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Some Comments on the ADHM Construction in $4k$ Dimensions

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Abstract. A class of completely solvable gauge field equations is investigated. These equations are shown to be closely analogous to the four-dimensional self-duality equations. A similar geometrical interpretation is exhibited, and a generalisation of the ADHM construction is shown to provide solutions.

1. Introduction

Recently there has been considerable interest shown in field theories defined on space times of dimension greater than four. Much of the activity has been in the area of Kaluza–Klein theories elaborating the ideas of Cremmer and Scherk, Witten and many others [1]. However, there has also been some investigation of the possibility of extending the idea of "self-duality" and applying it to pure Yang–Mills theories in higher dimensions. In other words, equations linear in the gauge field strengths $F_{\mu\nu}, \mu, \nu = 1 \dots D$, are sought which will, as a consequence of the Bianchi identities imply the full second order gauge field equations,

$$D^{\mu} F_{\mu\nu} = 0. \tag{1}$$

Linear relations of this type were studied in Ref. [2].

Ward [3] has pointed out that amongst these linear relationships implying the full field equations there are some, but by no means all, which arise as integrability conditions for certain sets of first order differential equations. Ward's first order equations generalise the pair introduced by Belavin and Zahkarov [4], whose integrability condition is the usual four dimensional self-duality equation,

$$F_{\mu\nu} = \pm {}^*F_{\mu\nu}, \quad {}^*F_{\mu\nu} = \tfrac{1}{2} \varepsilon_{\mu\nu\lambda\rho} F^{\lambda\rho}. \tag{2}$$

More explicitly, their first order equations may be written

$$e_{\mu}^{\dagger} \pi D^{\mu} \psi = 0, \tag{3}$$

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where $e_\mu^\dagger$ has components given by

$$e_4 = 1, \quad e_a = i\sigma_a, \quad a = 1,2,3, \tag{4}$$

and π is a complex two-vector. Notice that these equations are homogeneous in the components of π. Recall that the self-dual equation has been successfully tackled by twistor techniques [5], and in the case of finite action solutions by the algebraic construction of Atiyah, Drinfeld, Hitchin and Manin (ADHM) [6]. Ward's generalisation of Eq. (3) will be reviewed briefly in Sect. 2.

It appears that not all the integrability conditions discussed by Ward lead to linear relations amongst field strengths which actually imply the second order Yang–Mills equations. We wish to pay particular attention to the class of first order equations A_k described in Ref. (3), seemingly the one most amenable to a generalisation of the ADHM construction and also implying the full second order gauge equations. The relevant dimension of space for this class is $D = 4k$. The generalised ADHM construction and some of its properties are discussed in Sect. 3.

In four dimensions the interesting solutions to the self-dual Eq. (2) are those with finite action,

$$S = \int d^4x \tfrac{1}{2} \operatorname{tr}(F_{\mu\nu}^\dagger F_{\mu\nu}) = 8\pi^2 |k|, \tag{5}$$

where k is any integer. Indeed, any gauge potential, A_μ, yielding a finite action, though not necessarily satisfying Eq. (2), can be classified to a certain extent by computing its "topological charge" or "Pontryagin index"

$$P = \frac{1}{8\pi^2} \int d^4x \tfrac{1}{2} \operatorname{tr}(F_{\mu\nu}^\dagger {}^*F_{\mu\nu}). \tag{6}$$

This quantity is always integer valued for non-singular fields which tend asymptotically to pure gauges,

$$A_\mu \to g^{-1}\partial_\mu g, \quad g \in G \quad \text{as} \quad x^2 \to \infty \tag{7}$$

In Eq. (7), G is the compact gauge group. Clearly, (5) is a consequence of (6) when the potential is arranged to satisfy Eq. (2). For a given value of P Eq. (2) yields those fields minimising S.

One consequence of Eq. (7) and the conformal invariance of the four dimensional gauge theory is that we may just as well regard space conformally compactified to the four sphere, S^4. Taking that point of view the vector potentials A_μ are to be regarded as connections in G vector bundles over S^4. Such bundles are classified by the third homotopy group of the gauge group G, $\pi_3(G)$. The homotopy equivalence classes are labelled by integers which are analytically computed by the functional P, Eq. (6). In the more general situation, corresponding to the class A_k, we shall see that it is convenient to compactify the D dimensional space in a special way, in fact to $HP^{D/4}$, ($HP^1 \approx S^4$). Then, we shall need to understand the classification of G bundles over $HP^{D/4}$ and to discover to what extent the ADHM construction is able to cope with the variety of possibilities. For $D = 4$, of course, the ADHM construction is complete, yielding all the instantons. In the more general case it appears not to be. This investigation is the subject of Sect. 4.

2. Homogeneous Equations and Integrability Conditions

Ward's generalisation of Eq. (3) may be expressed as follows

$$V^\mu_a(\pi)D_\mu\psi = 0, \tag{8}$$

where $\mu = 1 \ldots D, a = 1 \ldots l$, and the quantities $V^\mu_a(\pi)$ are homogeneous of degree q in the components of π. The vector π itself is generalised to an $(m+1)$ component vector which, in view of the homogeneity of Eq. (8), we can regard as an element of CP^m. In other words, we may write

$$V^\mu_a(\pi) = V^\mu_{ab_1\ldots b_q}\pi^{b_1}\ldots\pi^{b_q}. \tag{9}$$

The class A_k is special in that l is taken to be even and the dimension D is taken to be twice l. Thus

$$l = 2k, \quad D = 4k. \tag{10}$$

Further, each of q and m is unity, in which case Eq. (9) becomes:

$$V^\mu_a(\pi) = V^\mu_{aP}\pi^P, \quad P = 1,2, \quad a = 1,\ldots,2k. \tag{11}$$

It is also supposed that the matrix V^μ_{aP}, regarded as a mapping from a space of dimension $2l$ to one of dimension $D = 2l$, is invertible.

Inevitably, Eq. (8) breaks the SO(D) invariance. We can think of the nonsingular matrix V^μ as reorganising the coordinates

$$x^\mu = V^\mu_{aP}x^{aP}, \tag{12}$$

and so the residual symmetry group will be a subgroup of SO(D) inside GL($2k,\mathbb{C}$) $\times$ GL($2,\mathbb{C}$), which subgroup depending on the precise choice of the components V^μ_{aP}. For the class A_k the choice of subgroup is Sp(1) $\times$ Sp(k)/Z_2 picked out by imposing suitable reality conditions on V^μ and x^{aP}. Notice that only for $D = 4$, $k = 1$ is the set of Eqs. (8) SO(D) invariant. Otherwise,

$$\mathrm{SO}(4k) \supsetneq \mathrm{Sp}(1) \times \mathrm{Sp}(k)/Z_2. \tag{13}$$

Setting

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \hat{\varepsilon} = \bigoplus_1^k \varepsilon, \tag{14}$$

we shall require

$$(\hat{\varepsilon} \times \varepsilon^{-1})^{aP} = \bar{x}^{aP} \tag{15}$$

and

$$(V^\mu)^\dagger_{Pa} = (\varepsilon V^{\mu\dagger}\hat{\varepsilon}^{-1})_{Pa} \tag{16}$$

as the reality condition. Compatible with these we may also pick V^μ to satisfy

$$(V^{\mu\dagger}V^\nu + V^{\nu\dagger}V^\mu)_{PQ} = 2\delta^{\mu\nu}\delta_{PQ} \tag{17}$$

For $k = 1$, Eq. (17) is satisfied by the quantities e_μ defined in Eq. (4).

Now let us turn to the integrability conditions arising from Eq. (8). They are:

$$(V^{\mu}_{aP} V^{\nu}_{bQ} + V^{\mu}_{aQ} V^{\nu}_{bP}) F_{\mu\nu} = 0 \tag{18}$$

To show that Eqs. (19) imply the Yang–Mills equations we demonstrate the existence [2] of a totally antisymmetric tensor $T_{\mu\nu\rho\sigma}$ so that Eq. (18) implies

$$\tfrac{1}{2} T_{\mu\nu\rho\sigma} F_{\rho\sigma} = \lambda F_{\rho\sigma}, \tag{19}$$

where λ is some eigenvalue. The second order equations follow directly from (19) and the Bianchi identities.

As a step to defining the tensor $T_{\mu\nu\rho\sigma}$ we first define a tensor $\theta^{\mu\nu\rho\sigma}$, not totally antisymmetric, by

$$\theta^{\mu\nu\rho\sigma} = \mathrm{tr}\,(V^{\mu\dagger} V^{\nu} V^{\dagger\rho} V^{\sigma}), \tag{20}$$

and note the identities

$$\theta^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0, \tag{21a}$$

$$\theta^{\mu\rho\nu\sigma} F_{\rho\sigma} = -4F^{\mu\nu}, \tag{21b}$$

$$\theta^{\mu\rho\sigma\nu} F_{\rho\sigma} = 8F^{\mu\nu}, \tag{21c}$$

which are direct consequences of Eq. (18). Thus, (21a) follows on contracting Eq. (18) with $(E^{\mu\dagger} E^{\nu} \varepsilon)_{QP} \hat{\varepsilon}_{ab}$ and rearranging, (21b) by contracting with $(E^{\mu\dagger})_{Qa} (E^{\nu\dagger})_{Pb}$, and (21c) by a rearrangement using the orthogonality relations (17).

Next, we manufacture a totally antisymmetric quantity from θ by setting

$$T^{\mu\nu\rho\sigma} = \tfrac{1}{12}(\theta^{\mu\nu\rho\sigma} + \theta^{\mu\sigma\nu\rho} + \theta^{\mu\rho\sigma\nu} - \theta^{\mu\rho\nu\sigma} - \theta^{\mu\nu\sigma\rho} - \theta^{\mu\sigma\rho\nu}). \tag{22}$$

Then using Eqs. (21) we find

$$\tfrac{1}{2} T^{\mu\nu\rho\sigma} F_{\rho\sigma} = F^{\mu\nu}. \tag{23}$$

The tensor $T^{\mu\nu\rho\sigma}$ does have two other eigenspaces, neither of which arises from the integrability condition. We can, however, construct bases for all the eigenspaces using the quantities V^{μ}. In four dimensions there are just two eigenspaces corresponding to the quantities $\eta, \bar{\eta}$ defined by

$$\begin{aligned} \eta_{\mu\nu} &= \tfrac{1}{2}(e^{\dagger}_{\mu} e_{\nu} - e^{\dagger}_{\nu} e_{\mu}), \\ \bar{\eta}_{\mu\nu} &= \tfrac{1}{2}(e_{\mu} e^{\dagger}_{\nu} - e_{\nu} e^{\dagger}_{\mu}); \end{aligned} \tag{24}$$

η is anti-dual and $\bar{\eta}$ self-dual, respectively [7]. By analogy we shall define

$$N_{\mu\nu} = \tfrac{1}{2}(V^{\dagger}_{\mu} V_{\nu} - V^{\dagger}_{\nu} V_{\mu}), \tag{25}$$

$$\bar{N}_{\mu\nu} = \tfrac{1}{2}(V_{\mu} V^{\dagger}_{\nu} - V_{\nu} V^{\dagger}_{\mu}), \tag{26}$$

and use the completeness relations

$$\tfrac{1}{2} V^{\mu\dagger} \otimes V^{\mu} = \tfrac{1}{2} V^{\mu} \otimes V^{\mu\dagger} = 1_2 \otimes 1_{2k}, \tag{27}$$

$$\tfrac{1}{2} V^{\mu} \otimes V^{\mu} = \tfrac{1}{2} V^{\mu\dagger} \otimes V^{\mu\dagger} = \varepsilon \otimes \hat{\varepsilon} \tag{28}$$

to compute the action of T on these tensors. We find

$$\tfrac{1}{2} T^{\mu\nu\rho\sigma} N_{\rho\sigma} = -\frac{2k+1}{3} N^{\mu\nu}, \tag{29}$$

$$\tfrac{1}{2}T^{\mu\nu\rho\sigma}\bar{N}_{\rho\sigma} = \bar{N}^{\mu\nu}. \tag{30}$$

There is a third tensor, orthogonal to N, $\bar{N}$ and vanishing for $k = 1$. An expression for it is

$$M^{\mu\nu} = \tfrac{1}{4}[m^{\mu\nu} - m^{\nu\mu}]$$

with

$$m^{\mu\nu}_{abPQ} = V^{\mu}_{aP}V^{\nu}_{bQ} + V^{\mu}_{aQ}V^{\nu}_{bP} - \frac{1}{k}\hat{\varepsilon}_{ab}\hat{\varepsilon}_{cd}V^{\mu}_{cP}V^{\nu}_{dQ}. \tag{31}$$

$M^{\mu\nu}$ satisfies

$$\tfrac{1}{2}T^{\mu\nu\rho\sigma}M_{\rho\sigma} = -\tfrac{1}{3}M^{\mu\nu} \quad k \geqq 2. \tag{32}$$

To summarise, from a purely algebraic point of view the eigenvalue λ could take the values $1, -\frac{1}{3}, -2k + 1/3$. However, from the point of view of the integrability conditions for Eq. (8), only $\lambda = 1$ occurs. A basis for the solutions to Eq. (19) with $\lambda = 1$ is given by the quantities $\bar{N}_{\mu\nu}$ which generalise the self-dual tensor $\bar{\eta}_{\mu\nu}$ in four dimensions. We shall find in the next section that only for this case does the ADHM construction generalise. This fact lends further support to Ward's idea that only those equations for the field strength which can be cast into the form of an integrability condition for a set of linear differential equations will be solvable, and have a nice geometrical interpretation analogous to that in four dimensions.

We note one final point. If we ask how many equations for the field strength there are implied by Eq. (23), the answer turns out to be $3k(2k-1)\dim(G)$, on simply counting up the contributions to Eq. (18). The number of potentials to be determined in $4k$ dimensions is $(4k-1)\dim(G)$. Hence, except when $k = 1$, Eqs. (24) overdetermine the gauge field.

3. Extended ADHM Construction

In this section we shall capitalise on the structure of Eq. (30) to propose a simple extension of the ADHM construction which also yields field strengths satisfying Eq. (23). At first we shall work locally, not worrying about the global properties of the procedure. We shall return to those in Sect. 4. We shall also restrict analysis to the gauge groups Sp(r) regarded as the set of unitary matrices whose entries are quaternions, that is 2×2 matrices of the form

$$a = e_{\mu}a_{\mu}. \tag{33}$$

This restriction is purely for notational convenience.

Recall that the ADHM prescription for Sp(r) may be described as a set of instructions for constructing an $(r + l) \times r$ quaternion matrix $V(x)$ satisfying

$$V^{\dagger}V = 1_r$$

and in terms of which the vector potential is given by

$$A_{\mu} = V^{\dagger}\partial_{\mu}V.$$

The matrix V itself must be chosen to be orthogonal to a set of l (quaternion) vectors each of which is a linear function of the coordinates x. These we assemble as a matrix $\Delta(x)$:

$$\Delta(x) = a + bx, \tag{34}$$

where a is an $(r+l)\times l$ constant quaternion matrix and the term bx is to be interpreted as

$$b_{iPR}x_{iRQ} \quad i=1,\ldots,k; \quad P,Q,R=1,2, \tag{35}$$

where the label a (in Eq. (12)) has been replaced by the pair (i,R), a natural substitution in view of the nature of the reality conditions. In fact we are regarding R^{4k} as H^k, taking the components four at a time and arranging them into quaternions in the usual way. Each matrix b_i, $i=1\ldots k$ is an $(r+l)\times l$ quaternion matrix. Then, the condition on V is

$$\Delta^\dagger(x)V = 0. \tag{36}$$

For those values of x for which the columns of Δ are linearly independent we shall have a completeness relation

$$1_{r+l} = VV^\dagger + \Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger \tag{37}$$

which we may use to construct an expression for the field strength:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = V^\dagger(\partial_\mu\Delta(\Delta^\dagger\Delta)^{-1}\partial_\nu\Delta^\dagger - \partial_\nu\Delta(\Delta^\dagger\Delta)^{-1}\partial_\mu\Delta^\dagger)V. \tag{38}$$

Finally, noting that

$$\partial_\mu x = V_\mu$$

and requiring that $\Delta^\dagger\Delta$ should commute with V_μ, we find

$$F_{\mu\nu} = 2V^\dagger b\bar{N}_{\mu\nu}(\Delta^\dagger\Delta)^{-1}b^\dagger V. \tag{39}$$

Hence, in view of Eq. (30) the field strength is "self-dual" in the extended sense, satisfying Eq. (23) automatically.

So far the argument is rather formal and we must now check carefully the statements we have made. Consider the statement that $\Delta^\dagger\Delta = f$ must commute with V_μ. We may think of V_μ as a $k\times 1$ column vector of unit quaternions e_μ in which case the statement that f commutes with V_μ can be cast into the form

$$(V_\mu)(f) = \begin{bmatrix} f & \cdot & \cdot \\ \cdot & f & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & f \end{bmatrix}(V_\mu), \tag{40}$$

indicating that f must commute with each component of V_μ. Thus, writing

$$f = \Delta^\dagger\Delta = a^\dagger a + a^\dagger b_i x_i + x_i^\dagger b_i^\dagger a + x_i^\dagger b_i^\dagger b_j x_j, \tag{41}$$

we see that this is possible only provided $a^\dagger a$, $a^\dagger b_i$, $b_i^\dagger b_j$ are all symmetric matrices *as quaternions* for each value of $i,j=1\ldots k$. These constraints are quite severe as we shall see. Supposing the constraints to be satisfied and $\Delta^\dagger\Delta$ to have maximal rank, $\bar{V}$ is determined for a given set of a's, b's up to a gauge transformation. Different sets of a's and b's will give the same vector potential provided they are related by

$$a = Qa'R, \quad b_i = Qb_i'R, \quad Q \in \mathrm{Sp}(l+r), \quad R \in \mathrm{GL}(l, R). \tag{42}$$

Taking into account the constraints on a, b_i and the equivalences (42) we can make a naive count of the degrees of freedom of the solution obtained. We find

$$N(k,l,r) = 4(r+1)kl - r(2r+1) - \tfrac{1}{2}l(l-1)(2k-1)(k-1), \tag{43}$$

at least for $r < kl$. Otherwise, not all of the equivalences (42) are effective and a more careful analysis leads to

$$N(k,l,r) = kl(2lk+3) - \tfrac{1}{2}l(l-1)(2k-1)(k-1) \tag{44}$$

for $r \geqq kl$. Of course, for $k=1$, $D=4$ and the result is the usual number of instanton degrees of freedom for an $\mathrm{Sp}(r)$ group in four dimensions. For $l > 1$ the dependence on k is such that the function $N(k,l,r)$ is eventually negative for fixed l, r. For $l = 1$, this is not so and Eqs. (43), (44) become, respectively

$$N(k,1,r) = 4(r+1)k - r(2r+1), \tag{45a}$$

$$N(k,1,r) = k(2k+3). \tag{45b}$$

Note that Eqs. (45) also describe precisely the number of degrees of freedom of an $\mathrm{Sp}(r)$ field in four dimensions with k the instanton number!

Before combining our analysis of cases, it is worth seeing what happens if we try to construct an ADHM procedure to match one of the other eigenspaces of $T^{\mu\nu\rho\sigma}$, say that corresponding to Eq. (29). Effectively, we would need to replace V_μ by $V_\mu^\dagger$ wherever it occurred. This means that the appropriate generalisation of $\Delta(x)$ is now a collection

$$\Delta_i = a_i + bx_i \quad i = 1, \ldots, k \tag{46}$$

each of the a_i and b being an $(l+r) \times l$ matrix of quaternions. Then, $(\Delta^\dagger\Delta)_{ij}$ would be a $2lk \times 2lk$ matrix which must commute with V_μ. This is an impossible stipulation considering the shape of the matrices involved unless

$$\begin{gathered} \Delta_i^\dagger\Delta_j = 0, \qquad i \neq j, \\ \Delta_1^\dagger\Delta_1 = \Delta_2^\dagger\Delta_2 = \cdots = \Delta_k^\dagger\Delta_k, \end{gathered} \tag{47}$$

which are much too strong. Conditions (47) are impossible to satisfy except for very special values of the coordinates. We cannot even imagine an ADHM prescription for the other eigenspace, Eq. (32). We are forced to the conclusion that only the case arising as an integrability condition can support an ADHM procedure and we shall concentrate on this for the remainder of the article.

4. Global Aspects of the Extended Construction

In Sect. 3 we saw how an extension of the ADHM procedure enabled us to construct certain field configurations which satisfy Eq. (24) in $\mathbb{R}^D$, where $D = 4k$. In the familiar case of $k = 1$, the self-duality equations in $\mathbb{R}^4$, the solutions to Eq. (24)

provide critical points of the action defined by Eq. (5), and all finite action solutions to the equations can be obtained from the ADHM construction. Now it is known [8] that there are no finite action solutions to the Yang–Mills equations if $D > 4$, and hence no such solutions to Eq. (23). Thus this requirement is not a useful way to impose boundary conditions on the solutions we are considering. On the other hand, it turns out that all finite action solutions to the Yang–Mills equations on $\mathbb{R}^4$ have an asymptotic behaviour which ensures that they are conformally equivalent to solutions defined on $S^4 = HP^1$, and we can impose suitable boundary conditions on the Eq. (48) by requiring that the solutions can be extended from $\mathbb{R}^D = H^k$ to HP^k in a suitable sense.

Equation (48) is invariant under the transformations,

$$x_i \to x_i' = (Q_{ij}x_j + Q_{i0})(Q_{0l}x_l + Q_{00})^{-1}, \tag{48}$$

where $Q_{\alpha\beta} \in H$, $0 \leqq \alpha, \beta \leqq k$, which generalise the conformal transformations on $\mathbb{R}^4 = H$. (There are implied sums in Eq. (48) over $j, l = 1$ to k.) It follows that it makes sense to regard the equation as being defined on HP^k, the space of "lines" in H^{k+1} [that is, the space of nonzero points of H^{k+1} identifying points of the form $(x_\alpha q)$ for different $q \in H$, $q \neq 0$]. This space is covered by $k+1$ open sets θ_β $(0 \leqq \beta \leqq k)$, each a copy of H^k, corresponding to the points (x_α) with $x_\beta \neq 0$. If we take coordinates on the sets θ_β so that $x_\beta \equiv 1$, the transformations (48) are induced on $\theta_0 \subset HP^k$ by

$$(x_\alpha) \to (x_\alpha') = (Q_{\alpha\beta} x_\beta), \tag{49}$$

summing over $\beta = 0$ to k. These transformations will also help map the open sets $\theta_\gamma \subset HP^k$ onto one another. A suitable generalisation of the finite action solutions on $\mathbb{R}^4$, which correspond to solutions defined in two patches, on $S^4 = HP^1$ and related by a gauge transformation in the overlap, is a solution to Eq. (23) defined in each of the patches U_α, the solutions being related by gauge transformations in the overlaps $U_\alpha \cap U_\beta, \alpha \neq \beta$.

A gauge field configuration (not necessarily a solution) defined in patches, but related by gauge transformations in the overlaps, in the way just described, constitutes a G-bundle over HP^k, G being the Yang–Mills gauge group. Such a bundle will have various topological invariants, that is quantities which do not change under continuous deformation of the field configurations.

For $k = 1$, i.e. S^4, there is essentially just one such quantity, the instanton number, the integral of the first Pontryagin class, given by

$$l = \frac{1}{16\pi^2} \int \mathrm{Tr}(F^*_{\alpha\beta} F_{\alpha\beta}) d^4x \equiv \frac{1}{16\pi^2} \int \mathrm{Tr}(F \wedge F), \tag{50}$$

with $F \equiv F_{\alpha\beta} dx^\alpha \wedge dx^\beta$. l must be an integer, and for each integral value of l there are possible field configurations. If $l \geqq 0$ there are solutions to the self-duality Eq. (24); further the ADHM construction yields all the solutions to these equations for each value of l, and for any gauge group.

In general, for $k > 1$, the situation is more complicated. In this case we can define k nonzero Pontryagin classes, $P_j(\xi)$, for a given G-bundle ξ over HP^k. Then $P_j(\xi)$ is an invariant polynomial in $F = F_{\alpha\beta} dx^\alpha \wedge dx^\beta$ of order $2j$. $P_j(\xi)$ is closed, so that by Stokes' theorem integrating it over any $4j$-dimensional closed surfaces in HP^k will give the same result, if the surfaces can be deformed into one another continuously. Moreover, as $P_j(\xi)$ is a topological invariant, such integrals are left unchanged by continuous variation of the potential. For a discussion of these and other characteristic classes see for example, Ref. (9). Thus integrating P_j over the obvious $HP^j \subset HP^k$ (obtained, e.g. by setting $k-j$ of the coordinates to zero) yields k Pontryagin indices

$$l = (-)^j \int_{HP^j} P_j(\xi). \tag{51}$$

Explicitly, for example,

$$l_2 = \frac{3}{4!(2\pi^4)} \left(\int_{\theta_0} \mathrm{Tr}(F \wedge F) \wedge \mathrm{Tr}(F \wedge F) - 2 \int_{\theta_0} \mathrm{Tr}(F \wedge F \wedge F \wedge F) \right). \tag{52}$$

It is no longer the case that there exist field configurations (G-bundles) over HP^k realising all possible values of the k-tuples of integers $(l_1, l_2, \ldots, l_k)$. Moreover the values of these integers do not always determine the configurations up to continuous deformation.

For definiteness let us consider the case where the gauge group $G = \mathrm{Sp}(r)$, as in Sect. 3. Then if $r \geqq k$ the topological classification of the field configuration is stable, that is unchanged if we enlarge G, by regarding $\mathrm{Sp}(r) \subset \mathrm{Sp}(r')$ with $r' > r$. Further the integers l_j completely determine it up to continuous deformation, even though not all combinations are realised. For example [10] if $k = 2$ and $r \geqq 2$, the condition that the Pontryagin indices l_1, l_2 arise for some $\mathrm{Sp}(r)$-bundle is that

$$l_2 \equiv \tfrac{1}{2} l_1(l_1 + 1) \qquad (\mathrm{mod}\ 12), \tag{53}$$

and such pairs (l_1, l_2) of integers label the classes of topologically equivalent $\mathrm{Sp}(r)$-bundles. However if we consider $\mathrm{Sp}(1)$-bundles a stable classification is not sufficient. Here l_2 vanishes identically so that (since every $\mathrm{Sp}(1)$-bundle defines an $\mathrm{Sp}(2)$ bundle) $\frac{1}{2} l_1(l_1 + 1) \equiv 0$, mod 12, implying

$$l_1 \equiv 0, 8, 15, \text{ or } 23, \qquad (\mathrm{mod}\ 24). \tag{54}$$

Indeed it is the case that such values of l_1 do arise for $\mathrm{Sp}(1)$-bundles over HP^2, but for each such l_1 there are *two* topological classes of bundles.

Of course it is a separate question as to whether field configurations corresponding to these bundles can satisfy the field equations, or, more particularly, the first order Eq. (23).

What we can see is that the bundles constructed by the generalised ADHM procedure of Sect. 3 only produces a subset of the possible topological quantum numbers. In fact it is rather difficult to see how the ADHM procedure could lead to Eq. (54) being satisfied other than by means of a straightforward equality. We can show that this is indeed the case by using the sum formula for Pontryagin classes: if we take the direct sum of two bundles ξ, η over HP^k,

$$P_r(\xi \oplus \eta) = \sum_{i+j=r} P_i(\xi) \wedge P_j(\eta), \tag{55}$$

with $P_0(\xi) = 1$ and $P_j(\xi) = 0$ for $j > k$ (in the sense that the forms on each side will give the same result when integrated over any closed $4k$-dimensional surface, i.e. each side is the same integer times τ^r, where τ is the generator of the cohomology ring of HP^k. This integer corresponds to the l_r defined in Eq. (52) as τ^r yields 1 when integrated over a standard HP^r.) Now in the ADHM construction for solutions to Eq. (23) for Sp(r) with $l_1 = l$, the solution is constructed as a bundle ξ orthogonal to a bundle η, defined by the image of $\Delta(x)$, in the trivial bundle $HP^k \times H^{l+r}$. The bundle η is a direct sum of l bundles corresponding to the l columns of $\Delta(x)$, one dimensional over H and having $l_0 = 1$, $l_1 = +1$, $P_j = 0$, $j > 1$. Thus

$$P_0(\eta) = 1, \quad P_1(\eta) = +l, \quad P_j(\eta) = \binom{l}{j}. \tag{56}$$

From these remarks and Eq. (55) it follows that

$$\begin{aligned} l_1 &= -P_1(\xi) = +l, \\ l_2 &= P_2(\xi) = l^2 - \tfrac{1}{2}l(l-1) = \tfrac{1}{2}l(l+1). \end{aligned} \tag{57}$$

Although it is not clear from the above whether all solutions to Eq. (23) are given by the ADHM construction, it is clear that there is an equivalence between solutions to these equations and certain holomorphic vector bundles, which generalises the Atiyah–Ward correspondence. We describe this equivalence below. The mathematical details do not differ essentially from those of the Atiyah–Ward correspondence, and we refer the interested reader to Ref. (5).

The idea is to introduce complex coordinates on the patches of HP^k, in other words to parametrize each k-quaternion vector in H^k by $2k$ complex numbers. To do this while respecting the invariance of Eq. (23) requires choosing the same parametrization by 2 complex numbers of each quaternionic component in H^k, so that the set of complex coordinates is defined by those of the first component. The possible choices of such coordinates are parametrized by $\mathbb{C}P^1$. Were the choices in different patches to fit together trivially, the space of such complex structures on (the tangent space of) HP^k would be parametrized by $HP^k \times \mathbb{C}P^1$. However, they do not, and the relevant space turns out to be $\mathbb{C}P^{2k+1}$.

The picture is that $\mathbb{C}P^{2k+1}$ contains the set of points p in HP^k together with complex coordinates at p. So there is a natural projection map:

$$\begin{gathered} \{\text{pt.} + \text{coords.}\} \to \{\text{pt.}\}, \\ \pi: \mathbb{C}P^{2k+1} \to HP^k. \end{gathered}$$

Any bundle E on HP^k can then be lifted to a bundle π^*E on $\mathbb{C}P^{2k+1}$: if E is thought of as attaching vector spaces to each point in HP^k then π^*E attaches the same vector space to a point q in $\mathbb{C}P^{2k+1}$ as E does to $\pi(q)$ in HP^k. Clearly π^*E is trivial on the $\mathbb{C}P^1$ "lines" in $\mathbb{C}P^{2k+1}$ which project to the same point in HP^k.

The point of this procedure is that, from any bundle E corresponding to a gauge configuration on HP^k, we can construct a bundle π^*E on the complex manifold $\mathbb{C}P^{2k+1}$. On a complex manifold, the notion of a homomorphic bundle makes sense, and one finds that π^*E is a holomorphic bundle precisely when the gauge configuration satisfies Eq. (23). This leads us to hope that a full classification of the possible sets of topological invariants for solutions to Eq. (23) and indeed an explicit construction of all such solutions, may be feasible, in the same way as the Atiyah–Ward correspondence leads to the proof of the completeness of the ADHM construction for instantons [5].

Appendix A: Failure of the 't Hooft Ansatz

By analogy with the self-duality equations, one might look for a simple class of solutions to Eq. (23) described by an ansatz extending that of 't Hooft [11]. This would be a natural attack on the problem even if no analogues of the ADHM equations were available, but can best be investigated within the ADHM framework, given that it exists.

The 't Hooft ansatz for Sp(1) solutions of Eq. (23) would be described by our ADHM ansatz of Eqs. (34), (35) [6, 12] by taking the matrices

$$b_j = \begin{bmatrix} b_j^0 \lambda_1 & \dots & b_j^0 \lambda_l \\ b_j^1 \lambda_0 & \dots & 0 \\ 0 & \dots & b_j^l \lambda_0 \end{bmatrix} \tag{A1}$$

($b_j^i, \lambda_i \in H$ for $0 \leqq i \leqq l$, $0 \leqq j \leqq k$). Thus

$$\Delta(x) = \begin{bmatrix} x^0 \lambda_l & \dots & x^0 \lambda_l \\ x^1 \lambda_0 & \dots & 0 \\ 0 & \dots & x^l \lambda_0 \end{bmatrix}, \tag{A2}$$

where

$$x^i = \sum_{j=0}^{k} b_j^i x_j . \tag{A3}$$

In fact, the solution given by Ward [3] is an example of such an ansatz, solving Eq. (23) over $\mathbb{R}^8$. This example cannot be extended to HP^2, and indeed no ADHM solution defined by (A2), (A3) can be valid on HP^k. The reason for this is that (for $k \geqq 2$) the first column (say) of $\Delta(x)$ in (A2) will vanish on the (non-zero) intersection of the hyperplanes $x^0 = x^1 = 0$ in HP^k, violating the requirement that the columns of $\Delta(x)$ be everywhere linearly independent.

Appendix B: An Explicit Solution

The simplest solution to Eq. (23), analogous to the basic instanton solution [13], is an Sp(2) gauge field configuration over HP^2 given in homogeneous coordinates by

$$\Delta(x, y, z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{B1}$$

On the patch $(z=1)$, Eqs. (36) imply

$$v^{\dagger}(x,y)=\begin{bmatrix}\frac{\bar{x}}{\sqrt{2}r} & \frac{\bar{y}}{\sqrt{2}r} & -\frac{1}{\sqrt{2}}\\ \frac{y}{r} & -\frac{x}{r} & 0\end{bmatrix}, \tag{B2}$$

where $r^2 = x\bar{x} + y\bar{y}$. Thus

$$\begin{aligned}A(x,y) &= v^{\dagger}(x,y)dv(x,y)\\ &=\begin{bmatrix}\frac{-d\bar{x}x-d\bar{y}y}{2r^2} & \frac{\bar{x}d\bar{y}-\bar{y}d\bar{x}+(\bar{y}\bar{x}-\bar{x}\bar{y})dr}{\sqrt{2}r^2}\\ \frac{ydx-xdy+(xy-yx)dr}{\sqrt{2}r^2} & \frac{-dy\bar{y}-dx\bar{x}}{r^2}\end{bmatrix}.\end{aligned} \tag{B3}$$

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