*(T^{-k_2}e_+(k_1)) \\ \psi_j^*(T^{-k_2}e_-(k_1)) \end{bmatrix} = \begin{bmatrix} \psi_j^*(T^{-k_2}A^*e_+(k_1)) \\ \psi_j^*(T^{-k_2}A^*e_-(k_1)) \end{bmatrix}.$$

(Note: $u \to \psi_j^*(u)$ is conjugate linear in $u \in H$.) Also

$$z^{\pm 1}\psi_j^*(k) = \psi_j^*(k_1 \mp 1, k_2) = \begin{bmatrix} \psi_j^*(T^{-k_2}e_+(k_1 \mp 1)) \\ \psi_j^*(T^{-k_2}e_-(k_1 \mp 1)) \end{bmatrix} = \begin{bmatrix} \psi_j^*(T^{-k_2}z^{\mp 1}e_+(k_1)) \\ \psi_j^*(T^{-k_2}z^{\mp 1}e_-(k_1)) \end{bmatrix}.$$

Thus

$$T(z)\psi_j^*(k) = \begin{bmatrix} \psi_j^*(T^{-k_2}T(z)^*e_+(k_1)) \\ \psi_j^*(T^{-k_2}T(z)^*e_-(k_1)) \end{bmatrix}, \quad \text{where } T(z)^* = T_-^*z + T_0^* + T_+^*z^{-1}.$$

Consulting (2.6) one sees that $T(z)^* = T(z)$. Thus $T(z)\psi_j^*(k) = \psi^*(k_1, k_2 - 1)$.

We next present the "wave function" construction of SMJ [17 IX] in a Euclidean incarnation. Let $M_i \in \mathrm{GL}(p, C)$ $(i = 1, \dots, n)$. Write $\tau(a) = \langle \vec{T}\sigma_{a_1}(M_1) \cdots \sigma_{a_n}(M_n) \rangle$, where the time ordering $\vec{T}$ puts $\sigma_{a_i}(M_i)$ in order of increasing second coordinates for the $a_i \in Z^2$ (we suppose there are no coincidences among the second coordinates). Write

$$W_{ij}(k, \alpha; l, \beta) = \frac{\langle \vec{T}\psi_i^*(k, \alpha)\psi_j(l, \beta)\sigma_{a_1}(M_1) \cdots \sigma_{a_n}(M_n) \rangle}{\tau(a)}, \tag{2.8}$$

where (k, α) and (l, β) are in $Z_{1/2} \times Z \times \{+, -\}$ and the time ordering $\vec{T}$ puts the operators in order of increasing second coordinates for k, l, $a_1, \dots, a_n$ with a sign change when $\psi_i^*(k, \alpha)$ is moved past $\psi_j(l, \beta)$, all other operators formally commuting. This prescription determines the value of $W_{ij}(k, \alpha; l, \beta)$ except when k lies on the horizontal ray to the right of one of the points a_i or when $k = l$. We first examine the ambiguity when $\pi_2(k) = \pi_2(a_r)$. For simplicity write $M_r = M$, $a_r = a$ and suppose $\pi_1(a) = 0$. The ambiguity at $\pi_2(k) = \pi_2(a)$ in (2.5) results because $\psi_j^*(k, \alpha)\sigma_a(M)$ and $\sigma_a(M)\psi_j^*(k, \alpha)$ are not equal. However, there is an interesting relation between the vectors formed from these operators for $j = 1, \dots, n$. To see this relation observe first that

$$\psi_1^*(k, \alpha)\sigma_a(M) = \Gamma(T^{k_2})\psi_i^*(e_\alpha(k_1))\sigma(M)\Gamma(T^{-k_2}),$$

$$\sigma_a(M)\psi_i^*(k, \alpha) = \Gamma(T^{k_2})\sigma(M)\psi_i^*(e_\alpha(k_1))\Gamma(T^{-k_2}),$$

where we used $k_2 = \pi_2(a)$ and $\pi_1(a) = 0$. Now

$$\sigma(M)\psi^*(e_\alpha(k_1) \otimes e_j) = \begin{cases} \psi^*(e_\alpha(k_1) \otimes M^{*-1}e_j)\sigma(M) & k_1 > 0 \\ \psi^*(e_\alpha(k_1) \otimes e_j)\sigma(M) & k_1 < 0 \end{cases}.$$

Thus

$$\sigma(M)\psi_i^*(e_\alpha(k_1)) = \sum_j M_{ij}^{-1}\psi_j^*(e_\alpha(k_1))\sigma(M), \quad k_1 > 0,$$

$$\sigma(M)\psi_i^*(e_\alpha(k_1)) = \psi_i^*(e_\alpha(k_1))\sigma(M), \qquad k_1 < 0. \tag{2.9}$$

Let $w(k)_+$ denote the column vector with components $\sigma(M)\psi_i^*(e_\alpha(k))$, $i = 1, \dots, p$. Let

$w(k)_-$ denote the column vector with components $\psi_i^*(e_\alpha(k))\sigma(M)$, $i=1,\dots,p$. The $\pm$ subscripts attached to $w(k)$ are intended to suggest "boundary values" arising when k_2 arrives $\pi_2(a)$ from above $(+)$ or from below $(-)$. The relation (2.9) becomes $w(\cdot)_+ = s(M)^{-1}w(\cdot)_-$ or

$$w(\cdot)_- = s(M)w(\cdot)_+. \tag{2.10}$$

A little thought shows that we may interpret (2.10) in the following manner. Let $W(k,\alpha)$ denote the $n\times n$ matrix with (i,j) entry $W_{ij}(k,\alpha;l,\beta)$. Because of (2.7) the entries of $W(k,\alpha)$ may be thought of as solutions to the finite difference equation (2.7) in the (k,α) variables. For k near a_r these solutions are multi-valued in the sense that if k makes a counterclockwise circuit of a_r starting on the ray to the right of a_r then when k returns to this ray the upper boundary value $W(k,\alpha)_+$ is transformed to $W(k,\alpha)_- = M_rW(k,\alpha)_+$.

We turn next to the ambiguity in (2.8) along the line $\pi_2(k)=\pi_2(l)$. Since $\psi_i^*(k,\alpha)\psi_j(l,\beta)+\psi_j(l,\beta)\psi_i^*(k,\alpha)=\delta_{ij}\delta_{kl}\delta_{\alpha\beta}$ when $\pi_2(k)=\pi_2(l)$, it follows that the time ordering convention with the sign adjustment described earlier results in an ambiguity only when $k=l$. Here we settle this ambiguity by choosing $-\psi_j(l,\beta)\psi_i^*(k,\alpha)$ in definition of W_{ij} at $k=l$. This choice has as a consequence that $W_{ij}(k,\alpha)$ satisfies the difference equation (2.7) for $k_2 \leqq l_2$. To be more specific, let $W_{ij}(k)=\begin{bmatrix} W_{ij}(k,+) \\ W_{ij}(k,-)\end{bmatrix}$. Then

$$W_{ij}(k_1,k_2)-T(z)W_{ij}(k_1,k_2+1) = -\delta_{ij}\delta_{l,k}\begin{bmatrix}\delta(\beta,+)\\ \delta(\beta,-)\end{bmatrix}. \tag{2.11}$$

The inhomogeneity in (2.11) arises from the two possible choices for W_{ij} at $k=l$. With one choice (the one we made) the homogeneous equation (2.7) is satisfied for $\pi_2(k)\leqq\pi_2(l)$. For the other choice (2.7) is satisfied for $\pi_2(k)\geqq\pi_2(l)$. The difference between the two choices is determined by the anti-commutator between $\psi_i^*(k,\alpha)$ and $\psi_j(l,\beta)$ given above for $\pi_2(k)=\pi_2(l)$ and leads to precisely the inhomogeneous term in (2.11).

The columns of $W_{ij}(k,\alpha)$ span a vector space which may be characterized in much the same way that SMJ [17, III] characterize a space of multivalued solutions to the Euclidean Dirac equation. Let w denote a map $w: Z_{1/2}\times Z\to C^2\otimes C^p$. Then if $w(k)$ satisfies

$$w(k)-T(z)\otimes I_p w(k+e_2)=\delta_{k,l}v \tag{2.12}$$

for some $l\in Z_{1/2}\times Z$ and some $v\in C^2\otimes C^p$, then we will say that the solution $w(k)$ to (2.12) has an inhomogeneity localized at $k=l$. Let

$$\|w(\cdot,k_2)\|^2=\sum_{k_1\in Z_{1/2}}\|w(k_1,k_2)\|^2,$$

where $\|w(k_1,k_2)\|$ is the usual Euclidean norm in $C^2\otimes C^p$. If $\|w(\cdot,k_2)\|<\infty$ for each $k\in Z$ and $\lim_{k_2\to\pm\infty}\|w(\cdot,k_2)\|=0$, we will say that $w(k)$ vanishes at ∞. In the following definition we write M for $(M_1,M_2,\dots,M_n)$ and 'a' for $(a_1\dots a_n)\in(Z^2)^n$.

Definition. Let $W_a(l, M)$ denote the space of maps $w: Z_{1/2} \times Z \to C^2 \otimes C^p$ satisfying the following 3 conditions:

(a) $w(k)$ is a solution to (2.12) with an inhomogeneity localized at $k = l$.

(b) $w(k)$ is multivalued with branch cuts to the right of the points $a_j \in Z^2$. The vector $w(k)$ transforms to $I \otimes M_j w(k)$ as k makes a counterclockwise circuit of a_j (near a_j) starting on the ray to the right of a_j.

(c) $w(k)$ vanishes at ∞.

Proposition 2.0. *If $\tau(a) \neq 0$ then the space $W_a(l, M)$ is finite dimensional with dimension $2p$. The columns of $W(k, \alpha)$ for $\beta = (+)$ and $\beta = (-)$ span $W_a(l, M)$.*

Proof. Write $a_j = (b_j, c_j)$. To simplify the discussion we suppose that the second coordinates of $a_1, a_2, \ldots, a_n$ occur in increasing order $c_1 < c_2 \cdots < c_n$ and that $\pi_2(l) = l_2 < c_1$. (We leave it to the reader to check that these assumptions are not really necessary.)

Suppose $w \in W_a(l, M)$. Then property (c) implies that $w(\cdot, k_2) \in l^2(Z_{1/2}, C^2) \otimes C^p$. Choose $k_2^0 \in Z$ such that $k_2^0 > c_n$. Since $w(k)$ satisfies (2.7) (the homogeneous version of (2.12)) for $k_2 \geqq k_2^0$ we have $w(\cdot, k_2) = T^{k_2^0 - k_2} w(\cdot, k_2^0)$ for $k_2 \geqq k_2^0$. However, since $\lim_{k_2 \to \infty} \| T^{k_2^0 - k_2} w(\cdot, k_2^0) \| = 0$ and $T = e^{-\gamma} Q_+ + e^{\gamma} Q_-$, it follows that $w(\cdot, k_2^0)$ must be in the range of Q_-. That is $Q_- w(\cdot, k_2^0) = w(\cdot, k_2^0)$. We may also use the transfer matrix to propagate $w(\cdot, k_2^0)$ down to the first branch cut at $k_2 = c_n$. Thus the upper boundary value is $w_+(\cdot, c_n) = T^{k_2^0 - c_n} w(\cdot, k_2^0)$. To obtain the correct lower boundary value we apply the monodromy map $s_{b_n, 0}(M_n)$. Hence $w_-(\cdot, c_n) = s_{b_n, 0}(M_n) T^{k_2^0 - c_n} w(\cdot, k_2^0)$. We proceed in this fashion until we encounter the inhomogeneity at $k_2 = l_2$. One finds:

$$w(\cdot, l_2 + 1) = T^{-l_2 - 1} s_{a_1}(M_1) \cdots s_{a_n}(M_n) T^{k_2^0} w(\cdot, k_2^0), \tag{2.13}$$

where $s_a(M) = T^{a_2} z^{a_1} s(M) z^{-a_1} T^{-a_2}$. At $k_2 = l_2$ one has:

$$w(\cdot, l_2) = T w(\cdot, l_2 + 1) + \delta(\cdot, l_1) v. \tag{2.14}$$

However, because $w(\cdot, k_2) = T^{l_2 - k_2} w(\cdot, l_2)$ for $k_2 < l_2$, $\lim_{k_2 \to -\infty} \| w(\cdot, k_2) \| = 0$, and $T = e^{-\gamma} Q_+ + e^{\gamma} Q_-$, it follows that $Q_- w(\cdot, l_2) = 0$. Substituting (2.13) into (2.14) and then applying Q_-, one finds

$$T^{-l_2} Q_- s_{a_1}(M_1) \cdots s_{a_n}(M_n) Q_- T^{k_2^0} w(\cdot, k_2^0) = -Q_-[\delta(\cdot - l_1) v], \tag{2.15}$$

where we've made use of $Q_- w(\cdot, k_2^0) = w(\cdot, k_2^0)$. From (2.15) one sees that $w(\cdot, k_2^0)$ is uniquely determined by v precisely when $d(s_{a_1}(M_1) \cdots s_{a_n}(M_n))$ is invertible. But $d(s_{a_1}(M_1) \cdots s_{a_n}(M_1))$ will be invertible precisely when $\tau(a) = \langle \sigma_{a_1}(M_1) \cdots \sigma_{a_n}(M_n) \rangle \neq 0$, since $\tau = \det(\underline{d})$ for some factorization $d = \underline{d} D$ with D invertible and $\underline{d}$ a trace class perturbation of the identity. We have seen that if $\tau(a) \neq 0$ then $\dim W_a(l, M) \leqq 2p$. But $Q_- \delta(\cdot, l_1) e_+$ and $Q_- \delta(\cdot, l_1) e_-$ are linearly independent, and by reversing the reasoning above one finds that solutions to (2.15) may be used to construct elements of $W_a(l, M)$. Thus $\dim W_a(l, M) = 2p$ if $\tau(a) \neq 0$.

From (2.9) and (2.11) it is clear that the columns of $W(k, \alpha)$ (for $\beta = +, -$) will span $W_a(l, M)$ provided the columns are elements of $W_a(l, M)$ (i.e., vanish at ∞). First we check that $W_{ij}(\cdot, k_2, \alpha) \in l^2(Z_{1/2}, C)$. Since $\psi_i^*(k, \alpha) = \psi^*(T^{-k_2} e_\alpha(k_1) \otimes e_i)$ and only the annihilation part of $\psi_i^*(k,\alpha)$ contributes to $W_{ij}(k,\alpha)$ for k_2 sufficiently negative it follows that:

$$W_{ij}(k,\alpha)=\int_{-\pi}^{\pi} f_\alpha(\theta)e^{k_2\gamma(\theta)}e^{ik_1\theta}d\theta,$$

where $f_\alpha(\cdot)\in L^2(S^1,C)$. Thus by the Plancherel Theorem $W_{ij}(\cdot,k_2,\alpha)\in l^2(Z_{1/2},C)$ and $\lim_{k_2\to-\infty}\|W_{ij}(\cdot,k_2)\|=0$.

In a similar fashion only the creation part of $\psi_i^*(k,\alpha)$ contributes to the inner product defining $W_{ij}(k,\alpha)$ when k_2 is sufficiently positive and one finds:

$$W_{ij}(k,\alpha)=\int_{-\pi}^{\pi} g_\alpha(\theta)e^{-k_2\gamma(\theta)}e^{ik_1\theta}d\theta,$$

where $g_\alpha(\cdot)\in L^2(S^1,C)$ and k_2 is sufficiently positive.

Thus $W_{ij}(\cdot,k_2,\alpha)$ has l^2 norm which vanishes in the limits $k_2\to\pm\infty$. Since $W_{ij}(k,\alpha)$ satisfies (2.11), it follows that $\|W_{ij}(\cdot,k_2,\alpha)\|<\infty$ for all k_2 and this finishes our proof of Proposition 2.0 ∎

It is instructive to think of this proposition in conjunction with the continuum results of SMJ [17 II], [17 IV], (see also [9]). If one "hides" the inhomogeneities in the difference equation next to the branch points and then scales to the critical point (as in [9]) with the lattice spacing going to zero and the correlation length fixed, one finds the wave functions for the Euclidean Dirac equation described in [17 III] (massive scaling regime). On the other hand one can sit at the critical point, leave the inhomogeneity "out in the open" and pass to the long range asymptotics to obtain a fundamental solutions to the Cauchy–Riemann equtions for the vector bundle over $C-\{a_1,\dots,a_n\}$ with holonomy M_j at a_j, which vanishes like $1/x$ at ∞ (massless regime).

In this paper we develop the massive scaling results to the point where the first scenario may be confirmed. The mass zero asymptotics are technically more difficult and we have only partial results which will be presented in a sequel to this paper.

To conclude this section we will examine what happens when the branch points $a_1,\dots,a_n$ are displaced. This will be used to establish identities for low order expansion coefficients in the continuum limit (see [9]), to establish continuity results for the τ-functions at the critical point, and to prove that $\sigma_a(M)$ and $\sigma_b(M)$ commute when $\pi_2(a)=\pi_2(b)$.

Comparing two monodromy fields at adjacent points one is lead to consider $\sigma_a(M)\sigma_{a+e_j}(M)^{-1}=V(a)\sigma_0(M)\sigma_{e_j}(M)^{-1}V(a)^{-1}$. But the induced rotation for $\sigma_0(M)\sigma_{e_j}(M)^{-1}$ is $s_0(M)s_{e_j}(M)^{-1}$, which is a finite dimensional perturbation of the identity. Thus

$$\sigma_0(M)\sigma_{e_j}(M)^{-1}=c_j\Gamma_Q(s_0(M)s_{e_j}(M)^{-1}),\quad j=1,2,\tag{2.16}$$

for some constants c_j. It is not hard to use the formalism of [10] to give formulas for c_j as determinants. We have $s(M)=\underline{s}(M)D(M)=\underline{s}D$. Thus $\sigma_0(M)=\Gamma_Q(\underline{s})\Gamma(D)$, and:

$$\sigma_{e_1}(M)=\Gamma(z)\Gamma_Q(\underline{s})\Gamma(D)\Gamma(z)^{-1},\quad \sigma_{e_2}(M)=\Gamma(T)\Gamma_Q(\underline{s})\Gamma(D)\Gamma(T)^{-1}.$$

Thus

$$\sigma_0(M)\sigma_{e_1}(M)^{-1} = \Gamma_Q[(szs^{-1}z^{-1})(zDz^{-1}D^{-1})]\Gamma(DzD^{-1}z^{-1}),$$
$$\sigma_0(M)\sigma_{e_2}(M)^{-1} = \Gamma_Q[(sTs^{-1}T^{-1})(TDT^{-1}D^{-1})]\Gamma(DTD^{-1}T^{-1}), \quad (2.17)$$

where we made use of (2.3) and the fact that $\Gamma_Q(\cdot)$ and $\Gamma(\cdot)$ are homomorphisms. Since $szs^{-1}z^{-1} = s(M)s_{e_1}(M)^{-1}$ and $sTs^{-1}T^{-1} = s(M)s_{e_2}(M)^{-1}$, it follows from (2.17) that

$$c_1 = \det(Q_-zDz^{-1}D^{-1}Q_-), \quad c_2 = \det(Q_-TDT^{-1}D^{-1}Q_-) \quad (2.18)$$

(an examination of the calculation leading to (2.17) shows that the arguments of the determinants in (2.18) are Id. + trace class on Q_-H).

To simplify the discussion we now concentrate on the scalar case $D(M) = D(\lambda)$. We calculate $d/d\lambda \log c_j(\lambda)$ to find

$$\frac{d}{d\lambda}\log c_1(\lambda) = \mathrm{Tr}\left(\frac{dD}{d\lambda}D^{-1} - z^{-1}\frac{dD}{d\lambda}D^{-1}z\right)_{Q_-H},$$
$$\frac{d}{d\lambda}\log c_2(\lambda) = \mathrm{Tr}\left(\frac{dD}{d\lambda}D^{-1} - T^{-1}\frac{dD}{d\lambda}D^{-1}T\right)_{Q_-H}.$$

Without difficulty one sees that $(dD/d\lambda)D^{-1} = \lambda^{-1}P_-$. Thus we are interested in calculating $\mathrm{Tr}(P_- - z^{-1}P_-z)_{Q_-H}$ and $\mathrm{Tr}(P_- - TP_-T^{-1})_{Q_-H}$. Using the Fourier expansion for $\csc(u - i\varepsilon)$ already recorded in Sect. 1, one finds

$$P_-f(x) = \lim_{\varepsilon\to 0+} \frac{1}{4iK}\int_{x-K}^{x+K} \frac{f(y)}{\sin[\pi(x-y-i\varepsilon)/2K]}dy.$$

It is appropriate to apply this integral operator to anti-periodic functions (these are the ones with "nice" expansions in $\exp(i\pi lx/K)$). For such a function f the formula for $P_-f(x)$ simplifies to:

$$P_-f(x) = \lim_{\varepsilon\to 0+} \frac{1}{4iK}\int_{-K}^{K} \frac{f(x)}{\sin[\pi(x-y-i\varepsilon)/2K]}dy.$$

The kernel for $P_- - z^{-1}P_-z$ is thus:

$$\frac{1}{4iK}z^{-1}(x)\frac{z(x)-z(y)}{\sin(\pi(x-y)/2K)}.$$

Denote the preceding kernel by $G(x,y)$. It is easy to check that $G(x,y)$ is a smooth function of (x,y) for $|x| < K$ and $|y| < K$. The function $G(x,y)$ is the restriction to $|x| \leqq K$, $|y| \leqq K$ of a smooth $2K$ antiperiodic function in x and y (which we continue to denote by $G(x,y)$). Antiperiodicity is just $G(x+2K,y) = -G(x,y)$ and $G(x,y+2K) = -G(x,y)$. Let $e_l(x) = (2K)^{-1/2}e^{-i\pi lx/2K}$ $(l \in \mathbb{Z}_{1/2})$ denote the standard antiperiodic basis for $L^2(-K,K)$. The trace of the integral operator with kernel G is the sum of the diagonal matrix elements of G relative to this basis and may be written in the form:

$$\sum_{l\in\mathbb{Z}_{1/2}} \int_{-K}^{K} \hat{G}(x,l)\overline{e_l(x)}dx,$$

where $\hat{G}(x,l) = \int_{-K}^{K} G(x,y)e_l(y)dy$. Integration by parts using the fact that $y \to G(x,y)e_l(y)$ is a smooth $2K$ periodic function shows that $|\hat{G}(x,l)| < C_N |l|^{-N}$ for each integer N with a constant that does not depend on x. Since $G(x,y)$ is smooth in y and antiperiodic, the Fourier series $\sum_{l \in \mathbb{Z}_{1/2}} \hat{G}(x,l)\overline{e_l(x)}$ converges pointwise to $G(x,x)$ (even at $x = \pm K$). The estimate just given for the Fourier coefficients $\hat{G}(x,l)$ ($N = 2$) shows that the Fourier series converges uniformly for $-K \leqq x \leqq K$. This is enough to interchange summation and integration in the formula for the trace, and we find the trace is given by:

$$\int_{-K}^{K} G(x,x)dx.$$

Thus:

$$\mathrm{Tr}(P_- - z^{-1}P_- z) = \frac{1}{2\pi i}\int_{-K}^{K} z^{-1}(x)\frac{dz}{dx}dx = \frac{1}{2\pi i}(\log z(K) - \log(z(-K)) = -1,$$

$$\mathrm{Tr}(P_- - w^{-1}P_- w) = \frac{1}{2\pi i}\int_{-K}^{K} w^{-1}(x)\frac{dw}{dx}dx = 0,$$

where the first equality follows from the fact that $z \in M_-$ and the second follows from the fact that $w(x)$ is a real period function on $[-K, K]$. We have then

$$\frac{d}{d\lambda}\log c_1(\lambda) = -\frac{1}{\lambda}, \quad \frac{d}{d\lambda}\log c_2(\lambda) = 0,$$

Since it is clear that $c_1(1) = c_2(1) = 1$, it follows that

$$c_1(\lambda) = \lambda^{-1} \quad \text{and} \quad c_2(\lambda) = 1. \tag{2.19}$$

The matrix case may be handled in a similar fashion. One finds:

$$c_1(\lambda) = (\lambda_1 \ldots \lambda_p)^{-1} = (\det M)^{-1}, \quad c_2(\lambda) = 1, \tag{2.20}$$

where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues for M. By translating and taking products (2.16) coupled with (2.20) may be rewritten more generally as:

$$\sigma_m(M)\sigma_n(M)^{-1} = (\det M)^{\pi_1(m-n)}\Gamma_Q(s_m(M)s_n(M)^{-1}). \tag{2.21}$$

It is a simple consequence of (2.21) that $\sigma_m(M)$ and $\sigma_n(M)$ commute when $\pi_2(m) = \pi_2(n)$. To see this substitute M^{-1} for M in (2.21), make use of $\sigma(M^{-1}) = \sigma(M)^{-1}$ ($M \to \sigma(M)$ is a homomorphism) and interchange the sites m and n. Then since $(\det M^{-1}) = (\det M)^{-1}$, $s(M^{-1}) = s(M)^{-1}$ and $s_m(M)$ and $s_n(M)$ commute when $\pi_2(m-n) = 0$, it follows that $\sigma_m(M)\sigma_n(M)^{-1} = \sigma_n(M)^{-1}\sigma_m(M)$, when $\pi_2(m-n) = 0$.

Next we calculate $s_0(M)s_{e_j}(M)^{-1}$. Let P_k denote the projection on the span of $e_\pm(k) \otimes e_j$ ($j = 1, \ldots, p$). Then without difficulty one may verify that

$$\begin{aligned} s(M)zs(M)^{-1} &= z + (I \otimes (M - I))P_{1/2}z, \\ s(M)z^{-1}s(M)^{-1} &= z^{-1} + (I \otimes (M^{-1} - I))P_{-1/2}z^{-1}. \end{aligned} \tag{2.22}$$

If we write $T(z) = T_+z + T_0 + T_-z^{-1}$ (see 2.6) and use the two preceding results, we find

$$s(M)T(z)s(M)^{-1}=T(z)+(T_+\otimes(M-I))P_{1/2}z+(T_-\otimes(M^{-1}-I))P_{-1/2}z^{-1}. \quad (2.23)$$

It is trivial to see from (2.22) and (2.23) that $s(M)zs(M)^{-1}z^{-1}$ and $s(M)Ts(M)^{-1}T^{-1}$ are finite rank perturbations of the identity. One may use formula (1.1) in [10] to calculate finite expansions for $\Gamma_Q(s_0(M)s_{e_j}(M)^{-1})$ in terms of the basis $\{\psi_j(e_\alpha(k)), \psi_j^*(e_\alpha(k))\}$ for the Clifford algebra. This may be used to establish difference identities useful for the analysis of the continuum limit along the lines developed in [9] for the Ising model. By taking products on the lattice one also finds formulas for $\Gamma_Q(s_a(M)s_b(M)^{-1})$ $(a, b\in Z^2)$ as finite sums of finite products of Clifford generators. In a sequel [22] we will use this to show that the vacuum expectation of a product of operators $\Gamma_Q(s_{a_j}(M)s_{b_j}(M)^{-1})$ has a limit as $k\uparrow 1$ (the "critical temperature"). We also show that $\lim_{k\uparrow 1}\langle\sigma_{a_1}(M_1)\ldots\sigma_{a_n}(M_n)\rangle$ exists provided $M_1\ldots M_n=I_p$ by making use of this last result and the homomorphism property for $M\to\sigma(M)$.

We have one further observation concerning (2.23). Note that conjugation of $T(z)$ by $s(M)$ produces an operator with the same (block) band difference structure as $T(z)$. This is reminiscent of isospectral deformations for Jacobi matrices and we suspect this is behind the intimate connection between s and T revealed in Sect. 1.

Before proceeding with the consideration of the massive scaling regime in the next section we make some more or less obvious remarks concerning the matrix $\begin{bmatrix} a(M) & b(M)\\ c(M) & d(M)\end{bmatrix}$ of $s(M)$ relative to the $H_+^p\oplus H_-^p$ decomposition of H^p. If $M=SJ_MS^{-1}$, where J_M is the Jordan normal form for M with eigenvalues $\lambda_1\ldots\lambda_p$ on the diagonal and 1's on the superdiagonal in elementary Jordan blocks, then clearly $a(M)=(I\otimes S)a(J_M)(I\otimes S^{-1})$, $b(M)=(I\otimes S)b(J_M)(I\otimes S^{-1})$ etc. The operators $b(J_M)$ and $c(J_M)$ have entries $b(\lambda_j)$ and $c(\lambda_j)$ on the diagonal ($b(\lambda)$, $c(\lambda)$ are the operators described in Sect. 1 and superdiagonal terms which are 0 or $Q_+\varepsilon_+Q_-\stackrel{\text{def}}{=}b_+$ or $Q_-\varepsilon_+Q_+\stackrel{\text{def}}{=}C_+$, $(\varepsilon_+=(1+\varepsilon)/2)$. The operators $a(J_M)$ and $d(J_M)$ have $a(\lambda_j)$ and $d(\lambda_j)$ on the diagonal with superdiagonal terms which are zero or $Q_\pm\varepsilon_+Q_\pm$; the terms $Q_+\varepsilon_+Q_+\stackrel{\text{def}}{=}a_+$ and $Q_-\varepsilon_+Q_-\stackrel{\text{def}}{=}d_+$ live in elementary Jordan blocks with diagonal entries $a(\lambda_j)$ or $d(\lambda_j)$ respectively (j fixed). If the eigenvalues $\lambda_1,\ldots,\lambda_p$ are non-exceptional so that $d(\lambda_j)$ is invertible for $j=1,\ldots,p$, then $d(M)$ is also invertible. The inversion of an elementary Jordan block $\begin{bmatrix} d(\lambda)\, d_+ & \ddots & 0\\ 0 & \ddots & d(\lambda)\, d_+\end{bmatrix}$ may be accomplished in the usual fashion by factoring out the diagonal and expanding the remainder in a finite geometric series:

$$\begin{bmatrix} d(\lambda) & d_+ & & 0\\ & \ddots & \ddots & \\ 0 & & d(\lambda) & d_+\end{bmatrix}^{-1}=\sum_{i=0}^{N}\begin{bmatrix} 0 & d(\lambda)^{-1}d_+ & & 0\\ & \ddots & \ddots & \\ & & \ddots & d(\lambda)^{-1}d_+\\ 0 & & & 0\end{bmatrix}^{i}(-1)^i d(\lambda)^{-1}. \quad (2.24)$$

Section 3

In this section we will consider the massive scaling regime for the correlations of the monodromy fields defined in Sect. 2. This regime is determined by sending the lattice spacing to zero and at the same time forcing the "temperature" to approach the critical "temperature" in such a fashion as to make the correlation length asymptotic to a non-zero constant (which we take to be 1 for simplicity). There is no "temperature" in our models, but since the correlation length is essentially determined by the transfer matrix, we will be able to fix the correlation length by sending $s \uparrow 1$, as this corresponds to sending $T \uparrow T_c$ in the Ising model. We proceed along the same lines as the proof in [9] of the convergence of the scaling limit for the Ising model. The appropriately scaled correlations at lattice spacing δ and modulus s_δ^2 are given by determinants $\det_2(1 + L_\delta R_\delta)$. We will embed the operators L_δ and R_δ in a fixed Hilbert space, and then prove that $L_\delta R_\delta$ converges in Schmidt norm as $\delta \to 0$.

Following the prescription in [9, p. 364], we set

$$s_\delta + s_\delta^{-1} = 2 + \frac{\delta^2}{2},$$

or

$$s_\delta = 1 - \frac{\delta}{\sqrt{2}} + O(\delta^2)\,(s_\delta \text{ is the root} < 1). \tag{3.1}$$

It is inconvenient to carry around the δ subscript. Throughout this section we write $s = s_\delta$ and $c = c_\delta = [s_\delta^2 + 1]^{1/2}$. We will also write $k = s_\delta^2$ and $K = K_\delta$ for the real quarter period associated with elliptic modulus k. Because it is hard to transform the kernels (1.15) back into the $z = e^{i\theta}$ variables, we will work instead in the uniformization variables $v_\pm$ given by (1.3) and (1.18). This requires a preliminary study of the substitutions (1.3) and (1.18) which was not required in the Ising case. Our first result concerns the translation of z and $w = e^{\mp\gamma}$ into the variables $v_\pm$. For simplicity we concentrate on v_+ which we write as v. The addition formula for $sn(\cdot)$ and the table of values 16.5 [20] imply that:

$$\sqrt{k}\,sn\left(v + K + \frac{iK'}{2}\right) = \frac{cn(v)dn(v) - i(1-k)sn(v)}{1 - k\,sn^2(v)}. \tag{3.2}$$

(This may be used, incidentally, to see that $v \to \sqrt{k}\,sn(v + K + iK'/2)$ winds around the unit circle as described earlier.) If we combine (3.2) with the first order (in $(1-k^2)$) asymptotics of $sn(\cdot)$, $cn(\cdot)$ and $dn(\cdot)$ in terms of hyperbolic functions near $k = 1$ (see (16.15.1), (16.15.2), (16.15.3) in [20]) then one finds:

$$\sqrt{k}\,sn\left(v + K + \frac{iK'}{2}\right) = 1 - i\delta\sqrt{2}\sinh(v)\cosh(v) + O(\delta^2),$$

which is to be understood as an asymptotic result at fixed v uniform for v in a fixed compact subset of $\mathbb{R}$. Combining this result with the substitution (1.3) one finds:

$$z = 1 + i\delta\sinh(2v) + O(\delta^2). \tag{3.3}$$

The second order terms for $\sqrt{k}sn(v+K+iK'/2)$ do not directly contribute to the lowest order for $z+z^{-1}$. The result is

$$\frac{z+z^{-1}}{2}=1-\frac{\delta^2}{2}\sinh^2(2v)+O(\delta^3),$$

from which it follows that

$$\cosh\gamma=s+s^{-1}-\frac{z+z^{-1}}{2}=1+\frac{\delta^2}{2}\cosh^2(2v)+O(\delta^3),$$

and finally

$$e^{-\gamma}=1-\delta\cosh(2v)+O(\delta^2). \tag{3.4}$$

Next we require a v-uniform estimate for $e^{-\gamma}$ in the v variables. Presumably one could substitute (3.2) into:

$$\cosh\gamma=-\alpha_1\frac{x^2-2k\beta x+k^2}{(k+\alpha_1 x)(x+\alpha_1 k)},$$

$$\sinh\gamma=2cs(1+s)\alpha_1\frac{x[(1-k^2x^{-1})(1-x)]^{1/2}}{(k+\alpha_1 x)(x+\alpha_1 k)},$$

and make the appropriate estimates. We did not see how to do this and so we will proceed differently. We will obtain the upper bound for $e^{-\gamma}$ which we desire by proving a lower bound for γ. The estimate (4.3) in [9] for γ is easily seen to be equivalent to

$$\gamma(\theta)>\omega_\delta(\theta)(1+\omega_\delta(\theta))^{-1},$$

where $\omega_\delta^2(\theta)=\delta^2+(1-\cos\theta)$. But $(1+\omega_\delta(\theta))^{-1}\geqq 1/3$, when $\delta<\sqrt{2}$ so that $\gamma(\theta)\geqq m\omega_\delta(\theta)$, where m is a constant independent of θ and δ. Now let $z=e^{i\theta}$ and $x=k(z-\alpha_1)/(1-\alpha_1 z)=e^{i\psi}$ $(-\pi<\psi\leqq\pi)$. Then from (1.4) one sees

$$d\theta=\frac{c(1+s^{-1})}{s+s^{-1}+1+\cos\psi}d\psi. \tag{3.5}$$

It follows from this that the graph of $\theta(\psi)$ is concave down between 0 and π. Since $\theta(0)=0$ and $\theta(\pi)=\pi$ it follows that $\psi\leqq\theta$ $(0\leqq\theta\leqq\pi)$. Hence

$$\gamma(\theta)\geqq m\omega_\delta(\theta)\geqq m\omega_\delta(\psi),\quad 0\leqq\theta\leqq\pi.$$

We turn our attention to the estimation of $\omega_\delta^2(\psi)=\delta^2+(1-\cos\psi)$ from below. The full elliptic substitution is rather awkward even at this stage so we introduce a hyperbolic substitution which is asymptotically correct (i.e. $=v$) as $\delta\to 0$. Let

$$x=k\frac{1+\frac{i\delta}{\sqrt{2}}\sinh(2u)}{1-\frac{i\delta}{\sqrt{2}}\sinh(2u)},$$

then one easily calculates

$$\omega_\delta^2(\psi) = \delta^2 \left(\frac{\cosh^2(2u) + \frac{\delta^2}{2}\sinh^2(2u)}{1 + \frac{\delta^2}{2}\sinh^2(2u)} \right). \tag{3.6}$$

Next we need the relationship between u and v. This is most easily seen by calculating the abelian differential (1.6) in the u coordinates. One finds

$$v = \frac{2}{1+k} \int_0^u \frac{\cosh(2u)du}{[\cosh^2(2u) + O(\delta)]^{1/2} \left[1 + \frac{\delta^2}{2}\sinh^2(2u) \right]^{1/2}},$$

where the term $O(\delta)$ is uniformly $O(\delta)$ in the u variables. It follows that

$$v \leqq M \int_0^u \frac{du}{\left(1 + \frac{\delta^2}{2}\sinh^2(2u) \right)^{1/2}},$$

where $M = 1 + O(\delta)$ is u-uniform. Now make the estimate $\sinh^2(2u) \geqq (e^{4u} - 2)/4$ in the preceding integral and calculate the resulting integral. One finds

$$v \leqq \frac{M}{2} \log \left[\frac{1 + [1 + a]^{1/2}}{1 + [1 + ae^{4u}]^{1/2}} \right] e^{2u}, \quad a = \frac{\delta^2}{2(1 - \delta^2)}.$$

Without difficulty one may use this to show that

$$e^{4v/M} \leqq \frac{2(2 + a)e^{4u}}{2 + \frac{\delta^2}{2}e^{4u}}.$$

The right-hand side is less than or equal to (const) $\delta^{-2}\omega_\delta^2(\psi)$, as one may see from (3.6), $(e^{2u})^2 \leqq 4(\cosh(2u))^2$ and $e^{4u} \geqq (\sinh(2u))^2$. Thus we have proved

$$\gamma(v) \geqq \delta m e^{2v/M} \geqq \delta m \cosh\left(\frac{2v}{M} \right), \quad 0 \leqq v \leqq K$$

for some constants $m, M > 0$ independent of δ and v. But $\gamma(v)$ is an even function of v so

$$\gamma(v) \geqq \delta m \cosh\left(\frac{2v}{M} \right) \quad |v| \leqq K. \tag{3.7}$$

The changes that are required in (3.3), (3.4) and (3.7) using the substitution $x = \sinh^{-2}(v_- + K - iK'/2)$ may all be deduced from $sn(u + iK') = k^{-1}sn^{-1}(u)$. The only one of these results that is changed is (3.3) which becomes

$$z = 1 - i\delta \sinh(2v_-) + O(\delta). \tag{3.8}$$

In taking the limit $\delta \to 0$ we will not be able to say anything for those values of λ at which $d(\lambda)$ fails to be invertible. These are the values $\lambda = q^{2l} = e^{-2\pi K' l/K}$ $(l \in Z_{1/2})$. Observe that $\delta \to 0$ we have $K \to \infty$ and $K' \to \pi/2$. Thus the exceptional values λ

become dense on the negative real axis. At any small non-zero value of δ these exceptional values cluster at $-\infty$ and 0. As δ is decreased they all march towards -1. The point -1 on the negative axis is unusual in that every other point on the negative axis is crossed infinitely often by exceptional values, but -1 is only "exceptional" in the limit $\delta \to 0$. In our discussion of scaling limits we will confine our attention to those values of λ not on the negative axis and $\lambda = -1$ (the Ising case).

Let $\sigma(M)$ denote the element of $\widehat{\mathrm{Gl}_Q}(H)$ whose induced rotation is $s(M)$ normalized as in Sect. 2. For $a \in \mathbb{R}^2$ define $\sigma_a(M) = V(a)\sigma(M)V(a)^{-1}$. Let $M_j \in \mathrm{GL}(p, \mathbb{C})$ $(j = 1, \ldots, n)$ be matrices whose spectrum intersects the negative axis at most at -1. We shall prove the convergence of:

$$\lim_{\delta \downarrow 0} \prod_{j=1}^{n} \langle \sigma(M_j) \rangle_{k=k_\delta}^{-1} \langle \vec{T} \sigma_{a_1/\delta}(M_1) \cdots \sigma_{a_n/\delta}(M_n) \rangle_{k=k_\delta},$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}^2$, the second coordinates are non-coincident $\pi_2(a_j) \neq \pi_2(a_k) (j \neq k)$, the "time" ordering $\vec{T}$ puts the operators $\sigma_{a_j/\delta}(M_j)$ in order of increasing second coordinate from left to right and $k = k_\delta$ refers to the fact that T and Q are evaluated at $k = k_\delta$. By relabeling we may assume $\pi_2(a_1) < \pi_2(a_2) \cdots < \pi_2(a_n)$. Then the expectation we are interested in is:

$$\prod_{j=1}^{n} \langle \sigma(M_j) \rangle_{k=k_\delta}^{-1} \langle \sigma_{a_1/\delta}(M_1) \cdots \sigma_{a_n/\delta}(M_n) \rangle_{k=k_\delta}. \tag{3.9}$$

As in the case of the Ising model it is convenient to take advantage of the smoothing properties of the transfer matrix. Instead of the product in (3.9) we introduce: $\tilde{\sigma}_j = V((a_j - a_{j-1})/2\delta)\sigma(M_j)V((a_{j+1} - a_j)/2\delta)$, where a_0 and a_{n+1} are arbitrary except for $\pi_2(a_0) < \pi_2(a_1)$ and $\pi_2(a_n) < \pi_2(a_{n+1})$. Using $V(a)1 = 1$ and $V^*(a)1 = 1$, one finds (3.9) can be written:

$$\prod_{j=1}^{n} \langle \sigma(M_j) \rangle_{k=k_\delta}^{-1} \langle \tilde{\sigma}_1 \cdots \tilde{\sigma}_n \rangle_{k=k_\delta}.$$

According to Theorem (3.2) in [10] this ratio of vacuum expectations is a determinant, $\det_2(I + L_\delta R_\delta)$, where the matrix L_δ is the $n \times n$ block matrix with entries

$$(i<k) \quad l_{ik} = \begin{cases} -Q_+ & k = i+1 \\ -a_{i+1}Q_+ & k = i+2 \\ -a_{i+1} \cdots a_{k-1} Q_+ & k > i+2, \end{cases}$$

$$(i>k) \quad l_{ik} = \begin{cases} Q_- & i = k+1 \\ d_{k+1}^{-1} Q_- & i = k+2 \\ d_{i-1}^{-1} \cdots d_{k+1}^{-1} Q_- & i > k+2, \end{cases}$$

$$(i=k) \quad l_{ii} = 0,$$

where $T(\tilde{\sigma}_j) = \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix}$
and $R_\delta = R(\tilde{\sigma}_1) \oplus \cdots \oplus R(\tilde{\sigma}_n)$, where

$$R(\tilde{\sigma}_j) = \begin{bmatrix} -b_j d_j^{-1} c_j & b_j d_j^{-1} \\ d_j^{-1} c_j & 0 \end{bmatrix}.$$

The transformation from $L^2(S^1,\mathbb{C}^2)$ to $L^2([-K,K],\mathbb{C}^2)$ described in Sect. 1 is a unitary equivalence. In an obvious fashion we may identify $L^2(S^1,\mathbb{C}^2)\otimes\mathbb{C}^p$ with $L^2([-K,K],\mathbb{C}^2)\otimes\mathbb{C}^p$. We now introduce the injection of H^p into a fixed Hilbert space that we will use for scaling. It is the natural isometric inclusion, $L^2([-K,K],\mathbb{C}^m)\xrightarrow{i}L^2(\mathbb{R},\mathbb{C}^m)$. We will show that $iL_\delta i^*$ converges in the strong topology on $L^2(\mathbb{R},\mathbb{C}^{2p})\oplus\cdots\oplus L^2(\mathbb{R},\mathbb{C}^{2p})$ (n summands) and that $iR_\delta i^*$ converges in Schmidt norm on the same space. Since i is the natural injection we will drop it in further considerations when confusion seems unlikely (an operator X on $L^2([-K,K],\mathbb{C}^m)$ we extend to $L^2(\mathbb{R},\mathbb{C}^m)$ as iXi^*, which vanishes on the orthogonal complement of $L^2([-K,K],\mathbb{C}^m)$ in $L^2(\mathbb{R},\mathbb{C}^m)$. A glance at the formulas for L_δ shows that we need only prove strong convergence ($\delta\to 0$) for a_j and d_j^{-1} on $L^2(\mathbb{R},\mathbb{C}^p)$ and Schmidt norm convergence ($\delta\to 0$) for $-b_jd_j^{-1}c_j, b_jd_j^{-1}$ and $d_j^{-1}c_j$ on $L^2(\mathbb{R},\mathbb{C}^p)$ to obtain the desired result. Now let $m_j=\pi_1((a_j-a_{j-1})/2)$ and $n_j=\pi_2((a_j-a_{j-1})/2)$. The results of Sect. 2 and the relation between $T(\tilde\sigma_j)$ and $T(\sigma(M_j))$ show that to prove the strong convergence of a_j and d_j^{-1} it suffices to establish the strong convergence of operators of the form:

$$
\begin{gathered}
z^{m_1/\delta}w^{n_1/\delta}a(\lambda)w^{n_2/\delta}z^{m_2/\delta},\\
z^{m_1/\delta}w^{n_1/\delta}a_+w^{n_2/\delta}z^{m_2/\delta},\\
z^{m_1/\delta}w^{-n_1/\delta}(d(\lambda)^{-1}d_+)^i d(\lambda)^{-1}w^{-n_2/\delta}z^{m_2/\delta},
\end{gathered}
\tag{3.10}
$$

where $z=e^{i\theta}$, $w=e^{\pm\gamma(\theta)}$, $n_j>0$ ($j=1,2$), and i is a non-negative integer. In a similar fashion one may reduce the proof of convergence for $b_jd_j^{-1}$, $d_j^{-1}c_j$, $-b_jd_j^{-1}c_j$ to a proof of convergence in Schmidt norm for the following sorts of operators:

$$
\begin{gathered}
z^{m_1/\delta}w^{n_1/\delta}b_+w^{-n_2/\delta}z^{m_2/\delta}\quad(\text{and } b_+\leftarrow c_+),\\
z^{m_1/\delta}w^{n_1/\delta}b(\lambda)(d(\lambda)^{-1}d_+)^i d(\lambda)^{-1}w^{-n_2/\delta}z^{m_2/\delta},\\
z^{m_1/\delta}w^{-n_1/\delta}d(\lambda)^{-1}(d_+d(\lambda)^{-1})^i c(\lambda)w^{n_2/\delta}z^{m_2/\delta},\\
z^{m_1/\delta}w^{n_1/\delta}b(\lambda)(d(\lambda)^{-1}d_+)^i d(\lambda)^{-1}c(\lambda)w^{n_2/\delta}z^{m_2/\delta}.
\end{gathered}
\tag{3.11}
$$

We now restrict our attention to those $\lambda\in\mathbb{C}$ not on the negative real axis. The case $\lambda=-1$ is interesting but its investigation is more complicated and will be reserved for a later treatment (see [9] for the scalar Ising case). We first illustrate a method for dealing with the operators in (3.11) by examining the last case; the other cases may be dealt with in a precisely analogous fashion. The results of Sect. 1 show that $b(\lambda)(d(\lambda)^{-1}d_+)^i\, d(\lambda)^{-1}\, c(\lambda)$ has eigenvalues $(1-\lambda)^2\,(\lambda+q^{-2l})^{-i-1}\,(1+q^{2l})^{-1}$, and is given by convolution with

$$
f(u,k)=\frac{(1-\lambda)^2}{2K}\sum_{l\in Z_{1/2}}(\lambda+q^{-2l})^{-i-1}(1+q^{2l})^{-1}e^{(il\pi/K)u},
\tag{3.12}
$$

where $q=e^{-\pi K'/K}$ and $k=s^2$ is the elliptic modulus. We wish to calculate $\lim_{k\uparrow 1}f(u,k)$ ($k\uparrow 1$ as $\delta\to 0$). Observe that with $x=\pi l/K$ the expression (3.12) is an infinite "Riemann sum" approximation to the following integral (we use $K'\uparrow\pi/2$ and $K\uparrow\infty$

as $k\uparrow 1$)

$$f(u)=\frac{(1-\lambda)^2}{2\pi}\int_{-\infty}^{\infty}(\lambda+e^{\pi x})^{-i-1}(1+e^{-\pi x})^{-1}e^{ixu}dx. \tag{3.13}$$

The exponential decrease of the summand in (3.12) makes it clear that dominated convergence applies and we have $\lim_{k\uparrow 1} f(u,k)=f(u)$. We will now show that this convergence takes place in L^2. The Plancheral Theorem implies that the square of the L^2 norm $\int_{-K}^{K}|f(u,k)|^2du$ is given by:

$$\frac{|1-\lambda|^4}{2\pi}\sum_{l\in Z_{1/2}}|\lambda+q^{-2l}|^{-2i-2}(1+q^{2l})^{-2}\frac{\pi}{K}. \tag{3.14}$$

Again using $q=e^{-\pi K'/K}$, one sees this is the "Riemann sum approximation" to an integral in the variable $x=l\pi/K$. Dominated convergence applies in (3.14) as $k\uparrow 1$ and one easily checks that the resulting limit is, by the Plancherel Theorem, equal to $\int_{-\infty}^{\infty}|f(u)|^2du$. Thus:

$$\lim_{k\uparrow 1}\int_{-K}^{K}|f(u,k)|^2du=\int_{-\infty}^{\infty}|f(u)|^2du.$$

It is an exercise in Rudin ([12] page 73) that this and pointwise convergence together imply $f(u,k)$ converges to $f(u)$ in L^2 (note $f(u,k)\equiv 0$ for $|u|\geqq K$!).

Next observe that (3.3), (3.4) and (3.8) imply that

$$\lim_{\delta\to 0} z^{m/\delta}=e^{\pm im\sinh(2v_\pm)},\quad \lim_{\delta\to 0} w^{n/\delta}=e^{\mp n\cosh(2v_\pm)}. \tag{3.15}$$

Furthermore the estimate $w^{\pm n/\delta}\leqq e^{-mn\cosh(2v_\pm)/M}$, which follows from (3.7) shows that multiplication by $w^{\pm n/\delta}$ (defined to be 0 outside $|v_\pm|\leqq K$) converges uniformly to multiplication by $e^{-n\cosh(2v_\pm)}$ when $n>0$ and $\delta\to 0$. The convergence of $z^{m/\delta}$ to $e^{\pm im\sinh(2v_\pm)}$ is uniform on fixed compact intervals for $v_\pm$ as $\delta\to 0$ and this shows (since $|z|=1$) that $z^{m/\delta}w^{\pm n/\delta}$ converges uniformly as a multiplication operator on $L^2(\mathbb{R},\mathbb{C}^2)$. Let u denote $v_\pm$ and write $g(u,k)=z^{m/\delta}w^{\pm n/\delta}$, where $k=k_\delta$. Consider the integral operator on $L^2(\mathbb{R},\mathbb{C})$ with kernel $G(u,v;k)=X_K(u)g(u,k)f(u-v,k)X_K(v)$, where $X_K(u)=\begin{cases}1 & |u|\leqq K\\ 0 & |u|>K\end{cases}$. Using the fact that g and f converge in L^2 as $k\to 1$ it is easy to see that $G(u,v,k)$ converges in $L^2(\mathbb{R}^2)$ as $k\to 1$. Thus the operator determined by the kernel $G(u,v,k)$ converges in Schmidt norm, and it follows that the same can be said for the last operator appearing in (3.11).

Next we turn to the strong convergence results needed for (3.10). For λ not on the negative real axis it is trivial to supply uniform (in δ) estimates for the operator norms of all the operators in (3.10). Thus it is sufficient to prove strong convergence on a dense set. We choose the dense set $C_0^\infty(\mathbb{R})$ for this purpose. We illustrate the method for the middle case in (3.10). Suppose $F\in C_0^\infty(\mathbb{R})$ and write $\hat{F}(x)=\int_{-\infty}^{\infty}F(u)e^{-ixu}du$. Then for K sufficiently large $\hat{F}(x)=\int_{-K}^{K}F(u)e^{-ixu}du$, and it follows that:

$$a_+F(u) = \sum_{l \in Z_{1/2}} \frac{1}{1+q^{2l}} \hat{F}\left(\frac{\pi l}{K}\right) e^{(i\pi l/K)u} \frac{1}{2K}.$$

Since $\hat{F}$ is rapidly decreasing, dominated convergence applies to this last "Riemann sum approximation" and we find pointwise convergence to

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+e^{\pi x}} \hat{F}(x) e^{ixu} dx,$$

as $\delta \to 0$. Since $(1+e^{\pi x})^{-1}$ is bounded, the L^2 norms converge in the appropriate sense and it follows that a_+ converges in the strong operator topology as $\delta \to 0$. Since the product of a strongly convergent sequence of operators and a sequence of operators converging in Schmidt norm also converges in Schmidt norm, it follows that $L_\delta R_\delta$ converges in Schmidt norm. But $\det_2(I+A)$ is continuous in A in the Schmidt norm. Thus we have proved:

Theorem 3.0. *Suppose $M_i \in \mathrm{Gl}(p, \mathbb{C})(i=1,\dots,n)$ and no M_i has an eigenvalue on the negative real axis. If $\pi_2(a_1) < \pi_2(a_2) \dots < \pi_2(a_n)$, then*

$$\lim_{\delta \to 0} \prod_{j=1}^{n} \langle \sigma(M_j) \rangle_{k=k_\delta}^{-1} \langle \sigma_{a_1/\delta}(M_1) \dots \sigma_{a_n/\delta}(M_n) \rangle_{k=k_\delta} \text{ exists.}$$

The same technique suffices to establish the convergence of suitably scaled wave functions (2.5) using the formulas from [7]. This will be taken up in another place where the local expansion results are also derived.

Bibliography

1. Baxter, R. J.: Exactly solved models in statistical mechanics. London: Academic Press 1982
2. Date, E., Jimbo, M., Kashiwara, M., Miwa, T.: Transformation groups for soliton equations:
   I. Proc. Jpn. Acad. **57A**, 342–347 (1981)
   II. Ibid. 387–392.
   III. J. Phys. Soc. Jpn. **50**, 3806–3812 (1981)
   IV. Physica 4D **1982**, 343–365
   V. RIMS preprint.
   VI. J. Phys. Soc. Jpn **50**, 3813–3818 (1981)
   VII. RIMS preprint
3. McCoy, B. M., Wu, T. T.: Nonlinear partial difference equations for the two spin correlation functions of the two dimensional Ising model. Nucl. Phys. **B180**, 89–115 (1981)
4. McCoy, B. M., Wu, T. T.: Nonlinear partial difference equations for the 2-dimensional Ising model. Phys. Rev. Letts. **45**, 675 (1980)
5. McCoy, B. M., Perk, J. H. H., Wu, T. T.: Ising field theory: Quadratic difference equations for the *n*-point Green's functions on the lattice. Phys. Rev. Letts. **46**, 757 (1981)
6. Malgrange, B.: Sur les deformations isomonodromiques. Université Scientifique et Medicale de Grenoble (1982) preprint
7. Palmer, J.: Products in spin representations. Adv. Appl. Math. **2**, 290–328 (1981)
8. Palmer, J., Tracy, C.: Two dimensional Ising correlations: Convergence of the scaling limit. Adv. Appl. Math. **2**, 329–388 (1981)
9. Palmer, J., Tracy, C.: Two dimensional Ising correlations: the S.M.J. analysis. Adv. Appl. Math., 46–102 (1982)
10. Palmer, J.: Product formulas for spherical functions. Preprint

11. Perk, J.H.H.: Quadratic identities for Ising correlations. Phys. Lett. **79A**, 3 (1980)
12. Rudin, W.: Real and complex analysis. New York: McGraw Hill 1966
13. Ruijsenaars, S.N.M.: The Wightman axioms for the fermionic Federbush model. Commun. Math. Phys. **87**, 181–228 (1982)
14. Ruijsenaars, S.N.M.: Integrable quantum field theories and Bogoliubov transformation. Ann. Phys. **132**, 328–382 (1981)
15. Ruijsenaars, S.N.M.: Scattering theory for the Federbush, massless Thirring and continuum Ising models. J. Funct. Anal. **48**, 135–171 (1982)
16. Ruijsenaars, S.N.M.: On the two-point functions of some integrable relativistic quantum field theories. J. Math. Phys. **24**, 922–31 (1983)
17. Sato, M., Miwa, T., Jimbo, M.: Holonomic quantum fields I–V. Publ. Res. Inst. Math. Sci. **14**, 223–267 (1978); **15**, 201–278 (1979); **15**, 577–629 (1979); **15**, 871–972 (1979); **16**, 531–584 (1980)
18. Segal, G., Wilson, G.: Loop groups and equations of KdV type. Oxford, preprint
19. Yang, C. N.: The spontaneous magnetization of the two-dimensional Ising model. Phys. Rev. **85**, 808–816 (1952)
20. Handbook of mathematical functions. National Bureau of Standards, Appl. Math. Ser. 65 (1968)
21. Henrici, P.: Applied and computational complex analysis, Vol. 2. New York: Wiley–Interscience 1974
22. Palmer, J.: Critical Scaling for monodromy fields (Preprint)

Communicated by K. Osterwalder

Received December 1, 1984; in revised form April 29, 1985