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Local Existence of the Borel Transform in Euclidean Massless Φ_4^4

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Abstract. We extend the methods of [1] to prove large order estimates on the renormalized Feynman amplitudes of massless Φ_4^4 euclidean field theory, at non-exceptional momenta. The Borel transform of the perturbative series is analytic in a disk centered at the origin of the complex plane. This result is a step towards the rigorous investigation of the infra-red singularities in the Borel plane, for theories containing massless particles, like the gauge theories.

I. Introduction

In [1], the large order behaviour of the perturbation series was rigorously investigated, for the renormalized massive Φ_4^4 field theory. The n^{th} order term for any Schwinger function at fixed external momenta was bounded by $K^n n!$, where K is a constant (depending on the external momenta). This implies the existence and analyticity of the Borel transform of the perturbative series, at least in a disk of the complex plane centered at the origin. This result has been rederived recently by other methods [2].

To extend the work of [1] to field theories containing massless particles is a non-trivial and interesting problem for several reasons. The renormalization scheme with subtractions at zero external momenta becomes ill-defined due to infra-red singularities in the amplitudes. Therefore, one should renormalize with subtractions at some intermediate energy scale μ, and the renormalized amplitudes are finite only at non-exceptional momenta. The structure of the corresponding renormalization operator, ensuring that the physical mass of the particles effectively vanishes, gets more complicated and the resulting structure is richer. In particular, varying the energy scale of the subtraction point, one recovers renormalization group equations, and interesting phenomena like the infra-red asymptotic freedom of the Φ_4^4 theory can be analyzed. More generally, the study of theories with massless particles, especially the analytic structure of their Borel transform, is certainly relevant to the program of rigorous construction of gauge theories, both in their complete version or in approximations like the $N \to \infty$ planar theories.

In this paper we limit ourselves to the case of massless Φ_4^4, but the method used in quite general. As in [1], we prove bounds on each Feynman amplitude at non-exceptional momenta, which depend both on the perturbation order and on the renormalization structure of the amplitude. Then we count the number of graphs with a given renormalization structure, and combining these estimates we obtain the local existence of the Borel transform in a disk whose radius shrinks to zero, both when an external invariant goes to zero or to infinity. This should be related to the existence of two different types of singularities in the Borel plane of the theory: infra-red renormalons on the left (negative) axis, and ultra-violet renormalons on the right (positive) axis.

At this stage one might exploit asymptotic freedom to go further. Combining the estimates of this paper with the dressing techniques introduced in [3], it should be possible to transform ordinary amplitudes into dressed amplitudes with improved estimates. Although we do not implement this program in the present paper, we can describe with good confidence the results one should get. The ordinary theory with positive coupling constant is asymptotically free in the infra-red region. (Using this fundamental property, the theory with an ultra-violet cut-off has been rigorously constructed recently [4, 5].) Therefore, after the dressing transformation of [3], the radius of convergence of the Borel transform should not shrink when external invariants go to zero, which indicates the harmless character of the infra-red renormalons in this theory. Of course, the radius should still shrink when external momenta go to infinity, since the ultra-violet renormalons should be harmful. Conversely, for the (unstable?) theory with negative coupling constant, after the dressing transformation, the radius should not shrink to zero when external invariants go to infinity, but still shrink when they go to zero: this is similar to the situation for non-abelian gauge theories.

Some familiarity of the reader with [1] is assumed since we use the same notations and several technical results. The paper is organized as follows: in Sect. II we give our notations and recall the definition of the modified renormalization operator in the α-parametric space; Sect. III contains the organization of the renormalization by classifying the forests in each Hepp's sector, and exhibits the result of the corresponding subtraction process; Sect. IV is devoted to the proof of the technical estimates which allow to derive our main theorems, given in Sect. V.

II. Definition of the Model

The perturbative definition of the renormalized massless Φ_4^4 model is given in the formulation of [6]. The Lagrangian is

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-\lambda\varphi^4+\text{counterterms}, \tag{2.1}$$

where the counterterms are defined by the renormalization rules given farther. The Schwinger functions are defined following the notations of [1], except the fact that the formal unrenormalized Feynman amplitude associated to a given graph G does not contain any mass term. That is, formula (II.2) of [1] is replaced by

$$I_G(\not{p})=\int_0^\infty\dots\int_0^\infty\prod_{i=1}^{\ell}d\alpha_i U^{-2}(\alpha)\exp\{-V(\not{p},\alpha)/U(\alpha)\}, \tag{2.2}$$

U and V being the standard Symanzik polynomials.

All the notations of graph theory, including the definition of the forests, reduced graphs, proper and closed subgraphs, follow exactly Sect. II of [1], and Lemma II.1 of [1] remains, of course, unchanged.

Now the definition of the renormalized massless amplitudes differs from that one which is used in the massive case. Following [6], the vanishing of the physical mass is ensured by a first step in the renormalization procedure: all self-energies (proper bipeds) are subtracted once at zero momentum. Then the remaining ultraviolet divergences (for bipeds and quadrupeds) are subtracted by a "μ-renormalization" which avoids the appearance of infra-red divergences. The final result of this renormalization scheme may be written in the parametric representation

$$I_G^R(\not{p})=\int_0^\infty \dots \int_0^\infty \prod_{i=1}^{\ell} d\alpha_i e^{-\mu^2 \sum_{i=1}^{\ell}\alpha_i} \mathscr{R}\left[e^{+\mu^2 \sum_{i=1}^{\ell}\alpha_i} \mathscr{R}'(U^{-2}e^{-V/U})\right]. \tag{2.3}$$

Since two proper bipeds can never overlap, the set of all proper bipeds of G is a forest, and the $\mathscr{R}'$ operation can be written

$$\mathscr{R}'=\prod_B (1-\mathscr{T}'_B), \tag{2.4}$$

where the product runs over all proper bipeds B of G, and $1-\mathscr{T}'_B$ subtracts only the first term in the ϱ_B Taylor expansion of $U^{-2}\exp\{-V/U\}$, if ϱ_B scales the α_i variables, $i\in B$. The $\mathscr{R}$ operation is given as usual by

$$\mathscr{R}=\sum_{\mathscr{F}} \prod_{F\in\mathscr{F}} (-\mathscr{T}_F), \tag{2.5}$$

where the sum runs over all proper divergent forests of G, including the empty one. After the $\mathscr{R}'$ operation is performed, the bipeds become logarithmically divergent instead of being linearly divergent in the α variables. Thus in (2.5) $\mathscr{T}_F$ retains only the first term in the ϱ_F Taylor expansion (for F biped or quadruped), if ϱ_F scales the α_i variables, $i\in F$.

From [6] it is known that (2.3) is an absolutely convergent integral representation for non-exceptional momenta (no vanishing partial sum of external momenta).

If U_F, $U_{\bar{F}}$, V_F, $V_{\bar{F}}$ are defined as in [1], and if we put

$$A=A_{\bar{G}}=\mu^2\sum_{i=1}^{\ell}\alpha_i; \qquad A_F=\mu^2\sum_{i\notin F}\alpha_i; \qquad A_{\bar{F}}=\mu^2\sum_{i\in F}\alpha_i, \tag{2.6}$$

we may write explicitly the renormalization corresponding to one subgraph F:

i) if F is a quadruped ($N_F=4$)

$$\mathscr{T}_F e^A U^{-2}\exp\{-V/U\}=U_F^{-2}\exp\left\{A_F-\frac{V_F}{U_F}\right\}, \tag{2.7}$$

$$\begin{aligned}(1-\mathscr{T}_F)e^A U^{-2}\exp\{-V/U\}=&\int_0^1 d\xi_F(U_F+\xi_F U_{\bar{F}})^{-2}\\ &\cdot\exp\left\{A_F+\xi_F A_{\bar{F}}-\frac{V_F+\xi_F V_{\bar{F}}}{U_F+\xi_F U_{\bar{F}}}\right\}\\ &\cdot\left[A_{\bar{F}}-2\frac{U_{\bar{F}}}{U_F+\xi_F U_{\bar{F}}}+\frac{V_F U_{\bar{F}}-V_{\bar{F}}U_F}{(U_F+\xi_F U_{\bar{F}})^2}\right],\end{aligned} \tag{2.8}$$

ii) if F is a biped ($N_F=2$)

$$e^A(1-\mathscr{T}_F)U^{-2}\exp\{-V/U\}=\int_0^1 d\psi_F(U_F+\psi_F U_{\bar F})^{-2}$$
$$\cdot\exp\left\{A-\frac{V_F+\psi_F V_{\bar F}}{U_F+\psi_F U_{\bar F}}\right\}$$
$$\cdot\left[-2\frac{U_{\bar F}}{U_F+\psi_F U_{\bar F}}+\frac{V_F U_{\bar F}-V_{\bar F}U_F}{(U_F+\psi_F U_{\bar F})^2}\right], \tag{2.9}$$

$$\mathscr{T}_F e^A(1-\mathscr{T}_F)U^{-2}\exp\{-V/U\}=U_F^{-2}\exp\left\{A_F-\frac{V_F}{U_F}\right\}$$
$$\cdot\left[-2\frac{U_{\bar F}}{U_F}+\frac{V_F U_{\bar F}-V_{\bar F}U_F}{U_F^2}\right], \tag{2.10}$$

$$(1-\mathscr{T}_F)e^A(1-\mathscr{T}_F)U^{-2}\exp\{-V/U\}$$
$$=\int_0^1 d\xi_F\int_0^1 d\psi_F(U_F+\xi_F\psi_F U_{\bar F})^{-2}\exp\left\{A_F+\xi_F A_{\bar F}-\frac{V_F+\xi_F\psi_F V_{\bar F}}{U_F+\xi_F\psi_F U_{\bar F}}\right\}$$
$$\cdot\left[A_{\bar F}\left(\frac{-2U_{\bar F}}{U_F+\xi_F\psi_F U_{\bar F}}+\frac{V_F U_{\bar F}-V_{\bar F}U_F}{(U_F+\xi_F\psi_F U_{\bar F})^2}\right)\right.$$
$$\left.+\frac{\psi_F}{U_F+\xi_F\psi_F U_{\bar F}}\left(6U_{\bar F}^2-6U_{\bar F}\frac{V_F U_{\bar F}-V_{\bar F}U_F}{U_F+\xi_F\psi_F U_{\bar F}}+\left(\frac{V_F U_{\bar F}-V_{\bar F}U_F}{U_F+\xi_F\psi_F U_{\bar F}}\right)^2\right)\right]. \tag{2.11}$$

From Lemma II.2 of [1], and formulae (2.7), (2.11) above, we see that if F is an open quadruped and F^* its closure, we have

$$\mathscr{T}_F(1-\mathscr{T}_{F^*})e^A(1-\mathscr{T}'_{F^*})U^{-2}\exp\{-V/U\}=0\,. \tag{2.12}$$

Thus in the same way as Lemma II.3 of [1] is proved, we may take as a more convenient definition of the $\mathscr{R}$ operation,

$$\mathscr{R}=\sum_{\mathscr{F}}\prod_{F\in\mathscr{F}}(-\mathscr{T}_F), \tag{2.13}$$

where now the sum is performed only over all closed divergent forests of G, including the empty one.

III. The Classes of Forests and the Subtraction Process

As in [1] we decompose the renormalized amplitude I_G^R into a sum of integrals in the Hepp's sectors,

$$h_\sigma=\{\alpha|0\leqq\alpha_{\sigma(1)}\leqq\alpha_{\sigma(2)}\leqq\ldots\leqq\alpha_{\sigma(\ell)}\}, \tag{3.1}$$
$$I_G^R=\sum_\sigma I_{G\sigma}^R, \tag{3.2}$$

where the sum runs over all permutations of $\{1,\ldots,\ell\}$, and

$$I_{G\sigma}^R=\int_{h_\sigma}\prod_{i=1}^{\ell}d\alpha_i e^{-A}\mathscr{R}[e^A\mathscr{R}'(U^{-2}e^{-V/U})]\,. \tag{3.3}$$

Then for each σ the various closed divergent forests are grouped into appropriate classes, in a way which is almost identical to the way defined in Sect. III.1 of [1]. Given a forest $\mathscr{F}$, and a graph F compatible with $\mathscr{F}$ (i.e. such that $\mathscr{F}\cup\{F\}$ is a forest), then

i) $x(F,\mathscr{F})$ is the smallest rank i such that $F\cap G_i^\sigma\cup A_{\mathscr{F}}(F)$ contains a proper component $X(F,\mathscr{F})$, the closure of which is F.

ii) Starting from the maximal elements of $\mathscr{F}$, we define

$$\mathscr{S}(\mathscr{F})=\{F|F\in\mathscr{F};\ x(F,\mathscr{F})>y(F,\mathscr{F})\}\,, \tag{3.4}$$

$$y(G,\mathscr{F})=\ell+1\,. \tag{3.5}$$

If F is a biped, $F\neq G$,

$$y(F,\mathscr{F})=\sup_{\sigma(i)\in E(F)} i\,, \tag{3.6}$$

where $E(F)$ is the set of external lines of F. If F is a quadruped, $F\neq G$,

$$y(F,\mathscr{F})=\inf_{\sigma(i)\in E'(F)} i\,, \tag{3.7}$$

where $E'(F)$ is the set external lines of F, internal in $B_{\mathscr{S}(\mathscr{F})}(F)$ and which are not external lines of any biped containing F.

By remarking that only one external line of a closed quadruped F can be an external line of a proper biped containing F, it is easy to check that Lemmas III.1, III.2, and III.3 of [1] remain true. Thus the final result of this classification is

$$I_G^R=\sum_{\mathscr{F}}\ \sum_{\sigma\in\mathfrak{S}_G^{\mathscr{F}}} I_{G\sigma}^{\mathscr{F}}\,, \tag{3.8}$$

where the first sum runs over all closed divergent forests of G, $\mathfrak{S}_G^{\mathscr{F}}$ is the set of permutations σ such that $\mathscr{F}$ is a skeleton forest for σ [i.e. $\mathscr{S}(\mathscr{F})=\mathscr{F}$], and

$$I_{G\sigma}^{\mathscr{F}}=\int_{h_\sigma}\prod_{i=1}^{\ell} d\alpha_i e^{-A}\prod_{F\in\mathscr{F}}(-\mathscr{T}_F)\prod_{H\in\mathscr{H}(\mathscr{F})}(1-\mathscr{T}_H)[e^A\mathscr{R}'(U^{-2}e^{-V/U})]\,, \tag{3.9}$$

$\mathscr{H}(\mathscr{F})$ being defined in [1].

In order to decouple the ultra-violet from the infra-red problems, we split again the sectors h_σ into subsectors $h_{\sigma j}$ $(0\leqq j\leqq\ell)$,

$$h_{\sigma j}=\{\alpha|0\leqq\alpha_{\sigma(1)}\leqq\ \dots\ \leqq\alpha_{\sigma(j)}\leqq 1\leqq\alpha_{\sigma(j+1)}\leqq\ \dots\ \leqq\alpha_{\sigma(\ell)}\}\,. \tag{3.10}$$

In each subsector, the infra-red problems do not appear for the variables $\alpha_{\sigma(i)}$, $i\leqq j$, which are not integrated up to $+\infty$. Conversely, the ultra-violet divergences corresponding to the $\alpha_{\sigma(i)}$, $i\geqq j+1$, are absent and the related $1-\mathscr{T}_H$ subtractions do not need to be performed. $\mathscr{F}$ being a skeleton forest, we define

$$\mathscr{H}_j=\{H|H\in\mathscr{H}(\mathscr{F});\ z(H,\mathscr{F})\leqq j\}\,, \tag{3.11}$$

where $z(H,\mathscr{F})$ is the smallest rank i such that $H\cap G_i^\sigma\cup A_{\mathscr{F}}(H)=H$. Then we write

$$\prod_{H\in\mathscr{H}(\mathscr{F})}(1-\mathscr{T}_H)=\prod_{H\in\mathscr{H}_j}(1-\mathscr{T}_H)\sum_{\mathscr{H}'\subseteq\mathscr{H}(\mathscr{F})-\mathscr{H}_j}\ \prod_{H'\in\mathscr{H}'}(-\mathscr{T}_{H'})\,. \tag{3.12}$$

Similarly, we do not perform the $1-\mathcal{T}'_B$ subtractions for the bipeds which are outside $\mathcal{F}\cup\mathcal{H}(\mathcal{F})$. Let $\mathcal{B}$ be the forest of all proper bipeds of G, and

$$\mathcal{B}_2=\mathcal{B}\cap(\mathcal{F}\cup\mathcal{H}(\mathcal{F}))\,;\qquad \mathcal{B}_1=\mathcal{B}-\mathcal{B}_2\,. \tag{3.13}$$

Given two arbitrary subforests

$$\mathcal{H}'\subseteqq\mathcal{H}(\mathcal{F})-\mathcal{H}_j\,;\qquad \mathcal{C}\subseteqq\mathcal{B}_1\,, \tag{3.14}$$

we put

$$\mathcal{F}\cup\mathcal{H}'=\mathcal{F}'\,;\qquad \mathcal{F}\cup\mathcal{H}'\cup\mathcal{C}=\mathcal{G}\,, \tag{3.15}$$

and we find

$$I^{\mathcal{F}}_{G\sigma}=\sum_{j=0}^{\ell}\;\sum_{\mathcal{H}'\subseteqq\mathcal{H}(\mathcal{F})-\mathcal{H}_j}\;\sum_{\mathcal{C}\subseteqq\mathcal{B}_1} I^{\mathcal{G}}_{G\sigma j}\,, \tag{3.16}$$

$$\begin{aligned} I^{\mathcal{G}}_{G\sigma j}=&\int_{h_{\sigma j}}\prod_{i=1}^{\ell} d\alpha_i e^{-A}\prod_{F\in\mathcal{F}'}(-\mathcal{T}_F)\prod_{F\in\mathcal{H}_j}(1-\mathcal{T}_F)e^{A}\\ &\cdot\prod_{F\in\mathcal{C}}(-\mathcal{T}'_F)\prod_{F\in\mathcal{B}_2}(1-\mathcal{T}'_F)U^{-2}e^{-V/U}\,. \end{aligned} \tag{3.17}$$

We will show that each integral (3.17) is absolutely convergent, and can be explicitly estimated.

In order to apply repeatedly the $\mathcal{T}_F$ or $1-\mathcal{T}_F$ operations, let us introduce some definitions. $\mathcal{I}$ and $\mathcal{J}$ being two disjoint subforests of $\mathcal{G}\cup\mathcal{H}_j$, we put

$$A_{\mathcal{I}\bar{\mathcal{J}}}=\mu^2\sum_{i\in E_{\mathcal{I}\bar{\mathcal{J}}}}\alpha_i\,, \tag{3.18}$$

where $E_{\mathcal{I}\bar{\mathcal{J}}}$ is the set of lines which belong to all graphs of $\mathcal{J}$, and to no graph of $\mathcal{I}$.

$$U_{\mathcal{I}\bar{\mathcal{J}}}=\sum_{S\in S_{\mathcal{I}\bar{\mathcal{J}}}}\prod_{i\notin S}\alpha_i\,, \tag{3.19}$$

where $S_{\mathcal{I}\bar{\mathcal{J}}}$ is the set of spanning trees S of G whose restriction to F is a spanning tree of F for any $F\in\mathcal{I}$, and is not a spanning tree of F for any $F\in\mathcal{J}$,

$$V_{\mathcal{I}\bar{\mathcal{J}}}=\sum_{T\in T_{\mathcal{I}\bar{\mathcal{J}}}} s_T\prod_{i\notin T}\alpha_i\,, \tag{3.20}$$

where $T_{\mathcal{I}\bar{\mathcal{J}}}$ is the set of two-trees T of G whose restriction to F is a spanning tree of F for any $F\in\mathcal{I}$, and is not a spanning tree of F for any $F\in\mathcal{J}$. Any two-tree T separates the external lines of G into two non-empty sets, one of which is T_1, and the corresponding cut-invariant is $s_T=\left(\sum_{a\in T_1} p\!\!\!/_a\right)^2$.

Finally, if $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ are three disjoint subforests of $\mathcal{G}\cup\mathcal{H}_j$, with $\mathcal{K}\subseteqq\mathcal{H}_j$, and $\mathcal{L}$ is a subforest of $\mathcal{B}_2$, we put

$$A^{\mathcal{K}}_{\mathcal{I}\bar{\mathcal{J}}}=\mu^2\sum_{\mathcal{K}'\subseteqq\mathcal{K}}\left(\prod_{F\in\mathcal{K}'}\xi_F\right)A_{\mathcal{I}\cup(\mathcal{K}-\mathcal{K}'),\,\overline{\mathcal{J}\cup\mathcal{K}'}}\,, \tag{3.21}$$

$$U^{\mathcal{K}\mathcal{L}}_{\mathcal{I}\bar{\mathcal{J}}}=\sum_{\substack{\mathcal{K}'\subseteqq\mathcal{K}\\ \mathcal{L}'\subseteqq\mathcal{L}}}\left(\prod_{F\in\mathcal{K}'}\xi_F\right)\left(\prod_{F\in\mathcal{L}'}\psi_F\right)U_{\mathcal{I}\cup(\mathcal{K}-\mathcal{K}')\cup(\mathcal{L}-\mathcal{L}'),\,\overline{\mathcal{J}\cup\mathcal{K}'\cup\mathcal{L}'}}\,, \tag{3.22}$$

$$V^{\mathcal{K}\mathcal{L}}_{\mathcal{I}\bar{\mathcal{J}}}=\sum_{\substack{\mathcal{K}'\subseteqq\mathcal{K}\\ \mathcal{L}'\subseteqq\mathcal{L}}}\left(\prod_{F\in\mathcal{K}'}\xi_F\right)\left(\prod_{F\in\mathcal{L}'}\psi_F\right)V_{\mathcal{I}\cup(\mathcal{K}-\mathcal{K}')\cup(\mathcal{L}-\mathcal{L}'),\,\overline{\mathcal{J}\cup\mathcal{K}'\cup\mathcal{L}'}}\,. \tag{3.23}$$

With these notations we find

Lemma III.1.

a) $$\prod_{F\in\mathscr{F}'}(-\mathscr{T}_F)\prod_{F\in\mathscr{H}_j}(1-\mathscr{T}_F)e^A\prod_{F\in\mathscr{C}}(-\mathscr{T}_F)\prod_{F\in\mathscr{B}_2}(1-\mathscr{T}_F)U^{-2}e^{-V/U}=\sum_\delta \mathscr{Z}_\delta\,. \quad (3.24)$$

b) $$\mathscr{Z}_\delta=\varepsilon\int_0^1\Big(\prod_{F\in\mathscr{K}}d\xi_F\Big)\Big(\prod_{F\in\mathscr{L}}d\psi_F\Big)\Big(\prod_{m=1}^{k+v+a}X_m\Big)$$
$$\cdot(U_{\mathscr{G}}^{\mathscr{K}\mathscr{L}})^{-k-v-2}\exp\{A_{\mathscr{F}'}^{\mathscr{K}}-V_{\mathscr{G}}^{\mathscr{K}\mathscr{L}}/U_{\mathscr{G}}^{\mathscr{K}\mathscr{L}}\}\,, \quad (3.25)$$

where ε, $\mathscr{K}$, $\mathscr{L}$, k, v, a *depend on the index* δ.

c) $$\varepsilon=\pm 1\,, \quad (3.26)$$
$$\mathscr{K}\subseteqq\mathscr{H}_j\,;\qquad \mathscr{L}\subseteqq\mathscr{B}_2\,, \quad (3.27)$$
$$k+v\leqq 2(|\mathscr{B}_2|+|\mathscr{H}_j|)\,. \quad (3.28)$$

d) $$X_m=U_{\mathscr{I}_m\mathscr{J}_m}^{\mathscr{K}_m\mathscr{L}_m}\quad \textit{for}\quad m=1,\ldots,k\,, \quad (3.29)$$
$$X_m=V_{\mathscr{I}_m\mathscr{J}_m}^{\mathscr{K}_m\mathscr{L}_m}\quad \textit{for}\quad m=k+1,\ldots,k+v\,, \quad (3.30)$$
$$X_m=A_{\mathscr{I}_m\mathscr{J}_m}^{\mathscr{K}_m}\quad \textit{for}\quad m=k+v+1,\ldots,k+v+a\,, \quad (3.31)$$
$$a(\delta)\leqq|\mathscr{H}_j|\,;\qquad v(\delta)+a(\delta)\leqq|\mathscr{B}_2|+|\mathscr{H}_j| \quad (3.32)$$

and the indices δ *satisfying* $v(\delta)=v$, $a(\delta)=a$, *run over a set of at most* $8^{|\mathscr{B}_2|+|\mathscr{H}_j|}(|\mathscr{B}_2|+|\mathscr{H}_j|-v-a)!$ *elements.*

e) $\mathscr{Q}$ *being the set of all proper closed quadrupeds of* G, *we have*

$$F\in\mathscr{Q}\cap\mathscr{G}=\mathscr{Q}\cap\mathscr{F}'\ \Rightarrow\ F\in\mathscr{I}_m\quad\forall m\,,\qquad 1\leqq m\leqq k+v+a\,, \quad (3.33)$$

$$F\in\mathscr{Q}\cap\mathscr{H}_j\ \Rightarrow\ F\in\mathscr{J}_m\quad \textit{for one and only one value } m_1(F) \textit{ of } m,$$
$$\textit{with}\quad 1\leqq m_1\leqq k+v+a\,, \quad (3.34)$$

$$F\in\mathscr{C}\ \Rightarrow\ F\in\mathscr{I}_m\quad\forall m\,,\qquad 1\leqq m\leqq k+v\,, \quad (3.35)$$

$$F\in\mathscr{B}\cap\mathscr{H}_j\ \Rightarrow\ F\in\mathscr{J}_m\quad \textit{for two and only two values } m_1(F),\ m_2(F) \textit{ of } m,$$
$$\textit{with}\quad 1\leqq m_1\leqq k+v+a\,,\quad 1\leqq m_2\leqq k+v\,, \quad (3.36)$$

$$F\in\mathscr{B}\cap\mathscr{F}'\ \Rightarrow\ F\in\mathscr{J}_m\quad \textit{for one and only one value } m_2(F) \textit{ of } m,$$
$$\textit{with}\quad 1\leqq m_2\leqq k+v\quad \textit{and}\quad F\in\mathscr{I}_m$$
$$\textit{for all the other values of } m,\quad 1\leqq m\leqq k+v+a\,, \quad (3.37)$$

$$F\in\mathscr{B}_2-(\mathscr{F}'\cup\mathscr{H}_j)=\mathscr{B}\cap[\mathscr{H}(\mathscr{F})-(\mathscr{H}'\cup\mathscr{H}_j)]\ \Rightarrow\ F\in\mathscr{J}_m$$
$$\textit{for one and only one value } m_2(F) \textit{ of } m,$$
$$\textit{with}\quad 1\leqq m_2\leqq k+v\,. \quad (3.38)$$

f)
$$\forall m, \quad 1 \leqq m \leqq k+v+a \begin{cases} \mathscr{K}_m \subseteqq \mathscr{K}, \\ \mathscr{L}_m \subseteqq \mathscr{L}, \end{cases} \tag{3.39}$$

$$\forall m, \quad 1 \leqq m \leqq k+v \begin{cases} \mathscr{G} \subseteqq \mathscr{I}_m \cup \mathscr{J}_m, \\ \mathscr{G} \cup \mathscr{K} \subseteqq \mathscr{I}_m \cup \mathscr{J}_m \cup \mathscr{K}_m, \\ \mathscr{G} \cup \mathscr{L} \subseteqq \mathscr{I}_m \cup \mathscr{J}_m \cup \mathscr{L}_m, \end{cases} \tag{3.40}$$

$$\forall m, \quad k+v+1 \leqq m \leqq k+v+a \begin{cases} \mathscr{J}_m \neq \emptyset, \\ \mathscr{J}_m \subseteqq \mathscr{H}_j. \end{cases} \tag{3.41}$$

g) *$\forall m$, $1 \leqq m \leqq k+v$, if $F \in \mathscr{J}_m$ and $F' \in \mathscr{A}_{\mathscr{G} \cup \mathscr{H}_j}(F)$, then $F' \in \mathscr{I}_m \cup \mathscr{J}_m$, except perhaps if $F \in \mathscr{B} \cap \mathscr{H}_j$ and $F' \in (\mathscr{B} \cap \mathscr{G}) \cup (\mathscr{Q} \cap \mathscr{H}_j)$; in the last case, m_1 and m_2 being the values of m defined in (3.38), $F \in \mathscr{J}_{m_1} \Rightarrow F' \in \mathscr{I}_{m_2} \cup \mathscr{J}_{m_2}$ and $F \in \mathscr{J}_{m_2} \Rightarrow F' \in \mathscr{I}_{m_1} \cup \mathscr{J}_{m_1}$. $\forall m$, $k+v+1 \leqq m \leqq k+v+a$, if $F \in \mathscr{J}_m$ and $F' \in \mathscr{A}_{\mathscr{H}_j}(F)$, then $F' \in \mathscr{I}_m \cup \mathscr{J}_m$.*

Proof. As for the proof of Lemma III.4 in [1], the proof is a simple matter of computation. We add the subgraphs of $\mathscr{G} \cup \mathscr{H}_j \cup \mathscr{B}_2$ one by one, starting from its maximal elements. Assuming formula (3.25) at a given step, we add a new subgraph F and perform the corresponding Taylor operation. Formulae (2.7), ..., (2.11), which give the first step, can be generalized to the following steps in a tedious but trivial way, and the various assertions of Lemma III.1 are recovered by simple inspection.

Let us now perform the usual change of variables, in each sector σ, defined by

$$\alpha_{i'} = \prod_{\substack{i \\ i' \in G_i^\sigma}} \beta_i, \tag{3.42}$$

where G_i^σ is the subgraph $\{\sigma(1), \sigma(2), \ldots, \sigma(i)\}$, made by taking i lines in the order of the sector. From (3.17) and Lemma III.1, we find

$$I_{G\sigma j}^{\mathscr{G}} = \int_0^1 \ldots \int_0^1 \left(\prod_{i=1}^{\ell-1} d\beta_i \beta_i^{i-1} \right) \sum_\delta \varepsilon \int_0^1 \left(\prod_{F \in \mathscr{K}} d\xi_F \right) \left(\prod_{F \in \mathscr{L}} d\psi_F \right) Y_\delta, \tag{3.43}$$

where

$$Y_\delta = (U_{\mathscr{G}}^{\mathscr{K}\mathscr{L}})^{-2} \left(\prod_{m=1}^{k} \frac{U_{\mathscr{I}_m \mathscr{J}_m}^{\mathscr{K}_m \mathscr{L}_m}}{U_{\mathscr{G}}^{\mathscr{K}\mathscr{L}}} \right) \left(\prod_{m=k+1}^{k+v} \frac{V_{\mathscr{I}_m \mathscr{J}_m}^{\mathscr{K}_m \mathscr{L}_m}}{U_{\mathscr{G}}^{\mathscr{K}\mathscr{L}}} \right)$$
$$\cdot \left(\prod_{m=k+v+1}^{k+v+a} A_{\mathscr{I}_m \mathscr{J}_m}^{\mathscr{K}_m} \right) \int_{b_1}^{b_2} d\beta_\ell \beta_\ell^{\omega_G + v + a - 1} \exp\{-\beta_\ell W\}, \tag{3.44}$$

$$\omega_G = \frac{N(G)}{2} - 2, \tag{3.45}$$

$$W = A - A_{\mathscr{F}'}^{\mathscr{K}} + \frac{V_{\mathscr{G}}^{\mathscr{K}\mathscr{L}}}{U_{\mathscr{G}}^{\mathscr{K}\mathscr{L}}}, \tag{3.46}$$

and it has to be understood that in $U_{\mathscr{G}}^{\mathscr{K}\mathscr{L}}$, $U_{\mathscr{I}_m \mathscr{J}_m}^{\mathscr{K}_m \mathscr{L}_m}$, $V_{\mathscr{I}_m \mathscr{J}_m}^{\mathscr{K}_m \mathscr{L}_m}$, $A_{\mathscr{I}_m \mathscr{J}_m}^{\mathscr{K}_m}$ and W, the β_ℓ variable is taken equal to 1, since the explicit β_ℓ dependence has been factorized by using the homogeneity properties of the integrand.

The bounds of the β_ℓ integration are given by:

$$\text{if} \quad 1 \leqq j \leqq \ell - 1, \quad b_1 = (\beta_{j+1} \dots \beta_{\ell-1})^{-1}; \quad b_2 = (\beta_j \dots \beta_{\ell-1})^{-1}, \tag{3.47}$$

$$\text{if} \quad j = 0, \quad b_1 = (\beta_1 \dots \beta_{\ell-1})^{-1}; \quad b_2 = +\infty, \tag{3.48}$$

$$\text{if} \quad j = \ell, \quad b_1 = 0; \quad b_2 = 1. \tag{3.49}$$

IV. Estimates

IV.1. Bounds on W. First it is evident that

$$A - A^{\mathscr{K}}_{\mathscr{F}'} \geqq A - A_{\mathscr{F}'} \geqq A - A_{\mathscr{F}} \geqq \mu^2 \prod_{i=i_1}^{\ell-1} \beta_i, \tag{4.1}$$

where i_1 is the highest rank of all the lines belonging to at least one $F \in \mathscr{F}$:

$$i_1 = \operatorname*{Sup}_{F \in \mathscr{F}} \operatorname*{Sup}_{\sigma(i) \in F} i. \tag{4.2}$$

Let us now find a lower bound on $V^{\mathscr{K}\mathscr{L}}_{\mathscr{G}} / U^{\mathscr{K}\mathscr{L}}_{\mathscr{G}}$. To each spanning tree S of $S_{\mathscr{G}}$ (i.e. such that $S \cap F$ is a spanning tree of F for every $F \in \mathscr{G}$) we associate a two-tree

$$T(S) = S - \{\lambda(S)\}, \tag{4.3}$$

where

i) $\lambda(S) \in S / \mathscr{F} \cup \mathscr{B}_1$.

ii) $S - \{\lambda(S)\}$ has two connected components, each of which contains a non-empty subset of the external vertices of G.

iii) $\forall F \in \mathscr{H}(\mathscr{F})$, $\lambda(S) \notin F - X(F, \mathscr{F})$.

iv) $\lambda(S)$ is the line with the above properties which has the maximal rank.

We define

$$i_2 = \operatorname*{Inf}_{\substack{S \in S_{\mathscr{G}} \\ \lambda(S) = \sigma(i)}} i. \tag{4.4}$$

Lemma IV.1. *If* $\omega_G \geqq 0$, *then*

$$V^{\mathscr{K}\mathscr{L}}_{\mathscr{G}} / U^{\mathscr{K}\mathscr{L}}_{\mathscr{G}} \geqq 2^{-\ell} s_{\inf} \xi_G^{\chi} \prod_{i=i_2}^{\ell-1} \beta_i, \tag{4.5}$$

where $s_{\inf}$ *is the minimal value of the cut-invariants of* G, *and* $\chi = 1$ *if* $G \in \mathscr{H}_j$, $\chi = 0$ *if* $G \notin \mathscr{H}_j$.

Proof. $\forall S$, $\lambda(S)$ is outside any $F \in \mathscr{H}(\mathscr{F})$, $F \neq G$. By writing (3.22) and (3.23) as

$$U^{\mathscr{K}\mathscr{L}}_{\mathscr{G}} = \sum_{S \in S_{\mathscr{G}}} u(S); \quad u(S) = \left(\prod_{i \notin S} \alpha_i\right) \left(\prod_{\substack{F \in \mathscr{K} \\ S \notin S_F}} \xi_F\right) \left(\prod_{\substack{F \in \mathscr{L} \\ S \notin S_F}} \psi_F\right), \tag{4.6}$$

$$V^{\mathscr{K}\mathscr{L}}_{\mathscr{G}} = \sum_{T \in T_{\mathscr{G}}} v(T); \quad v(T) = s_T \left(\prod_{i \notin T} \alpha_i\right) \left(\prod_{\substack{F \in \mathscr{K} \\ T \notin T_F}} \xi_F\right) \left(\prod_{\substack{F \in \mathscr{L} \\ T \notin T_F}} \psi_F\right), \tag{4.7}$$

and by using $\mathscr{K} \subseteqq \mathscr{H}_j \subseteqq \mathscr{H}(\mathscr{F})$, $\mathscr{L} \subseteqq \mathscr{B}_2$, we see that the product of ξ and ψ variables must be the same in $u(S)$ and $v(T(S))$, except eventually for ξ_G. The factor $2^{-\ell}$ takes into account the fact that many spanning trees (at most 2^ℓ) can be associated to the same two-trees. This achieves the proof of Lemma IV.1.

Let us put

$$m^2 = \mathrm{Inf}(s_{\mathrm{inf}}, \mu^2), \tag{4.8}$$

$$i_{01} = \mathrm{Sup}(i_1, i_2). \tag{4.9}$$

We note that $m^2 \neq 0$ at non-exceptional momenta, μ^2 being different from zero by definition. And we find finally

$$W \geqq W_{01} = 2^{-\ell} m^2 \xi_G^{\chi} \prod_{i=i_{01}}^{\ell-1} \beta_i. \tag{4.10}$$

Lemma IV.2.

$$W \leqq W_{02} = 2^\ell M^2, \tag{4.11}$$

where $M^2 = s_{\mathrm{sup}} + \mu^2$ *and* s_{sup} *is the maximal value of the cut-invariants of* G.

Proof. Trivially $A - A^{\mathscr{K}}_{\mathscr{F}'} \leqq A \leqq \ell\mu^2$. On the other hand, we may associate a spanning tree $S(T)$ to each two-tree T of G by adding an arbitrary line $\lambda'(T)$ such that $T \cup \{\lambda'(T)\}$ is a spanning tree. In each monomial $v(T)$ of $V^{\mathscr{K}\mathscr{L}}_{\mathscr{G}}$ the product of ξ and ψ variables contains at least as many variables as the corresponding monomial $u(S(T))$ of $U^{\mathscr{K}\mathscr{L}}_{\mathscr{G}}$, which achieves the proof of Lemma IV.2.

IV.2. Bound on the β_ℓ *integration.* i) For $1 \leqq j \leqq \ell - {}^1$ we bound the β_ℓ integration by

$$J = \int_{b_1}^{b_2} d\beta_\ell \beta_\ell^{\omega_G + v + a - 1} e^{-\beta_\ell W} \leqq b_2^{a - \frac{1}{2}} \int_{b_1}^{b_2} d\beta_\ell \beta_\ell^{\omega_G + v - \frac{1}{2}} e^{-\beta_\ell W}. \tag{4.12}$$

Therefore,

$$J \leqq b_2^{a - \frac{1}{2}} b_1^{\omega_G + v + \frac{1}{2}} \int_1^\infty dt t^{\omega_G + v - \frac{1}{2}} e^{-t b_1 W}. \tag{4.13}$$

We note that $v \geqq 1$ if G is a biped ($\omega_G = -1$), which allows us to replace the lower bound of the t integration by 0. Using the trivial bound $W \geqq V^{\mathscr{K}\mathscr{L}}_{\mathscr{G}} / U^{\mathscr{K}\mathscr{L}}_{\mathscr{G}}$, we find

$$J \leqq \left(\frac{U^{\mathscr{K}\mathscr{L}}_{\mathscr{G}}}{V^{\mathscr{K}\mathscr{L}}_{\mathscr{G}}} \right)^v b_2^a W^{-\omega_G} (b_2 W)^{-1/2} \Gamma(\omega_G + v + \tfrac{1}{2}). \tag{4.14}$$

By using (4.10), or (4.11) if $\omega_G = -1$, we get

$$J \leqq \left(\frac{U^{\mathscr{K}\mathscr{L}}_{\mathscr{G}}}{V^{\mathscr{K}\mathscr{L}}_{\mathscr{G}}} \right)^v b_2^a W_0^{-\omega_G} (b_2 W_0)^{-1/2} \Gamma(\omega_G + v + \tfrac{1}{2}), \tag{4.15}$$

where $W_0 = W_{01}$ if $\omega_G \geqq 0$, $W_0 = W_{02}$ if $\omega_G = -1$.

ii) In the particular subsector $j = 0$ we get no ultraviolet problem, $\mathscr{H}_j$ is empty and therefore $a = 0$. The β_ℓ integration is directly bounded by

$$J = \int_{b_1}^\infty d\beta_\ell \beta_\ell^{\omega_G + v - 1} e^{-\beta_\ell W} = b_1^{\omega_G + v} \int_1^\infty dt t^{\omega_G + v - 1} e^{-t b_1 W}. \tag{4.16}$$

Therefore,

$$J \leqq b_1^{\omega_G+v} \int_1^\infty dt\, t^{\omega_G+v-\frac{1}{2}} e^{-tb_1W} \leqq b_1^{\omega_G+v}(b_1W)^{-\omega_G-v-\frac{1}{2}}\Gamma(\omega_G+v+\tfrac{1}{2}), \quad (4.17)$$

and finally

$$J \leqq \left(\frac{U_{\mathscr{G}}^{\mathscr{K}\mathscr{L}}}{V_{\mathscr{G}}^{\mathscr{K}\mathscr{L}}}\right)^v W_0^{-\omega_G}(b_1W_0)^{-1/2}\Gamma(\omega_G+v+\tfrac{1}{2}), \quad (4.18)$$

which is identical to (4.15) for $a=j=0$ (with the convention $\beta_0=1$).

iii) In the particular subsector $j=\ell$, we get no infra-red problem since β_ℓ is bounded by 1. The β_ℓ integration is directly bounded by

$$J = \int_0^1 d\beta_\ell \beta_\ell^{\omega_G+v+a-1} e^{-\beta_\ell W} \leqq 1, \quad (4.19)$$

since $G \in \mathscr{H}_j$ if G is divergent, which implies $\omega_G+v+a \geqq 1$. This case is similar to the massive theory treated in [1]. Anyway it corresponds to a bound which is simpler than in other subsectors, and it may be included in the general case. From (3.44), (4.10), (4.11), (4.15), (4.18), and (4.19), we obtain

$$Y_\delta \leqq K_1^n (U_{\mathscr{G}}^{\mathscr{K}\mathscr{L}})^{-2} \left(\prod_{m=1}^{k} \frac{U_{\mathscr{I}_m\mathscr{J}_m}^{\mathscr{K}_m\mathscr{L}_m}}{U_{\mathscr{G}}^{\mathscr{K}\mathscr{L}}}\right) \left(\prod_{m=k+1}^{k+v} \frac{V_{\mathscr{I}_m\mathscr{J}_m}^{\mathscr{K}_m\mathscr{L}_m}}{V_{\mathscr{G}}^{\mathscr{K}\mathscr{L}}}\right) \left(\prod_{m=k+v+1}^{k+v+a} \frac{A_{\mathscr{I}_m\mathscr{J}_m}^{\mathscr{K}_m}}{\prod_{i=j}^{\ell-1}\beta_i}\right)$$

$$\cdot \left(\frac{\prod_{i=j}^{\ell-1}\beta_i}{\prod_{i=i_0}^{\ell-1}\beta_i}\right)^{1/2} \left(\prod_{i=i_0}^{\ell-1}\beta_i\right)^{-\omega_G} \Gamma(\omega_G+v+\tfrac{1}{2}), \quad (4.20)$$

where $i_0=i_{01}$ if $\omega_G \geqq 0$, $i_0=\ell$ if $\omega_G=-1$.

Here K_1, like $K_2, \ldots, K_6, K, K'$ in the following, is a number which may depend on m^2, M^2, and $N(G)$, but not on the order $n(G)$.

IV.3. Bound on $U_{\mathscr{G}}^{\mathscr{K}\mathscr{L}}$. We write $U_{\mathscr{G}}^{\mathscr{K}\mathscr{L}}$ like in (4.6), and we remark that the leading tree S_{lead} $\left(\text{i.e. the spanning tree } S \in S_{\mathscr{G}} \text{ for which } \prod_{i \notin S} \alpha_i \text{ is maximal}\right)$ must be a spanning tree inside each F belonging to $\mathscr{H}(\mathscr{F})$,

$$S_{\text{lead}} \in S_F, \qquad \forall F \in \mathscr{H}(\mathscr{F}). \quad (4.21)$$

This property follows from Sect. III.4 of [1] and may be also recovered from the correspondence between trees used again in the next subsection. Therefore, no ξ variable appears in the leading monomial of $U_{\mathscr{G}}^{\mathscr{K}\mathscr{L}}$. By taking $\psi_F=0\ \forall F$, and by noting that $S_{\mathscr{G}} \supseteqq S_{\mathscr{B}_1}$, since $\mathscr{B}_1 \supseteqq \mathscr{G}$, we see that the leading monomial of $U_{\mathscr{G}}^{\mathscr{K}\mathscr{L}}$ is bounded from below by the leading monomial of $U_{\mathscr{F}\cup\mathscr{B}_1}$. Therefore, we write

$$(U_{\mathscr{G}}^{\mathscr{K}\mathscr{L}})^{-2} \leqq \prod_{i=1}^{\ell-1} \beta_i^{q_i}, \quad (4.22)$$

with

$$q_i = -2 \sum_{F \in \mathscr{F}\cup\mathscr{B}_1\cup\{G\}} L(G_i^\sigma \cap F/\mathscr{F}\cup\mathscr{B}_1). \quad (4.23)$$

IV.4. Bound on $U^{\mathscr{K}_m \mathscr{L}_m}_{\mathscr{I}_m \mathscr{J}_m}/U^{\mathscr{K}\mathscr{L}}_{\mathscr{G}}$ *and* $V^{\mathscr{K}_m \mathscr{L}_m}_{\mathscr{I}_m \mathscr{J}_m}/V^{\mathscr{K}\mathscr{L}}_{\mathscr{G}}$. These ratios are bounded by introducing a correspondence φ_m between the spanning trees of $S_{\mathscr{I}_m \mathscr{J}_m}$ and the spanning trees of $S_{\mathscr{G}}$. If a given spanning tree S is such that $S \cap F/\mathscr{I}_m \cup \mathscr{J}_m$ is not a spanning tree for a given $F \in \mathscr{J}_m$, we add internal lines of $F/\mathscr{I}_m \cup \mathscr{J}_m$ and cut external lines of F until we obtain a spanning tree. This operation is made for each $F \in \mathscr{J}_m$, starting from the smallest graphs. Since φ_m is defined exactly like in Sect. III.4 of [1], we do not repeat here the analysis. Let us simply mention that in the particular case where $F \in \mathscr{H}(\mathscr{F})$, $\varphi_F(S) = S$, the set of graphs $F_1, \ldots, F_r$ which have a common external line λ with F is totally ordered by inclusion

$$F_1 \subset F_2 \subset \ldots \subset F_r \subset F. \tag{4.24}$$

From (3.6) and (3.7) it is true again that $F_i \in \mathscr{J}_m \Rightarrow F_i \in \mathscr{H}(\mathscr{F})$, and it is easy to see from our definitions of x and y that

$$\begin{aligned} &x(F_1, \mathscr{F}) < y(F_1, \mathscr{F}) \leqq x(F_2, \mathscr{F}) < y(F_2, \mathscr{F}) \leqq \ldots \\ &\ldots \leqq x(F_r, \mathscr{F}) < y(F_r, \mathscr{F}) \leqq x(F, \mathscr{F}) < y(F, \mathscr{F}) \leqq i(\lambda). \end{aligned} \tag{4.25}$$

Moreover, the same correspondence φ_m, with exactly the same properties, can be extended to a correspondence between the two-trees of $T_{\mathscr{I}_m \mathscr{J}_m}$ and $T_{\mathscr{G}}$. Thus we find the same bound, up to a factor M^2/μ^2, for the ratios $U^{\mathscr{K}_m \mathscr{L}_m}_{\mathscr{I}_m \mathscr{J}_m}/U^{\mathscr{K}\mathscr{L}}_{\mathscr{G}}$ and $V^{\mathscr{K}_m \mathscr{L}_m}_{\mathscr{I}_m \mathscr{J}_m}/V^{\mathscr{K}\mathscr{L}}_{\mathscr{G}}$.

If R_m denotes any such ratio, we find in the same way as in [1],

$$\frac{m^2}{M^2} R_m \leqq 2^{4|\mathscr{J}_m|} 2^{\sum_{F \in \mathscr{J}_m} \ell(F/\mathscr{I}_m \cup \mathscr{J}_m)} \frac{\prod_{F \in \mathscr{J}_m \cap \mathscr{H}(\mathscr{F})} \prod_{x(F,\mathscr{F}) \leqq i < y(F,\mathscr{F})} \beta_i}{\prod_{F \in \mathscr{J}_m \cap \mathscr{B} \cap \mathscr{F}} \prod_{y(F,\mathscr{F}) \leqq i < x(F,\mathscr{F})} \beta_i}. \tag{4.26}$$

By using Lemma III.1 and formula (III.33) of [1],

$$\begin{aligned} \prod_{m=1}^{k+v} R_m \leqq K_2^n &\left(\prod_{\substack{F \in \mathscr{H}_j \\ m_1(F) \leqq k+v}} \prod_{x(F,\mathscr{F}) \leqq i < y(F,\mathscr{F})} \beta_i \right) \\ &\cdot \left(\prod_{F \in \mathscr{H}(\mathscr{F}) \cap \mathscr{B}} \prod_{x(F,\mathscr{F}) \leqq i < y(F,\mathscr{F})} \beta_i \right) \left(\prod_{F \in \mathscr{F} \cap \mathscr{B}} \prod_{y(F,\mathscr{F}) \leqq i < x(F,\mathscr{F})} \beta_i^{-1} \right). \end{aligned} \tag{4.27}$$

IV.5. Bound on $A^{\mathscr{K}_m}_{\mathscr{I}_m \mathscr{J}_m} \left(\prod_{i=j}^{\ell-1} \beta_i \right)^{-1}$. We have

$$A^{\mathscr{K}_m}_{\mathscr{I}_m \mathscr{J}_m} \leqq A_{\mathscr{I}_m \mathscr{J}_m} = \mu^2 \sum_{i \in E_m} \alpha_i, \tag{4.28}$$

where E_m is the set of lines which belong to all $F \in \mathscr{J}_m$, and to no $F \in \mathscr{I}_m$. Therefore, $A^{\mathscr{K}_m}_{\mathscr{I}_m \mathscr{J}_m}$ is not vanishing only if there is no $F \in \mathscr{I}_m$ containing a graph $F \in \mathscr{J}_m$, and if the whole set of $F \in \mathscr{J}_m$ is totally ordered by inclusion. In this case, let us call F_1 the smallest graph of the nest $\mathscr{J}_m$. We find

$$A^{\mathscr{K}_m}_{\mathscr{I}_m \mathscr{J}_m} \leqq \mu^2 \ell(F_1/\mathscr{I}_m \cup \mathscr{J}_m) \alpha_{\sigma(z(F_1, \mathscr{I}_m \cup \mathscr{J}_m))}, \tag{4.29}$$

where $z(F_1, \mathscr{I}_m \cup \mathscr{J}_m)$ is defined in (3.11) as the maximal rank of the lines of $F_1/\mathscr{I}_m \cup \mathscr{J}_m$. From (3.33) and (3.37), $\mathscr{I}_m \cup \mathscr{J}_m \supseteq \mathscr{F}$, and we get

$$A^{\mathscr{K}_m}_{\mathscr{I}_m \mathscr{J}_m} \left(\prod_{i=j}^{\ell-1} \beta_i \right)^{-1} \leqq \mu^2 2^{\ell(F_1/\mathscr{I}_m \cup \mathscr{J}_m)} \prod_{i=z(F_1,\mathscr{F})}^{j-1} \beta_i, \tag{4.30}$$

Moreover, from (3.41), if $F\in\mathscr{J}_m$, then $F\in\mathscr{H}_j$, and thus $F\notin\mathscr{F}$. Let $\mathscr{J}_m$ be the nest $\{F_1,F_2,\dots,F_r\}$, with $F_1\subset F_2\subset\dots\subset F_r$. We may conclude

$$z(F_1,\mathscr{F})\leqq z(F_2,\mathscr{F})\leqq \dots \leqq z(F_r,\mathscr{F})\leqq j\,. \tag{4.31}$$

Using (3.34), (3.36) and formula (III.33) of [1], we find finally

$$\prod_{m=k+v+1}^{k+v+a}\frac{A^{\mathscr{K}_m}_{\mathscr{J}_m\bar{\mathscr{J}}_m}}{\prod\limits_{i=j}^{\ell-1}\beta_i}\leqq \mu^2 8^n \prod_{\substack{F\in\mathscr{H}_j\\ m_1(F)\geqq k+v+1}}\ \prod_{i\in Z_F}\beta_i\,, \tag{4.32}$$

where

$$Z_F=\{i|z(F,\mathscr{F})\leqq i\leqq \mathrm{Inf}(z(B_{\mathscr{H}_j}(F),\mathscr{F}),j)\}\,. \tag{4.33}$$

IV.6. Bound on the β_i integrations. From the preceding bounds we obtain an estimate of (3.43) by

$$I^{\mathscr{G}}_{G\sigma j}=\sum_\delta I_\delta\,, \tag{4.34}$$

$$|I_\delta|\leqq K_3^n\int_0^1\dots\int_0^1\left(\prod_{i=1}^{\ell-1}\beta_i^{i+\eta_i-1}d\beta_i\right)\Gamma(\omega_G+v+\tfrac{1}{2})\,, \tag{4.35}$$

where

$$\begin{aligned} i+\eta_i=&\sum_{F\in\mathscr{F}\cup\mathscr{B}_1\cup\{G\}}\omega(G_i^\sigma\cap F/\mathscr{F}\cup\mathscr{B}_1)+\sum_{\substack{F\in\mathscr{B}\cap\mathscr{H}(\mathscr{F})\\ x(F,\mathscr{F})\leqq i<y(F,\mathscr{F})}}1\\ &+\sum_{\substack{F\in\mathscr{H}_j\\ m_1(F)\leqq k+v\\ x(F,\mathscr{F})\leqq i<y(F,\mathscr{F})}}1+\sum_{\substack{F\in\mathscr{H}_j\\ m_1(F)\geqq k+v+1\\ i\in Z_F}}1-\sum_{\substack{F\in\mathscr{B}\cap\mathscr{F}\\ y(F,\mathscr{F})\leqq i<x(F,\mathscr{F})}}1\\ &+\tfrac{1}{2}\chi_{i_0j}-\omega_G\chi_0\,, \end{aligned} \tag{4.36}$$

and

$$\chi_{i_0j}=\begin{cases}1 & \text{if}\ \ j\leqq i<i_0\,,\\ -1 & \text{if}\ \ i_0\leqq i<j\,,\\ 0 & \text{otherwise}\,,\end{cases} \tag{4.37}$$

$$\chi_0=\begin{cases}1 & \text{if}\ \ i_0\leqq i\leqq \ell-1\,,\\ 0 & \text{otherwise}\,,\end{cases} \tag{4.38}$$

$$\omega(G_i^\sigma\cap F/\mathscr{F}\cup\mathscr{B}_1)=\sum_{\substack{C\ \text{connected component}\\ \text{of}\ G_i^\sigma\cap F/\mathscr{F}\cup\mathscr{B}_1}}\left(\frac{N(C)}{2}-2+b(C)\right), \tag{4.39}$$

$b(C)$ being the number of two-lines reduction vertices in C.

Lemma IV.3. $\forall F\in\mathscr{H}_j$, $i\in Z_F$, *if the following conditions are satisfied:*

$$i<j\,;\qquad z(F,\mathscr{F})\leqq i<y(F,\mathscr{F})\,;\qquad m_1(F)\geqq k+v+1\,.$$

Proof. i) Either $F'=B_{\mathscr{F}\cup\mathscr{H}_j}(F)\in\mathscr{H}_j$. Therefore, $F'\subset B_{\mathscr{F}}(F)$, and from the definition of $y(F,\mathscr{F})$, we must have $z(F',\mathscr{F})>i$, hence $i\in Z_F$.

ii) Or $F'=B_{\mathscr{F}\cup\mathscr{H}_j}(F)\in\mathscr{F}$. Then from (3.33) and (3.37), $F'\in\mathscr{J}_m$, $\forall m\geqq k+v+1$. From Subsect. IV.5, $A^{\mathscr{K}_m}_{\mathscr{J}_m\bar{\mathscr{J}}_m}$ would vanish for $m=m_1(F)$, which contradicts the hypothesis.

Lemma IV.4. *Defining* $y'(F,\mathscr{F})=\operatorname*{Inf}_{\sigma(i)\in E(F)} i$ *for* F *biped, we have*

$$\sum_{F\in\mathscr{F}\cup\mathscr{B}_1\cup\{G\}}\ \sum_{\substack{C \text{ connected component}\\ \text{of } G_i^\sigma\cap F/\mathscr{F}\cup\mathscr{B}_1}} b(C)-\sum_{\substack{F\in\mathscr{F}\cap\mathscr{B}\\ y(F,\mathscr{F})\leqq i<x(F,\mathscr{F})}} 1$$
$$\geqq \sum_{\substack{F\in\mathscr{F}\cap\mathscr{B}\\ x(F,\mathscr{F})\leqq i}} 1+\sum_{\substack{F\in\mathscr{B}_1\\ y'(F,\mathscr{F})\leqq i}} 1+\sum_{\substack{F\in\mathscr{F}\cap\mathscr{B}\\ y'(F,\mathscr{F})\leqq i<y(F,\mathscr{F})}} 1. \tag{4.40}$$

Proof. For any biped $F\in\mathscr{F}\cup\mathscr{B}_1$, there exists in $F'=B_{\mathscr{F}\cup\mathscr{B}_1}(F)$ a connected component of $G_i^\sigma\cap F'/\mathscr{F}\cup\mathscr{B}_1$, which contains the reduction vertex of F for $i\geqq y'(F,\mathscr{F})$. This proves Lemma IV.4.

We are now in position to estimate $i+\eta_i$. We do it first for $i<\operatorname{Inf}(j,i_0)$, by distinguishing the possible cases for the connected components C of $G_i^\sigma\cap F/\mathscr{F}\cup\mathscr{B}_1$. Defining $P(C)=\operatorname{Sup}\left(\frac{N(C)}{6},1\right)$ we find,

a) $N(C)\geqq 6$, then $\frac{N(C)}{2}-2\geqq P(C)$.

b) $\frac{N(C)}{2}-2\leqq 0$ and C is not proper. Then

i) Either C contains at least one (if C is a quadruped) or two (if C is a biped) two-lines reduction vertices coming from bipeds F' with $y'(F',\mathscr{F})\leqq i<y(F',\mathscr{F})$, and we find from Lemma IV.4,

$$\frac{N(C)}{2}-2+1\geqq P(C)\quad \text{if } C \text{ is a quadruped},$$

$$\frac{N(C)}{2}-2+2\geqq P(C)\quad \text{if } C \text{ is a biped}.$$

ii) Or C contains at least one proper biped B', with $B'=B/\mathscr{F}\cup\mathscr{B}_1$, and $i<y(B,\mathscr{F})$. In that case

– either $z(B,\mathscr{F})\leqq i$, $B\in\mathscr{H}_j$ and, if $m_1(B)\geqq k+v+1$, $i\in Z_B$ from Lemma IV.3;

– or $x(B,\mathscr{F})\leqq i<z(B,\mathscr{F})$, $B\in\mathscr{H}(\mathscr{F})$ and there must exist at least one reduction vertex inside $B/\mathscr{F}\cup\mathscr{B}_1$, coming from a biped $F'\in\mathscr{B}_1$ with $y(F',\mathscr{F})\leqq i$;

– or $i<x(B,\mathscr{F})$ and there must exist at least two reduction vertices inside $B/\mathscr{F}\cup\mathscr{B}_1$, coming from bipeds

$$F'_1\in\mathscr{B}_1,\quad F'_2\in\mathscr{B}_1,\quad \text{with}\quad y(F'_1,\mathscr{F})\leqq i,\quad y(F'_2,\mathscr{F})\leqq i.$$

Using Lemmas IV.3 and IV.4, we always find

$$\frac{N(C)}{2}-2+2\geqq P(C).$$

c) $\frac{N(C)}{2}-2=0$ and C is proper but not closed. Then C is a quadruped, the closure of which is a biped $B'=B/\mathscr{F}\cup\mathscr{B}_1$.

i) If $B\in\mathscr{H}(\mathscr{F})$, either $x(B,\mathscr{F})\leqq i<y(B,\mathscr{F})$, or there exists in C at least one reduction vertex coming from a biped $F'\in\mathscr{B}_1$ with $y(F',\mathscr{F})\leqq i$.

ii) If $B\in\mathscr{F}\cup\mathscr{B}_1$, either $x(B,\mathscr{F})\leqq i$, or there exists in C at least one reduction vertex coming from a biped $F'\in\mathscr{B}_1$, with $y(F',\mathscr{F})\leqq i$. In all these cases we find $\frac{N(C)}{2}-2+1\geqq P(C)$.

d) $\frac{N(C)}{2}-2=0$, C is proper and closed, $C\neq F/\mathscr{F}\cup\mathscr{B}_1$. Then $C=Q/\mathscr{F}\cup\mathscr{B}_1$. By using the fact that Q is closed, that $B_{\mathscr{F}}(Q)$ is proper and that F is a biped of $\mathscr{B}_1$ if $B_{\mathscr{F}}(Q)\neq F$, we see from the definition of y that $i<y(Q,\mathscr{F})$. The use of Lemma IV.3 again if $z(Q,\mathscr{F})\leqq i$, or Lemma IV.4 if $z(Q,\mathscr{F})>i$, gives $\frac{N(C)}{2}-2+1\geqq P(C)$.

e) $\frac{N(C)}{2}-2=-1$, C is proper and closed, $C\neq F/\mathscr{F}\cup\mathscr{B}_1$. Then $C=B/\mathscr{F}\cup\mathscr{B}_1$ with $B\in\mathscr{H}(\mathscr{F})$ and $i<y'(B,\mathscr{F})$. The analysis is identical to the analysis made in case b) above and gives $\frac{N(C)}{2}-2+2\geqq P(C)$.

f) $\frac{N(C)}{2}-2=0$, C is proper and closed, $C=F/\mathscr{F}\cup\mathscr{B}_1$, $F\neq G$. In that case, we keep $\frac{N(C)}{2}-2=0$.

g) $\frac{N(C)}{2}-2=-1$, C is proper and closed, $C=F/\mathscr{F}\cup\mathscr{B}_1$, $F\neq G$. Either $z(F,\mathscr{F})\leqq i$, or there exists in C at least one reduction vertex coming from a biped $F'\in\mathscr{B}_1$ with $y(F',\mathscr{F})\leqq i$. In both cases from Lemmas IV.3 or IV.4, $\frac{N(C)}{2}-2+1=0$.

h) $\frac{N(C)}{2}-2\leqq 0$, $C=F/\mathscr{F}\cup\mathscr{B}_1$ with $F=G$. In that case, since $G\in\mathscr{H}(\mathscr{F})$ and $i<y(G,\mathscr{F})=\ell+1$, we can do the same analysis as in case b) if G is a biped, giving $\frac{N(C)}{2}-2+2\geqq P(C)$, or as in case d) if G is a quadruped, giving $\frac{N(C)}{2}-2+1\geqq P(C)$.

All the cases from a) to h) are mutually exclusive and give finally

$$i<\mathrm{Inf}(i_0,j)\ \Rightarrow\ i+\eta_i\geqq\frac{\hat{N}_i}{6}, \tag{4.41}$$

with

$$\hat{N}_i=\sum_{\substack{F\in\mathscr{F}\cup\mathscr{B}_1\cup\{G\}\\ G_i^\sigma\cap F/\mathscr{F}\cup\mathscr{B}_1\neq F/\mathscr{F}\cup\mathscr{B}_1 \text{ if } F\neq G}} N(G_i^\sigma\cap F/\mathscr{F}\cup\mathscr{B}_1). \tag{4.42}$$

Lemma IV.5.

$$1\leqq i<\mathrm{Inf}(i_0,j)\ \Rightarrow\ i+\eta_i\geqq 1. \tag{4.43}$$

Proof. From the preceding analysis, we see that (4.43) is true unless we have $G_i^\sigma\cap F/\mathscr{F}\cup\mathscr{B}_1=\emptyset$ (emptiness) or $G_i^\sigma\cap F/\mathscr{F}\cup\mathscr{B}_1=F/\mathscr{F}\cup\mathscr{B}_1$ (fullness),

$\forall F\in\mathscr{F}\cup\mathscr{B}_1\cup\{G\}$. We cannot have always emptiness since $i\geqq 1$, nor always fullness since $i\leqq \ell-1$. Moreover, (4.43) is true from case h) above if we have fullness for $F=G$. Therefore, for (4.43) being violated, we must have at least one F_1 with fullness for F_1 and emptiness for $B_{\mathscr{F}\cup\mathscr{B}_1}(F_1)$. Since $x(F_1,\mathscr{F})>y(F_1,\mathscr{F})$, $\forall F_1\in\mathscr{F}\cup\mathscr{B}_1$, this is possible only if there is one (if F_1 is a quadruped) or two (if F_1 is a biped) $F'\in\mathscr{A}_{\mathscr{B}_1}(F_1)$ with $i\geqq y(F',\mathscr{F})$. From Lemma IV.4, we are left with at least one such F', not used in cases f) or g) above, and giving a further $+1$. This achieves the proof of Lemma IV.5.

Let us see now the case $i\geqq j$. From Subsect. IV.5, Lemma IV.3 remains true except for the maximal graphs F_r of the nests $\mathscr{J}_m$, $\forall m\geqq k+v+1$. Let us call $\mathscr{H}_M$ the subforest of $\mathscr{H}_j$ made from these F_r. We define a new forest

$$\mathscr{M}=\mathscr{F}\cup\mathscr{B}_1\cup(\mathscr{H}(\mathscr{F})-\mathscr{H}_j)\cup\mathscr{H}_M\,.$$

Lemma IV.6. $\forall i,\ 1\leqq i\leqq \ell-1$,

$$i+\eta_i\geqq\frac{\tilde N_i}{6}+\frac{1}{2}\chi_{i_0j}-\omega_G\chi_0\,,\tag{4.44}$$

with

$$\tilde N_i=\sum_{\substack{F\in\mathscr{M}\cup\{G\}\\ G_i^\sigma\cap F/\mathscr{M}\neq F/\mathscr{M}}} N(G_i^\sigma\cap F/\mathscr{M})\,.\tag{4.45}$$

Proof. First, as remarked in Subsect. IV.3, the leading tree of $S_{\mathscr{G}}$ is a spanning tree inside any $F\in\mathscr{H}(\mathscr{F})$. Therefore, $i+\eta_i$ may be estimated as well by replacing

$$\sum_{F\in\mathscr{F}\cup\mathscr{B}_1\cup\{G\}}\omega(G_i^\sigma\cap F/\mathscr{F}\cup\mathscr{B}_1)\quad\text{by}\quad\sum_{F\in\mathscr{M}\cup\{G\}}\omega(G_i^\sigma\cap F/\mathscr{M})\,.$$

Now the same analysis [cases a) to h) above] can be repeated, with the following modifications:

i) In Lemma IV.4 there appears a further $(+1)$ term for each

$$F'\in[(\mathscr{H}(\mathscr{F})-\mathscr{H}_j)\cup\mathscr{H}_M]\cap\mathscr{B}\,,\quad\text{if}\quad y'(F',\mathscr{F})\leqq i\,.$$

These terms are used everywhere like those coming from $F'\in\mathscr{B}_1$.

ii) Each time we consider some $F'\in\mathscr{H}(\mathscr{F})$, we may again conclude that $F'\in\mathscr{H}_j$ [now because all the graphs of $\mathscr{H}(\mathscr{F})-\mathscr{H}_j$ have been reduced], and that $i\in Z_{F'}$ (now because all the graphs of $\mathscr{H}_M$ have been reduced), except if $F'=G$.

iii) Each time we deduced from $i<z$ or $i<x$ the existence of one or two reduction vertices coming from $F'\in\mathscr{B}_1$, we may now deduce the same existence with

$$F'\in[(\mathscr{H}(\mathscr{F})-\mathscr{H}_j)\cup\mathscr{H}_M]\cap\mathscr{B}\,,$$

since

$$y(F',\mathscr{F})>x(F',\mathscr{F})\quad\text{if}\quad F'\in\mathscr{H}(\mathscr{F})\,.$$

iv) In case d) above, if $F\in\mathscr{H}(\mathscr{F})$, we may again conclude that $i<y(Q,\mathscr{F})$ from $y(F,\mathscr{F})>x(F,\mathscr{F})$.

v) In case h) above, we cannot conclude now that $G\in\mathscr{H}_j$. This corresponds to the fact that we maintain the restriction $G_i^\sigma\cap F/\mathscr{M}\neq F/\mathscr{M}$ even for $F=G$, and achieves the proof of Lemma IV.6.

Lemma IV.7. $\forall i,\ i_0 \leqq i \leqq \ell - 1$,

$$i+\eta_i \geqq 1+\tfrac{1}{2}\chi_{i_0 j}. \tag{4.46}$$

Proof. If $i \geqq i_0$, then $i \geqq i_2$. From the definition of i_2 in Subsect. IV.1, there is a spanning tree S_0 for which $\lambda(S_0)=\sigma(i_2)$. Let $\Lambda(S_0)$ be the set of lines λ' of $S_0/\mathcal{M}$ such that $S_0/\mathcal{M}-\{\lambda'\}$ has two connected components, each of which contains a non-empty subset of the external vertices of G. Then $\forall \lambda' \in \Lambda(S_0)$, either $\lambda'=\sigma(i')$ with $i' \leqq i_2$, or $\lambda' \in F - X(F,\mathcal{F})$ for some $F \in \mathcal{H}(\mathcal{F})$. In this last case, $i_2 \geqq y(F,\mathcal{F}) > x(F,\mathcal{F})$ and $X(F,\mathcal{F})/\mathcal{F} \subset G_i^\sigma$ for $i \geqq i_2$. Therefore, $\forall i \geqq i_2$, there must exist a connected component C_1 of $G_i^\sigma/\mathcal{M}$ which contains all the external vertices of G, and we have

$$\frac{N(C_1)}{2}-2-\omega_G\chi_0=\frac{N(C_1)-N(G)}{2} \geqq 0. \tag{4.47}$$

From the preceding analysis with the forest $\mathcal{M}$, we see that (4.46) is true unless there is fullness or emptiness for any $F \in \mathcal{M}$, $F \neq G$.

i) If there is not fullness for G, we find $\omega(G_i^\sigma/\mathcal{M}) \geqq \omega_G+1$.

ii) If there is fullness for G, we have emptiness for some $F_1 \in \mathcal{M}$, $F_1 \neq G$, with fullness for any $F \supset F_1$, $F \in \mathcal{M}$. But since $i \geqq i_0 \geqq i_1$, we have $F_1 \notin \mathcal{F}$ from Subsect. IV.1. We also have $F_1 \notin \mathcal{H}(\mathcal{F})$ because $i \geqq y(F_1,\mathcal{F}) > x(F_1,\mathcal{F})$ would contradict emptiness for F_1. Therefore, $F_1 \in \mathcal{B}_1$ and $i \geqq y(F_1,\mathcal{F})$. This F_1 is not used in the analysis and gives a further $+1$, which achieves the proof of Lemma IV.7.

If $\omega_G=-1$, $i_0=\ell$ and $i<i_0$. If $\omega_G=0$, we obtain directly the result (4.49) below. If $\omega_G \geqq 1$, we write, since $i+\eta_i \geqq \frac{1}{2}$,

$$\int_0^1 d\beta_i \beta_i^{i+\eta_i-1}=\frac{i+\eta_i+\omega_G}{i+\eta_i}\int_0^1 d\beta_i \beta_i^{i+\eta_i+\omega_G-1}<3\omega_G\int_0^1 d\beta_i \beta_i^{i+\eta_i+\omega_G-1}. \tag{4.48}$$

Using (4.48) and Lemmas IV.5, IV.6, and IV.7, we obtain finally from (4.35) the following bound on the β_i integrations:

$$|I_\delta| \leqq K_4^n \Gamma(\omega_G+v(\delta)+\tfrac{1}{2})\prod_{i=1}^{\ell-1} v_i^{-1}, \tag{4.49}$$

where

$$v_i=\mathrm{Sup}(\tfrac{1}{2},\tilde{N}_i), \tag{4.50}$$

and $\tilde{N}_i$ is defined in (4.45).

V. Results

Gathering the results of the preceding sections, we see from (3.8), (3.16), (4.35), and (4.49) that we are left with the summations over $\mathcal{F}$, $\mathcal{H}'$, $\mathcal{C}$, σ, j, and δ.

As a consequence of Lemma III.6 and Appendix B of [1], we may write, given a forest $\mathcal{M}$,

$$\sum_\sigma \frac{1}{S!}\prod_{i=1}^{\ell-1} v_i^{-1} \leqq K_5^n, \tag{5.1}$$

where

$$s=|\mathscr{M}|=|\mathscr{F}|+|\mathscr{B}_1|+|\mathscr{H}(\mathscr{F})-\mathscr{H}_j|+|\mathscr{H}_M|. \tag{5.2}$$

From the definition of $\mathscr{H}_M$, $|\mathscr{H}_M|=a(\delta)$. From item d) of Lemma III.1, the sum over δ gives at most a factor $8^n(|\mathscr{B}_2|+|\mathscr{H}_j|-v-a)!$, and the sum over j gives a factor $\ell+1\leqq 2n$. Finally, the sum over the disjoint forests $\mathscr{F}$, $\mathscr{H}'$, and $\mathscr{C}$ gives at most a factor 8^n, from Appendix A of [1]. We find

$$|I_G^R|\leqq K_6^n\Gamma(\omega_G+v+\tfrac{1}{2})s!\,(|\mathscr{B}_2|+|\mathscr{H}_j|-v-a)!, \tag{5.3}$$

or equivalently

$$|I_G^R|\leqq K^n(|\mathscr{F}|+|\mathscr{H}(\mathscr{F})|+|\mathscr{B}_1|+|\mathscr{B}_2|)!, \tag{5.4}$$

from which we deduce the following theorem, similar to the corresponding Theorem I in the massive case [1].

Theorem I. *For any graph G of the massless euclidean Φ_4^4 model, we have*

$$|I_G^R(p\!\!\!/)|\leqq K^n[f(G)]!, \tag{5.5}$$

where K depends only on μ, $N(G)$ and on the external momenta $p\!\!\!/_1,\ldots,p\!\!\!/_N$, and $f(G)$ is defined as

$$f(G)=\operatorname*{Sup}_{\substack{\text{closed divergent}\\ \text{forests } \mathscr{F} \text{ of } G}} f(\mathscr{F}), \tag{5.6}$$

$$f(\mathscr{F})=q(\mathscr{F})+2b(\mathscr{F}), \tag{5.7}$$

$q(\mathscr{F})$ and $b(\mathscr{F})$ being the number of quadrupeds and bipeds in $\mathscr{F}$.

On the other hand, the theorem proved in Appendix C of [1] remains, of course, unchanged:

Theorem II. *If $\gamma(N,n,f)$ is the number of labeled graphs G with $N(G)=N$, $n(G)=n$, $f(G)=f$, there exists a number K' depending only on N, such that*

$$\gamma(N,n,f)\leqq K'^n(n!)^2(f!)^{-1}. \tag{5.8}$$

As a simple corollary of Theorems I and II we have

Theorem III. *The Borel transformed perturbative series for any Schwinger function of the euclidean massless Φ_4^4 model, at fixed external momenta, has a non-vanishing radius of convergence.*

References

1. de Calan, C., Rivasseau, V.: Local existence of the Borel transform in euclidean Φ_4^4. Commun. Math. Phys. **82**, 69–100 (1981)
2. Feldman, J., Magnen, J., Rivasseau, V., Sénéor, R.: Bounds on renormalized Feynman graphs. Commun. Math. Phys. **100**, 23–55 (1985)
   Gallavotti, G., Nicolò, F.: Renormalization theory in four-dimensional scalar fields. I and II. Commun. Math. Phys. **100**, 545–590 (1985); and **101**, 247–282 (1985)

3. Rivasseau, V.: Construction and Borel summability of planar 4-dimensional euclidean field theory. Commun. Math. Phys. **95**, 445–486 (1984)
4. Gawedzki, K., Kupiainen, A.: Massless lattice Φ_4^4 theory. Commun. Math. Phys. **99**, 197–252 (1985)
5. Feldman, J., Magnen, J., Rivasseau, V., Sénéor, R.: Infrared Φ_4^4. Preprint C.P.T. Ecole Polytechnique A 640.1184 (1984), to appear in the proceedings of "Les Houches" summer school (1984)
6. Bergère, M.C., Lam, Y.M.P.: Zero mass limit in the perturbative quantum field theory. Commun. Math. Phys. **48**, 267–290 (1976)

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