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Massless Lattice ϕ_4^4 Theory: Rigorous Control of a Renormalizable Asymptotically Free Model

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Abstract. Using block spin renormalization group techniques, we rigorously control the functional integral of a weakly coupled critical lattice ϕ^4 theory in four euclidean dimensions proving the infrared asymptotic freedom of the model. This solves the infrared counterpart of and sheds some light on the problem of existence of continuum renormalizable ultraviolet asymptotically free models.

1. Introduction

One of the fundamental problems of Quantum Field Theory (QFT) is the existence of non-trivial models describing couplings of fields and scattering of particles. Such models do exist on the level of formal renormalized perturbation series, where renormalization removes the ultraviolet (UV) divergences of the naive perturbation expansion. The problem of non-perturbative existence can be viewed as equivalent to a non-perturbative understanding of renormalization. Up to now, the attempts at a non-perturbative control of the QFT models (constructive QFT [23], exactly soluble models [26]), although very instructive, have failed to produce quantum field theories in four space-time dimensions.

Much of our present understanding of the existence problem for QFT comes from the Renormalization Group (RG) approach. The RG, in its most full-fledged version [41] cast into the statistical mechanical framework in the euclidean space-time, replaces the static point of view of renormalized perturbation theory by a dynamical one. We try to see, mostly also perturbatively, how the local (euclidean) field theory may be obtained from its cut-off non-local versions in which the source of the troubles: the short distance (UV) singularities are regularized in order to guarantee the existence of the model. One of the crucial concepts arising from the RG approach is that of (UV) asymptotic freedom [25, 35]: a model is UV asymptotically free if its short distance asymptotics is non-interacting (free).

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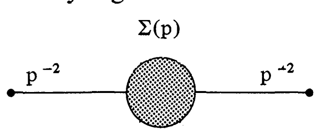
Asymptotic freedom provides a self-consistency condition on the perturbative approach to the local limit field theory. This is the reason behind a common belief (not quite shared by us) that only asymptotically free models (or their transforms) have a chance to exist in four dimensions.

Unfortunately, the simplest model of a renormalizable quantum field in four dimensions, the $\lambda\phi^4$ theory, turned out not to be (perturbatively) UV asymptotically free for $\lambda > 0$ [38]. This raised serious doubts about the existence of a non-trivial local limit for this case [41] confirmed by detailed studies [2, 3, 13, 14]. Today, although not completely excluded [15], positive coupling ϕ_4^4 quantum field theory is generally believed not to exist.

Fortunately, the four dimensional non-abelian gauge theories, pure as well as with not too many fermion flavors (QCD) provide text-book examples of renormalizable, UV asymptotically free models [25, 35, 27]. This gives importance to the non-perturbative study of renormalizability and asymptotic freedom.

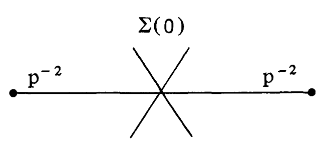
In the present paper, we consider the infrared (IR) counterpart of the UV problem presented above: a massless euclidean lattice theory which is IR renormalizable (has a dimensionless coupling constant and renormalizable IR divergences in every order of naive perturbation expansion) and is expected to be IR asymptotically free: its long distance asymptotics is perturbatively free (gaussian). Specifically, we consider the massless lattice $\lambda\phi_4^4$ theory with $\lambda > 0$. We could have also treated a model more realistic from the point of view of statistical mechanics: the Landau-Ginzburg theory of the uniaxial dipolar ferromagnet [10, 29] which exhibits a similar behavior, confirmed by experiment [1], in three dimensions.

The massless ϕ_4^4 theory presents in two ways non-trivial renormalization problems. In perturbation theory, the $\lambda\phi_4^4$ model with vanishing bare mass has IR divergences in arbitrarily high order due to the self-energy diagrams [32]



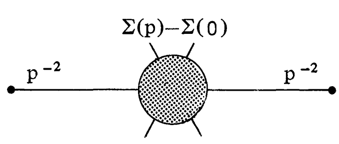
$$p \xrightarrow{\sim} 0 \frac{1}{p^4}. \tag{1}$$

However, if we fix the physical mass to zero (the correlation length to infinity) adjusting the bare mass appropriately (this may be viewed as IR renormalization), then the mass insertions



(2)

will subtract the self-energy diagrams turning them to



$$p \xrightarrow{\sim} 0 \frac{1}{p^2} \tag{3}$$

and will render the perturbative expansion finite. [In terms of the RG this is the problem of relevant perturbations or the naturality problem. In UV renormalization it corresponds to a careful adjustment of the bare mass at large cut-off to get an $\mathcal{O}(1)$ mass at the $\mathcal{O}(1)$ scale.]

The other, more fundamental, question is the IR divergence of perturbation theory at vanishing external momenta. The RG predicts that the (rescaled) Green's functions at large distances become those of the free theory.

The method by which we achieve a rigorous control of the weakly-coupled massless-ϕ_4^4-theory functional integral is the block spin (BS) version of the RG [28]. We split the integration into a sequence of steps. In step n, we fix the (rescaled) averages of fields over cubes of size L^n (n^{th} block spins) and integrate out the fluctuations of the averages over cubes of size L^{n-1}. At the critical point, the n^{th} BS effective theory can be viewed as statistical mechanics described essentially by a local ϕ^4 interaction (plus irrelevant, approximately local corrections) with the (running) coupling constant λ_n changing from scale to scale. The second order perturbative computation gives

$$\lambda_n = \lambda_{n-1} - \mathcal{O}(\lambda_{n-1}^2), \tag{4}$$

which results in

$$\lambda_n \sim \mathcal{O}\left(\frac{1}{n}\right). \tag{5}$$

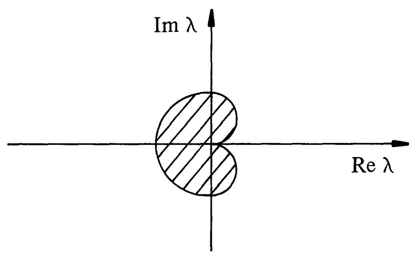
This expresses the perturbative IR asymptotic freedom of the massless ϕ_4^4 theory. The slow (logarithmic) decrease of λ_n is characteristic for the renormalizable models with dimensionless couplings and should be contrasted with the (super-renormalizable) behavior of the critical ϕ_d^4 theory with $d>4$, where $\lambda_n = \mathcal{O}(L^{n(4-d)})$.

Our aim is to show that the corrections to the perturbative analysis which evaluates the contribution of the fluctuations on the distance scale L^n by expanding it to the second order in the effective coupling λ_n of this scale are small and do not change the qualitative picture of the IR behavior. This is not obvious since the perturbation expansion is divergent. Note, however, that the λ_n-perturbation theory should be contrasted with the usual perturbative expansion in the initial coupling constant (on all scales) which lacks self-consistency of the latter and exhibits very non-uniform behavior of remainders. The bounds on the corrections are obtained by carefully weighting the essentially perturbative contributions of small fields against the non-perturbative ones of large fields. The big help in the analysis is provided by good analyticity properties of effective interactions of the BS fields in the field variables for which we can trade their bad analyticity properties in λ_n. Finally cluster expansion techniques [24] are used to exhibit the approximate locality of the BS effective interactions.

The present paper contains the proof of the convergence of the BS effective interactions to zero at the pace predicted by the perturbative argument $\left[\text{i.e. like } \mathcal{O}\left(\frac{1}{n}\right)\right]$. In the limit $n\to\infty$, the distribution of the BS fields becomes free (Gaussian). One can easily extend the present method to give the control of correlation functions and of their massless decay, compare [18]. Existence of logarithmic corrections to scaling [40] should be also confirmed by a straightforward analysis of the higher correlations. This way, we expect to bridge our work with the UV problem, as in [2] the triviality of positive coupling continuum ϕ_4^4

theory was deduced from the presence of the logarithmic corrections to scaling in the critical lattice theory (for all $\lambda > 0$).

However, there seems to exist a more direct UV application of our method. We expect to be able to show the existence of the continuum euclidian nontrivial UV asymptotically free ϕ_4^4 theory for the (physical) coupling in the following region of the complex (cut) plane



(6)

On the lattice, these theories may easily be written in terms of stable models which are perturbatively UV asymptotically free. The control of the continuum limit is very similar to what is done in the present paper: we only have to start very close to the free theory and study the departure of the RG flow from it. All these models seem, however, to lack the physical positivity (i.e. the quantum mechanical interpretation in the Minkowski space). On the negative axis, there exist two natural proposals for the theory. It is also easy to set up a stable physically positive lattice version of the $\lambda < 0$ ϕ_4^4 model. The latter probably does not have the UV asymptotically free continuum limit, however.

In conclusion, our work seems to provide a right approach to the construction of renormalizable asymptotically free models, both in IR and UV.

The non-perturbative control of a simple IR model having been achieved, we may ask questions about the relation of the (renormalized) perturbation expansion to the non-perturbative constructs. This is a more difficult problem. We hope to produce a proof of the Borel summability of the perturbative expansion for the IR ϕ_4^4 theory soon. The renormalon singularities [5, 30, 34, 39] of the Borel transform of the expansion sit in this case on the negative axis and do not obstruct the Borel summability although render its proof more complicated than in the super-renormalizable case [12, 19, 33]. In general, we expect the renormalon singularities due essentially to the slow change of the effective coupling λ_n from scale to scale to be much easier to treat than the instanton ones [31, 9] which encode a detailed information about the large field behavior of the effective interactions, see [20], where we apply Ecalle's theory [11] of resurgent functions to the study of the Borel-transform analytic structure for a simplified model.

These problems are not tackled in the present paper which is organised as follows:

Section 2 sets up the BS RG formalism.

Section 3 contains the perturbative analysis of a single BS transformation and studies its validity.

In Sect. 4, we discuss a general form of the effective interaction of a BS field exhibiting its leading terms, approximate locality and analyticity properties.

Section 5 describes how this form carries over from the BS interaction on scale L^n to the one on scale L^{n+1}. This is done by establishing a cluster expansion for a functional integral over the fluctuations of the L^n BS fields about fixed L^{n+1} BS values.

Section 6 states expected bounds on the contributions of various terms to the effective BS interactions inspired by the discussion of Sect. 3.

In Sects. 7–11, we show how these bounds carry inductively from one length scale to the next one. This is the technical core of the paper. We estimate in turn the local contributions to the new effective interactions (Sect. 7), the non-local small field ones (Sect. 8), the quartic term of the interaction and the new effective coupling constant (Sect. 9), the large field contributions (Sect. 10) and, at the end, the quadratic terms of the interaction and mass and wave function renormalizations (Sect. 11).

The technical work is done in finite periodic volumes but with volume independent estimates. This allows us to pass to the thermodynamic limit what we do in Sect. 12. For the infinite volume theory, we localize the critical values of the parameters (mass squared) of the initial theory. For the critical point of the model, the IR asymptotic freedom follows from our inductive bounds of Sect. 6.

Finally in Appendix 1, we estimate the contributions of the second order Feynman graphs to the change of the running coupling constant λ_n, crucial in obtaining the $\mathcal{O}\left(\frac{1}{n}\right)$ decrease of the latter.

In Appendix 2, we prove a simple fact, related to the Gleason's problem [21] about functions of complex variables, used in Sect. 11 to control the mass and field strength renormalizations.

The paper is essentially self-contained. Some of the earlier results concerning the BS formalism are, however, quoted here without proof. The general idea and much of the technical analysis we do is close to that of [17] (the renormalizable case proves to be in fact not much more difficult and quite similar to the super-renormalizable one, contrary to the gneral expectation). We try to avoid referring directly to [17] being, however, more sketchy in arguments which were worked out to greater detail there.

As we learn, J. Feldman, J. Magnen, V. Rivasseau, and R. Sénéor expect also to control the critical weakly coupled ϕ_4^4 theory using a version of the phase-space cell expansion of [22] strengthened by the RG type of analysis.

2. Block Spin Renormalization Group

Let us consider a scalar field on the *unit lattice* $\phi\equiv(\phi_x)$, $x\in\mathbb{Z}^4$. We shall define the Hamiltonian of the system for ϕ with compact support by

$$\begin{aligned}\mathcal{H}(\phi)&=\tfrac{1}{2}\sum_{\langle xy\rangle}(\phi_x-\phi_y)^2+\tfrac{1}{2}m_0^2\sum_x\phi_x^2-6\lambda_0\sum_x G_{0xx}\phi_x^2+\lambda_0\sum_x\phi_x^4\\&\equiv\tfrac{1}{2}\langle\phi|G_0^{-1}|\phi\rangle+V(\phi)\,,\end{aligned}\tag{1}$$

where $\langle xy\rangle$ runs through unordered pairs of nearest neighbor points (i.e. $|x-y|=1$) and $-G_0^{-1}=\Delta$ is the lattice Laplacian. Note that in (1) the quadratic contribution to the Wick ordering of the quartic term has been separated from the mass term. We would like to control the *Gibbs state* (states) corresponding to (1) with $m_0^2 = m_{\mathrm{crit}}^2(\lambda_0)$ such that the correlation length of the model is infinite (physical mass is zero). Our approach to the problem of the *thermodynamic limit* will be pragmatic. We shall first put our system in a finite *periodic box* $\Lambda = (\mathbb{Z}_{L^N})^4$ for L even, $N \geqq 0$ integers, and then, only after having done the whole analysis in finite volume, we shall pass to the thermodynamical limit. In principle other reasonable boundary conditions could be handled too at a cost of additional technical complications but we expect that they do not produce different states.

It is convenient to identify Λ with $(-\frac{1}{2}L^N, \frac{1}{2}L^N]^4 \subset \mathbb{Z}^4$, with the algebraic operations taken modulo L^N. Let us define the finite volume (periodic boundary condition) Hamiltonian by

$$\begin{aligned}\mathscr{H}^{\wedge}(\phi) &= \tfrac{1}{2} \sum_{\langle xy \rangle \subset \Lambda} (\phi_x - \phi_y)^2 + \tfrac{1}{2}\xi \Lambda^{-1} \Big(\sum_{x \in \Lambda} \phi_x\Big)^2 \\ &\quad + \tfrac{1}{2} m_0^2 \sum_{x \in \Lambda} \phi_x^2 - 6\lambda_0 \sum_{x \in \Lambda} G_{0xx}^{\wedge} \phi_x^2 + \lambda_0 \sum_{x \in \Lambda} \phi_x^4 \\ &\equiv \tfrac{1}{2} \langle \phi | G_0^{\wedge} | \phi \rangle + V^{\wedge}(\phi), \end{aligned} \tag{2}$$

where the periodic boundary condition inverse covariance is

$$(G_0^{\wedge})_{xy}^{-1} = 2d\delta_{xy} - \sum_{\substack{v \\ |v|=1}} \delta_{x+vy} + \xi \Lambda^{-1}, \tag{3}$$

$x, y \in \Lambda$ (Λ denotes also the number of points in Λ). For convenience, we have also regularized the zero mode of the periodic Laplacian by introducing $\xi > 0$ to make $(G_0^{\wedge})^{-1}$ strictly positive (in the thermodynamical limit the ξ dependence is wiped out). Note that for ϕ with compact support, $\mathscr{H}^{\wedge}(\phi) \to \mathscr{H}(\phi)$. In the periodic volume Λ, we define the Gibbs measure as

$$\frac{1}{Z_{\wedge}} \exp[-\mathscr{H}^{\wedge}(\phi)] D^{\wedge}\phi, \tag{4}$$

where

$$D^{\wedge}\phi = \prod_{x \in \Lambda} d\phi_x \tag{5}$$

and $Z_{\wedge}$ is the *partition function* in Λ normalizing (4). In what follows, we shall stay inside finite volume doing nevertheless *volume independent* estimates. This will be crucial for the final passage with Λ to $\mathbb{Z}^4$. To simplify the notation, we shall drop the superscript Λ on the finite volume expressions.

In order to study the contributions of various distance scales to (4), we introduce the *block spin* (BS) *fields* [28, 16] $\phi^n \equiv (\phi_x^n)$, $x \in \Lambda_n \equiv (L^{-n}\Lambda) \cap \mathbb{Z}^4$, $n = 1, \ldots, N$, by

$$\phi_x^n = z_n^{-1/2} L^{-3n} \sum_{\substack{y \\ -1/2L^n < y^\mu \leqq 1/2L^n}} \phi_{L^n x + y} \equiv z_n^{-1/2} (C^n \phi)_x, \tag{6}$$

where by C we have denoted the lattice operator with the matrix elements

$$C_{xy} = L^{-3} \sum_{\substack{v \\ -1/2L < v^\mu \leqq 1/2L}} \delta_{Lx+vy}. \tag{7}$$

The factors z_n giving the (finite) *wave function renormalization* will be determined later. The block spin fields are distributed with Gibbs measures given by *effective Hamiltonians*

$$\begin{aligned}\exp[-\mathscr{H}^n(\phi^n)] &= \exp[f_n\Lambda]\int \exp[-\mathscr{H}(\phi)]\delta(\phi^n - z_n^{-1/2}C^n\phi)D\phi \\ &= \exp[\delta f_{n-1}L^{-4(n-1)}\Lambda]\int \exp[-\mathscr{H}^{n-1}(\phi^{n-1})] \\ &\quad \delta(\phi^n - \zeta_{n-1}^{-1/2}C\phi^{n-1})D\phi^{n-1}. \end{aligned} \tag{8}$$

Here $\mathscr{H}^n$'s are normalized so that $\mathscr{H}^n(0)=0$, $\zeta_{n-1}=z_n/z_{n-1}$, $\zeta_0=z_1$, and f_n stands for the free energy up to the distance scale L^n,

$$f_n = -\frac{1}{\Lambda}\log\int \exp[-\mathscr{H}(\phi)]\delta(z_n^{-1/2}C^n\phi)D\phi. \tag{9}$$

It is a sum of contributions from scales L^k, $0\leqq k<n$,

$$f_n = \sum_{k=0}^{n-1} L^{-4k}\delta f_k, \tag{10}$$

$$\delta f_k = \frac{(-1)}{L^{-4k}\Lambda}\log\int \exp[-\mathscr{H}^k(\phi)]\delta(\zeta_k^{-1/2}C\phi^k)D\phi^k. \tag{11}$$

If the interaction $V=0$ in (2) then (for $z_n\equiv\zeta_n\equiv 1$),

$$\mathscr{H}^n(\phi^n) = \tfrac{1}{2}\langle\phi^n|G_n^{-1}|\phi^n\rangle, \tag{12}$$

where

$$G_n = C^n G_0 C^{+n} \tag{13}$$

($^+$ denotes the transposition). All the covariances G_n are versions of a massless one (see [16]) and in the thermodynamic limit $\Lambda\to\mathbb{Z}^4$ converge with n to

$$G_{\infty xy} = \int_{\square_x} d\mathscr{x}\int_{\square_y} d\mathscr{y}(-\Delta_{\text{cont}})^{-1}(\mathscr{x}-\mathscr{y}), \tag{14}$$

where Δ_{cont} is the continuum Laplacian and $\square_x$, $\square_y$ are unit cubes centered at $x, y\in\mathbb{Z}^d$.

$$\mathscr{H}_{\text{free}}^{\infty}(\phi) = \tfrac{1}{2}\langle\phi|G_\infty^{-1}|\phi\rangle \tag{15}$$

is the limit Hamiltonian, the *massless Gaussian fixed point* of the BS transformation.

Still in the case $V=0$, it is convenient to realize the random field ϕ distributed with the Gaussian probability $d\mu_{G_0}(\phi)$ (G_0 is the covariance of ϕ) as a sum of independent contributions from different scales.

One can write for $n=0,1,\ldots,N$ (see [16, 17] for the details)

$$G_{0xy} = L^{-2n}\mathscr{G}_{nL^{-n}xL^{-n}y} + \sum_{k=0}^{n-1} L^{-2k}\mathscr{T}_{kL^{-k}xL^{-k}y}, \tag{16}$$

where $\mathscr{G}_n\equiv(\mathscr{G}_{n\mathscr{x}\mathscr{y}})$, $\mathscr{x},\mathscr{y}\in L^{-n}\Lambda$, and $\mathscr{T}_k\equiv(\mathscr{T}_{k\mathscr{x}\mathscr{y}})$, $\mathscr{x},\mathscr{y}\in L^{-k}\Lambda$, are positive but not strictly positive operators (kernels). $\mathscr{G}_n$ has massless decay whereas $\mathscr{T}_k$ have massive ones (uniform in k). The decomposition (16) corresponds to the one used in the heuristic momentum space RG, see [41], where one splits the momentum region $|p|<K$ into layers and scales:

$$\int_{|p|<K} e^{ip(x-y)}p^{-2}d^4p = L^{-2n}\int_{|p|<K} e^{iL^{-n}p(x-y)}p^{-2}d^4p + \sum_{k=0}^{n-1} L^{-2k}\int_{L^{-1}K<|p|<K} e^{ipL^{-k}(x-y)}p^{-2}d^4p\,. \tag{17}$$

The degeneration of $\mathscr{G}_n$ and $\mathscr{T}_k$ is easily visible from the representations (see [16])

$$G_{n\mathscr{x}\mathscr{y}} = \sum_{x,y\in\Lambda_n} \mathscr{A}_{n\mathscr{x}x}G_{nxy}\mathscr{A}_{n\mathscr{y}y} \equiv (\mathscr{A}_n G_n \mathscr{A}_n^+)_{\mathscr{x}\mathscr{y}}\,, \tag{18}$$

$$\mathscr{T}_{k\mathscr{x}\mathscr{y}} = \sum_{\substack{x,y\in\Lambda_k\\ u,v\in\bar{\Lambda}_k}} \mathscr{A}_{k\mathscr{x}x}Q_{xu}\Gamma_{kuv}Q_{yv}\mathscr{A}_{k\mathscr{y}y} \equiv (\mathscr{A}_k Q\Gamma_k Q^+\mathscr{A}_k^+)_{\mathscr{x}\mathscr{y}}\,, \tag{19}$$

where $\bar{\Lambda}_k = \Lambda_k \backslash L\mathbb{Z}^4$ and

$$Q_{xu} = \begin{cases} \delta_{xu} & \text{if} \quad x\notin L\mathbb{Z}^4\,, \\ -1 & \text{if} \quad x\in L\mathbb{Z}^4 \quad \text{and} \quad -\tfrac{1}{2}L < x^\mu - u^\mu \leqq \tfrac{1}{2}L\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{20}$$

(Notice that $CQ=0$.) Kernels $\mathscr{A}_n$ are independent, translationally invariant by unit lattice vectors and have uniform exponential decay. G_n and Γ_k are strictly positive operators. Γ_k do not depend on ξ, are translationally invariant by vectors of $L\mathbb{Z}^4$ and also have uniform exponential decay. Moreover, there are two useful relations, see [16],

$$\int_{\square_x} d\mathscr{x}\,\mathscr{A}_{n\mathscr{x}y} = \delta_{xy}\,, \qquad x,y\in\Lambda_n\,, \tag{21}$$

where the integral stands for the Riemann sum on $L^{-n}\mathbb{Z}^4$ (here over the unit cube $\square_x$ centered at x) and

$$\sum_{x\in\Lambda_n} \mathscr{A}_{n\mathscr{x}x} = 1\,. \tag{22}$$

The decomposition (16) can be used to realize ϕ distributed with $d\mu_{G_0}$ as a sum of independent contributions

$$\phi_x = L^{-n}\psi^n_{L^{-n}x} + \sum_{k=0}^{n-1} L^{-k}\mathscr{Z}^k_{L^{-k}x}\,, \tag{23}$$

where $\psi^n \equiv (\psi^n_{\mathscr{x}})$, $\mathscr{x}\in L^{-n}\Lambda$, $\mathscr{Z}^k \equiv (\mathscr{Z}^k_{\mathscr{x}})$, $\mathscr{x}\in L^{-k}\Lambda$, are centered Gaussian fields distributed with covariances $\mathscr{G}_n$ and $\mathscr{T}_k$ respectively.

By (18) and (19) we can write

$$\psi^n_{\mathscr{x}} = \sum_{x\in\Lambda_n} \mathscr{A}_{n\mathscr{x}x}\phi^n_x \equiv (\mathscr{A}_n\phi^n)_{\mathscr{x}}\,, \tag{24}$$

and

$$\mathscr{Z}^k_{\mathscr{x}} = \sum_{\substack{x\in\Lambda_k\\ u,v\in\bar{\Lambda}_k}} \mathscr{A}_{k\mathscr{x}x}Q_{xu}\Gamma^{1/2}_{kuv}Z_v \equiv (\mathscr{A}_k Q\Gamma_k^{1/2}Z^k)_{\mathscr{x}} \equiv (M_k Z^k)_{\mathscr{x}}\,, \tag{25}$$

where ϕ^n is a Λ_n-lattice field distributed with covariance G_n and Z^k is a $\bar{\Lambda}_k$- one with covariance 1. In fact substituting (24) and (25) in to (23) and using (20) and (21) one can easily show that in the realization (23), ϕ^n as given by (24) coincides with the BS field (6) (for $z_n \equiv 1$) so that our notation is not abusive. As a byproduct, we obtain a relation inverse to (24):

$$\phi_x^n = \int_{\square_x} dx \psi_x^n . \tag{26}$$

From (23) it follows that

$$\psi_x^n = L^{-1} \psi_{L^{-1}x}^{n+1} + \mathscr{Z}_x^n , \tag{27}$$

and, since ψ^{n+1} and $\mathscr{Z}^n$ are independent as well as ϕ^{n+1} and Z^n, that the Gaussian measures split:

$$d\mu_{G_n}(\phi^n) = d\mu_{G_{n+1}}(\phi^{n+1}) \otimes d\mu_1(Z^n) . \tag{28}$$

Now let us pass to the perturbed case when $V \neq 0$. We shall rescale the fields by substituting $\phi^n \to z_n^{1/2}\phi^n$, $Z^n \to z_n^{1/2} Z^n$, $\psi^n \to z_n^{1/2}\psi^n$, $\mathscr{Z}^n \to z_n^{1/2}\mathscr{Z}^n$. Hence (27) and (28) become

$$\psi_x^n = L^{-1} \zeta_n^{1/2} \psi_{L^{-1}x}^{n+1} + \mathscr{Z}_x^n \tag{29}$$

and

$$d\mu_{G_n}(\phi^n) = d\mu_{\zeta_n^{-1} G_{n+1}}(\phi^{n+1}) \otimes d\mu_1(Z^n) . \tag{30}$$

For the first effective Hamiltonian, see (8), we obtain using (29) and (30):

$$\begin{aligned}
&\exp[-\mathscr{H}^1(\phi^1)] \\
&= \exp[\delta f_0 \Lambda] \int \exp[-\tfrac{1}{2}\langle \phi | G_0^{-1} | \phi \rangle - V(\phi)] \delta(\phi^1 - \zeta_0^{-1/2} C\phi) D\phi \\
&= \exp[\delta f_0 \Lambda + \tfrac{1}{2} \mathrm{Tr} \log(2\pi G_0)] \int \exp[-V(\phi)] \delta(\phi^1 - \zeta_0^{-1/2} C\phi) d\mu_{G_0}(\phi) \\
&= \exp[\delta f_0 \Lambda + \tfrac{1}{2} \mathrm{Tr} \log(2\pi G_0)] \\
&\quad \cdot \int \exp[-V(L^{-1}\zeta_0^{1/2} \tilde{\psi}_{L^{-1}\cdot}^1 + Z^0)] \delta(\phi^1 - \tilde{\phi}^1) d\mu_{\zeta_0^{-1} G_1}(\tilde{\phi}^1) d\mu_1(Z^0) \\
&= \exp[\delta f_0 \Lambda + \tfrac{1}{2} \mathrm{Tr} \log(2\pi G_0) - \tfrac{1}{2} \mathrm{Tr} \log(2\pi \zeta_0^{-1} G_1) - \tfrac{1}{2}\zeta_0 \langle \phi^1 | G_1^{-1} | \phi^1 \rangle] \\
&\quad \cdot \int \exp[-V(L^{-1}\zeta_0^{1/2} \psi_{L^{-1}\cdot}^1 + Z^0)] d\mu_1(Z^0) .
\end{aligned} \tag{31}$$

This way the first BS transformation has been expressed by the integral over the fluctuations Z^0.

From (31) we read off

$$\begin{aligned}
\exp[-\mathscr{H}^1(\phi^1)] = &\exp[-\tfrac{1}{2}\zeta_0 \langle \phi^1 | G_1^{-1} | \phi^1 \rangle] \\
&\cdot \int \exp[-V(L^{-1}\zeta_0^{1/2} \psi_{L^{-1}\cdot}^1 + \mathscr{Z}^0)] d\mu_1(Z^0) / (\phi^1 = 0)
\end{aligned} \tag{32}$$

and

$$\delta f_0 = -\frac{1}{2\Lambda} \mathrm{Tr} \log(2\pi G_0) + \frac{1}{2\Lambda} \mathrm{Tr} \log(2\pi \zeta_0^{-1} G_1) - \frac{1}{\Lambda} \log \int \exp[-V(\mathscr{Z}^0)] d\mu_1(Z^0) . \tag{33}$$

Let us define the effective interaction V^1 by

$$\exp[-V^1(\psi^1)]=\exp\left[\tfrac{1}{2}(1-\zeta_0)\sum_\mu\int dx(\partial_\mu\psi^1_x)^2\right]$$
$$\cdot\int\exp[-V(L^{-1}\zeta_0^{1/2}\psi^1_{L^{-1}\cdot}+\mathscr{Z}^0)]\,d\mu_1(Z^0)/(\psi^1=0)\,. \quad (34)$$

ζ_0 is chosen so that the subtraction in (34) removes the marginal $\int(\partial\psi^1)^2$ term from V^1, see Sect. 10. Since, as was shown in [16] and [17],

$$\sum_\mu\int dx(\partial_\mu\psi^1_x)^2=\langle\phi^1|G_1^{-1}|\phi^1\rangle-L^6\xi\Lambda^{-1}\left(\sum_x\phi^1_x\right)^2, \quad (35)$$

we obtain from (32) and (34):

$$\exp[-\mathscr{H}^1(\phi^1)]=\exp[-\tfrac{1}{2}\langle\phi^1|\bar{G}_1^{-1}|\phi^1\rangle-V^1(\psi^1)]\,, \quad (36)$$

where

$$\bar{G}_1^{-1}=E_1^\perp G_1^{-1}+L^2z_1\xi E_1\,, \quad (37)$$

with E_n being the orthogonal projection on functions constant on Λ_n. $\bar{G}_1$ differs from G_1 only by the value of the infrared regulator ξ.

Upon iteration of (31)–(37), we obtain

$$\exp[-\mathscr{H}^{n+1}(\phi^{n+1})]=\exp[-\tfrac{1}{2}\langle\phi^{n+1}|\bar{G}_{n+1}^{-1}|\phi^{n+1}\rangle-V^{n+1}(\psi^{n+1})]\,, \quad (38)$$

where

$$\exp[-V^{n+1}(\psi^{n+1})]=\exp\left[\tfrac{1}{2}(1-\zeta_n)\sum_\mu\int dx(\partial_\mu\psi^{n+1}_x)^2\right]$$
$$\cdot\int\exp[-V^n(L^{-1}\zeta_n^{1/2}\psi^{n+1}_{L^{-1}\cdot}+\mathscr{Z}^n)]\,d\mu_1(Z^n)/(\psi^{n+1}=0) \quad (39)$$

$$\bar{G}_{n+1}^{-1}=E_n^\perp G_{n+1}^{-1}+L^{2(n+1)}z_{n+1}\xi E_{n+1}\,. \quad (40)$$

Moreover

$$\delta f_n=-\frac{1}{2\Lambda_n}\operatorname{Tr}\log(2\pi\bar{G}_n)+\frac{1}{2\Lambda_n}\operatorname{Tr}\log(2\pi\zeta_n^{-1}C\bar{G}_nC^+)$$
$$-\frac{1}{\Lambda_n}\log\int\exp[-V^n(\mathscr{Z}^n)]\,d\mu_1(Z^n)\,. \quad (41)$$

This realizes the n^{th} BS transformation as an integral over the fluctuation field Z^n. Our aim will be to show that the effective interactions V^n in the thermodynamical limit converge with n to zero in a suitable sense.

Let us stress that the expression of the interactions V^n as functionals of ψ^n rather than ϕ^n is not accidental. Due to the locality of the relation (29), it is much more convenient to follow the dependence of V^n on ψ^n. In the next section we shall also see that the ψ^n dependence arises naturally in the perturbative approach. Before closing this section, let us list some of the (uniform in the volume) decay properties of the kernels $\mathscr{A}_n$ and Γ_n following directly by a momentum space analysis of their explicit form,

$$|\mathscr{A}_{nxx}|\leqq\mathcal{O}(1)e^{-\beta|x-x|}, \quad (42)$$

$$|\mathscr{A}_{n x x}-\mathscr{A}_{n y x}|/|x-y| \leqq \mathcal{O}(1)\left(e^{-\beta|x-x|}+e^{-\beta|y-x|}\right), \tag{43}$$

$$|(\partial_\mu \mathscr{A}_n \nabla_\mu^{-1})_{x x}| \leqq \mathcal{O}(1) e^{-\beta|x-x|} \tag{44}$$

(∂ denotes the gradient on $L^{-n}\Lambda$, ∇ the one on Λ_n),

$$|\partial_\mu \mathscr{A}_{n x x}-\partial_\mu \mathscr{A}_{n y x}|/|x-y|^{2/3} \leqq \mathcal{O}(1)\left(e^{-\beta|x-x|}+e^{-\beta|y-x|}\right), \tag{45}$$

$$|\Gamma_{kuv}^{1/2}| \leqq \mathcal{O}(1) e^{-\beta|u-v|} \tag{46}$$

for an L-dependent constant β. [In fact β can be chosen L independent in (42) to (45) whereas in (46) it is $\mathcal{O}(L^{-1})$.] Expressions (42), (44), and (46) were already stated and proven in Appendix to [16] and Appendix 1 to [17]. Expressions (43) and (45), which we find more convenient to use here, are proven the same way (separately for $|x-y| \leqq 1$ and $|x-y|>1$).

3. Perturbative Approach to the Renormalization Group and Its Significance

Here we shall study the effective interactions in the leading orders of the perturbation expansion. It has to be stressed that we *shall not* expand in the powers of the initial coupling constant λ_0 of the quartic term since such expansion is *not self-consistent* already in second order. Instead we shall perform a perturbative analysis (to the second order) of each BS transformation in terms of the *effective quartic coupling constant* λ_n of the corresponding scale.

Let us start with an ansatz

$$\begin{aligned} V^n(\psi^n) = & \tfrac{1}{2} m_n^2 \int dx (\psi_x^n)^2 - 6\lambda_n \int dx \mathscr{G}_{nxx} (\psi_x^n)^2 + \lambda_n \int dx (\psi_x^n)^4 \\ & - 8\lambda_n^2 \int dx dy \mathscr{Q}_{nxy} (\psi_x^n)^3 (\psi_y^n)^3 , \end{aligned} \tag{1}$$

where $m_n^2 \in [-\lambda_n^{3/2}, \lambda_n^{3/2}] \equiv I_n$, say, and

$$\mathscr{Q}_{nxy} = \sum_{k=0}^{n-1} L^{2(n-k)} \mathscr{T}_{kL^{n-k}xL^{n-k}y} = L^{2n} G_{0L^n x L^n y} - \mathscr{G}_{nxy} . \tag{2}$$

The second quadratic term in (1) contributes to the Wick ordering of the quartic one. The sixth order term corresponds to the diagram with the middle line being the sum of the propagators of the fluctuation fields $\mathscr{Z}^k$ with $k<n$. Although irrelevant (of negative dimension), this term has to be carefully taken into consideration because it gives an $\mathcal{O}(\lambda_n^2)$ feedback to the quartic term in the new interaction V^{n+1}, which has to be weighted against other $\mathcal{O}(\lambda_n^2)$ contributions. Thus for V^n given by (1), let us compute V^{n+1} to order λ_n^2 in the perturbation calculus. We shall show that, up to a mass term and irrelevant terms with negative dimensions which do not contribute to the $\int \psi^4$ term in the next step, (1) reproduces itself with $\lambda_n \to \lambda_{n+1} - \alpha_n \lambda_n^2$, where $\alpha_n = \mathcal{O}(\log L) > 0$. Hence the second order computation exhibits the dynamics of the quartic coupling which does not change under the linear approximation to the RG transformation: as opposed to the case of higher or lower dimension, in dimension 4, $\int \psi^4$ is a marginal (dimensionless) perturbation. The L dependence of α_n shows why the perturbation expansion in λ_0

is not self-consistent. Computing λ_n to the second order in λ_0 is equivalent to computing λ_1 to $\mathcal{O}(\lambda_0^2)$ for $L\to L^n$. Now, even for very small λ_0, this would give a large and negative result for n sufficiently large due to the logarithmic divergence of α_1 with $L\to\infty$.

To simplify the notation, let us drop the index n and let us replace $n+1$ by prime. (2.39) gives

$$\begin{aligned} V'(\psi') = & -\tfrac{1}{2}(1-\zeta)\sum_\mu \int dx(\partial_\mu \psi'_x)^2 + \tfrac{1}{2}L^2\zeta m^2 \int dx(\psi'_x)^2 - 6L^2\zeta\lambda \int dx \mathcal{G}_{LxLx}(\psi'_x)^2 \\ & + 6L^2\zeta\lambda \int dx \mathcal{T}_{LxLx}(\psi'_x)^2 + \mathcal{O}(\lambda^2) \text{ terms quadratic in } \psi' \\ & + \zeta^2\lambda \int dx (\psi'_x)^4 - 72L^4\zeta^2\lambda^2 \int dx dy \mathcal{Q}_{LxLy}\mathcal{T}_{LxLy}(\psi'_x)^2(\psi'_y)^2 \\ & - 48L^4\zeta^2\lambda^2 \int dx dy \mathcal{Q}_{LxLy}\mathcal{T}_{LyLy}(\psi'_x)^3\psi'_y \\ & - 36L^4\zeta^2\lambda^2 \int dx dy (\mathcal{T}_{LxLy})^2(\psi'_x)^2(\psi'_y)^2 \\ & - 48L^4\zeta^2\lambda^2 \int dx dy \mathcal{T}_{LxLy}\mathcal{T}_{LyLy}(\psi'_x)^3\psi'_y \\ & + 48L^4\zeta^2\lambda^2 \int dx dy \mathcal{G}_{LyLy}\mathcal{T}_{LxLy}(\psi'_x)^3\psi'_y \\ & - 8L^2\zeta^3\lambda^2 \int dx dy \mathcal{Q}_{LxLy}(\psi'_x)^3(\psi'_y)^3 \\ & - 8L^2\zeta^3\lambda^2 \int dx dy \mathcal{T}_{LxLy}(\psi'_x)^3(\psi'_y)^3 + \mathcal{O}(\lambda^{5/2}) . \end{aligned} \tag{3}$$

The third and the fourth term on the right-hand side of (3) make up

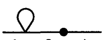
$$-6\zeta\lambda \int dx \mathcal{G}'_{xx}(\psi'_x)^2 , \tag{4}$$

since

$$\mathcal{G}'_{xy} = L^2(\mathcal{G}_{LxLy} - \mathcal{T}_{LxLy}) , \tag{5}$$

see (2.16). The rest of the quadratic terms may be written as a sum of a relevant mass term of dimension two $\frac{1}{2}(L^2\zeta m^2 + \mathcal{O}(\lambda^2)) \int (\psi')^2$, a marginal (dimensionless) kinetic term $\frac{1}{2}(\zeta - 1 + \mathcal{O}(\lambda^2)) \int (\partial\psi')^2$ and irrelevant (negative dimension) quadratic contributions. This step, as well as its analogue for the quartic terms discussed below, are in fact somewhat more tricky than usual since the kernels of the expressions on the right-hand side of (3) have only $\mathbb{Z}^d$ translation invariance while living on $L^{-n}\Lambda$. The precise way to circumvent this difficulty in extracting the "zero momentum" or "p^2" contributions to the diagrams will be discussed later. The wave function renormalization ζ of ψ' will be taken as to eliminate the $\int (\partial\psi')^2$ term from V'. Hence $1-\zeta = \mathcal{O}(\lambda^2)$ and can be dropped from the other terms of (3) in our approximation. The non-local terms of the fourth order in (3) can be written as the local marginal expression $\delta\lambda^1 \int (\psi')^4$, $\delta\lambda^1 = \mathcal{O}(\lambda^2)$, plus irrelevant contributions of negative dimensions,

$$\begin{aligned} \delta\lambda^1 = & -72L^4\lambda^2 \int_{\square_0} dx \int dy \mathcal{Q}_{LxLy}\mathcal{T}_{LxLy} - 48L^4\lambda^2 \int_{\square_0} dx \int dy \mathcal{Q}_{LxLy}\mathcal{T}_{LyLy} \\ & - 36L^4\lambda^2 \int_{\square_0} dx \int dy (\mathcal{T}_{LxLy})^2 - 48L^4\lambda^2 \int_{\square_0} dx \int dy \mathcal{T}_{LxLy}\mathcal{T}_{LyLy} \\ & + 48L^4\lambda^2 \int_{\square_0} dx dy \mathcal{T}_{LxLx}\mathcal{G}_{LyLy} . \end{aligned} \tag{6}$$

In Appendix 1, we show that the main contribution to the right-hand side for L large comes from the third term corresponding to the bubble with two $\mathscr{T}$ lines. The bubble diverges like $\mathcal{O}(\log L)$ for $L\to\infty$. This term dominates for $L \geqq \bar{L}$ the other ones corresponding to the bubble  with one $\mathscr{T}$ and one "harder" line or to  with $\mathscr{T}$ or a "harder" propagator on the open line (the latter terms vanish in fact). Hence

$$4C_+^{-1}\lambda^2 \leqq -\delta\lambda^1 \leqq \tfrac{1}{4}C_-^{-1}\lambda^2 \tag{7}$$

for some n-independent (but L dependent) constants C_-, $C_+>0$. Altogether the fourth order terms on the right-hand side of (3) may be written as

$$(\lambda+\delta\lambda^1)\int dx(\psi'_x)^4 + \text{irrelevant terms of order } \lambda^2 + \mathcal{O}(\lambda^{5/2}), \tag{8}$$

with $\delta\lambda^1$ satisfying (7).

Finally, since by (2)

$$\mathscr{Q}'_{xy} = L^2\mathscr{Q}_{LxLy} + L^2\mathscr{T}_{LxLy}, \tag{9}$$

the sixth order term of (3) is

$$-8\lambda^2\int dx dy \mathscr{Q}'_{xy}(\psi'_x)^3(\psi'_y)^3 . \tag{10}$$

Summarizing,

$$\begin{aligned} V'(\psi') = {} & \tfrac{1}{2}(L^2m^2 + \mathcal{O}((\lambda')^2))\int dx(\psi'_x)^2 - 6\lambda'\int dx\mathscr{G}'_{xx}(\psi'_x)^2 \\ & + \text{irrelevant quadratic terms of order } (\lambda')^2 \\ & + \lambda'\int dx(\psi'_x)^4 + \text{irrelevant quartic terms of order } (\lambda')^2 \\ & - 8(\lambda')^2\int dx dy \mathscr{Q}'_{xy}(\psi'_x)^3(\psi'_y)^3 + \mathcal{O}((\lambda')^{5/2}), \end{aligned} \tag{11}$$

where

$$\lambda' = \lambda + \delta\lambda^1 + \mathcal{O}(\lambda^{5/2}). \tag{12}$$

It is clear that we could have added to V irrelevant quadratic and quartic terms of order, say, $\mathcal{O}(\lambda^{3/2})$ and $\mathcal{O}(\lambda^{7/4})$ respectively and six or higher order $\mathcal{O}(\lambda^{13/6})$ terms without essentially changing the result of our analysis. From the form of the new mass term in (11), it is obvious that we can find a closed subinterval $J \subset I$ such that for m^2 going through it while other entries of V^n change continuously, $m'^2 = L^2m^2 + \mathcal{O}((\lambda')^2)$ sweeps $I' \equiv [-(\lambda')^{3/2}, (\lambda')^{3/2}]$. This will allow us to choose in an iterative procedure a sequence $I_0 \supset J'_0 \supset J''_0 \supset \ldots \supset J_0^{(n)} \supset \ldots$ of closed intervals such that for m_0^2 running through $J_0^{(n)}$, m_n^2 sweeps I_n. Following [6], we locate the critical mass of the infinite volume theory, $m^2_{\text{crit}}(\lambda_0)$, as the point of $\cap J_0^n$. For $m_0^2 = m^2_{\text{crit}}(\lambda_0)$ we can iterate our perturbative analysis. From (7),

$$\frac{1}{\lambda} + 3C_+^{-1} \leqq \frac{1}{\lambda+\delta\lambda^1} \leqq \frac{1}{\lambda} + \frac{1}{3}C_-^{-1} \tag{13}$$

for small λ, so that the assumption

$$\frac{C_-}{n_0+n} \leqq \lambda \leqq \frac{C_+}{n_0+n} \tag{14}$$

carries through: the effective fourth order couplings λ_n should decrease like $\mathcal{O}\left(\frac{1}{n}\right)$.

This shows the perturbative asymptotic freedom in IR of the critical ϕ_4^4 theory.

Summarizing: Our perturbative analysis carried over to the second order of the effective coupling constant at each distance scale (to be distinguished from the second order analysis in λ_0) tells us that the effective interactions vanish in the limit of long distances. The purpose of this paper is to prove this rigorously by providing appropriate estimates for the perturbative and non-perturbative corrections to the above arguments.

The problem with the perturbation expansion is that it is based on expanding the right-hand side of (2.39) (or of its logarithm) into powers of V which, even if given by (1) with small λ, becomes arbitrarily large in certain regions of the functional space of the fields $\psi = L^{-1}\zeta^{1/2}\psi'_{L^{-1}} + \mathscr{X}$, namely, where $|\psi| > \mathcal{O}(\lambda^{-1/4})$. Let us consider first small BS fields ψ', $|\psi'| < \mathcal{O}(\lambda^{-1/4})$. Still V becomes large if $|\mathscr{X}|$ is large and the perturbation expansion for V' in terms of powers of λ would diverge [we may expect the behavior of the type of $\mathcal{O}(n!)$ for large order n, typical for ϕ^4 integrals]. If we limit the functional integral in (2.39) to fields Z [and hence, by (2.25), (2.42), and (2.46) also fields $\mathscr{X}$] with absolute value smaller than $\mathcal{O}(\lambda^{-1/4})$, we may easily bound small corrections to the first perturbative orders using more or less standard cluster expansion arguments to control the volume dependence of the expressions. The corrections coming from the large values of Z will be shown to always carry non-perturbative small factors $\mathcal{O}(e^{-\varepsilon\lambda^{-1/2}})$ due to the small $d\mu_1$-probability of $|Z_x| \geqq \mathcal{O}(\lambda^{-1/4})$. However, we have to know that as well as the contributions of large fluctuations Z to $V'(\psi')$, also the ones to the first Taylor expansion coefficients of V' around $\psi' = 0$ (e.g. to λ') are small. We shall guarantee this by passing to fields ψ and ψ' with small imaginary parts, $|\mathrm{Im}\,\psi|$, $|\mathrm{Im}\,\psi'| < \mathcal{O}(\lambda^{-1/4})$ (i.e. in strips) and proving together with the bounds for small and large Z contributions to $V'(\psi')$ their analyticity in ψ'. Then the bounds for the contributions to the derivatives of V' at $\psi' = 0$ will follow by Cauchy estimates. As far as the properties of $V(\psi)$ for large ψ are concerned, we shall assume inductively stability bounds for the large ψ contribution to the Boltzmann factor $e^{-V(\psi)}$. They will guarantee the strip-analyticity of $e^{-V'(\psi')}$ and will assure that the small $d\mu_1$-probability of large fluctuations Z is not affected for small ψ' by the interaction.

Of course small or large field values may occur simultaneously in different space-time regions and our analysis has to take into account the space-time relations ignored in the above discussion. Here we shall follow the spirit of [4]. The crucial fact is that, although the recursion (2.39) does not preserve the locality of V, it may be expected to preserve its approximate locality because the only new non-locality present in (2.39) is due to the approximately local relation (2.25) between $\mathscr{X}$ and Z. Hence we expect coupling of different space-time regions in (2.39) to be exponentially decaying [with the correlation length being $\mathcal{O}(L)$] and shall use cluster (high temperature) expansion techniques to exhibit the decay. The expansion will allow us to perform the small field-large field analysis in an essentially local way. This will be the topic of the next section. Let us stress the contrast between (2.39) and the complete functional integral of a critical theory with infinite correlation length. The success of the RG lies just in the reduction of the latter problem to a sequence of high-temperature ones with bounded correlation lengths.

4. General Form of Effective Interactions

In this section the inductive assumptions concerning the analyticity properties of V^n are stated. Let us start by describing the sets of small fields ψ^n. As introduced, $\psi^n = \mathscr{A}_n \phi^n$, ϕ^n being the independent variables. Since the above relation is non-local and it will be essential to trace the local dependence (and analyticity) of V^n on ψ^n rather than on ϕ^n, we shall enlarge the space of small fields to a certain space of complex ψ^n's not necessarily of the form $\mathscr{A}_n \phi^n$.

Let us pave $L^{-n}\Lambda$ with the lattice of blocks Δ of size L^{N_0} centered at points of $L^{N_0}\mathbb{Z}^4$. L^{N_0} will later appear as the scale of the cluster expansion. A subset of $L^{-n}\Lambda$ being a union of blocks Δ will be called paved. For a paved set $X \subset L^{-n}\Lambda$, let

$$\begin{aligned}
\mathscr{K}(X) = \{&\text{complex } \psi^n \equiv (\psi^n_x),\ x \in L^{-n}\Lambda \cap X: \\
&|\psi_x| < C_1(n_0+n)^{1/4} \quad \text{for} \quad x \in X, \\
&|\psi^n_x - \psi^n_y|/|x-y| < C_0 C_1 (n_0+n)^{1/4} \quad \text{for} \quad x \neq y,\ x, y \in X, \\
&|\partial_\mu \psi^n_x - \partial_\mu \psi^n_y|/|x-y|^{2/3} < C_0 C_1 (n_0+n)^{1/4} \quad \text{for} \quad x \neq y, \\
&x, y, x+L^{-n}e_\mu, y+L^{-n}e_\mu \in X\}\,.
\end{aligned} \tag{1}$$

This will be the set of small fields on X. As we see, we have bounded not only the values of ψ^n but also those of its derivatives and of the exponent 2/3 Hölder derivatives of the derivatives of ψ^n. Let us notice that for $\psi^n = \mathscr{A}_n \phi^n$, the first bound of (1) with $X = L^{-n}\Lambda$ would imply the next ones [for $C_0 \geqq \bar{C}_0(L)$, as we assume] due to (2.26), (2.43), and (2.45). For more general small fields, we postulate these bounds in order to guarantee the smallness of irrelevant terms of V_n in which derivatives of ψ^n will appear, see below.

We shall inductively assume $V^n(\psi^n)$ to be an *even analytic functional on* $\mathscr{K}(L^{-n}\Lambda)$ *vanishing at zero*. We shall also assume V^n to possess *all* the euclidean symmetries of the *unit* lattice.

Let us denote by $V^n_k(\psi')$ the k^{th} order of the Taylor series for V^n at zero and by $V^n_{\geqq k}(\psi')$ the remainder of the expansion up to order $k-1$. We shall take the quadratic term of V^n of the form:

$$V^n_2(\psi^n) = \tfrac{1}{2} m_n^2 \int dx (\psi^n_x)^2 - 6\lambda_n \int dx\, G_{nxx} (\psi^n_x)^2 + \sum_{\mu,\nu} \int dx dy\, K^{\mu\nu}_{nxy} (\partial_\mu \psi^n_x - \partial_\mu \psi^n_y) \partial_\nu \psi^n_y\,. \tag{2}$$

This specifies the form of the second order irrelevant terms. Notice the absence of the $\int (\partial\psi)^2$ term entirely absorbed into the Gaussian measure by wave function renormalization, as discussed in the previous sections.

For the quartic term in V^n, we put

$$V^n_4(\psi^n) = \lambda_n \int dx (\psi^n_x)^4 + \sum_Y \tilde{V}^n_{4Y}(\psi^n)\,, \tag{3}$$

where $\tilde{V}^n_{4Y}(\psi^n)$ is the restriction to the diagonal of a quartic nonsymmetric form $\tilde{V}^n_{4Y}(\psi^n_1, \psi^n_2, \psi^n_3, \psi^n_4)$ depending on the fields ψ^n_i defined on the paved set Y. ψ^n_4 enters $\tilde{V}^n_{4Y}(\psi^n_1, \ldots, \psi^n_4)$ only through its differences at pairs of points. This guarantees the irrelevant character of $\sum_Y \tilde{V}^n_{4Y}(\psi^n)$, to be contrasted with the marginality of $\lambda_n \int (\psi^n)^4$.

We could have written the irrelevant contribution to V^n_4 in a more transparent way as $\int ds\,dt\,du\,dx\,dy\, N_{n_{stuxy}} \psi^n_s \psi^n_t \psi^n_u (\psi^n_x - \psi^n_y)$. The other form is preferable since we would not be able to extract satisfactory bounds for the kernels N_n. The twidle in $\tilde{V}_{4Y}$ signals that these expressions do not contain the whole quartic contribution to V^n.

Finally, for the sixth and higher order contributions to V^n, we assume

$$V^n_{\geqq 6}(\psi^n) = -8\lambda_n^2 \int dx dy \mathcal{Q}_{nxy}(\psi^n_x)^3(\psi^n_y)^3 + \sum_Y \tilde{V}^n_{\geqq 6y}(\psi^n), \tag{4}$$

see (3.1) and (3.2). Even functionals $\tilde{V}^n_{\geqq 6Y}(\psi^n)$ are assumed to depend only on ψ^n restricted to the paved set Y, to be analytic on $3\mathcal{K}(Y)$ and to have the Taylor series starting with sixth order terms.

In the next sections, we shall formulate inductive bounds for the building blocks of the interaction V^n introduced here. For the time being it is enough if we keep in mind that m_n^2, K_n, λ_n, λ_n^2, $\mathcal{Q}_n$, $\tilde{V}^n_{4Y}$, and $\tilde{V}^n_{\geqq 6Y}$ are small and exponentially decaying with the separation of the points of the kernels or of the points of the sets Y, the second property expressing the approximate locality of V^n.

Now let us discuss the large field contributions to the Boltzmann factor e^{-V^n}. First, notice that we may write

$$V^n_{\geqq 4}(\psi^n) = \lambda \int dx (\psi^n_x)^4 + \sum_Y \tilde{V}^n_{\geqq 4Y}(\psi^n) \equiv \lambda \int dx (\psi^n_x)^4 + \tilde{V}^n_{\geqq 4}(\psi^n), \tag{5}$$

where

$$\tilde{V}^n_{\geqq 4Y}(\psi^n) = \tilde{V}^n_{4Y}(\psi^n) - 8\lambda_n^2 \sum_{\substack{(\Delta_1,\Delta_2)\\ \Delta_1\cup\Delta_2 = Y'}} \int_{\Delta_1} dx \int_{\Delta_2} dy \mathcal{Q}_{nxy}(\psi^n_x)^3(\psi^n_y)^3 + \tilde{V}^n_{\geqq 6Y}(\psi^n). \tag{6}$$

This implies for the Boltzmann factor

$$\begin{aligned}\exp[-V^n_{\geqq 4}] &= \prod_Y (\exp[-\tilde{V}^n_{\geqq 4Y}] - 1 + 1)\exp[-\lambda_n \textstyle\int (\psi^n)^4] \\ &= \sum_{\{Y_\alpha\}} \prod_\alpha (\exp[-\tilde{V}^n_{\geqq 4Y_\alpha}] - 1)\exp[-\lambda_n \textstyle\int (\psi^n)^4].\end{aligned} \tag{7}$$

Let us introduce the following convenient terminology. A paved set X will be called connected with respect to a collection S of paved sets if each set of S lies either in X or outside X and if X cannot be divided into two proper paved subsets without dividing some of the sets of S. X will be called connected if it is connected with respect to pairs of nearest neighbor blocks Δ_1, Δ_2 in X.

Now let D be a paved set in $L^{-n}\Lambda$. Let S be composed of connected components (c.c.) of D and of sets of $\{Y_\alpha\}$. Let $\{X_i\}$ be the collection of the c.c. with respect to S of $D \cup \left(\bigcup_\alpha Y_\alpha\right)$ intersecting D. Let us fix $\{X_i\}$. It is clear that the sum over $\{Y_\alpha\}$ factorizes now to the sums inside each X_i and the outside sum. Thus we obtain from (7)

$$\exp[-V^n_{\geqq 4}(\psi^n)] = \sum_{\{X_i\}} \prod_i g^{nD}_{X_i}(\psi^n) \exp\left[-\lambda \int_{\sim D} dx (\psi^n_x)^4 - \sum_{Y \subset \sim \cup X_i} \tilde{V}_{\geqq 4Y}(\psi^n)\right], \tag{8}$$

where the sum over $\{X_i\}$ runs through collections of paved disjoint sets such that $D \subset \bigcup_i X_i$ and each $D \cap X_i$ is a non-empty union of c.c. of D. In (8),

$$g^{nD}_{X_i}(\psi^n) = \sum_{\{Y_\alpha\}} \prod_\alpha (\exp[-\tilde{V}^n_{\geqq 4Y_\alpha}(\psi^n)] - 1)\exp\left[-\lambda \int_{D\cap X_i} dx (\psi^n_x)^4\right], \tag{9}$$

with $\{Y_\alpha\}$ running through collections of paved subsets of X_i such that X_i is connected with respect to the c.c. of D and Y_α's. Notice that $g^D_{X_i}(\psi^n)$ depends only on $\psi^n|X_i$.

The main point of the partial resummation of the Mayer expansion (7) inside D is that (8) remains valid also for fields which become large inside D. Given $\psi^n = \mathscr{A}_n\phi^n$ for a real BS field ϕ^n, define $D(\psi^n)$ as the smallest paved set such that

$$|\psi^n_x| < 2C_1(n_0+n)^{1/4}\exp[\tfrac{1}{10}\alpha d(x, \sim D(\psi^n))] \tag{10}$$

for $\alpha = \varepsilon\beta$ [for β see (2.42) to (2.46)] with some sufficiently small $\varepsilon > 0$. Thus $D(\psi^n)$ is the set of points, where ψ^n becomes large but only exponentially fast with the distance from $\sim D(\psi^n)$ (the last condition will take care of the exponentially decreasing tails in the interfield coupling). The set of the large fields will be defined as

$$\mathscr{D}(D,X) = \bigcup_{\psi^n = \mathscr{A}_n\phi^n} (\psi^n|_X + \mathscr{K}(X)), \tag{11}$$

where the union is taken over real fields ϕ^n such that $D(\psi^n)$ is a subset of the paved set D. Notice that although we have admitted arbitrary small fields ψ^n, the large fields have the original form $\psi^n = \mathscr{A}_n\phi^n$, ϕ^n real, up to a small field correction. Notice also that

$$\mathscr{D}(D,X)|_{X\setminus D} \subset 3\mathscr{K}(X\setminus D). \tag{12}$$

Indeed, for $\psi^n = \mathscr{A}_n\phi^n$ with $D(\psi^n) \subset D$,

$$|\psi^n_x| \leqq 2C_1(n_0+n)^{1/4} \quad \text{if} \quad x \notin D \tag{13}$$

by (10). Moreover

$$\begin{aligned}|\psi^n_x - \psi^n_y|/|x-y| &\leqq \sum_{\mathrm{x}} (|\mathscr{A}_{nx\mathrm{x}} - \mathscr{A}_{ny\mathrm{x}}|/|x-y|)\,|\phi^n_{\mathrm{x}}| \\ &\leqq \mathscr{O}(1)\sum_{\mathrm{x}} (e^{-\beta|x-\mathrm{x}|} + e^{-\beta|y-\mathrm{x}|}) \int_{\square_{\mathrm{x}}} dx\,|\psi^n_x| \\ &\leqq \mathscr{O}(1)\,C_1(n_0+n)^{1/4}\sum_{\mathrm{x}} (e^{-\beta|x-\mathrm{x}|} + e^{-\beta|y-\mathrm{x}|})\,e^{\frac{1}{10}\alpha d(\mathrm{x}, \sim D)} \\ &\leqq \mathscr{O}(1)\,C_1(n_0+n)^{1/4} \leqq 2C_0C_1(n_0+n)^{1/4}\end{aligned} \tag{14}$$

for $x, y \notin D$ and $C_0 \geqq \bar{C}_0(L)$ by (2.43), (2.26), and (10). Similarly

$$|\partial_\mu\psi^n_x - \partial_\mu\psi^n_y|/|x-y|^{2/3} \leqq 2C_0C_1(n_0+n)^{1/4} \tag{15}$$

for $x, y \notin D$ and C_0 large enough. Hence $\psi^n \in 2\mathscr{K}(\sim D)$ [or more precisely $\psi^n|_{\sim D} \in 2\mathscr{K}(\sim D)$] and (12) follows from (11).

For large fields, we shall not insist on the analyticity (or even on the existence for complex ψ^n) of $V^n(\psi^n)$ but will simply assume the analyticity of the Boltzmann factor $\exp[-V^n(\psi^n)]$ on $\mathscr{D}(D, L^{-n}\Lambda)$ together with the representation (8) for $\exp[-V^n_{\geqq 4}(\psi^n)]$, where $g^{nD}_{X_i}(\psi^n)$ are even functionals depending on $\psi^n|_{X_i}$, analytic on $\mathscr{D}(D, X_i)$. The precise stability bounds on $g^{nD}_{X_i}$ will be stated in Sect. 6. Here we mention only that $g^{nD}_{X_i}$ decays exponentially with the separation of points of X_i and is bounded but not necessarily small.

For large fields, the relation (9) does not make sense any more but we shall still have a relation between g^{nD}_X and $g^{nD_1}_X$ for $D \supset D_1$ easily following from (8). Namely, on $\mathscr{D}(D_1, X)$,

$$g_X^{nD} = \sum_{\{X_i\},\{Y_\alpha\}} \prod_i g_{X_i}^{nD_1} \prod_\alpha (\exp[-\tilde{V}^n_{\geqq 4Y_\alpha}]-1)\exp\left[-\lambda_n \int_{(D\setminus D_1)\cap X} (\psi^n)^4\right], \quad (16)$$

where we sum over collections of disjoint sets $X_i \subset X$, $D_1 \cap X \subset \cup X_i$, $D_1 \cap X_i$ is a non-empty union of c.c. of D_1 and $Y_\alpha \subset (X \setminus \cup X_i)$ with X connected with respect to c.c. of D, X_i, and Y_α. Notice that (9) follows from (16) (and analytic continuation) if we put $D_1 = \emptyset$. Equation (16) also implies that if $D \cap X = D_1 \cap X$ then g_X^{nD} and $g_X^{nD_1}$ coincide on $\mathscr{D}(D_1, X)$.

5. Local Analysis of the Renormalization Group Recursion

The proof that the general form of the effective interaction discussed in the previous section is preserved by the RG transformation is based on a cluster expansion argument for the fluctuation integral of (2.39).

Again in order to simplify the notation, we drop the sub-(super-) script n and replace $n+1$ by prime. Define

$$\exp[-W'(\psi')] = \int \exp[-V(L^{-1}\psi'_{L^{-1}\cdot} + \mathscr{Z})]\, d\mu_1(Z). \quad (1)$$

As compared to V', see (2.39), W' contains a constant term as well as a $\int (\partial\psi')^2$ one and the field strength has not been renormalized in it.

The first step of the expansion for the right-hand side of (1) consists of localizing the regions in which $\mathscr{Z}$ field is large. This is done with the help of a partition of unity

$$1 = \sum_{\bar{p}} \chi_{\bar{p}}(Z), \quad (2)$$

where $\bar{p} = (p_u)$, $u \in \bar{\Lambda}_n \equiv \Lambda_n \setminus L\mathbb{Z}^4$, $p_u = 0, 1, \ldots$, and $\chi_{\bar{p}}$ is the following characteristic function

$$\chi_{\bar{p}}(Z) = \prod_{u \in \bar{\Lambda}_n} \chi_{\{(n_0+n)^{1/4} p_u \leqq |Z_u| < (n_0+n)^{1/4}(p_u+1)\}}\,. \quad (3)$$

Notice that $\bar{p} = 0$ selects small Z and hence (due to (2.25), (2.42), and (2.46)) small $\mathscr{Z}$. Given $\bar{p}$, let us define the large $\mathscr{Z}$ region R by

$$R = \bigcup_{u \in \bar{\Lambda}_n} \{L\Delta : \Delta \subset L^{-(n+1)}\Lambda,\, d(L\Delta, u) < 10\alpha^{-1} \log(p_u+1)\}. \quad (4)$$

Thus Z can become large in R but only exponentially fast in the distance from $\sim R$.

Given set D', where the BS field ψ' may get large and the set R of large fluctuations $\mathscr{Z}$, let us define the set D of large fields $\psi \equiv L^{-1}\psi'_{L^{-1}\cdot} + \mathscr{Z}$:

$$D = LD' \cup R. \quad (5)$$

Notice that if $\psi' \in \frac{1}{2} L\mathscr{D}(D', L^{-(n+1)}\Lambda)$ and Z is in the support of $\chi_{\bar{p}}$ then

$$\psi \in \tfrac{3}{4}\mathscr{D}(D, L^{-n}\Lambda). \quad (6)$$

Indeed

$$L^{-1}\psi'_{L^{-1}\cdot} \in \tfrac{2}{3}\mathscr{D}(LD', L^{-n}\Lambda) \subset \tfrac{2}{3}\mathscr{D}(D, L^{-n}\Lambda). \quad (7)$$

Moreover, by (2.25), (2.42), (2.46), (3), and (4),

$$\begin{aligned}|\mathscr{Z}_x| &= |(\mathscr{A}Q\Gamma^{1/2}Z)_x| \leqq \mathcal{O}(1) \sum_{u\in\bar{\Lambda}_n} e^{-\frac{3}{4}\beta|x-u|}|Z_n| \\ &\leqq \mathcal{O}(1)(n_0+n)^{1/4}\sum_u e^{-\frac{3}{4}\beta|x-u|}(p_u+1) \\ &\leqq \mathcal{O}(1)(n_0+n)^{1/4}\sum_u e^{-\frac{3}{4}\beta|x-u|+\frac{1}{10}\alpha d(u,\sim R)} \\ &\leqq \mathcal{O}(1)(n_0+n)^{1/4} e^{\frac{1}{10}\alpha d(x,\sim R)} \leqq \tfrac{1}{12}C_1(n_0+n)^{1/4} e^{\frac{1}{10}\alpha d(x,\sim R)} \end{aligned} \tag{8}$$

for $C_1 \geqq \bar{C}_1(L)$ so that $\mathscr{Z} \in \frac{1}{12}\mathscr{D}(R, L^{-n}\Lambda) \subset \frac{1}{12}\mathscr{D}(D, L^{-n}\Lambda)$. Thus after inserting (2) under the integral of the right-hand side of (1), we may express the integrand $\exp[-V(\psi)] = \exp[-V_2(\psi)]\exp[-V_{\geqq 4}(\psi)]$ using (4.2) and (4.8) with D given by (5).

This way we obtain

$$\begin{aligned}\exp[-W'(\psi')] = \sum_p \sum_{\{X_i\}} \int \prod_i g^D_{X_i}(\psi)\exp\Big[&-\tfrac{1}{2}m^2\int dx\psi_x^2 + 6\lambda\int dx G_{xx}\psi_x^2 \\ &- \sum_{\mu,\nu}\int dx dy K^{\mu\nu}_{xy}(\partial_\mu\psi_x - \partial_\mu\psi_y)\partial_\nu\psi_y - \lambda\int_{\sim D} dx\psi_x^4 \\ &- \sum_{Y\subset \sim UX_i} \tilde{V}_{\geqq 4Y}(\psi)\Big]\chi_{\bar{p}}(Z)d\mu_1(Z). \end{aligned} \tag{9}$$

Let us localize the irrelevant quadratic term of V, $\sum_{\mu,\nu}\int dx dy K^{\mu\nu}_{xy}(\partial_\mu\psi_x - \partial_\mu\psi_y)\partial_\nu\psi_y$ by writing it as $\sum_Y \tilde{V}_{2Y}(\psi)$, where Y is the smallest paved set containing $x, y, x+L^{-n}e_\mu$ and $y+L^{-n}e_\mu$. Mayer expanding

$$\exp\Big[-\sum_{Y\not\subset X_i}\tilde{V}_{2Y} - \sum_{Y\subset\sim UX_i}\tilde{V}_{\geqq 4Y}\Big] \text{ for all } i$$

under the integral of (9), we obtain

$$\begin{aligned}\exp[-W'(\psi')] &= \sum_{\bar{p}}\sum_{\{X_i\}}\sum_{\{Y_\alpha\}}\sum_{\{Y_\beta\}}\int\prod_i g^D_{X_i}(\psi) \\ &\cdot\exp\Big[-\tfrac{1}{2}m^2\int dx\psi_x^2 + 6\lambda\int dx\mathscr{G}_{xx}\psi_x^2 - \sum_i\sum_{Y\subset X_i}\tilde{V}_{2Y}(\psi) - \lambda\int_{\sim D}dx\psi_x^4\Big] \\ &\cdot\prod_\alpha(\exp[-\tilde{V}_{2Y_\alpha}(\psi)]-1)\prod_\beta(\exp[-\tilde{V}_{\geqq 4Y_\beta}(\psi)]-1)\chi_{\bar{p}}(Z)d\mu_1(Z), \end{aligned} \tag{10}$$

where Y_α are not contained in a single X_i and Y_β do not intersect $\cup X_i$.

We still have to decouple the non-locality of the right-hand side due to the kernels $\mathscr{A}Q\Gamma^{1/2} \equiv \mathscr{M}$ relating the fluctuation fields $\mathscr{Z}$ and Z, see (2.25). Let $\{U_k\}$ be the partition of the volume $L^{-n}\Lambda$ into unions of blocks $L\Delta(\Delta\subset L^{-(n+1)}\Lambda)$ connected with respect to X_i, Y_α, Y_β and the pairs of the nearest neighbor blocks $L\Delta_1$, $L\Delta_2$ such that $L\Delta_1\subset R$. The reason for taking this "collar" around R will become clear in a moment. Define

$$\psi^s = L^{-1}\psi'_{L^{-1}\cdot} + \mathscr{M}^s Z \equiv L^{-1}\psi'_{L^{-1}\cdot} + \mathscr{Z}^s, \tag{11}$$

where (for χ_{U_k} being the characteristic function of U_k)

$$\mathscr{M}^s = \sum_k \chi_{U_k}\mathscr{M}\chi_{U_k} + \sum_{k<k'} s_{kk'}(\chi_{U_k}\mathscr{M}\chi_{U_{k'}} + \chi_{U_{k'}}\mathscr{M}\chi_{U_k}). \tag{12}$$

Notice that for $(s_{kk'})=0$, $\mathcal{M}^s$ does not couple different U_k. We shall admit complex $s_{kk'}$ with

$$|s_{kk'}|\leqq 2r\exp[\tfrac{1}{2}\beta d(U_k, U_{k'})], \tag{13}$$

where r will be chosen later. Notice that for $x\in U_k$,

$$\mathcal{Z}^s_x=(\mathcal{M}\chi_{U_k}Z)_x+\sum_{k'\neq k} s_{kk'}(\mathcal{M}\chi_{U_{k'}}Z)_x\equiv\mathcal{Z}_{1x}+\mathcal{Z}^s_{2x}, \tag{14}$$

where for $k'<k$, we have put $s_{kk'}\equiv s_{k'k}$. Following (8), we show that the first term on the right-hand side is in $\frac{1}{12}\mathcal{D}(R_k, U_k)$, where $R_k\equiv R\cap U_k$. Let us consider the second term. Proceeding again like in (8), we notice after the fourth step that for $u\in U_{k'}$,

$$\tfrac{2}{3}\beta|x-u|\geqq\tfrac{1}{10}\alpha d(u, \sim R)+\tfrac{1}{2}\beta d(U_k, U_{k'}), \tag{15}$$

(since for $u\in R$ $|x-u|\geqq d(u, \sim R)$ by the construction of U_k's) and conclude that

$$|\mathcal{Z}^s_{2x}|\leqq\tfrac{1}{4}C_1(n_0+n)^{1/4} \tag{16}$$

for $C_1\geqq\bar{C}_1(L,r)$. Similarly, we show that

$$|\mathcal{Z}^s_{2x}-\mathcal{Z}^s_{2y}|/|x-y|\leqq\tfrac{1}{4}C_0C_1(n_0+n)^{1/4}\quad\text{for}\quad x,y\in U_k \tag{17}$$

and

$$|\partial_\mu\mathcal{Z}^s_{2x}-\partial_\mu\mathcal{Z}^s_{2y}|/|x-y|^{2/3}\leqq\tfrac{1}{4}C_0C_1(n_0+n)^{1/4}$$
$$\text{for}\quad x,y,x+L^{-n}e_\mu,y+L^{-n}e_\mu\in U_k \tag{18}$$

[note the crucial character of the limitation of the points to a single U_k in (17) and (18)].

Altogether, we infer [using also (7)] that for $\psi'\in\frac{1}{2}L\mathcal{D}(D',L^{-(n+1)}\Lambda)$ [or $\psi'\in\frac{1}{2}L\mathcal{D}(D',L^{-1}U_k)$],

$$\mathcal{Z}_1\in\tfrac{1}{12}\mathcal{D}(R_k, U_k),\ \mathcal{Z}^s_2\in\tfrac{1}{4}\mathcal{K}(U_k)\quad\text{and}\quad\psi^s\in\mathcal{D}(LD'\cup R_k, U_k). \tag{19}$$

We shall use the following formula [24] interpolating between $s=0$ and $s=1$ expressions:

$$\mathcal{F}(1)=\sum_\Gamma\int ds_\Gamma\partial_{s_\Gamma}\mathcal{F}(s), \tag{20}$$

where the sum runs over graphs Γ (collections of pairs $k<k'$: called in what follows lines), $ds_\Gamma=\prod_{\ell\in\Gamma}\chi_{[0,1]}(s_\ell)ds_\ell$, $\partial_{s_\Gamma}\mathcal{F}(s)=\prod_{\ell\in\Gamma}\frac{\partial}{\partial s_\ell}\mathcal{F}(s)$ and in the argument of $\mathcal{F}$ on the right-hand side of (20), s_ℓ are set to zero for $\ell\notin\Gamma$.

Splitting Γ into the connected components, we may rewrite (20) as

$$\mathcal{F}(1)=\sum_{\{\mathcal{U}_\gamma\}}\int\prod_\gamma S(\mathcal{U}_\gamma)\mathcal{F}(s), \tag{21}$$

where $\{\mathcal{U}_\gamma\}$ is a partition of $\{U_k\}$ and

$$S(\mathcal{U}_\gamma)=\sum_{\Gamma_c}\int ds_{\Gamma_c}\partial_{s_{\Gamma_c}}, \tag{22}$$

with the sum running through the connected graphs on $\mathcal{U}_\gamma$. $S(\mathcal{U}_\gamma)=1$ if $\mathcal{U}_\gamma$ is composed of one U_k.

Applying (21) to (10), with s-dependence introduced according to (11) and (12), we obtain

$$\exp[-W'(\psi')] = \sum_{\bar{p}} \sum_{\{X_i\}} \sum_{\{Y_\alpha\}} \sum_{\{Y_\beta\}} \sum_{\{\mathscr{U}_\gamma\}} \int \prod_\gamma S(\mathscr{U}_\gamma) \prod_i g^D_{X_i}(\psi^s)$$
$$\cdot \exp\left[-\tfrac{1}{2}m^2 \int dx(\psi^s_x)^2 + 6\lambda \int dx \mathscr{G}_{xx}(\psi^s_x)^2 - \sum_i \sum_{Y \subset X_i} \tilde{V}_{2Y}(\psi^s) - \lambda \int_{\sim D} dx(\psi^s_x)^4\right]$$
$$\cdot \prod_\alpha (\exp[-\tilde{V}_{2Y_\alpha}(\psi^s)]-1) \prod_\beta (\exp[-\tilde{V}_{\geqq 4Y_\beta}(\psi^s)]-1)\chi_{\bar{p}}(Z)d\mu_1(Z). \quad (23)$$

But the integral on the right-hand side of (23) decouples and the sums factorize over the sets $\mathscr{U}_\gamma = \bigcup_{U_k \in \mathscr{U}_\gamma} U_k$. For the later, one of the crucial observations is that for $X_i \subset U_k$ we may replace $g^D_{X_i}(\psi^s)$ by $g^{LD' \cup R_k}_{X_i}(\psi^s)$, since $\psi^s \in \mathscr{D}(LD' \cup R_k, U_k)$ and $D \cap U_k = (LD' \cup R_k) \cap U_k$, see the remark following (4.16). Also, for the factorization of $\sum_{\bar{p}}$ over $\mathscr{U}_\gamma$ it is important that together with $L\Delta_1 \subset R_k$, we have included into U_k all the nearest neighbor blocks $L\Delta_2$. Otherwise we would encounter the non-factorizing condition that $\mathscr{U}_\gamma \cap R$ is a c.c. of R.

Let us introduce paved sets $X'_\gamma \subset L^{-(n+1)}\Lambda$ such that $LX'_\gamma = \mathscr{U}_\gamma$ but admitting $X'_\gamma = \Delta$ only inside D'. We can rewrite (23) as a partition function of a polymer system with (disjoint) polymers X'_γ:

$$\exp[-W'(\psi')] = \sum_{\{X'_\gamma\}} \prod_\gamma \varrho^{D'}_{X'_\gamma}(\psi') \exp\left[-\sum_{\Delta \subset \sim D'} W'_\Delta(\psi')\right]. \quad (24)$$

The polymer activities are given by

$$\varrho^{D'}_{X'}(\psi') = \sum_{\bar{p}} \sum_{\{X_i\}} \sum_{\{Y_\alpha\}} \sum_{\{Y_\beta\}} \int S(\mathscr{U}) \prod_j g^D_{X_i}(\psi^s)$$
$$\cdot \exp\left[-\tfrac{1}{2}m^2 \int_{LX'} dx(\psi^s_x)^2 + 6\lambda \int_{LX'} dx \mathscr{G}_{xx}(\psi^s_x)^2 - \sum_i \sum_{Y \subset X_i} \tilde{V}_{2Y}(\psi^s) - \lambda \int_{LX' \backslash D} dx(\psi^s_x)^4\right]$$
$$\cdot \prod_\alpha (\exp[-\tilde{V}_{2Y_\alpha}(\psi^s)]-1) \prod_\beta (\exp[-\tilde{V}_{\geqq 4Y_\beta}(\psi^s)]-1)\chi_{\bar{p}}(Z_{LX'})d\mu_1(Z_{LX'})$$
$$\cdot \exp\left[\sum_{\Delta \subset X' \backslash D'} W'_\Delta(\psi')\right], \quad (25)$$

where $\bar{p} = 0$ outside LX', $R \equiv R(\bar{p}) \subset LX'$ together with all blocks $L\Delta$ having nearest neighbors in R, $D = LD' \cup R$, X_i are disjoint subsets of LX' and each $D \cap X_i$ is a non-empty union of c.c. of D, $D \cap LX' \subset \cup X_i$, $Y_\alpha \subset LX'$ but not in a single X_i, $Y_\beta \subset LX' \backslash (\cup X_i)$ and $\mathscr{U}$ is the partition of LX' into unions of blocks $L\Delta$ connected with respect to X_i, Y_α, Y_β and pairs of nearest neighbor blocks $L\Delta_1$, $L\Delta_2$, $L\Delta_1 \subset R$. In (25) $\psi^s = L^{-1}\psi'_{L^{-1}\cdot} + \mathscr{M}^s Z_{LX'}$, where $Z_{LX'}$ vanishes outside LX'. We recall that X' can be equal to a single block Δ only of it lies inside D'.

The contributions of the single Δ clusters outside D' are gathered in

$$\exp[-W'_\Delta(\psi')] = \int \exp\left[-\tfrac{1}{2}m^2 \int_{L\Delta} dx(\psi^0_x)^2 + 6\lambda \int_{L\Delta} dx \mathscr{G}_{xx}(\psi^0_x)^2\right.$$
$$\left. - \sum_{Y \subset L\Delta} \tilde{V}_{2Y}(\psi^0) - \lambda \int_{L\Delta} dx(\psi^0_x)^4 - \sum_{Y \subset L\Delta} \tilde{V}_{\geqq 4Y}(\psi^0)\right] \chi_0(Z_{L\Delta})d\mu_1(Z_{L\Delta}). \quad (26)$$

It should be clear (see the precise estimates in the next sections), that $W'_\Delta(\psi')$ as well as the polymer activities $\varrho^{D'}_{X'}(\psi')$ exist and are analytic for $\psi' \in \frac{3}{2}L\mathscr{K}(\Delta)$ and $\psi' \in \frac{1}{2}L\mathscr{D}(D', X')$ respectively.

For ψ' small ($D'=\emptyset$) in $\frac{1}{2}L\mathscr{K}(X')$ put $\varrho^{\emptyset}_{X'}(\psi') \equiv \varrho_{X'}(\psi')$. In this case the activities $\varrho_{X'}$ will be small and exponentially decaying with the separation of points of X'. We may exponentiate the right-hand side of (24), see e.g. Sect. I.3 of [37], obtaining for $\psi' \in \frac{1}{2}L\mathscr{K}(L^{-(n+1)}\Delta)$

$$W'(\psi') = \sum_Y W'_Y(\psi'), \tag{27}$$

where for Y bigger than Δ,

$$W'_Y(\psi') = -\sum_{\substack{(X_\xi)_{\xi=1}^{\Xi} \\ \cup X_\xi = Y}} \frac{1}{\Xi!} \sum_{\gamma_c} \prod_{\ell \in \gamma_c} A(\ell) \prod_\zeta \varrho_{X_\zeta}(\psi'). \tag{28}$$

In (28) γ_c runs over connected graphs of lines ℓ joining vertices $\{1, \ldots, \Xi\}$, $A(\ell) = -1$ if $X'_{\ell_-} \cap X'_{\ell_+} \neq \emptyset$, where $\ell = (\ell_-, \ell_+)$ and zero otherwise. The sum in (28) will be shown to converge and give again small $W'_Y(\psi')$ with exponential decay in Y. Using the Taylor expansion, we write

$$\begin{aligned} W'(\psi') &= W'(0) + W'_2(\psi') + W'_4(\psi') + W'_{\geqq 6}(\psi') \\ &= \sum_Y (W'_Y(0) + W'_{2Y}(\psi') + W'_{4Y}(\psi') + W'_{\geqq 6Y}(\psi')) . \end{aligned} \tag{29}$$

For a general set $S \subset L^{-n}\Delta$, denote by $\bar{S}$ the smallest paved set containing S. Let us write for the quartic terms of W'

$$\begin{aligned} W'_{4Y'}(\psi') &= \sum_{\Delta = Y'} \lambda \int_\Delta dx (\psi'_x)^4 + \sum_{Y:(L^{-1}Y)^- = Y'} \tilde{V}_{4Y}(L^{-1}\psi'_{L^{-1}\cdot}) + \tilde{\tilde{W}}'_{4Y'}(\psi') \\ &\equiv \sum_{\Delta = Y'} \lambda \int dx (\psi'_x)^4 + \tilde{W}'^1_{4Y'}(\psi') + \tilde{\tilde{W}}'_{4Y'}(\psi') . \end{aligned} \tag{30}$$

$\tilde{\tilde{W}}'_Y(\psi')$ contains the $\mathcal{O}(V_{\geqq 6})$ and $\mathcal{O}((V)^2)$ contributions to $W'_{4Y}(\psi')$. As we shall show below,

$$\sum_Y \tilde{\tilde{W}}'_{4Y}(\psi') = \delta\lambda \int dx (\psi'_x)^4 + \sum_Y \tilde{W}'^2_{4Y}(\psi') , \tag{31}$$

where $\tilde{W}'^2_{4Y}$, as $\tilde{W}'^1_{4Y}$, possess the expected properties of $\tilde{V}'_{4Y}(\psi')$ (in particular, they are irrelevant).

The extraction of the marginal term $\delta\lambda \int (\psi')^4$ will be described in detail in Sect. 9. Here let us only mention that (31) cannot be written as $\tilde{\tilde{W}}'_{4Y}(\psi') = \int dx \delta\lambda_Y(x) (\psi'_x)^4 + \tilde{W}'^2_{4Y}(\psi')$, if only this still holds approximately up to exponentially decaying tails.

Let us come back to (24) for $\psi' \in \frac{1}{2}L\mathscr{D}(D', L^{-(n+1)}\Delta)$. We shall now extract from $W'(\psi')$ the constant and the quadratic term as well as the part of the non-local quartic one. The latter, after reshuffling it according to (31), will be reinstated back in a moment.

Let us then introduce

$$\exp[-\bar{W}'_{\geqq 4}] \equiv \exp[-W'] \exp\left[W'(0) + W'_2 + \sum_Y \tilde{\tilde{W}}'_{4Y}\right]. \tag{32}$$

By (24) and (29),

$$\exp[-\bar{W}'_{\geqq 4}] = \sum_{\{X'_\gamma\}} \prod_\gamma \varrho^{D'}_{X'_\gamma} \exp\left[-\sum_{\Delta \subset \sim D'} W'_\Delta + \sum_Y (W'_Y(0) + W'_{2Y}) + \sum_Y \tilde{\tilde{W}}'_{4Y}\right]. \tag{33}$$

Let us define the interior $\mathring{X}$ of a paved set X as the union of $\Delta \subset X$ which do not touch blocks $\Delta \subset \sim X$. Mayer expanding $\exp\left[\sum_Y (W'_Y(0)+W'_{2Y})+\sum_Y \tilde{\tilde{W}}'_{4Y}\right]$, the sum running over the paved sets Y which do not lie entirely in the interior $\mathring{D}'_i$ of a single c.c. D'_i of $D'\cap\bar{X}$ and resumming the clusters of this expansion outside the large field region D', we obtain

$$\exp[-\bar{W}'_{\geqq 4}] = \sum_{\{\bar{X}_l\}} \prod_l \bar{g}'_{X_l} \sum_{\{X'_\gamma\}} \prod_\gamma \varrho_{X'_\gamma} \cdot \exp\left[-\lambda \int_{\cup \bar{X}_l \setminus D'} (\psi')^4 - \sum_{\Delta \subset \sim \cup \bar{X}_l} W'_\Delta + \sum_{Y\subset \sim \cup \bar{X}_l} (W'_Y(0)+W'_{2Y}) + \sum_{Y\subset \sim\cup \bar{X}_l} W'_{4Y}\right]. \quad (34)$$

Here $\bar{X}_l$ runs through the collections of disjoint paved sets such that $D'\cap\bar{X}_l$ is a non-empty union of c.c. of D', $D'\subset\cup\bar{X}_l$, X'_γ are disjoint and $X'_\gamma\cap\bar{X}_l=\emptyset$.

$$\begin{aligned}\bar{g}'^{D'}_{\bar{X}} = \sum_{\{X'_\gamma\}}\sum_{\{Y_\alpha\}}\sum_{\{Y_\beta\}} \prod_\gamma \varrho^{D'}_{X'_\gamma} \prod_\alpha (\exp[W'_{Y_\alpha}(0)+W'_{2Y_\alpha}]-1) \\ \cdot \prod_\beta (\exp[\tilde{\tilde{W}}'_{4Y_\beta}]-1)\exp\left[-\sum_{\Delta\subset\bar{X}\setminus D'} W'_\Delta + \lambda \int_{\bar{X}\setminus D'} (\psi')^4\right. \\ \left. + \sum_{D'_i}\sum_{Y\subset \mathring{D}'_i} (W'_Y(0)+W'_{2Y}+\tilde{\tilde{W}}'_{4Y})\right],\end{aligned} \quad (35)$$

where X'_γ are disjoint, $\emptyset \neq D'\cap\bar{X}\subset\cup X'_\gamma$, no Y_α and no $Y_\beta(\subset\bar{X})$ lie entirely in the interior of a single c.c. D'_i of $D'\cap\bar{X}$ and $\bar{X}$ is connected with respect to X'_γ, Y_α, and Y_β.

In (34) we may exponentiate the sum over $\{X'_\gamma\}$:

$$\sum_{\{X'_\gamma\}} \prod_\gamma \varrho_{X'_\gamma} \exp\left[-\sum_{\Delta\subset\sim\cup\bar{X}_l} W'_\Delta\right] = \exp\left[-\sum_{Y\subset\sim\cup\bar{X}_l} W'_Y\right], \quad (36)$$

see (28). Inserting (36) to (34) and using (29) and (30), we obtain

$$\exp[-\bar{W}'_{\geqq 4}] = \sum_{\{\bar{X}_l\}} \prod_l \bar{g}'^{D'}_{\bar{X}_l} \exp\left[-\lambda \int_{\sim D'} (\psi')^4 - \sum_{Y\subset\sim\cup\bar{X}_l} \tilde{W}'^1_{4Y} - \sum_{Y\subset\sim\cup\bar{X}_l} W'_{\geqq 6Y}\right]. (37)$$

Now, using (31), we shall reinstate the factor $\exp\left[-\sum_Y \tilde{\tilde{W}}'_{4Y}(\psi')\right]$ removed in $\exp[-\bar{W}'_{\geqq 4}]$, see (32).

$$\begin{aligned}\exp[-W'_{\geqq 4}] &= \exp[-\bar{W}'_{\geqq 4}]\exp\left[-\delta\lambda\int(\psi')^4 - \sum_Y \tilde{W}'^2_{4Y}\right] \\ &= \sum_{\{\bar{X}_l\}}\sum_{\{Y_\alpha\}}\sum_{\{Y_\beta\}}\sum_{\{Y_\gamma\}} \prod_l \bar{g}'^{D'}_{\bar{X}_l} \prod_\alpha (\exp[-\tilde{W}'^1_{4Y_\alpha}]-1) \\ &\quad\cdot \prod_\beta (\exp[-W'_{\geqq 6Y_\beta}]-1)\prod_\gamma(\exp[-W'^2_{4Y_\gamma}]-1) \\ &\quad\cdot \exp\left[-\lambda\int_{\sim D'}(\psi')^4 - \delta\lambda\int(\psi')^4 - \sum_{D'_i}\sum_{Y\subset\mathring{D}'_i}\tilde{W}'^2_{4Y}\right],\end{aligned} \quad (38)$$

where we have partly Mayer expanded the non-local terms in the exponential. In (38), Y_α, $Y_\beta \subset \sim\cup\bar{X}_l$ and no Y_γ lies entirely in the interior of a single c.c. of D'. Fixing the clusters of the expansion on the right-hand side of (38) intersecting D' and exponentiating the outside sum, we shall obtain

$$\exp[-W'_{\geqq 4}] = \sum_{\{X_j\}} \prod_j \tilde{g}'^{D'}_{X_j} \exp\left[-(\lambda+\delta\lambda)\int_{\sim D'}(\psi')^4 - \sum_{Y\subset\sim\cup X_j}(\tilde{W}'^1_{4Y}+\tilde{W}'^2_{4Y}+W'_{\geqq 6Y})\right], \quad (39)$$

where $\{X_j\}$ runs through the collections of disjoint paved sets such that $D' \cap X_j$ is a non-empty union of c.c. D'_{ji} of D' and $D' \subset X_j$.

$$\tilde{g}_X'^{D'} = \sum_{\{\bar{X}_l\}} \sum_{\{Y_\alpha\}} \sum_{\{Y_\beta\}} \sum_{\{Y_\gamma\}} \prod_l \tilde{g}_{\bar{X}_l}'^{D'} \prod_\alpha (\exp[-\tilde{W}_{4Y_\alpha}'^1] - 1) \prod_\beta (\exp[-W_{\geqq 4}'] - 1)$$
$$\cdot \prod_\gamma (\exp[-\tilde{W}_{4Y_\gamma}'^2] - 1) \exp\left[-6\lambda \int_{D'X} (\psi')^4 - \sum_{D'_i} \sum_{Y \subset \mathring{D}'_i} \tilde{W}_{4Y}'^2\right] \quad (40)$$

with $\bar{X}_l$, Y_α, Y_β, and Y_γ as in (38) except that they lie in X and X has to be connected with respects to them.

We still have to deal with the second order contributions to W_2'. In fact we shall show that for $\psi' = \mathscr{A}'\phi'$,

$$W_2'(\psi') + 6(\lambda + \delta\lambda)\zeta \int dx \mathscr{G}_{xx}'(\psi_x')^2 = \tfrac{1}{2}(L^2m^2 + \delta m^2) \int dx (\psi_x')^2 + \tfrac{1}{2}\delta c \sum_\mu \int dx (\partial_\mu \psi_x')^2$$
$$+ \sum_{\mu,\nu} \int dx dy (L^4 K_{LxLy}^{\mu\nu} + \delta K_{xy}^{\mu\nu})(\partial_\mu \psi_x' - \partial_\mu \psi_y')\partial_\nu \psi_y' . \quad (41)$$

Now choosing the wave function renormalization

$$\zeta = (1 + \delta c)^{-1} \quad (42)$$

[as will be shown $|\delta c| \leqq \mathcal{O}(\lambda^{7/4})$ and hence $|1-\zeta| \leqq \mathcal{O}(\lambda^{7/4})$], we may rewrite (41) as

$$W_2'(\zeta^{1/2}\psi') = \tfrac{1}{2}(L^2m^2 + \delta m^2)\zeta \int dx (\psi_x')^2$$
$$- 6(\lambda + \delta\lambda)\zeta^2 \int dx \mathscr{G}_{xx}'(\psi_x')^2 + \tfrac{1}{2}(1-\zeta) \sum_\mu \int dx (\partial_\mu \psi_x')^2$$
$$+ \zeta \sum_{\mu,\nu} \int dx dy (L^4 K_{LxLy}^{\mu\nu} + \delta K_{xy}^{\mu\nu})(\partial_\mu \psi_x' - \partial_\mu \psi_y')\partial_\nu \psi_y' . \quad (43)$$

Now for $V_2'(\psi')$ with $\psi' = \mathscr{A}'\phi'$, we shall get from (2.39) and (1),

$$V_2'(\psi') = \tfrac{1}{2}m^{2\prime} \int dx (\psi_x')^2 - 6\lambda' \int dx \mathscr{G}_{xx}'(\psi_x')^2 + \sum_{\mu,\nu} \int dx dy K_{xy}'^{\mu\nu}(\partial_\mu \psi_x' - \partial_\mu \psi_y')\partial_\nu \psi_y', \quad (44)$$

provided that we define

$$m^{2\prime} \equiv (L^2m^2 + \delta m^2)\zeta , \quad (45)$$

$$\lambda' = (\lambda + \delta\lambda)\zeta^2 , \quad (46)$$

$$K_{xy}'^{\mu\nu} = (L^4 K_{LxLy}^{\mu\nu} + \delta K_{xy}^{\mu\nu})\zeta . \quad (47)$$

By definition, we shall extend (44) to any $L^{-(n+1)}\Lambda$ fields ψ' obtaining this way (4.2) with $n \mapsto n+1$.

For the quartic term V_4', we obtain from (2.39), (1), (30), and (31)

$$V_4'(\psi') = W_4'(\zeta^{1/2}\psi') = (\lambda + \delta\lambda)\zeta^2 \int dx (\psi_x')^4 + \zeta^2 \sum_Y \tilde{W}_{4Y}'^1(\psi')$$
$$+ \zeta^2 \sum_Y \tilde{W}_{4Y}'^2(\psi') \equiv \lambda' \int dx (\psi_x')^4 + \sum_Y \tilde{V}_{4Y}'(\psi') , \quad (48)$$

where we have set

$$\tilde{V}_{4Y'}' = \zeta^2(\tilde{W}_{4Y}'^1 + \tilde{W}_{4Y}'^2) \equiv \zeta^2 \tilde{W}_{4Y}' . \quad (49)$$

Equation (48) reproduces (4.3) for $n \mapsto n+1$. Similarly

$$V'_{\geqq 6}(\psi') = W'_{\geqq 6}(\zeta^{1/2}\psi') = -8(\lambda')^2 \int dx dy \mathcal{Q}'_{xy}(\psi'_x)^3(\psi'_y)^3 + \sum_Y \tilde{V}'_{\geqq 6Y}(\psi'), \quad (50)$$

where

$$V'_{\geqq 6Y}(\psi') = W'_{\geqq 6Y}(\zeta^{1/2}\psi') + 8(\lambda')^2 \sum_{\substack{(\Delta_1,\Delta_2) \\ \Delta_1 \cup \Delta_2 = Y}} \int_{\Delta_1} dx \int_{\Delta_2} dy \mathcal{Q}'_{xy}(\psi'_x)^3(\psi'_y)^3. \quad (51)$$

This gives (4.4) for $n \mapsto n+1$. As we shall see, (49) holds e.g. on $3\mathcal{K}(L^{-(n+1)}\Lambda)$ and the contributions on the right-hand side have the desired analycity properties (which should be clear already now).

For large ψ' e.g. in $\mathcal{D}(D', L^{-(n+1)}\Lambda)$, (2.39), (4.6), (1), (39), (49), and (51) give

$$\exp[-V'_{\geqq 4}(\psi')] = \sum_{\{X_j\}} \prod_j g'^{D'}_{X_j}(\psi') \exp\left[-\lambda' \int_{\sim D'} dx (\psi'_x)^4 - \sum_{Y \subset \sim \cup X_j} \tilde{V}'_{\geqq 4Y}(\psi')\right], \quad (52)$$

where

$$g'^{D'}_X(\psi') = \tilde{g}'^{D'}_X(\zeta^{1/2}\psi'). \quad (53)$$

(52) is (4.8) for $n \mapsto n+1$. Again $g'^{D'}_X$ are analytic on $\mathcal{D}(D', X)$.

Equations (44), (48), (50), and (52) show that the general form of the effective interactions described in Sect. 4 reproduces itself under (2.39). It is a straightforward exercise left to the reader to show that (4.16) also holds for $n \mapsto n+1$, compare Appendix 2 of [17]. In the following sections, we shall see how inductive bounds on various contributions to the effective interactions carry through (2.39), and shall show that they behave in accordance with predictions of Sect. 3 [e.g. $\lambda_n = \mathcal{O}((n_0+n)^{-1})$]. This will establish the IR asymptotic freedom of the weakly coupled lattice critical ϕ_4^4 theory.

6. Inductive Bounds for the Effective Interactions

Small Fields

For a paved set X, by $\mathcal{L}(X)$ we shall denote the length of the shortest tree on the centers of the Δ blocks building X and, possibly, other points of the (periodic) continuum.

The quadratic irrelevant term of the effective interactions will be controlled by means of the estimate

$$\int_{\Delta_1} dx \int_{\Delta_2} dy |K^{\mu\nu}_{nxy}|/|x-y|^{2/3} \leqq (n_0+n)^{-3/2} \exp[-\alpha\mathcal{L}(\Delta_1 \cup \Delta_2)], \quad (1)$$

where $\alpha = \varepsilon\beta$ is the same constant which appeared in (4.10).

The effective coupling constant will be bounded by

$$C_-(n_0+n)^{-1} \leqq \lambda_n \leqq C_+(n_0+n)^{-1} \quad (2)$$

for $0 < C_- < C_+$, $C_+ = C_+(L)$.

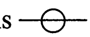
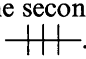
For the irrelevant quartic contributions, we assume

$$|\tilde{V}^n_{4Y}(\psi^n_1, \psi^n_2, \psi^n_3, \psi^n_4)| \leqq (n_0+n)^{-3/4} \exp[-\alpha\mathcal{L}(Y)] \quad (3)$$

for $\psi_1^n, \psi_2^n, \psi_3^n \in \mathscr{K}(Y)$ and ψ_4^n satisfying the bounds defining $\mathscr{K}(Y)$, see (4.1), except the first one for its absolute value [we recall that $\tilde{V}_{4Y}(\psi_1^n, \ldots, \psi_4^n)$ depends only on the differences of ψ_4^n at pairs of points of Y].

Finally $\tilde{V}_{\geqq 6Y}(\psi^n)$ are analytic on $3\mathscr{K}(Y)$ satisfying

$$|\tilde{V}_{\geqq 6Y}(\psi^n)| \leqq (n_0+n)^{-2/3} \exp[-\alpha \mathscr{L}(Y)] . \tag{4}$$

The choice of powers of $(n_0+n)^{-1} = \mathcal{O}(\lambda_n)$ in (1), (3), and (4) has been dictated by convenience. We could have respectively chosen $\mathcal{O}((n_0+n)^{-2})$, $\mathcal{O}((n_0+n)^{-2})$, $\mathcal{O}((n_0+n)^{-1})$, and $\mathcal{O}((n_0+n)^{-1})$ as well since the main contributions to K_n and $\tilde{V}_{4Y}^n$ come from the second order graphs  and  and to $\tilde{V}_{\geqq 6Y}$ from the third order graph . The factors $\exp[-\alpha \mathscr{L}(Y)]$ in our bounds exhibit the approximate locality of V^n.

Large Fields

We assume that the large field contributions $g_X^{nD}(\psi^n)$ to the Boltzmann factor $\exp[-V_{\geqq 4}^n(\psi^n)]$ are analytic on $\mathscr{D}(D, X)$ satisfying there

$$|g_X^{nD}(\psi^n)| \leqq \exp\left[C_2 |D \cap X| - \lambda_n^{1/2} \int_{D\cap X} dx |\psi_x^n|^2 + 20\lambda_n \int_{D\cap X} dx (\operatorname{Im} \psi_x^n)^4 - \alpha \mathscr{L}(X)\right], \tag{5}$$

where $C_2 = C_2(L, N_0)$ and for a paved set Y, $|Y|$ denotes the number of the Δ blocks building Y.

The choice of (5) is one of the crucial contributions into the analysis of the non-perturbative corrections to the effective interactions (besides the idea to use the analyticity in the field variables instead of the one in coupling constants). Let us notice that (5) has a chance to iterate: $\int (\operatorname{Im} \psi^n)^4$ goes more or less through the RG recursion unchanged and $\lambda_n^{1/2} \int |\psi^n|^2 \mapsto (L^2 \lambda_n^{1/2} + \mathcal{O}(\lambda_n)) \int |\psi^{n+1}|^2$, which in the large field region and for large C_1 [see (4.10)] should match the increase of the constant term $C_2 |D \cap X|$ providing even a contractive factor which will be used to control the combinatories of the cluster expansion.

At the initial step of the iteration when $V(\phi)$ is given by (2.2), $K_{0xy}^{\mu\nu} = 0$, $\tilde{V}_{4Y} = 0$, $\mathscr{Q}_{0xy} = 0$, $\tilde{V}_{\geqq 6Y} = 0$ and

$$g_X^{0D} = \begin{cases} \exp\left[-\lambda_0 \sum_{X \in X} \phi_x^4\right] & \text{if } X \text{ is a c.c. of } D, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Choice of n_0 corresponds by (2) to a choice of λ_0 (the bigger n_0, the smaller λ_0). Writing $\lambda_0^{1/4} \phi_x = a + ib$ with real a, b, we obtain

$$\begin{aligned} \lambda_0 \operatorname{Re} \phi_x^4 &= a^4 - 6a^2b^2 + b^4 \geqq \tfrac{1}{2}a^4 - 17b^4 \geqq \tfrac{1}{2}(a^4+b^4) - 20b^4 \\ &\geqq -C + 2(a^2+b^2) - 20b^4 = -C + 3\lambda_0^{1/2} |\phi_x|^2 - 20\lambda_0 (\operatorname{Im} \phi_x)^4 , \end{aligned} \tag{7}$$

so that (5) for $n=0$ clearly follows for $C_2 \geqq \bar{C}_2(L^{4N_0})$.

We shall always assume that $L \geqq \bar{L}$, $N_0 \geqq \bar{N}_0(L)$, $C_0 \geqq \bar{C}_0(L)$, $C_2 \geqq \bar{C}_2(L, N_0)$, $C_1 \geqq C_1(L, N_0, C_2)$ and $n_0 \geqq \bar{n}_0(L, N_0, C_0, C_1, C_2)$. The initial ϕ^4 interaction given by (2.2) has the form described in Sect. 4 and for small $\lambda_0 > 0$ satisfies the inductive bounds.

The Main Technical Result

that we want to prove is the following: Suppose that V^n has the properties described in Sect. 4 and fulfills the bounds of the present section with $m_n^2 \in I_n \equiv [-(n_0+n)^{-3/2}, (n_0+n)^{3/2}]$. Then V^{n+1} has again the same properties with $n \mapsto n+1$ except for the new mass squared which satisfies

$$|m_{n+1}^2 - L^2 m_n^2| \leqq \mathcal{O}((n_0+n)^{-7/4}). \tag{8}$$

7. Inductive Step: Estimation of the Local Small Field Contribution to the New Effective Interaction

We shall start the proof of the above stated result from the analysis of W'_Δ, see (5.26). W'_Δ carries the main contribution to V'.

Let us assume that $\psi' \in \frac{3}{2} L \mathcal{K}(\Delta)$ and let $\mathcal{V}_{L\Delta}(\psi^0)$ denote the argument of the exponential on the right-hand side of (5.26). We recall that

$$\psi_x^0 = L^{-1}\psi'_{L^{-1}x} + \sum_{u \in L\Delta} \mathcal{M}_{xu} Z_u \quad \text{for} \quad x \in L\Delta. \tag{1}$$

Let

$$d\nu(Z_{L\Delta}) = \chi_0(Z_{L\Delta}) d\mu_1(Z_{L\Delta}) / \int \chi_0(Z_{L\Delta}) d\mu_1(Z_{L\Delta}). \tag{2}$$

It is straightforward to see that for Z in the support of $d\nu_{L\Delta}$, $\psi^0 \in 3\mathcal{K}(L\Delta)$. We shall compute W'_Δ perturbatively. Let us denote

$$\langle - \rangle_t = \int - \exp[-t\mathcal{V}_{L\Delta}(\psi^0)] d\nu(Z_{L\Delta}) / \int \exp[-t\mathcal{V}_{L\Delta}(\psi^0)] d\nu(Z_{L\Delta}). \tag{3}$$

Thus

$$\begin{aligned} W'_\Delta(\psi') &= -\log \int \exp[-\mathcal{V}_{L\Delta}(\psi^0)] d\nu(Z_{L\Delta}) - \log \int \chi_0(Z_{L\Delta}) d\mu_1(Z_{L\Delta}) \\ &= -\int_0^1 \frac{d}{dt} \log \int \exp[-t\mathcal{V}_{L\Delta}(\psi^0)] d\nu(Z_{L\Delta}) + \mathcal{O}_0(e^{-\varepsilon(n_0+n)^{1/2}}) \\ &= \langle \mathcal{V}_{L\Delta} \rangle_0 - \tfrac{1}{2} \langle \mathcal{V}_{L\Delta}; \mathcal{V}_{L\Delta} \rangle_0^T + \tfrac{1}{2} \int_0^t dt (1-t)^2 \langle \mathcal{V}_{L\Delta}; \mathcal{V}_{L\Delta}; \mathcal{V}_{L\Delta} \rangle_t^T \\ &\quad + \mathcal{O}_0(e^{-\varepsilon(n_0+n)^{1/2}}), \end{aligned} \tag{4}$$

where $\langle ; \ldots ; \rangle_t^T$ denotes the truncated expectation and the subscript at $\mathcal{O}(-)$ points to the order of the expression in ψ' [we use the weak definition of $\mathcal{O}(-)$ which does not exclude $o(-)$].

In estimating the expectations on the right-hand side of (4), we shall use the fact that if $\mathcal{U}_{\geqq kL\Delta}(\psi)$ is an expression of order $\geqq k$ in ψ which is bounded on $\mathcal{K}(L\Delta)$ by C, then, since $\mathcal{M}_{\cdot u}|_{L\Delta} \in \mathcal{O}((n_0+n)^{-1/4})\, \mathcal{K}(L\Delta)$,

$$\begin{aligned} \int |\mathcal{U}_{\geqq kL\Delta}(\mathcal{M}Z_{L\Delta})| d\nu(Z_{L\Delta}) &\leqq \frac{1}{(k-1)!} \int_0^1 dt (1-t)^{k-1} \int \left| \frac{d^k}{dt^k} \mathcal{U}_{\geqq kL\Delta}(t\mathcal{M}Z_{L\Delta}) \right| d\nu(Z_{L\Delta}) \\ &\leqq \frac{1}{(k-1)!} \int dt (1-t)^{k-1} \sum_{u_1, \ldots, u_k \in L\Delta} \\ &\quad \cdot \int \left| \prod_{i=1}^k \frac{\partial}{\partial t_i} \right|_{t_i \equiv 0} \mathcal{U}_{\geqq kL\Delta}\left(t\mathcal{M}Z_{L\Delta} + \sum_{i=1}^k t_i \mathcal{M}_{\cdot u_i}\right) \prod_{i=1}^k Z_{u_i} \Bigg| d\nu(Z_{L\Delta}) \leqq C\mathcal{O}((n_0+n)^{-k/4}), \end{aligned} \tag{5}$$

where we have used the Cauchy estimate to bound the t derivatives. Loosely speaking, the integral on the right-hand side of (5) contains at least $k/2$ contractions of pairs of Z fields and each such contraction yields an additional $\mathcal{O}((n_0+n)^{-1/2})$ factor as compared to the bound for the expression on $\mathcal{K}(L\Delta)$.

Let us start the estimation with the first term on the right-hand side of (4)

$$\begin{aligned}
\langle \mathcal{V}_{L\Delta}\rangle_0 &= \tfrac{1}{2}L^2m^2\int_\Delta dx(\psi'_x)^2 - 6L^2\lambda\int_\Delta dx\mathcal{G}_{LxLx}(\psi'_x)^2 \\
&+\lambda\int_\Delta dx(\psi'_x)^4 + 6\lambda L^2\sum_{u\in L\Delta}\int_\Delta dx(\mathcal{M}_{Lxu})^2\langle Z_u^2\rangle_0(\psi'_x)^2 \\
&+\sum_{Y\subset L\Delta}\Big[\tilde{V}_{2Y}(L^{-1}\psi'_{L^{-1}\cdot})+V_{4Y}(L^{-1}\psi'_{L^{-1}\cdot}) \\
&+\sum_{1\leqq i<j\leqq 4}\sum_{u\in L\Delta}\langle Z_u^2\rangle_0\tilde{V}_{4Y}(L^{-1}\psi_{L^{-1}\cdot},\ldots,\underset{i}{\mathcal{M}}_{\cdot u},\ldots,\underset{j}{\mathcal{M}}_{\cdot u},\ldots,L^{-1}\psi'_{L^{-1}\cdot}) \\
&+\langle\tilde{V}_{\geqq 6Y}(\psi^0)\rangle_0 - \langle\tilde{V}_{\geqq 6Y}(\psi^0)\rangle_0|_{\psi'=0}\Big] - 8L^2\lambda^2\int_\Delta dx\int_\Delta dy\mathcal{Q}_{LxLy}(\psi'_x)^3(\psi'_y)^3 \\
&-72L^4\lambda^2\sum_{u\in L\Delta}\langle Z_u^2\rangle_0\int_\Delta dx\int_\Delta dy\mathcal{Q}_{LxLy}\mathcal{M}_{Lxu}\mathcal{M}_{Lyu}(\psi'_x)^2(\psi'_y)^2 \\
&-48L^4\lambda^2\sum_{u\in L\Delta}\langle Z_u^2\rangle_0\int_\Delta dx\int_\Delta dy\mathcal{Q}_{LxLy}\mathcal{M}_{Lxu}\mathcal{M}_{Lyu}(\psi'_x)^3\psi'_y \\
&-72L^6\lambda^2\sum_{u_1,\ldots,u_4\in L\Delta}\langle Z_{u_1}\ldots Z_{u_4}\rangle_0\int_\Delta dx\int_\Delta dy\mathcal{Q}_{LxLy}\mathcal{M}_{Lxu_1} \\
&\cdot\mathcal{M}_{Lxu_2}\mathcal{M}_{Lyu_3}\mathcal{M}_{Lyu_4}\psi'_x\psi'_y \\
&-48L^6\lambda^2\sum_{u_1,\ldots,u_4\in L\Delta}\langle Z_{u_1}\ldots Z_{u_4}\rangle_0\int_\Delta dx\int_\Delta dy\mathcal{Q}_{LxLy}\mathcal{M}_{Lyu_1} \\
&\cdot\mathcal{M}_{Lyu_2}\mathcal{M}_{Lyu_3}\mathcal{M}_{Lyu_4}(\psi'_x)^2 + \langle\mathcal{V}_{L\Delta}\rangle_0|_{\psi'=0}\,. \qquad (6)
\end{aligned}$$

We shall estimate various terms of (6) with the use of (5). Note first that the kernel $\mathcal{Q}$ satisfies the following bound

$$\begin{aligned}
\int_{\Delta_1} dx\int_{\Delta_2} dy|\mathcal{Q}_{xy}| &\leqq \sum_{k=0}^{n-1}L^{2(n-k)}\int_{\Delta_1} dx\int_{\Delta_2} dy|\mathcal{T}_{kL^{-n}xL^{n-k}y}| \\
&\leqq \mathcal{O}(1)\exp[-\tfrac{1}{4}\beta\mathcal{L}(\Delta_1\cup\Delta_2)]\sum_{k=0}^{n-1}L^{2(n-k)}\int_{\Delta_1} dx\int_{\Delta_2} dy\exp[-\tfrac{1}{4}\beta L^{n-k}|x-y|] \\
&\leqq \mathcal{O}(1)\exp[-\tfrac{1}{4}\beta\mathcal{L}(\Delta_1\cup\Delta_2)]\sum_{k=0}^{n-1}L^{-2(n-k)}\int dy\, e^{-1/4\beta|y|} \\
&\leqq \mathcal{O}(1)\exp[-\tfrac{1}{4}\beta\mathcal{L}(\Delta_1\cup\Delta_2)]\,, \qquad (7)
\end{aligned}$$

where we have used the uniform exponential decay of $\mathcal{T}_k$ following from (2.19), (2.42), and (2.46). Inequality (7) and the inductive bounds of Sect. 6 give

$$\mathcal{V}_{L\Delta} = \mathcal{O}_2((n_0+n)^{-1/2}) + \mathcal{O}_4(1) + \mathcal{O}_{\geqq 6}((n_0+n)^{-1/2}) \qquad (8)$$

on small fields. Thus, by virtue of (5), the constant in (6) satisfies

$$\langle\mathcal{V}_{L\Delta}\rangle|_{\psi'=0} = \mathcal{O}((n_0+n)^{-1})\,. \qquad (9)$$

Let us pass to the quadratic contributions to $\langle \mathscr{V}_{L\Delta}\rangle_0$. Using (6.3), we obtain ($\psi' \in \frac{3}{2}L\mathscr{K}(\Delta)$!)

$$\sum_{Y\subset L\Delta}\sum_{i<j}\sum_{u\in L\Delta}\langle Z_u^2\rangle_0 \tilde{V}_{4Y}(L^{-1}\psi'_{L^{-1}\cdot},\dots,\underset{i}{\mathscr{M}_{\cdot u}},\dots,\underset{j}{\mathscr{M}_{\cdot u}},\dots,L^{-1}\psi'_{L^{-1}\cdot})| \leqq \mathcal{O}((n_0+n)^{-5/4}) \tag{10}$$

(one Z contraction provides additional $\mathcal{O}((n_0+n)^{-1/2})$). The quadratic contribution from $\sum_{Y\subset L\Delta}\langle \tilde{V}_{\geqq 6Y}(\psi^0)\rangle_0$ is

$$\frac{1}{2}\sum_{Y\subset L\Delta}\frac{d^2}{dt^2}\bigg|_{t=0}\langle \tilde{V}_{\geqq 6Y}(tL^{-1}\psi'_{L^{-1}\cdot}+\mathscr{M}Z_{L\Delta})\rangle_0 \tag{11}$$

and by virtue of (6.4) and (5) is bounded by $\mathcal{O}((n_0+n)^{-5/3})$ (there are at least two Z contractions here). The last but one and last but two terms on the right-hand side of (6) are, with the use of (8), bounded by $\mathcal{O}((n_0+n)^{-3/2})$.

The quartic contribution from $\sum_{Y\subset\Delta}\langle \tilde{V}_{\geqq 6Y}(\psi^0)\rangle_0$ is

$$\frac{1}{4!}\sum_{Y\subset L\Delta}\frac{d^4}{dt^4}\bigg|_{t=0}\langle \tilde{V}_{\geqq 6Y}(tL^{-1}\psi'_{L^{-1}\cdot}+\mathscr{M}Z_{L\Delta})\rangle_0 \tag{12}$$

and is bounded by $\mathcal{O}((n_0+n)^{-7/6})$ (at least one Z contraction).

The remainder of $\sum_{Y\subset\Delta}\langle \tilde{V}_{\geqq 6Y}(\psi^0)\rangle_0$ is

$$\frac{1}{5!}\sum_{Y\subset L\Delta}\int_0^1 dt(1-t)^5\frac{d^6}{dt^6}\langle \tilde{V}_{\geqq 6Y}(tL^{-1}\psi'_{L^{-1}\cdot}+\mathscr{M}Z_{L\Delta})\rangle_0 \equiv \tilde{W}'^{1}_{\geqq 6\Delta}(\psi')\,. \tag{13}$$

Since by (6.4)

$$\begin{aligned}\left|\sum_{Y\subset L\Delta}\langle \tilde{V}_{\geqq 6Y}(tL^{-1}\psi'_{L^{-1}\cdot}+\mathscr{M}Z_{L\Delta})\rangle_0\right| &\leqq (n_0+n)^{-2/3}\sum_{Y\subset L\Delta}\exp[-\alpha\mathscr{L}(Y)]\\ &\leqq (L^4+1)(n_0+n)^{-2/3}\end{aligned} \tag{14}$$

for $N_0 \geqq \bar{N}_0(L)$ and since the constant, quadratic and quartic contributions to this expressions were bounded [using (5)] by $\mathcal{O}((n_0+n)^{-13/6})$, $\mathcal{O}((n_0+n)^{-5/3})$ and $\mathcal{O}((n_0+n)^{-7/6})$ respectively,

$$|\tilde{W}'^{1}_{\geqq 6\Delta}(\psi')| \leqq (L^4+2)(n_0+n)^{-2/3} \tag{15}$$

(all the time for $\psi' \in \frac{3}{2}L\mathscr{K}(\Delta)$).

Finally, let us notice that $\langle \prod Z_{u_i}\rangle_0$ are given by their gaussian values with $\mathcal{O}(e^{-\varepsilon(n_0+n)^{1/2}})$ corrections. Denoting

$$\sum_{Y\subset L\Delta}\tilde{V}_{LY}(L^{-1}\psi'_{L^{-1}\cdot}) \equiv \tilde{W}'^{1}_{2\Delta}(\psi')\,, \tag{16}$$

$$\sum_{Y\subset L\Delta}\tilde{V}_{4Y}(L^{-1}\psi'_{L^{-1}\cdot}) \equiv \tilde{W}'^{1}_{4\Delta}(\psi')\,, \tag{17}$$

we may summarize the above discussion by writing

$$\begin{aligned}\langle \mathscr{V}_{L\Delta}\rangle_0 = {} & \mathscr{O}_0((n_0+n)^{-1}) + \tfrac{1}{2}L^2m^2 \int_\Delta dx(\psi'_x)^2 - 6L^2\lambda \int_\Delta dx\, \mathscr{G}_{LxLx}(\psi'_x)^2 \\ & + 6\lambda L^2 \sum_{u\in L\Delta} \int dx(\mathscr{M}_{Lxu})^2(\psi'_x)^2 + \tilde{W}'^{1}_{2\Delta}(\psi') + \mathscr{O}_2((n_0+n)^{-5/4}) \\ & + \lambda \int_\Delta dx(\psi'_x)^4 - 72L^4\lambda^2 \sum_{u\in L\Delta} \int_\Delta dx \int_\Delta dy\, \mathscr{Q}_{LxLy}\mathscr{M}_{Lxu}\mathscr{M}_{Lyu}(\psi'_x)^2(\psi'_y)^2 \\ & - 48L^4\lambda^2 \sum_{u\in L\Delta} \int_\Delta dx \int_\Delta dy\, \mathscr{Q}_{LxLy}\mathscr{M}_{Lyu}\mathscr{M}_{Lyu}(\psi')^3_x\psi'_y \\ & + \tilde{W}'^{1}_{4\Delta}(\psi') + \mathscr{O}_4((n_0+n)^{-7/6}) - 8L^2\lambda^2 \int_\Delta dx \int_\Delta dy\, \mathscr{Q}_{LxLy}(\psi'_x)^3(\psi'_y)^3 \\ & + \tilde{W}_{\geqq 6\Delta}(\psi')\,. \end{aligned} \tag{18}$$

The second term on the right-hand side of (4), $-\frac{1}{2}\langle \mathscr{V}_{L\Delta}; \mathscr{V}_{L\Delta}\rangle_0^T$, is treated similarly as the first one with the additional use of the fact that $\mathscr{V}_{L\Delta}(L^{-1}\psi'_{L^{-1}\cdot})$ (i.e. $Z=0$ term) does not contribute to the truncated expectation. A straightforward analysis yields

$$\begin{aligned} -\tfrac{1}{2}\langle \mathscr{V}_{L\Delta}; \mathscr{V}_{L\Delta}\rangle_0^T = {} & \mathscr{O}_0((n_0+n)^{-2}) + \mathscr{O}_2((n_0+n)^{-3/2}) \\ & - 36L^4\lambda^2 \sum_{u_1,u_2\in L\Delta} \int_\Delta dx \int_\Delta dy\, \mathscr{M}_{Lxu_1}\mathscr{M}_{Lyu_1}\mathscr{M}_{Lxu_2}\mathscr{M}_{Lyu_2}(\psi'_x)^2(\psi'_y)^2 \\ & - 48L^4\lambda^2 \sum_{u_1,u_2\in L\Delta} \int_\Delta dx \int_\Delta dy\, \mathscr{M}_{Lxu_1}\mathscr{M}_{Lyu_1}\mathscr{M}_{Lyu_2}\mathscr{M}_{Lyu_2}(\psi'_x)^3\psi'_y \\ & + 48L^4\lambda^2 \sum_{u\in L\Delta} \int_\Delta dx \int_\Delta dy\, \mathscr{G}_{LyLy}\mathscr{M}_{Lxu}\mathscr{M}_{Lyu}(\psi'_x)^3\psi'_y + \mathscr{O}_4((n_0+n)^{-3/2}) \\ & - 8L^2\lambda^2 \sum_{u\in L\Delta} \int_\Delta dx \int_\Delta dy\, \mathscr{M}_{Lxu}\mathscr{M}_{Lyu}(\psi'_x)^3(\psi'_y)^3 + \mathscr{O}_{\geqq 6}((n_0+n)^{-1})\,. \end{aligned} \tag{19}$$

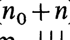
Equation (19) has been obtained by writing $\mathscr{V}_{L\Delta}$ as

$$\mathscr{O}_2((n_0+n)^{-1/2}) + \mathscr{O}_4(1) + \mathscr{O}_2((n_0+n)^{-1}) + \mathscr{O}_4((n_0+n)^{-3/4}) + \mathscr{O}_{\geqq 6}((n_0+n)^{-1/2})\,,$$

accounting explicitly for some contributions involving the first two terms and estimating the others with the use of (5) which implies that $\langle \mathscr{O}_k((n_0+n)^p); \mathscr{O}_l((n_0+n)^q)\rangle_0^T$ is a sum of $\mathscr{O}_m((n_0+n)^{p+q-1/4(k+l-m)})$.

Finally, let us consider the contribution of the third term on the right-hand side of (4). Let us notice that since for $\psi' \in \frac{3}{2}L\mathscr{K}(\Delta)$ and for Z in the support of dv, $|\mathscr{V}_{L\Delta}| \leqq \mathscr{O}(1)$, when estimating non-vanishing terms $\langle i; j; k\rangle_t^T$, we may undo the truncation and replace the $\langle\ \rangle_t$ expectations by the $t=0$ ones, provided we replace the integrands by their absolute values and multiply the whole expression by $\mathscr{O}(1)$. Thus we easily see that

$$\begin{aligned} \langle \mathscr{V}_{L\Delta}; \mathscr{V}_{L\Delta}; \mathscr{V}_{L\Delta}\rangle_t^T = {} & \mathscr{O}_0((n_0+n)^{-3}) + \mathscr{O}_2((n_0+n)^{-5/2}) \\ & + \mathscr{O}_4((n_0+n)^{-2}) + \mathscr{O}_{\geqq 6}((n_0+n)^{-3/4}) \end{aligned} \tag{20}$$

(in fact the last term can be replaced by $\mathscr{O}_{\geqq 6}((n_0+n)^{-1})$ since the leading third order contribution to it comes from the diagram  which can be exhibited by integration by parts).

Gathering (4), (18)–(20) and denoting

$$\mathcal{T}_{\Delta xy} \equiv \sum_{u \in L\Delta} \mathcal{M}_{xy} \mathcal{M}_{yu} \tag{21}$$

[cf. (2.19) and (2.25)], we obtain

$$\begin{aligned} W'_\Delta(\psi') = {} & \mathcal{O}_0((n_0+n)^{-1}) + \tfrac{1}{2}L^2 m^2 \int_\Delta dx (\psi'_x)^2 - 6L^2\lambda \int_\Delta dx \mathcal{G}_{LxLx}(\psi'_x)^2 \\ & + 6L^2\lambda \int dx \mathcal{T}_{\Delta LxLx}(\psi'_x)^2 + \tilde{W}'^1_{2\Delta}(\psi') + \mathcal{O}_2((n_0+n)^{-5/4}) \\ & + \lambda \int_\Delta dx (\psi'_x)^4 - 72L^4\lambda^2 \int_\Delta dx \int_\Delta dy \mathcal{Q}_{LxLy} \mathcal{T}_{\Delta LxLy}(\psi'_x)^2(\psi'_y)^2 \\ & - 48L^4\lambda^2 \int_\Delta dx \int_\Delta dy \mathcal{Q}_{LxLy} \mathcal{T}_{\Delta LyLy}(\psi'_x)^3 \psi'_y \\ & - 36L^4\lambda^2 \int_\Delta dx \int_\Delta dy (\mathcal{T}_{\Delta LxLy})^2 (\psi'_x)^2(\psi'_y)^2 \\ & - 48L^4\lambda^2 \int_\Delta dx \int_\Delta dy \mathcal{T}_{\Delta LxLy} \mathcal{T}_{\Delta LyLy}(\psi'_x)^3 \psi'_y \\ & + 48L^4\lambda^2 \int_\Delta dx \int_\Delta dy \mathcal{G}_{LyLy} \mathcal{T}_{\Delta LxLy}(\psi'_x)^3 \psi'_y \\ & + \tilde{W}'^1_{4\Delta}(\psi') + \mathcal{O}_4((n_0+n)^{-7/6}) - 8L^2\lambda^2 \int_\Delta dx \int_\Delta dy \mathcal{Q}_{LxLy}(\psi'_x)^3(\psi'_y)^3 \\ & - 8L^2\lambda^2 \int_\Delta dx \int_\Delta dy \mathcal{T}_{\Delta LxLy}(\psi'_x)^3(\psi'_y)^3 + \tilde{W}'_{\geqq 6\Delta}(\psi') \end{aligned} \tag{22}$$

for $\psi' \in L\mathcal{K}(\Delta)$. In (22)

$$\tilde{W}'_{\geqq 6\Delta} = \tilde{W}'^1_{\geqq 6\Delta} + \mathcal{O}_{\geqq 6}((n_0+n)^{-3/4}). \tag{23}$$

Clearly analyticity of W'_Δ on $\frac{3}{2}L\mathcal{K}(\Delta)$ also follows.

By (23) and (15),

$$|\tilde{W}'_{\geqq 6\Delta}(\psi')| \leqq (L^4+3)(n_0+n)^{-2/3} \tag{24}$$

on $\frac{3}{2}L\mathcal{K}(\Delta)$.

This ends the estimation of W'_Δ. Let us notice that (22) contains the same terms of the second order in λ as (3.3) except for the localization of the "hard" propagators, $\mathcal{T} \equiv \mathcal{A}Q\Gamma Q^+\mathcal{A}^+ \mapsto \chi_{L\Delta}\mathcal{A}Q\Gamma^{1/2}\chi_{L\Delta}\Gamma^{1/2}Q^+\mathcal{A}^+\chi_{L\Delta}$. We shall recover the missing non-local parts of them in the non-local contributions W'_Y, $Y \neq \Delta$.

8. Inductive Step: Estimation of the Non-Local Small Field Contributions to the New Effective Interactions

In the present section, we shall estimate the small field ($D' = \emptyset$) polymer activities $\varrho_{X'}(\psi')$ for $\psi' \in \frac{1}{2}L\mathcal{K}(X')$ ($\subset \frac{1}{2}L\mathcal{D}(\emptyset, X')$), as given by (5.25). We shall be somewhat sketchy, since the analysis is straightforward and has been done in detail for the $(\nabla\phi)^4$ model in Sect. 5 of [17] with comparison to which the present case is only slightly more tedious.

First notice that for $D' = \emptyset$, $D = R$ in $g^D_{X_i}(\psi^s)$ appearing under the integral on the right-hand side of (5.25): ψ^s can be large only if the fluctuation field $Z_{LX'}$ is large.

Consider the terms of the expansion with $R \neq \emptyset$. For $g^D_{X_i}(\psi^s)$ we shall use the bound (6.5). Since we would like to bound the s-derivatives occuring in $S(U)$ by the Cauchy estimate, we have admitted complex values of s, see (5.13). This might seem dangerous as the factor $\exp\left[20\lambda \int\limits_{R\cap X_i} (\mathrm{Im}\,\psi^s)^4\right]$ gives a term $\exp[\mathcal{O}(Z^4)]$ potentially non-integrable in R. But [see (5.11), (5.12), and (5.14)]

$$\mathrm{Im}\,\psi^s = L^{-1}\,\mathrm{Im}\,\psi'_{L^{-1}\cdot} + \mathrm{Im}\,\mathscr{Z}^s_2\,, \tag{1}$$

and hence by virtue of (5.19) is bounded,

$$|\mathrm{Im}\,\psi^s_x| \leqq \tfrac{3}{4}C_1(n_0+n)^{-1/4}\,. \tag{2}$$

Thus (6.5) implies

$$|g^R_{X_i}(\psi^s)| \leqq \exp[\mathcal{O}(1)|R\cap X_i| - \alpha\mathscr{L}(X_i)]\,. \tag{3}$$

For the mass and Wick ordering quadratic terms in (5.25), we easily get

$$\left|\exp\left[-\tfrac{1}{2}m^2 \int\limits_{LX'} dx(\psi^s_x)^2 + 6\lambda \int\limits_{LX'} dx\,\mathscr{G}_{xx}(\psi^s_x)^2\right]\right|$$
$$\leqq \exp\left[\mathcal{O}((n_0+n)^{-1/2})|X'| + \mathcal{O}((n_0+n)^{-1}) \sum_{u\in LX'} Z^2_u\right]. \tag{4}$$

By (4.12) and (5.19), $\mathscr{Z}^s$ (as well as $\psi'_{L^{-1}\cdot}$) is small, i.e. bounded by $\mathcal{O}((n_0+n)^{1/4})$ outside R. Hence

$$\left|\exp\left[-\lambda \int\limits_{LX'\setminus R} (\psi^s)^4\right]\right| \leqq \left|\exp\left[-\lambda \int\limits_{X'} (\psi')^4 + \mathcal{O}(1)|R|\right.\right.$$
$$\left.\left. + \mathcal{O}((n_0+n)^{-1/4}) \int\limits_{LX'\setminus R} |\mathscr{Z}^s|\right]\right| \leqq \left|\exp\left[-\lambda \int\limits_{X'} (\psi')^4\right]\right| \exp\left[\mathcal{O}(1)|R|\right.$$
$$\left. + \mathcal{O}((n_0+n)^{-1/4})|X'| + \mathcal{O}((n_0+n)^{-1/4}) \sum_{u\in LX'} Z^2_u\right]. \tag{5}$$

Also, by (7.22),

$$\left|\exp\left[\sum_{\Delta\subset X'} W'_\Delta\right]\right| \leqq \left|\exp\left[\lambda \int\limits_{X'} (\psi')^4\right]\right| \exp[\mathcal{O}((n_0+n)^{-1/2})|X'|]\,. \tag{6}$$

Furthermore, since $Y_\beta \subset LX'\setminus R$, and consequently (again by (4.12) and (5.19)) $\psi^s \in 3\mathscr{K}(LX'\setminus R)$,

$$|\tilde{V}_{\geqq 4Y_\beta}(\psi^s)| \leqq \mathcal{O}((n_0+n)^{-1/2})\exp[-\alpha\mathscr{L}(Y)] \tag{7}$$

and

$$|\exp[-\tilde{V}_{\geqq 4Y_\beta}(\psi^s)] - 1| \leqq \mathcal{O}((n_0+n)^{-1/2})\exp[-\alpha\mathscr{L}(Y_\beta)]\,. \tag{8}$$

Finally, we have to treat the $\tilde{V}_{2Y}(\psi^s)$ contributions under the integral of (5.25). Since Y might intersect R, they need a careful treatment. It is convenient to use the following general relation.

Lemma. *If $\psi \in \mathscr{D}(D,X)$ and $Y \subset X$, then*

$$\psi \in \left(\mathcal{O}(1) + \mathcal{O}((n_0+n)^{-1/4}) \sum_{\substack{x\in D\cap X\\ x\in\mathbb{Z}^4}} \exp[-\beta d(Y,x)] \int\limits_{\square_x} |\psi|\right) \mathscr{K}(Y)\,, \tag{9}$$

where, as always, $\square_x$ denotes the unit cube centered at x.

Notice that in particular if $Y \cap D = \emptyset$, (9) implies that $\mathscr{D}(D,X)|_Y \subset \mathcal{O}(1)\mathscr{K}(Y)$ which is a weaker version of (4.12).

Proof of Lemma. By the definition (4.11) of $\mathscr{D}(D,X)$,

$$\psi = \mathscr{A}\phi + \tilde{\psi}, \tag{10}$$

where $D(\mathscr{A}\phi) \subset D$ and $\tilde{\psi} \in \mathscr{K}(X)$. Now for $x \in Y$,

$$\begin{aligned}|(\mathscr{A}\phi)_x| &\leqq \mathcal{O}(1) \sum_{x \in \Lambda_n} e^{-\beta|x-x|}|\phi_x| \\ &\leqq \mathcal{O}(1) \sum_x e^{-\beta|x-x|} \int_{\square_x} |\mathscr{A}\phi| \leqq \mathcal{O}(1) \sum_{x \in D \cap X} e^{-\beta|x-x|} \int_{\square_x} |\mathscr{A}\phi| \\ &\quad + \mathcal{O}(1) \sum_{x \notin D \cap X} e^{-\beta|x-x| + \frac{1}{10}\alpha|x-x|}(n_0+n)^{1/4} \\ &\leqq \mathcal{O}(1) \sum_{x \in D \cap X} e^{-\beta d(Y,x)} \int_{\square_x} |\psi| + \mathcal{O}((n_0+n)^{1/4}),\end{aligned} \tag{11}$$

where we have used in turn: (2.42), (2.21), (4.10) together with the fact that for $x \notin D \cap X$, $d(x, \sim D) \leqq d(x,X)$ and smallness of $\tilde{\psi}$. Similarly we show that for $x, y \in Y$, $|(\mathscr{A}\phi)_x - (\mathscr{A}\phi)_y|/|x-y|$ is bounded by the right-hand side of (9) as well as $|(\partial_\mu \mathscr{A}\phi)_x - (\partial_\mu \mathscr{A}\phi)_y|/|x-y|^{2/3}$ if additionally $x + L^{-n}e_\mu$ and $y + L^{-n}e_\mu$ are in Y. Equation (9) follows now from the definition (4.1) of $\mathscr{K}(Y)$.

Let us see how (9) works. Since on $\mathscr{K}(Y)$

$$|\tilde{V}_{2Y}(\psi)| \leqq \mathcal{O}((n_0+n)^{-1}) \exp[-\alpha\mathscr{L}(Y)], \tag{12}$$

see (4.1) and (6.1) and since, by (5.19), $\psi^s \in \mathscr{D}(R_k, U_k)$ then, for $Y \subset U_k$, (9) implies

$$\begin{aligned}|\tilde{V}_{2Y}(\psi^s)| &\leqq \left[\mathcal{O}((n_0+n)^{-1}) + \mathcal{O}((n_0+n)^{-3/2})\left(\sum_{x \in R_k} \exp[-\beta d(Y,x)] \cdot \int_{\square_x} |\psi^s|\right)^2\right] \exp[-\alpha\mathscr{L}(Y)] \\ &\leqq \mathcal{O}((n_0+n)^{-1}) \exp[-\alpha\mathscr{L}(Y)] + \mathcal{O}((n_0+n)^{-3/2}) \exp[-\alpha\mathscr{L}(Y)] \\ &\quad \cdot \sum_{x,y \in R_k} \exp[-\beta d(Y,x) - \beta d(Y,y)] \int_{\square_x} |\psi^s| \int_{\square_y} |\psi^s|.\end{aligned} \tag{13}$$

Now if Y_α is not in a single X_i (or in a simple c.c. of R), then

$$d(x, \sim R_k) \leqq d(Y,x) + \mathscr{L}(Y_\alpha) + \mathcal{O}(1) \tag{14}$$

and

$$d(y, \sim R_k) \leqq d(Y,y) + \mathscr{L}(Y_\alpha) + \mathcal{O}(1). \tag{15}$$

Hence, since $\psi^s \in \mathscr{D}(R_k, U_k)$,

$$\exp[-\tfrac{1}{10}\alpha d(Y_\alpha,x) - \tfrac{1}{10}\alpha d(Y_\alpha,y) - \tfrac{1}{5}\alpha\mathscr{L}(Y_\alpha)] \int_{\square_x} |\psi^s| \int_{\square_y} |\psi^s| \leqq \mathcal{O}((n_0+n)^{1/4}) \tag{16}$$

and

$$|\tilde{V}_{2Y_\alpha}(\psi^s)| \leqq \mathcal{O}((n_0+n)^{-1}) \exp[-\tfrac{3}{4}\alpha\mathscr{L}(Y_\alpha)], \tag{17}$$

$$|\exp[-\tilde{V}_{2Y_\alpha}(\psi^s)] - 1| \leqq \mathcal{O}((n_0+n)^{-1}) \exp[-\tfrac{3}{4}a\mathscr{L}(Y_\alpha)]. \tag{18}$$

For other terms $\tilde{V}_{2Y}(\psi^s)$ appearing in (5.25), namely, those with $Y\subset X_i$, we get from (11) applying the Schwartz inequality

$$|\tilde{V}_{2Y_\alpha}(\psi^s)|\leqq \mathcal{O}((n_0+n)^{-1})\exp[-\alpha\mathcal{L}(Y)]+\mathcal{O}((n_0+n)^{-3/2})\exp[-\tfrac{3}{4}\alpha\mathcal{L}(Y)] \cdot \sum_{x\in R}\exp[-\beta d(Y,x)]\int_{\square_x}|\psi^s|^2 \tag{19}$$

and

$$\left|\sum_i\sum_{Y\subset X_i}\tilde{V}_{2Y}(\psi^s)\right|\leqq \mathcal{O}((n_0+n)^{-1})|X'|+\mathcal{O}((n_0+n)^{-3/2})\sum_{x\in R}\int_{\square_x}|\psi^s|^2$$
$$\leqq \mathcal{O}((n_0+n)^{-1})|X'|+\mathcal{O}((n_0+n)^{-3/2})\sum_{u\in LX'}Z_u^2\,. \tag{20}$$

Gathering (3)–(8), (18), and (20) and using the Cauchy estimates to bound the s-derivatives of $S(\mathcal{U})$ of (5.25), we obtain

$$|\varrho_{X'}|\leqq \sum_p\sum_{\{X_i\}}\sum_{\{Y_\alpha\}}\sum_{\{Y_\beta\}}\sum_{\Gamma_c=\{(k,k')\}}\prod_{(k,k')\in\Gamma_c}(r^{-1}\exp[-\tfrac{1}{2}\beta d(U_k,U_{k'})])$$
$$\cdot\exp\Big[\mathcal{O}(1)|R|+\mathcal{O}((n_0+n)^{-1/4})|X'|-\alpha\sum_i\mathcal{L}(X_i)$$
$$-\tfrac{3}{4}\alpha\sum_\alpha\mathcal{L}(Y_\alpha)-\alpha\sum_\beta\mathcal{L}(Y_\beta)\Big]\prod_\alpha\mathcal{O}((n_0+n)^{-1})\prod_\beta\mathcal{O}((n_0+n)^{-1/2})$$
$$\cdot\int\exp\Big[\mathcal{O}((n_0+n)^{-1/4})\sum_{u\in LX'}Z_u^2\Big]\chi_{\bar{p}}(Z_{LX'})d\mu_1(Z_{LX'})\,. \tag{21}$$

It is straightforward, compare formula (5.38) of [17], to show that the sum of terms with $p\neq 0$ on the right-hand side of (21) [for $r>\bar{r}_0(L,N_0)$] is bounded by

$$\mathcal{O}(e^{-\varepsilon(n_0+n)^{1/2}})\exp[-7\alpha\mathcal{L}(X')]\,. \tag{22}$$

The non-perturbatively small factor $\mathcal{O}(e^{-\varepsilon(n_0+n)^{1/2}})$ comes from small Gaussian probability of large Z. For $p\neq 0$ the integral on the right-hand side of (21) is bounded by $\mathcal{O}\left(\exp\left[-\varepsilon(n_0+n)^{1/2}\sum_u p_u^2\right]\right)$. This allows us also to control $\sum_{\bar{p}}$ and provides a decay factor in the size of R which together with the other decay factors allows us to extract, say, $\exp[-\tfrac{1}{2}\alpha\mathcal{L}(LX')]$ out of the whole expression. This factor is bounded in turn by $\mathcal{O}(1)\exp[-\mathcal{O}(L)\alpha\mathcal{L}(X')]$, where, by taking $L\geqq L_0$, we have replaced $\mathcal{O}(L)$ by 7. Sums over $\{X_i\}$, $\{Y_\alpha\}$, $\{Y_\beta\}$, and Γ_c are controlled by the remaining decay, and other small factors.

Let us consider now the terms of (5.25) with $p=0$. Their analysis is very similar to that of (7.4). The only substantial difference is that we usually stay in a big volume, with $\prod_{k,k'}\exp[-\tfrac{1}{4}\beta d(U_k,U_{k'})]$ coming from the s-derivatives through the Cauchy estimates and the tree decay factors of $\tilde{V}_{2Y_\alpha}$ or $\tilde{V}_{\geqq 4Y_\beta}$ [together with the connectivity constraints imposed on the sums in (2.25)] providing the desired decay factor $\exp[-7\alpha\mathcal{L}(X')]$ for the contributions to $\varrho_{X'}(\psi')$. As in (7.22), some of the terms of the lowest order will be listed explicitly. Thus we list

a) the quadratic terms linear in λ produced by a single s-derivative (no Y_α, no Y_β),

b) the quadratic irrelevant terms coming from a single Y_α contributing $\tilde{V}_{2Y_\alpha}(L^{-1}\psi'_{L^{-1}\cdot})$ (no s-derivatives, no Y_β's),

c) the quartic terms coming from a single Y_β contributing $\tilde{V}_{6Y_\beta}(\psi^s)$ with at most two s-derivatives and no Y_α,

d) the quartic terms proportional to λ^2 coming from the contributions with no Y_α and no Y_β and s-derivatives hitting at most twice the exponential of the local part of $V(\psi^s)$, partially cancelled by terms of a) multiplied by the terms $6L^2\lambda \int dx \mathcal{T}_{\Delta Lx Lx}(\psi'_x)^2$ coming from the expansion of exp $[W'_\Delta]$,

e) the quartic irrelevant terms coming from a single Y_β contributing $\tilde{V}_{4Y_\beta}(L^{-1}\psi'_{L^{-1}\cdot})$ (no s-derivatives, no Y_α's),

f) the sixth order terms proportional to λ^2 coming from the contributions with no Y_α, no Y_β and s-derivatives hitting at most twice the exponential of the local part of $V(\psi^s)$,

g) the sixth order terms coming from a single Y_β contributing $\tilde{V}_{6Y_\beta}(L^{-1}\psi'_{L^{-1}\cdot})$ (no s-derivatives, no Y_α's).

The terms which are not listed are of higher order and we shall extract from their sums explicit powers of $(n_0+n)^{-1}$ using an analogue of (7.5) (one contraction again produces an extra $\mathcal{O}((n_0+n)^{-1/2})$. Let us denote for $i=2, 4$

$$\tilde{W}'^{1}_{iY'}(\psi') = \sum_{Y:(L^{-1}Y)^- = Y'} \tilde{V}_{iY}(L^{-1}\psi'_{L^{-1}\cdot}), \tag{23}$$

compare (5.30), (7.16), and (7.17). A straightforward analysis sketched above (we leave the details as an exercise) results in the following representation for $\varrho_{X'}(\psi')$ into which the $p \neq 0$ contribution bounded by (22) has been also absorbed:

$$\begin{aligned}
\varrho_{X'}(\psi') = {} & \mathcal{O}((n_0+n)^{-1})\exp[-7\alpha\mathcal{L}(X')] \\
& -6L^2\lambda \sum_{(\Delta_1,\Delta_2)} \int_{\Delta_1} dx \mathcal{T}_{\Delta_2 Lx Lx}(\psi'_x)^2 - \tilde{W}'^{1}_{2X'}(\psi') + \mathcal{O}_2((n_0+n)^{-5/4}) \\
& \cdot \exp[-7\alpha\mathcal{L}(X')] + 72L^4\lambda^2 \sum_{(\Delta_1,\Delta_2,\Delta_3)} \int_{\Delta_1} dx \int_{\Delta_2} dy \mathcal{Q}_{LxLy}\mathcal{T}_{\Delta_3 Lx Ly}(\psi'_x)^2(\psi'_y)^2 \\
& +48L^4\lambda^2 \sum_{(\Delta_1,\Delta_2,\Delta_3)} \int_{\Delta_1} dx \int_{\Delta_2} dy \mathcal{Q}_{LxLy}\mathcal{T}_{\Delta_3 Ly Ly}(\psi'_x)^3\psi'_y \\
& +18L^4\lambda^2 \sum_{\substack{(\Delta_1,\dots,\Delta_4) \\ \Delta_1 \neq \Delta_3, \Delta_2 \neq \Delta_4 \\ (\Delta_1\cup\Delta_3)\cap(\Delta_2\cup\Delta_4) \neq \emptyset}} \int_{\Delta_1} dx \int_{\Delta_2} dy \mathcal{T}_{\Delta_3 Lx Lx}\mathcal{T}_{\Delta_4 Ly Ly}(\psi'_x)^2(\psi'_y)^2 \\
& +36L^4\lambda^2 \sum_{(\Delta_1,\dots,\Delta_4)} \int_{\Delta_1} dx \int_{\Delta_2} dy \mathcal{T}_{\Delta_3 Lx Ly}\mathcal{T}_{\Delta_4 Lx Ly}(\psi'_x)^2(\psi'_y)^2 \\
& +48L^4\lambda^2 \sum_{(\Delta_1,\dots,\Delta_4)} \int_{\Delta_1} dx \int_{\Delta_2} dy \mathcal{T}_{\Delta_3 Lx Ly}\mathcal{T}_{\Delta_4 Ly Ly}(\psi'_x)^3\psi'_y \\
& -48L^4\lambda^2 \sum_{(\Delta_1,\Delta_2,\Delta_3)} \int_{\Delta_1} dx \int_{\Delta_2} dy \mathcal{G}_{LyLy}\mathcal{T}_{\Delta_3 Lx Ly}(\psi'_x)^3\psi'_y \\
& -\tilde{W}'^{1}_{4X'}(\psi') + \mathcal{O}_4((n_0+n)^{-7/6})\exp[-7\alpha\mathcal{L}(X')] \\
& +8L^2\lambda^2 \sum_{(\Delta_1,\Delta_2)} \int_{\Delta_1} dx \int_{\Delta_2} dy \mathcal{Q}_{LxLy}(\psi'_x)^3(\psi'_y)^3 \\
& +8L^2\lambda^2 \sum_{(\Delta_1,\Delta_2,\Delta_3)} \int_{\Delta_1} dx \int_{\Delta_2} dy \mathcal{T}_{\Delta_3 Lx Ly}(\psi'_x)^3(\psi'_y)^3 \\
& -\tilde{W}'^{1}_{\geq 6X'}(\psi') + \mathcal{O}_{\geq 6}((n_0+n)^{-3/4})\exp[-7\alpha\mathcal{L}(Y)]
\end{aligned} \tag{24}$$

for $\psi' \in \frac{1}{2} L \mathscr{K}(X')$, where in the sums (Δ_i) are such that $\cup \Delta_i = X'$ [if there are none such (Δ_i), the term does not appear],

$$\tilde{W}'^{1}_{\geqq 6Y'}(\psi') \equiv \frac{1}{5!} \sum_{Y:(L^{-1}Y)^{-}=Y'} \int_0^1 dt (1-t)^5 \frac{d^6}{dt^6} \int \tilde{V}_{\geqq 6Y}(tL^{-1}\psi'_{L^{-1}\cdot} + \mathscr{M} Z_{LY'}) \chi_0(Z_{LY'}) d\mu_1(Z_{LY'}) . \tag{25}$$

As in (7.14) and (7.15), for $\psi' \in \frac{1}{2} L \mathscr{K}(Y)$, we have

$$|\tilde{W}'^{1}_{\geqq 6Y}(\psi')| \leqq \begin{cases} (L^3+2)(n_0+n)^{-2/3} \exp[-\alpha \mathscr{L}(Y)] , \\ \mathcal{O}((n_0+n)^{-2/3}) \qquad \exp[-7\alpha \mathscr{L}(Y)] , \end{cases} \tag{26}$$

where the factor L^3 is due to the presence of at most L^3 Y's in the sum of (25) with $\mathscr{L}(Y) = \mathscr{L}(Y')$.

It is instructive to compare (24) with (7.22) or (3.3) to find out that it contains, besides the non-local terms of the second order in λ missing in (7.22), also a term corresponding to a disconnected second order perturbative diagram (the third quartic term).

Given (24), (5.28) implies a similar representation for W'_Y (with $Y \neq \Delta$), compare (5.54) of [17]. The difference, besides the change $X' \mapsto Y$, consists of the disappearance of the perturbative quartic term corresponding to the disconnected diagram cancelled by the $\Xi = 2$ contribution of (5.28) and of replacement $7\alpha \to 6\alpha$ [a fraction of the tree decay factors is used to bound the sums on the right-hand side of (5.28)].

Thus for $\psi' \in \frac{1}{2} L \mathscr{K}(Y)$,

$$\begin{aligned} W'_Y(\psi') = & \, \mathcal{O}_0((n_0+n)^{-1}) \exp[-6\alpha \mathscr{L}(Y)] \\ & + 6L^2 \lambda \sum_{(\Delta_1,\Delta_2)} \int_{\Delta_1} dx \mathscr{T}_{\Delta_2 Lx Lx} (\psi'_x)^2 \\ & + \tilde{W}'^{1}_{2Y}(\psi') + \mathcal{O}_2((n_0+n)^{-5/4}) \exp[-6\alpha \mathscr{L}(Y)] \\ & - 72L^4\lambda^2 \sum_{(\Delta_1,\dots,\Delta_3)} \int_{\Delta_1} dx \int_{\Delta_2} dy \mathscr{Q}_{LxLy} \mathscr{T}_{\Delta_3 Lx Ly} (\psi'_x)^2 (\psi'_y)^2 \\ & - 48L^4\lambda^2 \sum_{(\Delta_1,\dots,\Delta_3)} \int_{\Delta_1} dx \int_{\Delta_2} dy \mathscr{Q}_{LxLy} \mathscr{T}_{\Delta_3 Ly Ly} (\psi'_x)^3 \psi'_y \\ & - 36L^4\lambda^2 \sum_{(\Delta_1,\dots,\Delta_4)} \int_{\Delta_1} dx \int_{\Delta_2} dy \mathscr{T}_{\Delta_3 Lx Ly} \mathscr{T}_{\Delta_4 Lx Ly} (\psi'_x)^2 (\psi'_y)^2 \\ & - 48L^4\lambda^2 \sum_{(\Delta_1,\dots,\Delta_4)} \int_{\Delta_1} dx \int_{\Delta_2} dy \mathscr{T}_{\Delta_3 Lx Ly} \mathscr{T}_{\Delta_4 Ly Ly} (\psi'_x)^3 \psi'_y \\ & + 48L^4\lambda^2 \sum_{(\Delta_1,\dots,\Delta_3)} \int_{\Delta_1} dx \int_{\Delta_2} dy \mathscr{G}_{LyLy} \mathscr{T}_{\Delta_3 Lx Ly} (\psi'_x)^3 (\psi'_y)^3 \\ & + \tilde{W}'^{1}_{4Y}(\psi') + \mathcal{O}_4((n_0+n)^{-7/6}) \exp[-6\alpha \mathscr{L}(Y)] \\ & - 8L^2\lambda^2 \sum_{(\Delta_1,\Delta_2)} \int_{\Delta_1} dx \int_{\Delta_2} dy \mathscr{Q}_{LxLy} (\psi'_x)^3 (\psi'_y)^3 \\ & - 8L^2\lambda^2 \sum_{(\Delta_1,\dots,\Delta_3)} \int_{\Delta_1} dx \int_{\Delta_2} dy \mathscr{T}_{\Delta_3 Lx Ly} (\psi'_x)^3 (\psi'_y)^3 \\ & + \tilde{W}'_{\geqq 6Y}(\psi') , \end{aligned} \tag{28}$$

where the sums over (Δ_i) are restricted by requiring that $\cup \Delta_i = Y$.

Notice that the quadratic terms linear in λ, quartic ones linear and quadratic in λ as well as the sixth order contributions proportional to λ^2 on the right-hand side of (7.22) and (28) are localizations of the corresponding terms of (3.3) (with $\zeta = 1$). For the last term of (28), we have

$$\tilde{W}'_{\geqq 6Y} = \tilde{W}'^{1}_{\geqq 6Y} + \mathcal{O}_{\geqq 6}((n_0+n)^{-3/4}) \exp[-6\alpha \mathcal{L}(Y)] \tag{29}$$

on $\frac{1}{2}L\mathcal{K}(Y)$. Thus, by (26),

$$|\tilde{W}'_{\geqq 6Y}| \leqq \begin{cases} (L^3+3)(n_0+n)^{-2/3} \exp[-\alpha \mathcal{L}(Y)], \\ \mathcal{O}((n_0+n)^{-2/3} \exp[-6\alpha \mathcal{L}(Y)]. \end{cases} \tag{30}$$

Of course the analyticity of W'_Y on $\frac{1}{2}L\mathcal{K}(Y)$, and hence also of $\tilde{W}'_{\geqq 6Y}$ follows. Since $z \mapsto z^{-6} \tilde{W}'_{\geqq 6Y}(z\psi')$ is an analytic function, we infer from the maximum principle that by restricting ourselves to $\psi' \in 4\mathcal{K}(Y)$ we gain an additional factor $(4L^{-1})^6$ on the right-hand side of (30). So, for L big enough,

$$|\tilde{W}'_{\geqq 6Y}| \leqq \tfrac{1}{2}(n_0+n)^{-2/3} \exp[-\alpha \mathcal{L}(Y)] \tag{31}$$

on $4\mathcal{K}(Y)$. By virtue of (7.24), this holds also for $Y = \Delta$. Estimate (31) expresses the irrelevance of the sixth and higher order contributions to the effective interactions.

9. Inductive Step: Extraction of the New Effective Coupling Constant

Here, we present the treatment of the fourth order contributions W' to W'_4. This is the most important but also the most technical part of the argument.

By virtue of (5.30), (7.22), and (8.28), $\tilde{\tilde{W}}'_{4Y}(\psi')$ [on $\frac{1}{2}L\mathcal{K}(Y)$] are given by the quartic terms of second order in λ on the right-hand side of (7.22) or (8.28) respectively, plus $\mathcal{O}_4((n_0+n)^{-7/6}) \exp[-6\alpha \mathcal{L}(Y)]$. Let us denote these contributions by $\tilde{\tilde{W}}'^{1}_{4Y}$ and $\tilde{\tilde{W}}'^{2}_{4Y}$ respectively. It is easy to see that

$$|\tilde{\tilde{W}}'^{1}_{4Y}(\psi')| \leqq \mathcal{O}((n_0+n)^{-1}) \exp[-6\alpha \mathcal{L}(Y)] \tag{1}$$

on $\frac{1}{2}L\mathcal{K}(Y)$. We have to represent $\tilde{\tilde{W}}'_4 \equiv \sum_Y \tilde{\tilde{W}}'_{4Y}$ as a sum of a local quartic interaction and of approximately local irrelevant quartic terms as in (5.31). This will be done in several steps which might look technically complicated and not quite natural but which guarantee good bounds on the obtained expressions. At the begining, let us polarize $\tilde{\tilde{W}}'^{(i)}_{4Y}(\psi')$, $i = 1, 2$, producing a symmetric quartic form

$$\tilde{\tilde{W}}'^{(i)}_{4Y}(\psi_1, \ldots, \psi_4) = \frac{1}{4!} \prod_{i=1}^{4} \frac{\partial}{\partial t_i}\bigg|_{t=0} \tilde{\tilde{W}}'^{(i)}_{4Y}\left(\sum_{i=1}^{4} t_i \psi_i\right). \tag{2}$$

By the Cauchy estimate,

$$|\tilde{\tilde{W}}'^{1}_{4Y}(\psi_1, \ldots, \psi_4)| \leqq \mathcal{O}((n_0+n)^{-1}) \exp[-6\alpha \mathcal{L}(Y)] \tag{3}$$

and

$$|\tilde{\tilde{W}}'^{2}_{4Y}(\psi_1, \ldots, \psi_4)| \leqq \mathcal{O}((n_0+n)^{-7/6}) \exp[-6\alpha \mathcal{L}(Y)] \tag{4}$$

for $\psi_i \in \mathcal{K}(Y)$. Introduce symmetric kernels

$$\Lambda^{(i)}_{Y x_1, \ldots, x_4} = \tilde{\tilde{W}}'^{(i)}_{Y}(\mathcal{A}'_{\cdot x_1}, \ldots, \mathcal{A}'_{\cdot x_4}). \tag{5}$$

Since by virtue of (2.42), (2.43), (2.45), and (4.1),

$$\mathscr{A}'_{\cdot x}|_Y \in \mathcal{O}((n_0+n)^{-1/4}) \exp[-\beta d(x, Y)] \mathscr{K}(Y), \tag{6}$$

$$|\Lambda^1_{Yx_1 \dots x_4}| \leqq \mathcal{O}((n_0+n)^{-2}) \exp[-6\alpha \mathscr{L}(Y \cup \overline{\{x_i\}})] \tag{7}$$

and

$$|\Lambda^2_{Yx_1 \dots x_4}| \leqq \mathcal{O}((n_0+n)^{-13/6}) \exp[-6\alpha \mathscr{L}(Y \cup \overline{\{x_i\}})]. \tag{8}$$

If $\psi' = \mathscr{A}'\phi'$, then $\Lambda^{(i)}_Y$ are the ϕ'-kernels of $\tilde{\tilde{W}}{}'^{(i)}_{4Y}(\psi')$. We also recall that in this case $\phi'_x = \int_{\square_x} \psi'$, see (2.26). For a general ψ', we shall write

$$\psi' = \mathscr{A}'\bar{\psi}' + (\psi' - \mathscr{A}'\bar{\psi}'), \tag{9}$$

where for $x \in \Lambda_n$

$$\bar{\psi}_x = \int_{\square_x} \psi. \tag{10}$$

Notice that because of (2.22)

$$\psi'_{x} - (\mathscr{A}'\bar{\psi}')_{x} = \psi'_{x} - \sum_x \mathscr{A}'_{xx} \int_{\square_x} dy \psi'_y = \sum_x \mathscr{A}'_{xx} \int_{\square_x} dy (\psi'_x - \psi'_y), \tag{11}$$

so that $\psi' - \mathscr{A}'\bar{\psi}'$ depends on ψ' through its differences at pairs of points only. We may write

$$\begin{aligned} \tilde{\tilde{W}}{}'_{4Y}(\psi') = & \sum_{x_1, \dots, x_4} \Lambda_{Yx_1, \dots, x_4} \bar{\psi}'_{x_1} \dots \bar{\psi}'_{x_4} \\ & + \sum_{i=1}^4 \tilde{\tilde{W}}{}'_{4Y}(\mathscr{A}'\bar{\psi}', \dots, \mathscr{A}'\bar{\psi}', \psi' \underset{\hat{i}}{-} \mathscr{A}'\bar{\psi}', \dots, \psi' - \mathscr{A}'\bar{\psi}). \end{aligned} \tag{12}$$

The second term on the right-hand side of (12) will be shown to be irrelevant due to the dependence of $\psi' - \mathscr{A}'\bar{\psi}'$ on differences of ψ' only. The first term will be transformed into

$$\sum_x \delta\lambda_{Yx} (\bar{\psi}'_x)^4 + \sum_{i=2}^4 \sum_{(x_i)} \Lambda_{Yx_1, \dots, x_4} \bar{\psi}'_{x_1} \dots \bar{\psi}'_{x_1} (\bar{\psi}'_{x_i} - \bar{\psi}'_{x_1}) \bar{\psi}'_{x_{i+1}} \dots \bar{\psi}'_{x_4}, \tag{13}$$

where

$$\delta\lambda^{(i)}_{Yx} = \sum_{x_2, x_3, x_4} \Lambda^{(i)}_{Yxx_2x_3x_4}. \tag{14}$$

Proceeding further, we rewrite the first term of (13) as

$$\sum_x \delta\lambda_{Yx} \int_{\square_x} (\psi')^4 + \sum_{i=1}^4 \binom{4}{i} \sum_x \delta\lambda_{Yx} \int_{\square_x} (\psi')^{4-i} (\bar{\psi}'_x - \psi')^i. \tag{15}$$

Thus, gathering (12), (13), and (15), we obtain

$$\begin{aligned} \tilde{\tilde{W}}{}'_{4Y}(\psi') = & \sum_x \delta\lambda_{Yx} \int_{\square_x} (\psi')^4 + \sum_{i=1}^4 \Bigg[\tilde{\tilde{W}}{}'_{4Y}(\mathscr{A}'\bar{\psi}'_1, \dots, \mathscr{A}'\bar{\psi}'_{i-1}, \psi'_i - \mathscr{A}'\bar{\psi}'_i, \dots, \psi'_4 - \mathscr{A}'\bar{\psi}'_4) \\ & + \sum_{x_1, \dots, x_4} \Lambda_{Yx_1 \dots x_4} \bar{\psi}'_{1x_1} \dots \bar{\psi}'_{i-1x_1} \bar{\psi}'_{ix_{i+1}} \dots \bar{\psi}'_{3x_4} (\bar{\psi}'_{4x_i} - \bar{\psi}'_{4x_1}) \\ & + \binom{4}{i} \sum_x \delta\lambda_{Yx} \int_{\square_x} \psi'_1 \dots \psi'_{4-i} (\bar{\psi}'_{4-i+1x} - \psi'_{4-i+1}) \dots (\bar{\psi}'_{4x} - \psi'_4) \Bigg] \Bigg|_{\psi'_i \equiv \psi'}, \end{aligned} \tag{16}$$

where we have written the irrelevant quartic term as a four-linear (non-symmetric) form restricted to the diagonal into which ψ_4 enters only through its differences at pairs of points. The different irrelevant terms on the right-hand side of (16) depend also on fields ψ_i' outside Y. We shall localize this dependence by writing

$$\psi_i' = \psi_i'\chi_Y + \sum_{\Delta \cap Y = \emptyset} \psi_i'\chi_\Delta, \tag{17}$$

$$\psi_4' - \mathscr{A}'\bar{\psi}_4' = (\psi_4'\mathscr{A}'\chi_Y - \mathscr{A}'(\bar{\psi}_4'\chi_Y)) + \sum_{\Delta \cap Y = \emptyset} (\psi_4'\mathscr{A}'\chi_\Delta - \mathscr{A}'(\bar{\psi}_4'\chi_\Delta)) \tag{18}$$

(note special localization of $\psi_4' - \mathscr{A}'\bar{\psi}_4'$ which uses (2.22)). After localizing the irrelevant terms on the right-hand side of (16), we shall gather those in which the union of the localization squares Δ is P into $\tilde{W}'^2_{4YY\cup P}(\psi_1', \ldots, \psi_4')$ (do not get confused about the superscripts: $\tilde{W}_4'^2$ gathers non-local contributions from $\tilde{\tilde{W}}_4' \equiv \tilde{\tilde{W}}_4'^1 + \tilde{\tilde{W}}_4'^2$). With $\tilde{W}'^2_{4YY'}(\psi') \equiv \tilde{W}'^2_{4YY'}(\psi', \ldots, \psi')$, (16) becomes

$$\tilde{\tilde{W}}'_{4Y}(\psi') = \sum_x \delta\lambda_{Yx} \int_{\square_x} (\psi')^4 + \sum_{Y' \supset Y} \tilde{W}'^2_{4YY'}(\psi'). \tag{19}$$

It is easy to notice the x-independence of

$$\sum_Y \delta\lambda_{Yx}^{(i)} \equiv \delta\lambda^{(i)}. \tag{20}$$

Indeed since W' possesses all the unit lattice euclidean symmetries, so does $\tilde{\tilde{W}}_4'$, and hence $\sum_Y \Lambda_{Yx_1\ldots x_4}$. Since $\sum_Y \tilde{\tilde{W}}'^1_{4Y}$ (given by the quartic $0(\lambda^2)$ terms of (3.3) with $\zeta = 1$) is also symmetric, so are $\sum_Y \Lambda^i_{Yx_1\ldots x_4}$. But $\sum_Y \delta\lambda^{(i)}_{Yx} = \sum_{x_2, x_3, x_4, Y} \Lambda^{(i)}_{Yxx_2x_3x_4}$, which is x independent due to the translation invariance of $\sum_Y \Lambda^{(i)}_{Yx_1\ldots x_4}$. Using also (2.22), it is easy to show that $\delta\lambda^1$ defined by (20) coincides with $\delta\lambda^1$ of (3.6). For $\delta\lambda^2$, (8) yields

$$|\delta\lambda^2| \leq \mathcal{O}((n_0+n)^{-13/6}), \tag{21}$$

so that $\delta\lambda^2$ is small correction to $\delta\lambda^1 = \mathcal{O}((n_0+n)^{-2})$ in $\delta\lambda = \delta\lambda^1 + \delta\lambda^2$.

Upon defining

$$\tilde{W}'^2_{4Y'}(\psi_1', \ldots, \psi_4') = \sum_{Y \subset Y'} \tilde{W}'^2_{4YY'}(\psi_1', \ldots, \psi_4') \tag{22}$$

and

$$\tilde{W}'^2_{4Y}(\psi') \equiv \tilde{W}'^2_{4Y}(\psi', \ldots, \psi'), \tag{23}$$

(19) and (20) imply (5.31).

We still have to bound $\tilde{W}'^2_{4Y'}(\psi_1', \ldots, \psi_4')$. Suppose that ψ_1', ψ_2' and $\psi_3' \in \mathscr{K}(Y')$ and that ψ_4' fulfills all the bounds of (4.1) defining $\mathscr{K}(Y')$ except the first one for $|\psi_4'|$. Notice that by virtue of (2.42), (2.43), and (2.45), for $i = 1, 2, 3$,

$$\mathscr{A}'\chi_Y\bar{\psi}_i' \in \mathcal{O}(1)\mathscr{K}(Y) \tag{24}$$

and

$$\mathscr{A}'\chi_\Delta\bar{\psi}_i' \in \mathcal{O}(1)\exp[-\beta d(Y, \Delta)]\mathscr{K}(Y). \tag{25}$$

For $\mathscr{x} \in Y$ and

$$\omega_{\mathscr{x}} \equiv \psi'_{4\mathscr{x}}(\mathscr{A}'\chi_Y)_{\mathscr{x}} - (\mathscr{A}'(\bar{\psi}_4'\chi_Y))_{\mathscr{x}} = \sum_{x \in Y} \mathscr{A}'_{\mathscr{x}x} \int_{\square_x} d\mathscr{z}(\psi'_{4\mathscr{x}} - \psi'_{4\mathscr{z}}),$$

$$|\omega_{\mathscr{x}}| \leq \sum_{x \in Y} |\mathscr{A}'_{\mathscr{x}x}| (|\mathscr{x} - x| + 1) \int_{\square_x} d\mathscr{y} |\psi'_{4\mathscr{x}} - \psi'_{4\mathscr{y}}| / |\mathscr{x} - \mathscr{y}| \leq \mathcal{O}((n_0+n)^{1/4}). \tag{26}$$

Furthermore, for x, $y \in Y$ and $|x-y| \leqq 1$,

$$|\omega_x - \omega_y|/|x-y| = \left|\sum_{x\in Y} \int_{\square_x} dz(\mathscr{A}'_{xx}(\psi'_{4x}-\psi'_{4z}) - \mathscr{A}'_{yx}(\psi'_{4y}-\psi'_{4z}))\right| / |x-y|$$
$$\leqq \sum_{x\in Y} (|\mathscr{A}'_{xx} - \mathscr{A}'_{yx}|/|x-y|) \int_{\square_x} dz|\psi'_{4x}-\psi'_{4z}|$$
$$+ \sum_{x\in Y} |\mathscr{A}'_{yx}|\,|\psi'_{4x}-\psi'_{4y}|/|x-y| \leqq \mathcal{O}((n_0+n)^{1/4}). \tag{27}$$

The same holds for $|x-y|>1$ due to (26).

Finally for x, y, $x+L^{-(n+1)}e_\mu$, $y+L^{-(n+1)}e_\mu \in X$, since

$$(\partial_\mu \omega)_x = \sum_{x\in Y} (\partial_\mu \mathscr{A}'_{\cdot x})_x \int_{\square_x} dz(\psi'_{4x}-\psi'_{4z}) + \sum_{x\in Y} \mathscr{A}'_{xx}\partial_\mu \psi'_{4x}$$
$$+L^{-(n+1)} \sum_{x\in Y} (\partial_\mu \mathscr{A}'_{\cdot x})_x \partial_\mu \psi'_{4x}, \tag{28}$$

$$|(\partial_\mu\omega)_x - (\partial_\mu\omega)_y|/|x-y|^{2/3} \leqq \mathcal{O}((n_0+n)^{1/4}). \tag{29}$$

Thus

$$\psi'_4(\mathscr{A}'\chi_Y) - \mathscr{A}'(\bar{\psi}'_4\chi_Y) \in \mathcal{O}(1)\mathscr{K}(Y), \tag{30}$$

and similarly we show that

$$\psi'_4(\mathscr{A}'\chi_\Delta) - \mathscr{A}'(\bar{\psi}'_4\chi_\Delta) \in \mathcal{O}(1)\exp[-\tfrac{1}{2}\beta d(Y,\Delta)]\mathscr{K}(Y). \tag{31}$$

From (3), (4), the assumptions on ψ'_i, (24), (25), (30), and (31), it follows that the contribution to $\tilde{W}'^2_{4YY'}(\psi'_1, \ldots, \psi'_4)$ coming from the first term in the brackets on the right-hand side of (16) is bounded by $\mathcal{O}((n_0+n)^{-1})\exp[-6\alpha\mathscr{L}(Y')]$. From (7) and (8) in turn, we infer that the contributions of the second and the third term in the brackets are estimated by $\mathcal{O}((n_0+n)^{-1})\exp[-5\alpha\mathscr{L}(Y')]$.

Altogether

$$|\tilde{W}'^2_{4YY'}(\psi'_1, \ldots, \psi'_4)| \leqq \mathcal{O}((n_0+n)^{-1})\exp[-5\alpha\mathscr{L}(Y')] \tag{32}$$

and

$$|\tilde{W}'^2_{4Y'}(\psi'_1, \ldots, \psi'_4)| \leqq \mathcal{O}((n_0+n)^{-1})\exp[-4\alpha\mathscr{L}(Y')] \tag{33}$$

for $\psi'_i \in \mathscr{K}(Y')$, $i=1,2,3$, and ψ'_4 satisfying the bounds defining $\mathscr{K}(Y')$ except the one for $|\psi'_4|$.

For the other irrelevant quartic contributions to W'_4, one has

$$|\tilde{W}'^1_{4Y'}(\psi'_1, \ldots, \psi'_4)| = \left|\sum_{Y:(L^{-1}Y)^- = Y'} \tilde{V}_{4Y}(L^{-1}\psi'_{1L^{-1}\cdot}, \ldots, L^{-1}\psi'_{4L^{-1}\cdot}\right|$$
$$\leqq \begin{cases} (L^4+1)L^{-5}(n_0+n)^{-3/4}\exp[-\alpha\mathscr{L}(Y')], \\ \mathcal{O}((n_0+n)^{-3/4})\exp[-4\alpha\mathscr{L}(Y')] \end{cases} \tag{34}$$

where in the first bound we have used the fact that there are at most L^4 Y's with $\mathscr{L}(Y) = \mathscr{L}(Y')$, and for ψ'_4 satisfying the second and the third inequality of (4.1) for $n+1$, $L\psi'_{L^{-1}\cdot}$ almost satisfies it for n (this produces the additional L^{-1}). In $\tilde{W}'_{4Y} \equiv \tilde{W}'^1_{4Y} + \tilde{W}'^2_{4Y}$ the second term is a small correction. Expression (33) and the first inequality of (34) show that the bound for $\tilde{W}'_{4Y}$ contracts with respect to that for $\tilde{V}_{4Y}$ if L is big enough:

$$|\tilde{W}'_{4Y}(\psi'_1, \ldots, \psi'_4)| \leqq \tfrac{1}{2}(n_0+n+1)^{-3/4}\exp[-\alpha\mathscr{L}(Y)] \tag{35}$$

for $\psi'_i \in \mathcal{K}(Y)$ for $i=1,2,3$ and ψ'_4 satisfying the defining bounds of $\mathcal{K}(Y)$ except for the first one. This demonstrates the irrelevant character of these terms.

As for the marginal contribution, (21) shows that our perturbative analysis of the change of the effective quartic coupling performed in Sect. 3, see (3.13), essentially remains valid:

$$\frac{1}{\lambda}+2C_+^{-1} \leqq \frac{1}{\lambda+\delta\lambda} \leqq \frac{1}{\lambda}+\frac{1}{2}C_-^{-1}. \tag{36}$$

10. Inductive Step: The Large Field Bound

In this section we shall show how the large field bound (6.5) iterates. Let us start with the estimation of $\varrho_{X'}^{D'}(\psi')$ as given by (5.25) with $D' \neq \emptyset$ [for $\psi' \in \frac{1}{2}L\mathcal{D}(D', X')$]. The main difference as compared to the $D'=\emptyset$ case treated in Sect. 8 is in the contribution of $\prod_i g_{X_i}^D(\psi^s)$, which becomes more subtle to estimate when ψ' is large. On the other hand, we shall need only a rather rough upper bound and tracing of powers of $(n_0+n)^{-1}$ contributed by the expansion outside the large field region will not be necessary.

First let us notice that, since $D=LD'UR$,

$$C_2 \sum_i |D \cap X_i| \leqq L^4 C_2 |D' \cap X'| + C_2 |R|. \tag{1}$$

Furthermore

$$\begin{aligned} &-\lambda^{1/2} \sum_i \int_{D\cap X_i} dx |\psi_x^s|^2 \\ &\leqq -\tfrac{1}{2} L^{-2} \lambda^{1/2} \sum_i \int_{D\cap X_i} dx |\psi'_{L^{-1}x}|^2 + \lambda^{1/2} \sum_i \int_{D\cap X_i} dx |\mathcal{Z}_x^s|^2 \\ &\leqq -\tfrac{1}{2} L^2 \lambda^{1/2} \int_{D'\cap X'} dx |\psi'_x|^2 + \mathcal{O}(1)|R| + \mathcal{O}((n_0+n)^{-1/2}) \sum_{u\in LX'} Z_u^2 , \end{aligned} \tag{2}$$

and, by (5.11), (5.12), (5.14),

$$\begin{aligned} 20\lambda \sum_i \int_{D\cap X_i} dx (\operatorname{Im}\psi_x^s)^4 \leqq{}& 20 L^{-4} \lambda \sum_i \int_{D\cap X_i} dx (\operatorname{Im}\psi'_{L^{-1}x})^4 \\ &+ \mathcal{O}((n_0+n)^{-1/4}) \int_{LX'} dx |\operatorname{Im}\mathcal{Z}_x^s| , \end{aligned} \tag{3}$$

where we have used the smallness of $\operatorname{Im}\psi'$ and $\operatorname{Im}\mathcal{Z}^s = \operatorname{Im}\mathcal{Z}_2^s$, see (5.19). Thus

$$\begin{aligned} 20\lambda \sum_i \int_{D\cap X_i} (\operatorname{Im}\psi^s)^4 \leqq{}& 20\lambda \int_{D'\cap X'} (\operatorname{Im}\psi')^4 + \mathcal{O}(1)|R| + \mathcal{O}((n_0+n)^{-1/4})|X'| \\ &+ \mathcal{O}((n_0+n)^{-1/4} \sum_{u\in LX'} Z_u^2 . \end{aligned} \tag{4}$$

Inequalities (1), (2), and (4), together with (6.5) give

$$\begin{aligned} \left|\prod_i g_{X_i}^D(\psi^s)\right| \leqq \exp\Big[& L^4 C_2 |D'\cap X'| + \mathcal{O}(1)|R| + \mathcal{O}((n_0+n)^{-1/4})|X'| \\ & -\tfrac{1}{2} L^2 \lambda^{1/2} \int_{D'\cap X'} |\psi'|^2 + \mathcal{O}((n_0+n)^{-1/4}) \\ & \cdot \sum_{u\in LX'} Z_u^2 + 20\lambda \int_{D'\cap X'} (\operatorname{Im}\psi')^4 - \alpha \sum_i \mathcal{L}(X_i) \Big] , \end{aligned} \tag{5}$$

which replaces (8.3). In turn (8.4) is replaced by

$$\left|\exp\left[-\tfrac{1}{2}m^2 \int_{LX'} dx(\psi_x^s)^2 + 6\lambda \int_{LX'} dx G_{xx}(\psi_x^s)^2\right]\right|$$
$$\leqq \exp\left[\mathcal{O}((n_0+n)^{-1}) \int_{D'\cap X'} dx |\psi_x'|^2 + \mathcal{O}((n_0+n)^{-1/2})|X'| + \mathcal{O}((n_0+n)^{-1}) \sum_{u\in LX'} Z_u^2\right], \tag{6}$$

(8.5) by

$$\left|\exp\left[-\lambda \int_{LX'\setminus D} (\psi^s)^4\right]\right| \leqq \left|\exp\left[-\lambda \int_{X'\setminus D'} (\psi')^4\right]\right| \exp\left[\mathcal{O}(1)|R| + \mathcal{O}((n_0+n)^{-1/4})|X'| + \mathcal{O}((n_0+n)^{-1/4}) \sum_{u\in LX'} Z_u^2\right], \tag{7}$$

and (8.6) by

$$\left|\exp\left[\sum_{\Delta\subset X'\setminus D'} W_\Delta'\right]\right| \leqq \left|\exp\left[\lambda \int_{X'\setminus D'} (\psi')^4\right]\right| \exp[\mathcal{O}((n_0+n)^{-1/2})|X'|]. \tag{8}$$

For $\tilde{V}_{\geqq 4Y_\beta}$ terms, (8.7) and (8.8) still hold since $Y_\beta \subset LX'\setminus D$. The $\tilde{V}_{2Y}$ terms are also estimated identically as in Sect. 8. Namely (8.13)–(8.20), except for the last inequality of (20), remain valid if we change R_k to $D_k = (LD'\cup R)\cap U_k$ and R to $(LD' \cup R)\cap LX'$. Hence (8.17) and (8.18) still hold and

$$\left|\sum_i \sum_{Y\subset X_i} \tilde{V}_{2Y}(\psi^s)\right| \leqq \mathcal{O}((n_0+n)^{-1})|X'| + \mathcal{O}((n_0+n)^{-3/2}) \int_{D'\cap X'} |\psi'|^2 + \mathcal{O}((n_0+n)^{-3/2}) \sum_{u\in LX'} Z_u^2. \tag{9}$$

Substitution of (5)–(8), (8.8), (8.18), and (9) to (5.25) gives [on $\tfrac{1}{2}L\mathcal{D}(D',X')$],

$$\begin{aligned}|\varrho_{X'}^{D'}| \leqq &\sum_{\bar{p}} \sum_{\{X_i\}} \sum_{\{Y_\alpha\}} \sum_{\{Y_\beta\}} \sum_{\Gamma_c} \prod_{(k,k')\in\Gamma_c} (r^{-1}\exp[-\tfrac{1}{2}\beta d(U_k, U_{k'})])\\ &\cdot \exp\Big[L^4 C_2 |D'\cap X'| + \mathcal{O}(1)|R| + \mathcal{O}((n_0+n)^{-1/4})|X'|\\ &-\tfrac{1}{3}L^2\lambda^{1/2} \int_{D'\cap X'} |\psi'|^2 + 20\lambda \int_{D'\cap X'} (\mathrm{Im}\,\psi')^4\\ &-\alpha\sum_i \mathcal{L}(X_i) - \alpha\sum_\alpha \mathcal{L}(Y_\alpha) - \alpha\sum_\beta \mathcal{L}(Y_\beta)\Big] \prod_\alpha \mathcal{O}((n_0+n)^{-1}) \prod_\beta \mathcal{O}((n_0+n)^{-1/2})\\ &\cdot \int \exp\left[\mathcal{O}((n_0+n)^{-1/4}) \sum_{u\in LX'} Z_u^2\right] \chi_{\bar{p}}(Z_{LX'}) d\mu_1(Z_{LX'})\\ \leqq &\exp\Big[(L^4+1)C_2|D'\cap X'| - \tfrac{1}{3}L^2\lambda^{1/2} \int_{D'\cap X'} |\psi'|^2\\ &+20\lambda \int_{D'\cap X'} (\mathrm{Im}\,\psi')^4 - 7\alpha\mathcal{L}(X')\Big]\end{aligned} \tag{10}$$

for $C_2 \geqq \bar{C}_2(L, N_0)$, the sums being controlled as for $\varrho_{X'}^{\emptyset}$.

Let us pass to the estimation of the large field contributions $\bar{g}_X'^{D'}$ to $\exp[-\bar{W}_{\geqq 4}']$, see (5.34) and (5.35). By (7.22) and (8.28),

$$|W_Y'(0)| \leqq \mathcal{O}((n_0+n)^{-1}) \exp[-6\alpha\mathcal{L}(Y)], \tag{11}$$

$$|W_{2Y}'| \leqq \mathcal{O}((n_0+n)^{-1/2}) \exp[-6\alpha\mathcal{L}(Y)], \tag{12}$$

the last on $\mathscr{K}(Y)$. Proceeding as in (8.13)–(8.20), we show that for $\psi' \in \frac{1}{2}L\mathscr{D}(D', X')$,

$$|W'_{2Y_\alpha}| \leqq \mathcal{O}((n_0+n)^{-1/2}) \exp[-5\alpha\mathscr{L}(Y)] \tag{13}$$

for Y_α not in the interior of a single c.c. D'_i of $D' \cap \bar{X}$ and

$$\left|\sum_{D'_i} \sum_{Y \subset \mathring{D}'_i} W'_{2Y}\right| \leqq \mathcal{O}((n_0+n)^{-1/2}) |\bar{X}| + \mathcal{O}((n_0+n)^{-1}) \int_{D' \cap \bar{X}} |\psi'|^2 . \tag{14}$$

Now, on $\mathscr{K}(Y)$,

$$|\tilde{\tilde{W}}'_{4Y}| \leqq \mathcal{O}((n_0+n)^{-1}) \exp[-6\alpha\mathscr{L}(Y)] . \tag{15}$$

Hence by virtue of (8.9), on $\frac{1}{2}L\mathscr{D}(D', \bar{X})$,

$$\begin{aligned} |\tilde{\tilde{W}}'_{4Y}| \leqq & \mathcal{O}((n_0+n)^{-1}) \exp[-6\alpha\mathscr{L}(Y)] + \mathcal{O}((n_0+n)^{-2}) \sum_{x_1, \dots, x_4 \in D' \cap \bar{X}} \\ & \cdot \exp[-6\alpha\mathscr{L}(Y \cup \overline{\{x_i\}})] \prod_{i=1}^{4} \int_{\square_{x_i}} |\psi'| . \end{aligned} \tag{16}$$

Again for Y not in the interior of a single c.c. of $D' \cap \bar{X}$, the exponentially decaying factor $\exp[-\frac{1}{2}\alpha\mathscr{L}(Y \cup \overline{\{x_i\}})]$ can be used to match the growth of $\prod_{i=1}^{4} \int_{\square_{x_i}} |\psi'|$, and another one to bound the sum over x_i. Thus on $\frac{1}{2}L\mathscr{D}(D', \bar{X})$,

$$|\tilde{\tilde{W}}'_{4Y_\beta}| \leqq \mathcal{O}((n_0+n)^{-1}) \exp[-5\alpha\mathscr{L}(Y)] . \tag{17}$$

Using (8), (10), (8.24), (13), (14), and (17), we immediately infer from (5.35) that on $\frac{1}{2}L\mathscr{D}(D', \bar{X})$,

$$\begin{aligned} |\bar{g}_X'^{D'}| \leqq \exp\Big[& (L^4+2)C_2|D' \cap \bar{X}| - \tfrac{1}{4}L^2\lambda^{1/2} \int_{D' \cap \bar{X}} |\psi'|^2 \\ & + 20\lambda \int_{D' \cap \bar{X}} (\operatorname{Im}\psi')^4 - 4\alpha\mathscr{L}(\bar{X})\Big] \left|\exp\Big[\sum_{D'_i} \sum_{Y \subset \mathring{D}'_i} \tilde{\tilde{W}}'_{4Y}\Big]\right| . \end{aligned} \tag{18}$$

We have not estimated the quartic contributions $\tilde{\tilde{W}}'_{4Y}$ with Y within $\mathring{D}'_i$ since their main part will be canceled in the next step reinstating the quartic terms of W' missing in $\bar{W}'_{\geqq 4}$.

Let us then pass to the estimation of $\tilde{g}_X'^{D'}$ given by (5.40). Since Y_α and Y_β lie outside D',

$$|\tilde{W}'^{1}_{4Y_\alpha}| \leqq \mathcal{O}((n_0+n)^{-3/4}) \exp[-4\alpha\mathscr{L}(Y_\alpha)] , \tag{19}$$

[see (9.34)] and [see (7.8), (7.22), (7.44), (8.28), and (8.30)]

$$|W'_{\geqq 6Y_\beta}| \leqq \mathcal{O}((n_0+n)^{-1/2}) \exp[-6\alpha\mathscr{L}(Y_\beta)] \tag{20}$$

on $\frac{1}{2}L\mathscr{D}(D', X)$. Furthermore, (9.33) implies

$$|\tilde{W}'^{2}_{4Y_\gamma}| \leqq \mathcal{O}((n_0+n)^{-1}) \exp[-3\alpha\mathscr{L}(Y_\gamma)] , \tag{21}$$

again on $\frac{1}{2}L\mathscr{D}(D', X)$, in the same way as (15) implied (17). Finally, we shall have to estimate

$$\sum_{Y \subset \mathring{D}'_i} \tilde{\tilde{W}}'_{4Y} - \delta\lambda \int_{\mathring{D}'_i} (\psi')^4 - \sum_{Y \subset \mathring{D}'_i} \tilde{W}'^{2}_{4Y} \tag{22}$$

on $\frac{1}{2}L\mathscr{D}(D', X)$. Let $\psi' = \mathscr{A}'\phi' + \tilde{\psi}' \in \frac{1}{2}L\mathscr{D}(D', X)$ with ϕ' real, $D(2L^{-1}\mathscr{A}'\phi') \subset D'$ and $\tilde{\psi}' \in \frac{1}{2}L\mathscr{K}(X)$, see (4.11). Via multiplication by an appropriate smooth function being 1 on $\mathring{D}_i'$ and zero outside D_i, we may replace $\tilde{\psi}'$ by a globally defined function $\tilde{\tilde{\psi}}' \in \mathcal{O}(1)\mathcal{K}(\Lambda_{n+1})$ coinciding with $\tilde{\psi}'$ on $\mathring{D}_i'$. Clearly

$$\omega' = \mathcal{A}'\phi' + \tilde{\tilde{\psi}}' \in \mathcal{O}(1)\mathcal{D}(D', \Lambda_{n+1}), \tag{23}$$

and coincides with ψ' and $\mathring{D}_i'$. Thus the values of (22) on ψ' and ω' are equal. Now, by virtue of (9.20) and (9.22),

$$(22)(\omega') = \sum_{Y \subset \mathring{D}_i'} \tilde{\tilde{W}}_{4Y}'(\omega') - \sum_{Y} \sum_{x \in \mathring{D}_i} \delta\lambda_{Yx} \int_{\square_x} (\omega')^4 - \sum_{Y \subset Y' \subset \mathring{D}_i'} \tilde{W}_{4YY'}'^2(\omega'). \tag{24}$$

Insertion of (9.19) into (24) gives

$$(22)(\omega') = \sum_{Y \subset \mathring{D}_i'} \sum_{x \notin \mathring{D}_i'} \delta\lambda_{Yx} \int_{\square_x} (\omega')^4 - \sum_{Y \not\subset \mathring{D}_i'} \sum_{x \in \mathring{D}_i'} \delta\lambda_{Yx} \int_{\square_x} (\omega')^4 + \sum_{\substack{Y \subset \mathring{D}_i' \\ Y' \supset Y : Y' \subset \mathring{D}_i'}} \tilde{W}_{YY'}'^2(\omega'). \tag{25}$$

On the right-hand side, we have only tail terms localized in bigger sets than $\mathring{D}_i'$. As by (9.7), (9.8), and (9.14),

$$|\delta\lambda_{Yx}| \leqq \mathcal{O}((n_0+n)^{-2}) \exp[-5\alpha\mathcal{L}(YU\overline{\{x\}})], \tag{26}$$

we can use part of the tree decay factor to match the growth of ω' inside D' in the first two terms on the right-hand side of (25). The third term is estimated as in (21) with the use of (9.32), and the constraint that Y' does not lie in the interior of a single c.c. of D'. This way we show that on $\frac{1}{2}L\mathcal{D}(D', X)$,

$$|(22)| \leqq \mathcal{O}((n_0+n)^{-1}) |\mathring{D}_i|. \tag{27}$$

Expressions (18) to (21) and (27), when inserted to (5.40), yield

$$|\tilde{g}_X'^{D'}| \leqq \exp\left[(L^4+3)C_2|D' \cap X| - \tfrac{1}{4}L^2\lambda^{1/2} \int_{D' \cap X} |\psi'|^2 + 20\lambda \int_{D' \cap X} (\mathrm{Im}\,\psi')^4 - 2\alpha\mathcal{L}(X)\right] \tag{28}$$

on $\frac{1}{2}L\mathcal{D}(D', X)$. The analyticity of $\tilde{g}_X'^{D'}$ on this set clearly also follows.

Inequality (28) is almost what we want, see (6.5), except the growth of the constant in front of $|D' \cap X|$ (field independent perturbation is relevant). In the remainder of this section we shall show how to bring this constant down to C_2 using the extra strength of the quadratic expression. Roughly speaking, on $D' \cap \bar{X}$, ψ' is at least of order $\mathcal{O}(C_1(n_0+n)^{1/4})$ so that $-\frac{1}{10}L^2\lambda^{1/2} \int_{D' \cap \bar{X}} |\psi'|^2$ should provide $-\mathcal{O}(C_1^2|D' \cap \bar{X}|)$ constant cancelling the growth of C_2 for large C_1. The problem with this argument is that first ψ' may be big in D' but does not have to, and second, even if it is big in absolute value, at some point it could contribute little to the L^2 norm.

Let us fix $\psi' \in \frac{1}{2}L\mathcal{D}(D', X)$ with $\psi' = \mathcal{A}'\phi' + \tilde{\psi}'$, ϕ' real, $D(SL^{-1}\mathcal{A}'\phi') \equiv D_1' \subset D'$, $\tilde{\psi}' \in \frac{1}{2}L\mathcal{K}(X)$. If we take D_1' as a new D', the first difficulty will be avoided. By a version of (4.16) for $\tilde{g}_X'^{D'}$, which the reader will easily establish following the lines of Appendix 2 of [17],

$$\tilde{g}_X'^{D'}(\psi') = \sum_{\{X_i\}, \{Y_\alpha\}} \prod_i \tilde{g}_{X_i}'^{D_1'}(\psi') \prod_\alpha (\exp[-\tilde{W}_{\geqq 4Y_\alpha}'(\psi')] - 1) \cdot \exp\left[-(\lambda+\delta\lambda) \int_{(D' \setminus D_1') \cap X} (\psi')^4\right], \tag{29}$$

where

$$\tilde{W}'_{\geqq 4Y} \equiv \tilde{W}'_{4Y} + W'_{\geqq 6Y} \equiv \tilde{W}'^{1}_{4Y} + \tilde{W}'^{2}_{4Y} + W'_{\geqq 6Y}, \tag{30}$$

and X_i are disjoint, $D'_1 \cap X \subset \cup X_i$, $Y_\alpha \subset X \backslash \cup X_i$ and X is connected with respect to c.c. of D', X_i and Y_α. Notice that using (29) would be dangerous for establishing the analyticity of $\tilde{g}'^{D'}_X(\psi')$ since D'_1 depends on ψ', but once this is done, we shall employ it to improve (28). Inequality (28) for $\tilde{g}^{D'_1}_{X_i}(\psi')$ and the W'_Y estimates easily imply

$$|\tilde{g}'^{D'}_X(\psi')| \leqq \exp[(L^4+3)C_2|D'_1 \cap X| + \tfrac{1}{3}C_2|D' \cap X| - \tfrac{1}{4}L^2\lambda^{1/2} \int_{D'_1 \cap X} |\psi'|^2$$
$$+ 20\lambda \int_{D'_1 \cap X} (\mathrm{Im}\,\psi')^4 - \alpha\mathscr{L}(X)] \left|\exp\left[-\lambda \int_{(D'\backslash D'_1)\cap X} (\psi')^4\right]\right|. \tag{31}$$

But by (6.7),

$$\left|\exp\left[-(\lambda+\delta\lambda) \int_{(D'\backslash D'_1)\cap X} (\psi')^4\right]\right| \leqq \exp\left[\tfrac{1}{3}C_2|(D'\backslash D'_1)\cap X| - 2\lambda^{1/2} \int_{(D'\backslash D'_1)\cap X} |\psi'|^2\right.$$
$$\left. + 20\lambda \int_{(D'\backslash D'_1)\cap X} (\mathrm{Im}\,\psi')^4\right]. \tag{32}$$

Hence

$$|\tilde{g}'^{D'}_X(\psi')| \leqq \exp\left[\tfrac{2}{3}C_2|D' \cap X| - 2\lambda^{1/2} \int_{D' \cap X} |\psi'|^2 + 20\lambda \int_{D' \cap X} (\mathrm{Im}\,\psi')^4 - \alpha\mathscr{L}(X)\right]$$
$$\cdot \exp\left[(L^4+3)C_2|D'_1 \cap X| - (\tfrac{1}{4}L^2 - 2)\lambda^{1/2} \int_{D'_1 \cap X} |\psi'|^2\right]. \tag{33}$$

We still have to show that the second exponential on the right-hand side of (33) is bounded by 1, i.e. that whenever $|\psi'|$ gets large it contributes sufficiently to the L^2 norm. Here we shall use crucially the fact that ψ' is equal to $\mathscr{A}'\phi'$ plus a small correction $\tilde{\psi}'$ and that $\partial_\mu \mathscr{A}'$ is a bounded exponentially decaying kernel.

Let us rescale C_1 and n_0+n out of the problem by introducing

$$\psi'' = \mathscr{A}'\phi'' + \tilde{\psi}'' = 2L^{-1}C_1^{-1}(n_0+n)^{-1/4}\psi'. \tag{34}$$

Notice that D'_1 is by definition the smallest paved set such that

$$|\mathscr{A}'\phi''| < 2\exp[\tfrac{1}{10}\alpha d(x, \sim D'_1)], \tag{35}$$

see (4.10). Define for $\Delta \subset X$

$$A_\Delta = \sup_\Delta |\psi''|, \tag{36}$$

$$B_\Delta = \left(\int_\Delta |\psi''|^2\right)^{1/2}, \tag{37}$$

$$C_\Delta = \sup_\Delta |\partial\psi''|. \tag{38}$$

Let

$$\tilde{D}' = \{\Delta \subset D'_1 \cap X : A_\Delta \geqq 1\}. \tag{39}$$

Since somewhere in $D'_1 \cap X$, $|\mathscr{A}'\phi''| \geqq 2$ (otherwise $D'_1 \cap X$ would be empty and we would be done), and since $|\tilde{\psi}''| < 1$, $\tilde{D}' \neq \emptyset$. We have the following estimate for C_Δ

$$\begin{aligned} C_\Delta &\leqq \sup_\Delta (|\partial \mathscr{A}'\phi''| + |\partial \tilde{\psi}''|) \\ &\leqq \mathcal{O}(1) \sum_x \exp[-\beta d(\Delta, x)]\, |\phi''_x| + C_0 \\ &\leqq \mathcal{O}(1) \sum_{x \in D'_1 \cap X} \exp[-\beta \mathrm{d}(\Delta, \mathrm{x})]\, |\phi''_x| + \mathcal{O}(1) \\ &= \mathcal{O}(1) \sum_{x \in D'_1 \cap X} \exp[-\beta d(\Delta, x)] \left| \int_{\square_x} \mathscr{A}'\phi'' \right| + \mathcal{O}(1) \\ &\leqq \mathcal{O}(1) \int_{x \in D'_1 \cap X} \exp[-\beta d(\Delta, x)] \int_{\square_x} |\psi''| + \mathcal{O}(1) \\ &\leqq \mathcal{O}(1) \int_{\Delta' \subset D'_1 \cap X} \exp[-\beta d(\Delta, \Delta')] B_{\Delta'} + C_3 , \end{aligned} \tag{40}$$

where we have used (2.43) [or (2.44)], (2.21) and the smallness of $\tilde{\psi}''$. Let

$$\tilde{D}'_1 = \bigcup_{\Delta \in \tilde{D}' : C_\Delta \leqq 2C_3 A_\Delta} \Delta , \tag{41}$$

$$\tilde{D}'_2 = \tilde{D}' \setminus \tilde{D}'_1 . \tag{42}$$

Since on $\Delta \subset \tilde{D}'_1$ the derivatives of ψ'' are bounded by the supremum of $|\psi''|$, ψ'' has to contribute sizably to the L^2 norm:

$$A_\Delta \leqq \mathcal{O}(1) B_\Delta . \tag{43}$$

Hence

$$\mathcal{O}(1) \sum_{\Delta \subset D'_1 \cap X} B_\Delta^2 \geqq \sum_{\Delta \subset \tilde{D}'_1} A_\Delta^2 . \tag{44}$$

For $\Delta \subset \tilde{D}'_2$, we use (40):

$$A_\Delta \leqq 2A_\Delta - 1 < C_3^{-1} C_\Delta - 1 \leqq \mathcal{O}(1) \sum_{\Delta' \subset D'_1 \cap X} \exp[-\beta d(\Delta, \Delta')] B_{\Delta'} . \tag{45}$$

Upon squaring and summation over $\Delta \subset \tilde{D}'_2$, this gives

$$\sum_{\Delta \subset \tilde{D}'_2} A_\Delta^2 \leqq \mathcal{O}(1) \sum_{\substack{\Delta \subset \tilde{D}'_2 \\ \Delta', \Delta'' \subset D'_1 \cap X}} \exp[-\beta (d(\Delta, \Delta') + d(\Delta, \Delta''))] B_{\Delta'} B_{\Delta''} \leqq \mathcal{O}(1) \sum_{\Delta \subset D'_1 \cap X} B_\Delta^2 . \tag{46}$$

Since by (35)

$$|D'_1 \cap X| \leqq \mathcal{O}(1) \sum_{\Delta \subset \tilde{D}'} \log^4(1 + A_\Delta) , \tag{47}$$

(44) and (46) imply

$$|D'_1 \cap X| \leqq \mathcal{O}(1) \sum_{\Delta \subset D'_1 \cap X} B_\Delta^2 = \mathcal{O}(1) C_1^{-2} (n_0 + n)^{-1/2} \int_{D'_1 \cap X} |\psi'|^2 , \tag{48}$$

where $\mathcal{O}(1)$ is C_1 independent. Thus for $C_1 \geqq \bar{C}_1(L, N_0, C_2)$, we obtain from (33)

$$|\tilde{g}_X'^{D'}| \leqq \exp\left[\tfrac{2}{3} C_2 |D' \cap X| - 2\lambda^{1/2} \int_{D' \cap X} |\psi'|^2 + 20\lambda \int_{D' \cap X} (\mathrm{Im}\, \psi')^4 - \alpha \mathscr{L}(Y)\right] \tag{49}$$

on $\frac{1}{2} L \mathscr{D}(D', X)$.

Since $\tilde{g}_X'^{D'}(\psi')$ differs from the final $g_X'^{D'}(\psi')$ only by the wave function renormalization $\psi' \mapsto \zeta^{1/2}\psi'$, see (5.53), (49) will yield directly the inductive bound (6.5) for $n \mapsto n+1$.

11. Inductive Step: Mass and Wave Function Renormalization

By (7.22) and (8.28), with the use of (3.5), the second order contribution to W' can be written as

$$W_2'(\psi') = \tfrac{1}{2}L^2 m^2 \int d\mathfrak{x}(\psi_{\mathfrak{x}}')^2 - 6\lambda \int d\mathfrak{x} G_{\mathfrak{x}\mathfrak{x}}'(\psi_{\mathfrak{x}}')^2 + \sum_Y (\tilde{W}_{2Y}'^1(\psi') + \tilde{W}_{2Y}'^2(\psi')), \tag{1}$$

where on $\mathscr{K}(Y)$,

$$|\tilde{W}_{2Y}'^2(\psi')| \leqq \mathcal{O}(n_0+n)^{-5/4})\exp[-6\alpha\mathscr{L}(Y)]. \tag{2}$$

Notice that by virtue of (7.16) and (8.23)

$$\sum_Y \tilde{W}_{2Y}'^1(\psi') = L^4 \sum_{\mu,\nu} \int d\mathfrak{x} d\mathfrak{y} K_{L\mathfrak{x}L\mathfrak{y}}^{\mu\nu}(\partial_\mu \psi_{\mathfrak{x}}' - \partial_\mu \psi_{\mathfrak{y}}')\partial_\nu \psi_{\mathfrak{y}}'. \tag{3}$$

Let us introduce

$$\bar{W}_2'(\psi') \equiv \sum_Y \tilde{W}_{2Y}'^2(\psi') + 6(\lambda(\zeta-1) + \delta\lambda\zeta) \int d\mathfrak{y} G_{\mathfrak{x}\mathfrak{x}}'(\psi_{\mathfrak{x}}')^2 \tag{4}$$

for

$$|\zeta - 1| \leqq \mathcal{O}((n_0+n)^{-7/4}). \tag{5}$$

We shall limit ourselves to $\psi' = \mathscr{A}'\phi'$ only.
It is easy to see using (2) that

$$\bar{W}_2'(\psi') = \sum_{x,y \in \Lambda_{n+1}} I_{xy}\phi_x'\phi_y', \tag{6}$$

where $I_{xy} = I_{yx}$ possesses the unit-lattice euclidean symmetries and

$$|I_{xy}| \leqq \mathcal{O}((n_0+n)^{-7/4})\exp[-5\alpha|x-y|]. \tag{7}$$

As shown in Appendix 2, we may rewrite (6) as

$$\bar{W}_2'(\psi') = \tfrac{1}{2}\delta m^2 \sum_x (\phi_x')^2 + \sum_{\mu,\nu}\sum_{x,y} J_{1xy}^{\mu\nu} \nabla_\mu \phi_x' \nabla_\nu \phi_y', \tag{8}$$

where

$$|\delta m^2| \leqq \mathcal{O}((n_0+n)^{-7/4}) \tag{9}$$

and J_{1xy} is again symmetric and satisfies the bound

$$|J_{1xy}^{\mu\nu}| \leqq \mathcal{O}((n_0+n)^{-7/4})\exp[-2\alpha|x-y|]. \tag{10}$$

as $\phi_x' = \int_{\square_x} \psi'$,

$$\tfrac{1}{2}\delta m^2(\phi_x')^2 = \tfrac{1}{2}\delta m^2 \int_{\square_x} (\psi')^2 - \tfrac{1}{2}\delta m^2 \int_{\square_x} (\psi' - \phi_x')^2. \tag{11}$$

But for $x \in \square_x$,

$$\psi'_x - \phi'_x = \psi'_x - \int_{\square_x} \psi' = \sum_\mu \int dy\, r^\mu_{xy} \partial_\mu \psi'_y, \tag{12}$$

where we can choose r^μ_{xy} supported by $y \in \square_x$ and with all the unit-lattice euclidean symmetries. Hence

$$\begin{aligned}
&-\tfrac{1}{2}\delta m^2 \sum_x \int_{\square_x} (\psi' - \phi'_x)^2 \\
&= -\tfrac{1}{2}\delta m^2 \sum_{\mu,\nu} \int dx\,dy\,dz\, r^\mu_{xy} r^\nu_{xz} \partial_\mu \psi'_y \partial_\nu \psi'_z \\
&= -\tfrac{1}{2}\delta m^2 \sum_{\mu,\nu} \sum_{x,y} \int dx\,dy\,dz\, r^\mu_{xy} r^\nu_{xz} (\partial_\mu \mathscr{A}'_\mu \nabla_\mu^{-1})_{yx} (\partial_\mu \mathscr{A}' \nabla_\mu^{-1})_{zy} \nabla_\mu \phi'_x \nabla_\nu \phi'_y \\
&\equiv \sum_{\mu,\nu} \sum_{x,y} J^{\mu\nu}_{2xy} \nabla_\mu \phi'_x \nabla_\nu \phi'_y,
\end{aligned} \tag{13}$$

where (by (9) and (2.44))

$$|J^{\mu\nu}_{2xy}| \leqq \mathcal{O}((n_0+n)^{-7/4}) \exp[-\beta|x-y|], \tag{14}$$

and $J^{\mu\nu}_{2xy}$ is also euclidean symmetric. Setting

$$J = J_1 + J_2, \tag{15}$$

we can rewrite (8) as

$$\bar{W}'_2(\psi') = \tfrac{1}{2}\delta m^2 \int (\psi')^2 + \sum_{\mu,\nu} \sum_{x,y} J^{\mu\nu}_{xy} \nabla_\mu \phi'_x \nabla_\nu \phi'_y. \tag{16}$$

Now

$$\nabla_\mu \phi'_x = \sum_\nu \int dx\, h^\nu_{\mu x x} \partial_\nu \psi'_x, \tag{17}$$

where

$$h^\nu_{\mu x x} = 0 \quad \text{for} \quad \mu \neq \nu, \tag{18}$$

and for $\mu = \nu$, $h^\mu_{\mu x x}$ is a lattice version of a function whose μ-derivatives is 1 on $\square_x$ and -1 on $\square_{x+e_\mu}$:

$$\partial^{(x)}_\mu h^\mu_{\mu x x} = \chi_{\square_x}(x + L^{-(n+1)} e_\mu) - \chi_{\square_{x+e_\mu}}(x + L^{-(n+1)} e^\mu), \tag{19}$$

$$h^\mu_{\mu x x} = 0 \quad \text{for} \quad x \notin \square_x \cup \square_{x+e_\mu}. \tag{20}$$

These (tensor) functions are again euclidean symmetric and moreover

$$\int dx\, h^\mu_{\mu x x} = 1 = \sum_x h^\mu_{\mu x x}. \tag{21}$$

Substituting (17) to (16), we obtain

$$\bar{W}_2(\psi') = \tfrac{1}{2}\delta m^2 \int dx (\psi'_x)^2 + \sum_{\mu,\nu} \int dx\,dy\, \delta K^{\mu\nu}_{xy} \partial_\mu \psi'_x \partial_\nu \psi'_y, \tag{21}$$

where

$$\delta K^{\mu\nu}_{xy} = \sum_{x,y} J^{\mu\nu}_{xy} h^\mu_{\mu x x} h^\nu_{\nu y y}. \tag{23}$$

$\delta K^{\mu\nu}_{xy}$ has all the unit-lattice euclidean symmetries. By virtue of (10), (14), (15), and (23),

$$|\delta K^{\mu\nu}_{xy}| \leqq \mathcal{O}((n_0+n)^{-7/4}) \exp[-2\alpha|x-y|]\,. \tag{24}$$

Using (21) and the symmetries of J, we obtain

$$\int dx \delta K^{\mu\nu}_{xy} = \sum_{x,y} J^{\mu\nu}_{xy} h^{\nu}_{\nu y y} = \sum_{x} J^{\mu\nu}_{x0} \equiv \tfrac{1}{2}\delta c \delta^{\mu\nu} \tag{25}$$

(for non-symmetric interactions we could get a matrix which is not proportional to identity on the right-hand side). Clearly

$$|\delta c| \leqq \mathcal{O}((n_0+n)^{-7/4})\,. \tag{26}$$

Thus, we may give (22) the following form

$$\begin{aligned} \bar{W}_2'(\psi') = {} & \tfrac{1}{2}\delta m^2 \int dx (\psi'_x)^2 + \tfrac{1}{2}\delta c \sum_{\mu} \int dx (\partial_\mu \psi'_x)^2 \\ & + \sum_{\mu,\nu} \int dx dy \delta K^{\mu\nu}_{xy} (\partial_\mu \psi'_x - \partial_\mu \psi'_y) \partial_\nu \psi'_y\,. \end{aligned} \tag{27}$$

Equations (1), (3), (4), and (27) imply (5.40). Due to (26), (5), and (5.42) are compatible.

Now (5.46), (6.1) for n, (5) and (24) yield easily (6.1) for $n \mapsto n+1$. Relations (5.45), (9.36), and (5) produce (6.2) for $n \mapsto n+1$ as well as (5.49), (9.35), and (5) do (6.3) for $n \to n+1$. Form the definition (5.50) and (7.22), (8.28), (3.2), (8.31), with the use of (5), (6.4) with $n \mapsto n+1$ also follows.

This proves the small field inductive assumptions for the new effective interactions. Similarly (10.49), by virtue of (5.52) and (5) do the large field ones given by (6.5) with $n \mapsto n+1$. Finally from (44), (5), and (9), (6.8) follows. This completes the proof of the Main Technical Result (see the end of Sect. 6).

12. Thermodynamic Limit and Infrared Asymptotic Freedom of the Critical Point Theory

Up to now, we have worked in finite periodic volumes. Nevertheless our estimates were *volume indepenent.* This makes the thermodynamical limit fairly simple. Define the sets $\mathcal{K}(X)$ and $\mathcal{D}(D,X)$ of small and large $\Lambda = \mathbb{Z}^4$ fields as before but whith finite sets X in (4.1) and with finite sets D, X and compact support ϕ in (4.11). Since the kernels $\mathcal{A}_n$ and Γ_n in finite volumes are periodizations of the infinite volume ones satisfying the bounds of the end of Sect. 2, it is easy to see that infinite volume $(1-\varepsilon)\mathcal{D}(D,X)$ is contained in the finite volume $\mathcal{D}(D,X)$'s for large volumes (we recall that periodic Λ has been identified with an L^N-block in $\mathbb{Z}^4$). Suppose now inductively, that m_n^2, $K^{\mu\nu}_{nxy}$, λ_n, $\tilde{V}^n_{4Y}(\psi^n)$, $\tilde{V}^n_{\geqq 6Y}(\psi^n)$ and $g^{nD}_X(\psi^n)$ satisfy our inductive assumptions and converge when $\Lambda \nearrow \mathbb{Z}^4$, the latter two for $\psi^n \in 3\mathcal{K}(Y)$ and $\psi^n \in (1-\varepsilon)\mathcal{D}(D,X)$ respectively. From analycity of $\tilde{V}^n_{4Y}$, $\tilde{V}^n_{\geqq 6Y}$ and g^{nD}_X and the uniform bounds, also their almost uniform convergence follows. $W^{n+1}_Y(\psi^{n+1})$ and $\tilde{g}^{n+1D}_X(\psi^{n+1})$ depend on the volume through V^n_Y and g^{nD}_X and through the kernels $\mathcal{M}_n$ entering their arguments. Using the almost uniform convergence of V^n_Y and g^{nD}_X together with the convergence of $\mathcal{M}_n$ and the dominated convergence theorem, we infer the convergence of $W_Y^{n+1}(\psi^{n+1})$ and $\tilde{g}_X^{n+1D}(\psi^{n+1})$. From the latter, the desired convergence of m_{n+1}^2, $K_{n+1xy}^{\mu\nu}$, $\tilde{V}_{4Y}^{n+1}(\psi^{n+1})$, $\tilde{V}_{\geqq 6Y}^{n+1}(\psi^{n+1})$ and $g_X^{n+1D}(\psi^{n+1})$ with the volume follows. We also see easily that these objects are continuous functions of the initial parameters m_0^2 and λ_0 provided that the previous scale ones where.

Let us notice that, under our inductive assumptions, for $\psi^n = \mathscr{A}_n\phi^n$ and ϕ^n with compact support, $V^n(\psi^n)$ exists in the thermodynamical limit and satisfies (4.2)–(4.4) or (4.8) depending on whether ψ^n is small or big. For the infinite volume weakly coupled ϕ^4 theory, we shall choose the critical point by setting $m_0^2 = m_{\text{crit}}^2(\lambda_0)$ with the latter in $\cap J_0^{(n)}$, where

$$J_0^{(n+1)} \subset J_0^{(n)} \subset \ldots \subset J_0' \subset I_0 = [-n_0^{-3/2}, n_0^{-3/2}]$$

is a sequence of the closed intervals of the values of m_0^2 chosen inductively [with the use of (6.8) and the continuity in m_0^2] so that m_n^2 sweeps $I_n = [-(n_0+n)^{-3/2}, (n_0+n)^{-3/2}]$ when m_0^2 runs through $J_0^{(n)}$. At the critical point, all our effective interactions satisfy the bounds of sect. 6, and clearly

$$V^n(\psi^n) \xrightarrow[n\to\infty]{} 0 \tag{1}$$

for $\psi^n = \mathscr{A}_n\phi$ and ϕ with compact support. This establishes the IR asymptotic freedom of the weakly coupled critical lattice ϕ_4^4 theory which is the main result of the present paper.

A reader of [18] will also understand that an easy extension of the present method allows rigorous treatment of correlation functions of the model which have a massless decay. We plan to come back to this problem in order to exhibit the logarithmic corrections to scaling [40, 7, 8] and in the context of the UV asymptotically free negative coupling theory.

As to whether our work provides a general tool to treat renormalizable asymptotically free models (first of all the UV problem of the gauge theories), we leave the jugement to the reader (and the future).

Appendix 1

Here we shall establish (3.7) for $\delta\lambda^1$ given by (3.6). Consider first the third term on the right-hand side of (3.6). We shall derive lower and upper bounds for

$$L^4 \int_{\square_0} dx \int dy (\mathscr{T}_{nLxLy})^2 . \tag{1}$$

For a lower bound (recall (3.5))

$$\begin{aligned}(1) &= L^{-4} \int_{L\square_0} dx \int dy (\mathscr{T}_{nxy})^2 \\ &\geqq L^{-4} \int_{L\square_0} dx \int_{L\square_0} dy (\mathscr{T}_{nxy})^2 = L^{-4} \int_{L\square_0} dx \int_{L\square_0} dy (\mathscr{G}_{nxy} - L^{-2}\mathscr{G}_{n+1L^{-1}xL^{-1}y})^2 \\ &\geqq L^{-4} \int_{L\square_0} dx \int_{L\square_0} dy (\mathscr{G}_{nxy})^2 - 2\left[L^{-4} \int_{L\square_0} dx \int_{L\square_0} dy (\mathscr{G}_{nxy})^2\right]^{1/2} \\ &\quad \cdot \left[L^{-4} \int_{L\square_0} dx \int_{L\square_0} dy L^{-4} (\mathscr{G}_{n+1L^{-1}xL^{-1}y})^2\right]^{1/2} .\end{aligned} \tag{2}$$

But

$$L^{-8}\int_{L\square_0} dx \int_{L\square_0} dy(\mathcal{G}_{n+1L^{-1}xL^{-1}y})^2 = \int_{\square_0} dx \int_{\square_0} dy(\mathcal{G}_{n+1xy})^2 \leqq \mathcal{O}(1) \tag{3}$$

with $\mathcal{O}(1)$ L independent. On the other hand [recall (2.21)]

$$\begin{aligned}
L^{-4}\int_{L\square_0} dx \int_{L\square_0} dy(G_{nxy})^2 &\geqq L^{-4}\sum_{x,y\in L\square_0}\left(\int_{\square_x} dx \int_{\square_y} dy\mathcal{G}_{nxy}\right)^2 \\
&= L^{-4}\sum_{x,y\in L\square_0}(G_{nxy})^2 = L^{-4}\sum_{x,y\in L\square_0}\frac{1}{\Lambda_n^2}\sum_{\substack{0\neq p,q\in 2\pi L^{-N+n}\mathbb{Z}^d \\ -\pi<p_\mu,q_\mu\leqq\pi}} \\
&\quad\cdot e^{-ip(x-y)}\hat{G}_n(p-q)\hat{G}_n(q) \\
&= L^{-4}\frac{1}{\Lambda_n^2}\sum_{0\neq p,q}\prod_\mu\left(\frac{\sin Lp_\mu/2}{\sin p_\mu/2}\right)^2 \sum_{\substack{0\neq p,q \\ -\pi<p_\mu,q_\mu\leqq\pi}}\hat{G}_n(p-q)\hat{G}_n(q) \\
&\geqq \mathcal{O}(1)L^4\frac{1}{\Lambda_n^2}\sum_{\substack{0\neq p,q \\ |p_\mu|<\frac{\pi}{L} \\ -\pi<q_\mu\leqq\pi}}\frac{1}{|p-q|^2}\frac{1}{|q|^2} \\
&\geqq \mathcal{O}(1)L^4\frac{1}{\Lambda_n}\sum_{\substack{0\neq p \\ |p_\mu|<\frac{\pi}{L}}}|\log p^2| \geqq \mathcal{O}(1)\log L,
\end{aligned} \tag{4}$$

where in the fifth step we have used the uniform bounds for the Fourier transforms $\hat{G}_n(p)$, see Appendix of [16]. Equations (2)–(4) give immediately

$$(1) \geqq \mathcal{O}(1)\log L. \tag{5}$$

To get a similar upper bound, take for simplicity $L = L_0^m$ with $L_0 = \mathcal{O}(1)$ (say $L_0 = 2$) and m large. Denoting explicitly the L dependence of $\mathcal{T}$, we have

$$\mathcal{T}^{(L)}_{kxy} = \sum_{l=0}^{m-1} L_0^{-2l}\mathcal{T}^{(L_0)}_{km+lL_0^{-1}xL_0^{-1}y}. \tag{6}$$

Thus

$$\begin{aligned}
(1) &= L^{-4}\int_{L\square_0} dx \int dy\left(\sum_{l=0}^{m-1} L_0^{-2l}\mathcal{T}^{(L_0)}_{nm+lL_0^{-1}xL_0^{-1}y}\right)^2 \\
&\leqq \mathcal{O}(1)\sum_{l=0}^{m-1} L_0^{-4l}\int dy \exp[-\varepsilon L_0^{-l}|y|] \\
&\quad + \mathcal{O}(1)\sum_{l=0}^{m-1}\sum_{k=l+1}^{m-1} L_0^{-2(l+k)}\int dy \exp[-\varepsilon(L_0^{-l}+L_0^{-k})|y|] \\
&\leqq \mathcal{O}(1)\sum_{l=0}^{m-1} L_0^{-4l}L_0^{4l} = \mathcal{O}(1)m = \mathcal{O}(1)\log L.
\end{aligned} \tag{7}$$

This establishes the desired upper bound for (1).

As for the other contributions to $\delta\lambda^1$, the first term on the right-hand side of (3.6) becomes for $L \mapsto L_0^m$,

$$-72L^{-4}\lambda^2 \int_{L\Box_0} d\mathscr{x} \int d\mathscr{y} \left(\sum_{k=0}^{n-1} L_0^{2(n-k)m} \sum_{l=0}^{m-1} L_0^{-2l} \mathscr{T}^{(L_0)}_{km+lL_0^{(n-k)m-l}\mathscr{x}L_0^{(n-k)m-l}\mathscr{y}} \right)$$
$$\cdot \sum_{l=0}^{m-1} L_0^{-2l} \mathscr{T}^{(L_0)}_{nm+lL_0^{-1}\mathscr{x}L_0^{-1}\mathscr{y}}, \tag{8}$$

$$|(8)| \leqq \mathcal{O}(1)\lambda^2 \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} L_0^{2[(n-k)m-l]} \int d\mathscr{y} \exp[-\varepsilon L_0^{(n-k)m-l}|\mathscr{y}|]$$
$$\leqq \mathcal{O}(1)\lambda^2 \sum_{l=1}^{\infty} L_0^{-2l} \leqq \mathcal{O}(1)\lambda^2 . \tag{9}$$

The other terms on the right-hand side of (3.6) vanish. Indeed by (2.19) and (2.21),

$$\int_{\Box_x} d\mathscr{x} \mathscr{T}_{kL\mathscr{x}L\mathscr{y}} = L^{-4} \sum_{\substack{y \\ -\frac{1}{2}L < y^\mu - Lx^\mu \leqq \frac{1}{2}L}} (Q\Gamma_k Q^+ \mathscr{A}_k^+)_{yL\mathscr{y}} = L^{-1}(CQ\Gamma_k Q^+ \mathscr{A}_k^+)_{xL\mathscr{y}} = 0 \tag{10}$$

as $CQ=0$. Equations (3.2) and (10) imply also that

$$\int_{\Box_x} d\mathscr{x} \mathscr{Q}_{nL\mathscr{x}L\mathscr{y}} = 0 . \tag{11}$$

This way, (5), (7), and (9) show that for $L = L_0^m$ with m large,

$$-\mathcal{O}(1)m\lambda^2 \leqq \delta\lambda^1 \leqq -\mathcal{O}(1)m\lambda^2 ,$$

from which (3.7) follows.

Appendix 2

For I_{xy} given by (11.6) define

$$\varphi(p) = \sum_{x \in \Lambda_n} e^{-ipx} I_{0x} .$$

$\varphi(p)$ is periodic in each p_μ with the period 2π and, due to (11.7) is analytic for $|\mathrm{Im}\, p_\mu| < \frac{5}{2}\alpha$. Equations (11.8)–(11.10) are corollaries of the following result giving a solution to the Gleason problem [36].

Lemma. *Let $f(p_1, \ldots, p_d)$ be a function analytic for $|\mathrm{Im}\, p_j| < a$, periodic in each p_μ with period 2π such that*

$$\frac{\partial^l f(0)}{\partial p_{\mu_1} \ldots \partial p_{\mu_l}} = 0 \quad \textit{for} \quad l = 0, 1, \ldots, k-1 .$$

Then there exist functions $f_{\mu_1 \ldots \mu_k}(p)$ with the same analyticity and periodicity properties satisfying

$$f(p) = \sum_{\mu_1, \ldots, \mu_k} \prod_{j=1}^{k} (e^{-ip_{\mu_j}} - 1) f_{\mu_1 \ldots \mu_k}(p) .$$

$f_{\mu_1 \ldots \mu_k}$ can be taken linearly depending on f and such that for each $0 < a_1 < a_2 < a$,

$$\sup_{|\mathrm{Im}\, p_\mu| \leqq a_1} |f_{\mu_1 \ldots \mu_k}(p)| \leqq C(d, k, a_1, a_2) \sup_{|\mathrm{Im}\, p_\mu| \leqq a_2} |f(p)| .$$

Proof. Define $\tilde{f}(z_1, \ldots, z_d)$ analytic for $e^{-a} < |z_j| < e^a$ by

$$\tilde{f}(e^{-ip_1}, \ldots, e^{-ip_d}) \equiv f(p_1, \ldots, p_d). \tag{5}$$

Again

$$\frac{\partial^l \tilde{f}(1, \ldots, 1)}{\partial z_{\mu_1} \ldots \partial z_{\mu_l}} = 0 \quad \text{for} \quad l = 0, 1, \ldots, k-1. \tag{6}$$

Notice that by the Cauchy formula,

$$\begin{aligned} \tilde{f}(z) = & \sum_{\substack{\sigma = (\sigma_1, \ldots, \sigma_d) \\ \sigma_\mu = \pm 1}} \frac{1}{(2\pi i)^d} \int_{|\zeta_1| = e^{\sigma_1(a-\varepsilon)}} d\zeta_1 \\ & \ldots \int_{|\zeta_d| = e^{\sigma_d(a-\varepsilon)}} d\zeta_d \frac{\tilde{f}(\zeta_1, \ldots, \zeta_d)}{(\zeta_1 - z_1) \ldots (\zeta_d - z_d)} \\ \equiv & \sum_\sigma \tilde{f}_\sigma(z_1^{\sigma_1}, \ldots, z_d^{\sigma_d}). \end{aligned} \tag{7}$$

$$\begin{aligned} \tilde{f}_\sigma(v_1, \ldots, v_d) = & \frac{1}{(2\pi i)^d} \int_{|\eta_1| = e^{a-\varepsilon}} d\eta_1 \ldots \int_{|\eta_d| = e^{a-\varepsilon}} d\eta_d \\ & \cdot \frac{\tilde{f}(\eta_1^{\sigma_1}, \ldots, \eta_d^{\sigma_d})}{(\eta_1 - v_1), \ldots, (\eta_d - v_d)} \prod_{\mu=1}^d (v_\mu / \eta_\mu)^{\frac{1-\sigma_\mu}{2}} \end{aligned} \tag{8}$$

(the integrals over $|z| = e^{a-\varepsilon}$ are counter-clockwise and the ones over $|z| = e^{-a+\varepsilon}$ are clockwise). Clearly $\tilde{f}_\sigma(v)$ are analytic functions for $|v_\mu| < e^a$ and for $l = 0, 1, \ldots, k$,

$$\sup_{|v_j| \leq e^{a_1}} \left| \frac{\partial^l \tilde{f}_\sigma(v)}{\partial v_{\mu_1} \ldots \partial v_{\mu_l}} \right| \leq \bar{C}(d, k, a_1, a_2) \sup_{e^{-a_2} \leq |z_\mu| \leq e^{a_2}} |\tilde{f}(z_1, \ldots, z_d)|. \tag{9}$$

But, Taylor expanding, we may write

$$\begin{aligned} \tilde{f}_\sigma(v) = & \sum_{l=0}^{k-1} \frac{1}{l!} \sum_{\mu_1, \ldots, \mu_l} \frac{\partial^l \tilde{f}_\sigma(1, \ldots, 1)}{\partial v_{\mu_1} \ldots \partial v_{\mu_l}} \prod_{j=1}^l (v_{\mu_j} - 1) \\ & + \frac{1}{(k-1)!} \sum_{\mu_1, \ldots, \mu_k} \int_0^1 dt (1-t)^{k-1} \\ & \cdot \frac{\partial^k \tilde{f}_\sigma}{\partial v_{\mu_1} \ldots \partial v_{\mu_k}} (1 - t + t v_1, \ldots, 1 - t + t v_d) \prod_{j=1}^k (v_{\mu_j} - 1) \end{aligned} \tag{10}$$

or

$$\begin{aligned} \tilde{f}_\sigma(z_1^{\sigma_1}, \ldots, z_d^{\sigma_d}) = & \sum_{l=0}^{k-1} \frac{1}{l!} \sum_{\mu_1, \ldots, \mu_l} \frac{\partial^l \tilde{f}_\sigma(1, \ldots, 1)}{\partial v_{\mu_1} \ldots \partial v_{\mu_l}} \prod_{j=1}^l (-z_{\mu_j})^{\frac{\sigma_{\mu_j} - 1}{2}} (z_{\mu_j} - 1) \\ & + \frac{1}{(k-1)!} \sum_{\mu_1, \ldots, \mu_k} \int_0^1 dt (1-t)^{k-1} \\ & \cdot \frac{\delta^k \tilde{f}_\sigma}{\partial v_{\mu_1} \ldots \partial v_{\mu_k}} (1 - t + t z_1^{\sigma_1}, \ldots, 1 - t + t z_d^{\sigma_d}) \prod_{j=1}^k (-z_{\mu_j})^{\frac{\sigma_{\mu_j} - 1}{2}} (z_{\mu_j} - 1). \end{aligned} \tag{11}$$

In the first sum on the right-hand side of (11), we may expand

$$(-z_{\mu_j})^{-1} = -(1 + z_{\mu_j} - 1)^{-1} = \sum_{n=0}^{k-l-1} (-1)^{n+1} (z_{\mu_j} - 1)^n - (-1)^{k-l} z_{\mu_j}^{-1} (z_{\mu_j} - 1)^{k-l}. \tag{12}$$

Hence (grouping the terms in a somewhat arbitrary way), we obtain

$$\tilde{f}_\sigma(z_1^{\sigma_1}, \ldots, z_d^{\sigma_d}) = \sum_{l=0}^{k-1} \sum_{\mu_1, \ldots, \mu_l} a_{\sigma\mu_1\ldots\mu_l} \prod_{j=1}^{l} (z_{\mu_j} - 1)$$
$$+ \sum_{\mu_1, \ldots, \mu_k} \tilde{f}_{\sigma\mu_1\ldots\mu_k}(z_1, \ldots, z_d) \prod_{j=1}^{k} (z_{\mu_j} - 1) \tag{13}$$

with $\tilde{f}_{\sigma\mu_1\ldots\mu_k}(z)$ analytic for $e^{-a} < z_\mu < e^a$ linear in $\tilde{f}$ and satisfying (4) with $C \to 2^{-d}C$, say. Upon the summation over σ of (13) the lower order terms ($l < k$) must vanish and hence by (7), we obtain

$$\tilde{f}(z) = \sum_{\mu_1, \ldots, \mu_k} \tilde{f}_{\mu_1\ldots\mu_k}(z) \prod_{j=1}^{k} (z_{\mu_j} - 1) \tag{14}$$

for $\tilde{f}_{\mu_1\ldots\mu_k} \equiv \sum_\sigma \tilde{f}_{\sigma\mu_1\ldots\mu_k}$. This completes the proof of Lemma.

We apply Lemma with $k=2$ to $f(p) = \varphi(p) - \varphi(0)$ where φ is given by (1). Let $\varphi_{\mu\nu}(p) = -f_{\mu\nu}(p)e^{-ip_\nu}$. We have

$$\varphi(p) = \sum_{\mu,\nu} (e^{-ip_\mu} - 1)(e^{ip_\nu} - 1)\varphi_{\mu\nu}(p) \tag{15}$$

and put

$$\tfrac{1}{2}\delta m^2 = \varphi(0)\,, \tag{16}$$

$$J_{1xy}^{\mu\nu} = \frac{1}{\Lambda_{n+1}} \sum_{\substack{p \in 2\pi L^{-N+n+1}\mathbb{Z}^4 \\ -\pi < p_\mu \leqq \pi}} e^{ip(x-y)} \varphi_{\mu\nu}(p)\,. \tag{17}$$

The symmetry properties of $J_{1xy}^{\mu\nu}$ may be guaranteed by averaging

$$e^{-\frac{i}{2}p_\mu} e^{\frac{i}{2}p_\nu} \varphi_{\mu\nu}(p)$$

over the lattice rotations and reflections. Equation (11.10) follows from the uniform bound for $\varphi_{\mu\nu}(p)$ for $|\mathrm{Im}\, p_\mu| < 2\alpha$.

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