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# A Generalized Fluctuation-Dissipation Theorem for the One-Dimensional Diffusion Process

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**Abstract.** The $[\alpha, \beta, \gamma]$-Langevin equation describes the time evolution of a real stationary process with $T$-positivity (reflection positivity) originating in the axiomatic quantum field theory. For this $[\alpha, \beta, \gamma]$-Langevin equation a generalized fluctuation-dissipation theorem is proved. We shall obtain, as its application, a generalized fluctuation-dissipation theorem for the one-dimensional non-linear diffusion process, which presents one solution of Ryogo Kubo's problem in physics.

## 1. Introduction

In order to clarify a probabilistic meaning of the concept of $T$-positivity (reflection positivity) with its origin in axiomatic quantum field theory [3, 14], we have investigated a real stationary Gaussian process $X$ having $T$-positivity from the viewpoint of the theory of stochastic differential equations [10, 11, 13]. In the previous paper [11], we characterized a class of stochastic differential equations describing the time evolution of $X$ as a $[\alpha, \beta, \gamma]$-Langevin equation and then obtained a fluctuation-dissipation theorem for this $[\alpha, \beta, \gamma]$-Langevin equation as a generalized fluctuation-dissipation theorem in the theory of Ornstein-Uhlenbeck Brownian motion in statistical physics [2, 6–8, 15].

The purpose of the present paper is to refine the results of [11] and then make them serve to get a generalized second fluctuation-dissipation theorem for the one-dimensional non-linear diffusion process, which presents one solution of Kubo's problem in physics [6–8]. Before reformulating Kubo's problem stated in [7], we shall recall briefly a second fluctuation-dissipation theorem for Ornstein-Uhlenbeck Brownian motion. Let $\mathscr{X} = (\mathscr{X}(t), P_x; t \in [0, \infty), x \in \mathbb{R})$ be an Ornstein-Uhlenbeck Brownian motion whose time evolution is governed by the following stochastic differential equation:

$$\left.\begin{aligned} d\mathscr{X}(t) &= -\beta\mathscr{X}(t)dt + \alpha dB(t) \quad (t \in (0, \infty)) \\ \mathscr{X}(0) &= x . \end{aligned}\right\} \tag{1.1}$$

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Here $\alpha$ and $\beta$ are positive numbers and $(B(t); t\in[0,\infty))$ is a one-dimensional standard Brownian motion. We know that

$$\lim_{t\to\infty} P_x(\mathscr{X}(t)\in dy)=(\pi\alpha^2\beta^{-1})^{-1/2}\exp(-y^2/\alpha^2\beta^{-1})dy=m_\infty(dy), \tag{1.2}$$

and the probability measure $m_\infty$ is a unique invariant measure for the process $\mathscr{X}$. Let $N=(N(t); t\in\mathbb{R})$ be a stationary Markov process representing the stationary state of the process $\mathscr{X}$ with $m_\infty$ as its initial distribution. Then it follows that the time evolution of the process $N$ is governed by the following stochastic differential equation:

$$dN(t)=-\beta N(t)dt+\alpha dW(t) \quad (t\in\mathbb{R}). \tag{1.3}$$

Here $(W(t); t\in\mathbb{R})$ is a one-dimensional standard Brownian motion. Denoting by $R$ the covariance function of the process $N$, we have

$$R(t)=\frac{\alpha^2}{2\beta}\exp(-\beta|t|) \quad (t\in\mathbb{R}). \tag{1.4}$$

From (1.4), we immediately obtain

$$\frac{\alpha^2}{2}=R(0)\beta. \tag{1.5}$$

For the purpose of understanding the physical meaning of formula (1.5), we shall consider the motion of a Brownian particle moving with velocity $N(t)$ at time $t$ in a viscous fluid with friction coefficient $\beta$ whose equation of motion is described by the stochastic differential Eq. (1.3). In that case we can regard the left-hand side in (1.5) as the power of random force causing the zigzag motion of a Brownian particle. On the other hand, (1.2) and (1.5) make us notice that the constant $R(0)$ is a variance of the equilibrium measure $m_\infty$, and so we can regard $R(0)=kT$, where $k$ is a Boltzman constant and $T$ is an absolute temperature in the system under consideration. Therefore formula (1.5) stands for a relation between the power of a random force and the friction coefficient of a viscous fluid. And it is to be called a *second fluctuation-dissipation theorem*. The theoretical ground for a physical understanding of formula (1.5) lies in the stochastic differential Eq. (1.3). That is, it is important, not only from the viewpoint of statistical mechanics, but also from that of probability, to derive a stochastic differential equation describing the time evolution of a stationary process only by using its qualitative nature. The key is how to extract a random force.

Now we shall reformulate Kubo's problem stated in [7] as follows. Let

$$\mathscr{X}=(\mathscr{X}(t), P_x; t\in[0,\infty), x\in\mathbb{R})$$

be a one-dimensional diffusion process whose time evolution is governed by the following stochastic differential equation:

$$\left.\begin{aligned} &d\mathscr{X}(t)=b(\mathscr{X}(t))dt+\sigma(\mathscr{X}(t))dB(t) \quad (t\in(0,\infty))\\ &\mathscr{X}(0)=x.\end{aligned}\right\} \tag{1.6}$$

Here $(B(t); t\in[0,\infty))$ is a one-dimensional standard Brownian motion, $b$ and $\sigma$ are continuous functions on $\mathbb{R}$. Then we know that the Fokker-Planck equation associated with the stochastic differential Eq. (1.6) is

$$\left.\begin{aligned} &\frac{\partial}{\partial t}u(t,x)=\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}u(t,x)+b(x)\frac{\partial}{\partial x}u(t,x) && (t\in(0,\infty), x\in\mathbb{R}) \\ &u(0,x)=f(x) && (x\in\mathbb{R}). \end{aligned}\right\} \tag{1.7}$$

Here $a=\sigma^2$ and $f$ is a given initial function and the solution $u$ for Eq. (1.7) is given by

$$u(t,x)=E_x(f(\mathscr{X}(t))) \quad (t\in[0,\infty), x\in\mathbb{R}). \tag{1.8}$$

Here $E_x$ denotes an integration with respect to a probability measure $P_x$. We define a Borel measure $m$ on $\mathbb{R}$ by

$$m(dx)=\frac{1}{a(x)}\exp(C(x))dx, \tag{1.9}$$

where $C$ is a function given by

$$C(x)=\int_0^x \frac{b(y)}{a(y)}dy. \tag{1.10}$$

Let us suppose the following conditions:

$$m(\mathbb{R})<\infty, \tag{1.11}$$

$$\int_{-\infty}^0\left(\int_y^0 m(dx)\right)e^{-C(y)}dy=\int_0^\infty\left(\int_0^y m(dx)\right)e^{-C(y)}dy=\infty. \tag{1.12}$$

We note that condition (1.12) implies that both boundary points $\pm\infty$ are entrance or natural in the sense of Feller. We define a probability Borel measure $m_\infty$ on $\mathbb{R}$ by

$$m_\infty(dx)=\frac{1}{m(\mathbb{R})}m(dx). \tag{1.13}$$

Then we know from [9] that $m_\infty$ is a unique invariant probability measure for the diffusion process $\mathscr{X}$. Let $N=(N(t); t\in\mathbb{R})$ be a stationary Markov process on a probability space $(\Omega,\mathscr{B},P)$ representing the stationary state of the diffusion process $\mathscr{X}$ with $m_\infty$ as its initial distribution. In the present paper, Kubo's problem takes the form of deriving stochastic differential equation describing the time evolution of the process $N$ by extracting an appropriate white noise as a random force so as to obtain, as a result, a relation between a diffusion coefficient and a drift coefficient in such a stochastic differential equation as a generalization of a second fluctuation-dissipation theorem (1.5) for the Ornstein-Uhlenbeck Brownian motion.

In Sect. 5, we shall prove main Theorems 5.2 and 5.3.

**Theorem 5.2.** *Besides conditions* (1.11) *and* (1.12), *we suppose the following conditions:*

$$\int_{\mathbb{R}}(x^2+b(x)^2)m(dx)<\infty, \tag{1.14}$$

$$\int_{\mathbb{R}} xm(dx)=0. \tag{1.15}$$

*Then there exists a unique quadruplet $[\alpha, \beta, \gamma, I]$ appearing in the case that the time evolution of N is governed by the following stochastic differential equation:*

$$dN(t) = -\left(\beta N(t) + \int_{-\infty}^{0} N(t+s)\gamma(s)ds\right) + \alpha dI(t) \quad (t \in \mathbb{R}). \tag{1.16}$$

*Here the triple $[\alpha, \beta, \gamma]$ satisfies the following conditions:*

$$\alpha > 0 \quad and \quad \beta > 0, \tag{1.17}$$

$$\gamma(s) = -\chi_{(-\infty,0)}(s) \int_0^\infty e^{s\lambda}\mu(d\lambda), \tag{1.18}$$

*with a Borel measure $\mu$ on $[0, \infty)$ satisfying the conditions $\mu(\{0\}) = 0$ and $\int_0^\infty \lambda^{-1}\mu(d\lambda) \leqq \beta$, and the stochastic process $I = (I(t); t \in \mathbb{R})$ satisfies the following conditions: for any $s, t \in \mathbb{R}$, $s < t$,*

$$E(|I(t) - I(s)|^2) = t - s, \tag{1.19}$$

*the closed linear hull of $\{N(u); u \leqq t\}$ equals the closed linear hull of $\{I(u) - I(v); u, v \leqq t\}$ in $L^2(\Omega, \mathscr{B}, P)$.* (1.20)

We shall call the stochastic differential Eq. (1.16) and random force $I$ respectively, $[\alpha, \beta, \gamma]$-*Langevin equation* and *innovation process* with causal condition associated with the stationary Markov process $N$.

**Theorem 5.3.** (i) *The following relation holds between the diffusion coefficient and the drift coefficient in the stochastic differential Eq.* (1.16);

$$\frac{\alpha^2}{2} = R(0)C_{\beta,\gamma}. \tag{1.21}$$

*Here R is a covariance function of N and $C_{\beta,\gamma}$ is given by*

$$C_{\beta,\gamma} = \pi\left(\int_{\mathbb{R}} |\beta + \hat{\gamma}(\xi) - i\xi|^{-2} d\xi\right)^{-1}. \tag{1.22}$$

(ii) *The triple $\left[\frac{\alpha^2}{2}, R(0), C_{\beta,\gamma}\right]$ can be written in terms of the diffusion coefficient a and drift coefficient b in the Fokker-Planck Eq.* (1.7) *as follows:*

$$\left.\begin{aligned} \frac{\alpha^2}{2} &= \left(-\int_{\mathbb{R}} \frac{xb(x)}{a(x)} e^{C(x)}dx\right)\left(\int_{\mathbb{R}} \frac{1}{a(x)} e^{C(x)}dx\right)^{-1} \\ R(0) &= \left(\int_{\mathbb{R}} \frac{x^2}{a(x)} e^{C(x)}dx\right)\left(\int_{\mathbb{R}} \frac{1}{a(x)} e^{C(x)}dx\right)^{-1} \\ C_{\beta,\gamma} &= \left(-\int_{\mathbb{R}} \frac{xb(x)}{a(x)} e^{C(x)}dx\right)\left(\int_{\mathbb{R}} \frac{x^2}{a(x)} e^{C(x)}dx\right)^{-1}. \end{aligned}\right\} \tag{1.23}$$

We call formula (1.21) and constant $C_{\beta,\gamma}$ *a generalized second fluctuation-dissipation theorem* and *a generalized friction coefficient*, respectively. Here we note that for the Ornstein-Uhlenbeck Brownian motion governed by the stochastic differential Eq. (1.1), formula (1.21) along with (1.23) is reduced to the second fluctuation-dissipation theorem (1.5).

The fundamental idea in proving Theorems 5.2 and 5.3 is to regard the stationary Markov process $N$ as a stationary process having $T$-positivity and then apply the results in [11] to this process $N$. For that purpose we have to refine the results of [11]. Apart from the diffusion process $\mathscr{X}$ and stationary Markov process $N$, let $X=(X(t); t\in\mathbb{R})$ be a real stationary Gaussian process with $T$-positivity. Then we know from [4] that the covariance function $R$ of $X$ is represented as

$$R(t)=\int_0^\infty e^{-\lambda|t|}\sigma(d\lambda). \tag{1.24}$$

Here $\sigma$ is a Borel measure on $[0,\infty)$. In [11] we have treated the case where $\sigma$ satisfies $\sigma(\{0\})=0$ and $\int_0^\infty(\lambda^{-1}+\lambda^2)\sigma(d\lambda)<\infty$. In Sect. 2, we shall reproduce main results in the first half of [11] only under the condition $\int_0^\infty \lambda^2\sigma(d\lambda)<\infty$. Apart from a probabilistic structure, we shall in Sect. 3 introduce a Langevin data by giving the analytical viewpoint of the results in Sect. 2. Furthermore we shall obtain a formula by which the Langevin data can be directly calculated in terms of the measure $\sigma$. By taking the same consideration as the second half of [11], we shall in Sect. 4 derive a stochastic differential equation describing the time evolution of $X$ and then obtain generalized fluctuation-dissipation theorems for $X$. Furthermore we shall obtain a best estimate for a generalized drift coefficient appearing in a generalized second fluctuation-dissipation theorem. In Sect. 5, we shall reformulate the results in Sects. 2–4 to a stationary curve with $T$-positivity in a Hilbert space, which covers two examples of a homogeneous random field with $T$-positivity [3] and a stationary symmetric Markov process. As a realization, we shall apply the results in this section to the one-dimensional non-linear diffusion process in order to obtain main Theorems 5.2 and 5.3. We follow the notation and terminology in [11].

## 2. $[\alpha,\beta,\gamma]$-Langevin Equation

Let $X=(X(t); t\in\mathbb{R})$ be a real stationary Gaussian process such that $X(t)$ is continuous in the mean and its expectation is zero. Furthermore we suppose that $X$ has $T$-positivity, that is, its covariance function $R$ can be represented in the form

$$R(t)=\int_{[0,\infty)} e^{-\lambda|t|}\sigma(d\lambda)\quad (t\in\mathbb{R}), \tag{2.1}$$

where $\sigma$ is a bounded Borel measure on $[0,\infty)$ [4]. Moreover we assume the following condition

$$\sigma(\{0\})=0,\qquad 0<\sigma([0,\infty))<\infty\quad\text{and}\quad \int_0^\infty \lambda^2\sigma(d\lambda)<\infty. \tag{2.2}$$

We then see that $X$ has such a spectral density $\Delta$ that

$$R(t)=\int_{\mathbb{R}} e^{-it\xi}\Delta(\xi)d\xi\quad (t\in\mathbb{R}), \tag{2.3}$$

$$\Delta(\xi)=\frac{1}{\pi}\int_0^\infty \frac{\lambda}{\xi^2+\lambda^2}\sigma(d\lambda)\quad (\xi\in\mathbb{R}). \tag{2.4}$$

Since it follows from Theorem 2.1 in [10] that the function $\Delta$ satisfies the Hardy condition

$$\int_{\mathbb{R}} \frac{|\log \Delta(\xi)|}{1+\xi^2} d\xi < \infty , \tag{2,5}$$

we can define the outer function $h$ of $X$ by

$$h(\zeta) = \exp\left(\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1+\lambda\zeta}{\lambda-\zeta} \frac{\log \Delta(\lambda)}{1+\lambda^2} d\lambda\right) \quad (\zeta \in \mathbb{C}^+). \tag{2.6}$$

We recall the well-known facts from the theory of $H^2$-space and the spectral theory of $X$ [1]; Since by (2.5) the function $h$ satisfies

$$h \in \mathcal{O}(\mathbb{C}^+) \quad \text{and} \quad \sup_{\eta>0} \int_{\mathbb{R}} |h(\xi+i\eta)|^2 d\xi < \infty , \tag{2.7}$$

it follows that for almost all $\xi \in \mathbb{R}$

$$\lim_{\eta \downarrow 0} h(\xi+i\eta) \equiv h(\xi+i0) \equiv h(\xi) \tag{2.8}$$

exists and it satisfies

$$h \in L^2 , \quad \overline{h(\xi)} = h(-\xi) \quad \text{and} \quad |h(\xi)|^2 = \Delta(\xi) . \tag{2.9}$$

Next we denote by $E$ the Fourier transform of $h$:

$$E(t) = \int_{\mathbb{R}} e^{-it\xi} h(\xi) d\xi \equiv \hat{h}(t) . \tag{2.10}$$

We then see that

$$E \in L^2 \quad \text{and} \quad E=0 \quad \text{in} \quad (-\infty, 0), \tag{2.11}$$

$$R(t) = \frac{1}{2\pi} \int_0^\infty E(t+s)E(s) ds \quad \text{for any} \quad t \in \mathbb{R}. \tag{2.12}$$

Furthermore it follows that there exists a one-dimensional Brownian motion $B=(B(t); t \in \mathbb{R})$ such that for any $t \in \mathbb{R}$

$$X(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} E(t-s) dB(s) , \tag{2.13}$$

$$\sigma(X(s); s \leqq t) = \sigma(B(s_1) - B(s_2); s_1, s_2 \leqq t) . \tag{2.14}$$

The function $E$ and condition (2.14) are said to be a canonical representation kernel of $X$ and a causal condition, respectively.

Now we remine the main results of [11]. By noting Step 1 to Step 4 in the proof of Theorem 6.1 in [11], we have

**Theorem 2.1.** (i) *There exists a unique triple* $[\alpha, \beta, \gamma]$ *such that*

$$h(\zeta) = \frac{\alpha}{\sqrt{2\pi}} \frac{1}{\beta + \hat{\gamma}(\zeta) - i\zeta} \quad (\zeta \in \mathbb{C}^+) . \tag{2.15}$$

*Here the triple* $[\alpha, \beta, \gamma]$ *satisfies the following conditions:*

$$\alpha > 0 \quad \textit{and} \quad \beta > 0, \tag{2.16}$$

$$\gamma(s) = -\chi_{(-\infty,0)}(s) \int_{[0,\infty)} e^{s\lambda}\mu(d\lambda) \tag{2.17}$$

*with a Borel measure* $\mu$ *on* $[0,\infty)$ *satisfying the conditions*

$$\mu(\{0\}) = 0 \quad \textit{and} \quad \beta \geqq \int_0^\infty \lambda^{-1}\mu(d\lambda).$$

(ii) *The process X satisfies the following* $[\alpha, \beta, \gamma]$*-Langevin equation:*

$$dX(t) = -\left(\beta X(t) + \int_{-\infty}^{0} X(t+s)\gamma(s)ds\right)dt + \alpha dB(t). \tag{2.18}$$

*Remark 2.1.* It follows from Step 5 in the proof of Theorem 6.1 in [11] that

$$\beta > \int_0^\infty \lambda^{-1}\mu(d\lambda) \tag{2.19}$$

if and only if

$$\int_0^\infty \lambda^{-1}\sigma(d\lambda) < \infty. \tag{2.20}$$

Conversely we have

**Theorem 2.2** (Theorem 6.2 in [11]). *For any triple* $[\alpha, \beta, \gamma]$ *satisfying conditions* (2.16), (2.17), *and* (2.19), *there exists a unique real stationary Gaussian process X satisfying T-positivity with* (2.1), (2.2), *and* (2.20) *such that the outer function h of X has the form* (2.15). *Furthermore the process X is a unique stationary solution of the* $[\alpha, \beta, \gamma]$*-Langevin Eq.* (2.18).

Next, by modifying the proof of Theorem 7.1 in [11], we shall show

**Theorem 2.3.** (i) *For any* $z \in \{z \in \mathbb{C}; \mathrm{Re}\, z > 0\}$

$$\int_0^\infty e^{-zt}E(t)dt = \sqrt{2\pi}\,\alpha \frac{1}{\beta + z - \int_0^\infty \frac{1}{z+\lambda}\mu(d\lambda)}. \tag{2.21}$$

(ii) *There exists a unique bounded Borel measure* $\nu$ *on* $[0,\infty)$ *with* $\nu(\{0\}) = 0$, $0 < \nu([0,\infty))$, *and*

$$\int_0^\infty \left(\lambda + \int_0^\infty \frac{1}{\lambda + \lambda'}\nu(d\lambda')\right)\nu(d\lambda) < \infty$$

*such that*

$$E(t) = \chi_{[0,\infty)}(t) \int_0^\infty e^{-t\lambda}\nu(d\lambda). \tag{2.22}$$

(iii)

$$\sigma(d\lambda)=\frac{1}{2\pi}\left(\int_0^\infty \frac{1}{\lambda+\lambda'}\nu(d\lambda')\right)\nu(d\lambda). \tag{2.23}$$

*Proof.* By the result of Sect. 2.3 in [1], we see that

$$e^{-\eta t}E(t)=(h(\cdot+i\eta))^\wedge(t) \text{ for any } \eta\in(0,\infty) \text{ and a.e. } t\in\mathbb{R}. \tag{2.24}$$

Therefore, by taking the inverse Fourier transform of both sides of (2.24) and then noting (2.11), we have (i). Let $\sigma$ be any bounded Borel measure on $[0,\infty)$ satisfying conditions (2.1) and (2.2), and then define for each $n\in\mathbb{N}$ a bounded Borel measure $\sigma_n$ by $\sigma_n(d\lambda)=\sigma\left(\left[\frac{1}{n},n\right]\cap d\lambda\right)$. Furthermore, we denote by $h_n$ and $E_n$ the outer function and canonical representation kernel of a real stationary Gaussian process $X_n$, respectively, whose covariance function $R_n$ is given by (2.1) with $\sigma=\sigma_n$. Then it follows from (2.24), Step 2 in Sect. 5 of [11] and (i) proved just above that for any $\eta\in(0,\infty)$,

$$\int_0^\infty e^{-\eta t}E(t)dt=2\pi h(i\eta)=2\pi\lim_{n\to\infty}h_n(i\eta)=\lim_{n\to\infty}\int_0^\infty e^{-\eta t}E_n(t)dt. \tag{2.25}$$

Since $\sigma_n$ satisfies condition (2.20), we can apply Theorem 7.1 in [11] to see that there exists a bounded Borel measure $\nu_n$ on $[0,\infty)$ such that $E_n(t)=\int_0^\infty e^{-t\lambda}\nu_n(d\lambda)$. Moreover, since by Proposition 3.2 in [11]

$$\nu_n([0,\infty))=\left(4\pi\int_0^\infty \lambda\sigma_n(d\lambda)\right)^{1/2}$$

converges to $\left(4\pi\int_0^\infty \lambda\sigma(d\lambda)\right)^{1/2}$, we can get a subsequence $(n_k)_{k\in\mathbb{N}}$ and a bounded Borel measure $\bar{\nu}$ on $[0,\infty]$ such that $\lim_{k\to\infty}\nu_{n_k}=\bar{\nu}$. We define a bounded Borel measure $\nu$ on $[0,\infty)$ by $\nu(d\lambda)=\bar{\nu}([0,\infty)\cap d\lambda)$, and then a bounded function $F$ on $[0,\infty)$ by $F(t)=\int_{[0,\infty)} e^{-t\lambda}\nu(d\lambda)$. Since $E_n$ converges boundedly to $F$, we see from (2.25) that for any $\eta\in(0,\infty)$

$$\int_0^\infty e^{-\eta t}E(t)dt=\int_0^\infty e^{-\eta t}F(t)dt,$$

which implies that $E=F$ and so (2.22) holds. Similarly as Theorem 7.3 in [11], (iii) follows from (2.1), (2.12), and (2.22). Finally we shall prove the regularity condition of the measure $\nu$. By (2.23),

$$\begin{aligned}\int_0^\infty \lambda^2\sigma(d\lambda)&=\frac{1}{2\pi}\int_0^\infty \lambda\left(\int_0^\lambda \frac{\lambda}{\lambda+\lambda'}\nu(d\lambda')\right)\nu(d\lambda)\\&\geq\frac{1}{4\pi}\left(\int_0^\infty \lambda\nu([0,\lambda))\right)\nu(d\lambda).\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_0^\infty \lambda\nu([\lambda,\infty))\nu(d\lambda) &= \int_0^\infty \left(\int_0^{\lambda'} \lambda\nu(d\lambda)\right)\nu(d\lambda')\\ &\leqq 2\int_0^\infty \left(\int_0^{\lambda'}\left(\frac{(\lambda')^2}{\lambda+\lambda'}\nu(d\lambda)\right)\right)\nu(d\lambda')\\ &\leqq 2\int_0^\infty \left(\int_0^\infty \frac{(\lambda')^2}{\lambda+\lambda'}\nu(d\lambda)\right)\nu(d\lambda')\\ &= 4\pi\int_0^\infty \lambda^2\sigma(d\lambda).\end{aligned}$$

Therefore, we find from (2.2) that $\int_0^\infty \lambda\nu(d\lambda)<\infty$. The other condition for $\nu$ follows from the boundedness of $\sigma$ and (2.23). Q.E.D.

By using Theorem 2.3, we shall prove directly the following Theorem 2.4, which gives another proof of Propositions 3.2 and 3.3 in [11].

**Theorem 2.4.**

(1) (i) $$E'(t)=-\left(\beta E(t)+\int_{-\infty}^0 E(t+s)\gamma(s)ds\right) \quad in \quad (0,\infty),$$

(ii) $$E(0+)=\sqrt{2\pi}\alpha.$$

(2) (i) $$R'(t)=-\left(\beta R(t)+\int_{-\infty}^0 R(t+s)\gamma(s)ds\right) \quad in \quad (0,\infty),$$

(ii) $$R'(0+)=\frac{\alpha^2}{2}.$$

*Proof.* (1) By (2.21) and (2.11), for any $x\in(0,\infty)$,

$$\int_0^\infty \frac{1}{x+\lambda}\nu(d\lambda)=\sqrt{2\pi}\alpha\frac{1}{\beta+x-\int_0^\infty \frac{1}{x+\lambda}\mu(d\lambda)}. \tag{2.26}$$

Multiplying by $x$ both sides of (2.26) and then letting $x$ to infinity, we have (ii). By (2.22) and (1) (ii) proved just above,

$$\int_0^\infty e^{-xt}E'(t)dt=\sqrt{2\pi}\alpha+x\int_0^\infty \frac{1}{x+\lambda}\nu(d\lambda) \quad \text{for any} \quad x\in(0,\infty).$$

On the other hand, it follows from (2.10), (2.17), and (2.22) that for any $x\in(0,\infty)$

$$\int_0^\infty e^{-xt}\left(\beta E(t)+\int_{-\infty}^0 E(t+s)\gamma(s)ds\right)dt=\int_0^\infty \frac{1}{x+\lambda}\nu(d\lambda)\int_0^\infty \frac{1}{x+\lambda}\mu(d\lambda).$$

Therefore, we have (i) by (2.26) and the uniqueness of the Laplace transform. (2)(i) follows from (2.12) and (1)(i). By (2.22), (2.23), and (1)(ii),

$$\begin{aligned}\int_0^\infty \lambda\sigma(d\lambda) &= \frac{1}{2\pi}\int_0^\infty\int_0^\infty \nu(d\lambda)\nu(d\lambda')\frac{\lambda}{\lambda+\lambda'}\\ &= \frac{1}{2}\frac{1}{2\pi}\int_0^\infty\int_0^\infty \nu(d\lambda)\nu(d\lambda')\frac{\lambda+\lambda'}{\lambda+\lambda'}\\ &= \frac{1}{2}\frac{1}{2\pi}(\nu([0,\infty)))^2\\ &= \frac{1}{2}\alpha^2 .\end{aligned}$$

Therefore, we have (2)(ii) by (2.1). Q.E.D.

We will in Sect. 4 give a relation between the value of $R(0)$ and the triple $[\alpha, \beta, \gamma]$ which is the second fluctuation-dissipation theorem [6–8, 11].

## 3. Langevin Data

We will rewrite the results in Sect. 2 analytically apart from the probabilistic structure. We define $\Sigma$ and $\mathscr{L}$ by

$$\Sigma = \Big\{\sigma; \sigma \text{ is a Borel measure on } [0,\infty) \text{ such that } \sigma(\{0\})=0,\ 0<\sigma([0,\infty))<\infty, \text{ and } \int_0^\infty \lambda^2\sigma(d\lambda)<\infty\Big\}, \tag{3.1}$$

$$\mathscr{L} = \Big\{(\alpha,\beta,\mu); \alpha>0,\ \beta>0, \text{ and } \mu \text{ is a Borel measure on } [0,\infty) \text{ such that } \mu(\{0\})=0, \text{ and } \beta \geqq \int_0^\infty \lambda^{-1}\mu(d\lambda)\Big\}. \tag{3.2}$$

Furthermore we define $\Sigma_0$ and $\mathscr{L}_0$ by

$$\Sigma_0 = \Big\{\sigma\in\Sigma; \int_0^\infty \lambda^{-1}\sigma(d\lambda)<\infty\Big\}, \tag{3.3}$$

$$\mathscr{L}_0 = \Big\{(\alpha,\beta,\mu)\in\mathscr{L}; \beta > \int_0^\infty \lambda^{-1}\mu(d\lambda)\Big\}. \tag{3.4}$$

Then we shall show

**Theorem 3.1.** (i) *There exists an injective mapping $L$ from $\Sigma$ into $\mathscr{L}$ such that for any* $x\in(0,\infty)$,

$$\int_0^\infty \frac{\lambda+\beta-\int_0^\infty \frac{1}{\lambda+\lambda'}\mu(d\lambda')}{x+\lambda}\sigma(d\lambda) = \alpha^2\frac{1}{\beta+x-\int_0^\infty \frac{1}{x+\lambda}\mu(d\lambda)}. \tag{3.5}$$

(ii) $L(\Sigma_0)=\mathscr{L}_0$.

*Proof.* (i) Let $\sigma \in \Sigma$ be fixed arbitrarily and consider a real stationary Gaussian process $X$ with covariance function $R$ given by (2.1). Then we claim that the triple $[\alpha, \beta, \mu]$ in Theorem 2.1 satisfies Eq. (3.5); by (2.21), (2.22), (2.23), and (2.26),

$$\left(\lambda+\beta-\int_0^\infty \frac{1}{\lambda+\lambda'}\mu(d\lambda')\right)\sigma(d\lambda)=\frac{\alpha}{\sqrt{2\pi}}\nu(d\lambda). \tag{3.6}$$

By taking Stieltjes transform of both sides of (3.6) and noting (2.21) and (2.22) again, we have Eq. (3.5). Since the measure $\sigma$ satisfying Eq. (3.5) is uniquely determined by the uniqueness of Stieltjes transform, we find that the mapping $L$ defined by $L(\sigma)=(\alpha, \beta, \mu)$ is injective. (ii) follows from Remark 2.1 and Theorem 2.2. Q.E.D.

*Definition 3.1.* We call a triple $(\alpha, \beta, \mu)=L(\sigma)$ *a Langevin data* associated with $\sigma$.

Next we shall obtain a formula concerning the Langevin data $[\alpha, \beta, \mu]$ associated with $\sigma$.

**Theorem 3.2.**

(i)
$$\frac{\alpha^2}{2}=\int_0^\infty \lambda\sigma(d\lambda),$$

(ii)
$$\beta=\frac{1}{\pi}\int_{\mathbb{R}}\frac{\int_0^\infty \frac{\lambda^3}{(\xi^2+\lambda^2)^2}\sigma(d\lambda)}{\int_0^\infty \frac{\lambda}{\xi^2+\lambda^2}\sigma(d\lambda)}d\xi,$$

(iii) *For any* $x\in(0,\infty)$

$$\int_0^\infty \frac{1}{x+\lambda}\mu(d\lambda)=\beta+x-\frac{\alpha}{\sqrt{2\pi}}\exp\left(-\frac{1}{2\pi}\int_{\mathbb{R}}\frac{x}{x^2+\xi^2}\log\Delta(\xi)d\xi\right).$$

*Here* $\Delta(\xi)$ *is given by* (2.4).

*Proof.* (i) follows from (2.1) and Theorem 2.4 (2) (ii). (ii) can be shown as follows; by (2.26)

$$\beta=\lim_{x\to\infty}\left(\frac{\sqrt{2\pi}\alpha}{\int_0^\infty \frac{1}{x+\lambda}\nu(d\lambda)}-x\right), \tag{3.7}$$

and moreover by (2.6), (2.22), and (2.24), we have for any $x\in(0,\infty)$

$$(\sqrt{2\pi}\alpha)^{-1}\int_0^\infty \frac{1}{x+\lambda}\nu(d\lambda)=(\sqrt{2\pi}/\alpha)^{-1}\exp\left(\frac{1}{2\pi}\int_{\mathbb{R}}\frac{x}{x^2+\xi^2}\log\Delta(\xi)d\xi\right). \tag{3.8}$$

Therefore, defining a function $f$ on $(0,\infty)$ by

$$\begin{aligned} f(x)&=\log x+\frac{1}{2\pi}\int_{\mathbb{R}}\frac{x}{x^2+\xi^2}\log\Delta(\xi)d\xi-\log\frac{\alpha}{\sqrt{2\pi}}\\ &=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{1+\xi^2}\log(\Delta(x\xi)x^2)d\xi-\log\frac{\alpha}{\sqrt{2\pi}}, \end{aligned} \tag{3.9}$$

we see from (3.7) that

$$\beta=\lim_{x\to\infty} x(\exp(-f(x))-1). \tag{3.10}$$

Noting that for any $x\in(0,\infty)$ and $\xi\in\mathbb{R}-\{0\}$

$$\frac{\partial}{\partial x}\log(\Delta(x\xi)x^2)=\frac{2\int_0^\infty \frac{\lambda^3}{(\lambda^2+x^2\xi^2)^2}\sigma(d\lambda)}{x\int_0^\infty \frac{\lambda}{\lambda^2+x^2\xi^2}\sigma(d\lambda)} \tag{3.11}$$

and

$$\left|\frac{\partial}{\partial x}\log(\Delta(x\xi)x^2)\right|\leqq\frac{2}{x}, \tag{3.12}$$

we can see from (3.9) that for any $x\in(0,\infty)$

$$\begin{aligned} f'(x)&=\frac{1}{\pi x}\int_{\mathbb{R}}\frac{1}{1+\xi^2}\frac{\int_0^\infty \frac{\lambda^3}{(\lambda^2+x^2\xi^2)^2}\sigma(d\lambda)}{\int_0^\infty \frac{\lambda}{\lambda^2+x^2\xi^2}\sigma(d\lambda)}d\xi \\ &=\frac{1}{\pi}\int_{\mathbb{R}}\frac{1}{x^2+\xi^2}\frac{\int_0^\infty \frac{\lambda^3}{(\xi^2+\lambda^2)^2}\sigma(d\lambda)}{\int_0^\infty \frac{\lambda}{\xi^2+\lambda^2}\sigma(d\lambda)}d\xi. \end{aligned} \tag{3.13}$$

Defining $g(x)=f\left(\frac{1}{x}\right)$, we find from (3.10) that $\beta=\lim_{x\to\infty}\frac{\exp(-g(x))-1}{x}$. Since $g\in C^1([0,\infty))$, $g(0+)=0$, and $g'(x)=\frac{-1}{x^2}f'\left(\frac{1}{x}\right)$, it follows from Ropital's theorem and Lebesque's monotone convergence theorem that

$$\begin{aligned} \beta&=\lim_{x\to 0}(-g'(x)) \\ &=\lim_{x\to\infty}x^2f'(x) \\ &=\lim_{x\to\infty}\frac{1}{\pi}\int_{\mathbb{R}}\frac{1}{1+\frac{\xi^2}{x^2}}\cdot\frac{\int_0^\infty \frac{\lambda^3}{(\xi^2+\lambda^2)^2}\sigma(d\lambda)}{\int_0^\infty \frac{\lambda}{\xi^2+\lambda^2}\sigma(d\lambda)}d\xi \\ &=\frac{1}{\pi}\int_{\mathbb{R}}\frac{\int_0^\infty \frac{\lambda^3}{(\xi^2+\lambda^2)^2}\sigma(d\lambda)}{\int_0^\infty \frac{\lambda}{\xi^2+\lambda^2}\sigma(d\lambda)}d\xi. \end{aligned}$$

(iii) follows from (2.26) and (3.8). Q.E.D.

## 4. A Generalized Fluctuation-Dissipation Theorem

In this section we shall consider a real stationary Gaussian process $X=(X(t);t\in\mathbb{R})$ having $T$-positivity with covariance function $R$ of the form (2.1) and (2.2). Then we know from Theorem 2.1 (ii) that the time evolution of $X$ is described by the following $[\alpha,\beta,\gamma]$-Langevin equation:

$$dX(t)=-\left(\beta X(t)+\int_{-\infty}^{0}X(t+s)\gamma(s)ds\right)dt+\alpha dB(t)\,. \tag{4.1}$$

By taking the same consideration as in (9.12) and (9.13) of [11], we have the following generalized fluctuation-dissipation theorem.

**Theorem 4.1.** (i) *(a generalized first fluctuation-dissipation theorem)*

$$\frac{1}{\beta+\hat{\gamma}(\zeta)-i\zeta}=2\pi\frac{h(\zeta)}{\lim\limits_{\varepsilon\downarrow 0}\int_{\mathbb{R}}e^{-i\varepsilon\xi}h(\xi+i0)d\xi}\quad(\zeta\in\mathbb{C}^{+})\,. \tag{4.2}$$

(ii) *(a generalized second fluctuation-dissipation theorem)*

$$\frac{\alpha^2}{2}=R(0)C_{\beta,\gamma}\,. \tag{4.3}$$

Here $C_{\beta,\gamma}$ is given by

$$C_{\beta,\gamma}=\pi\left(\int_{\mathbb{R}}|\beta+\hat{\gamma}(\xi)-i\xi|^{-2}d\xi\right)^{-1}. \tag{4.4}$$

We call the constant $C_{\beta,\gamma}$ *a generalized friction coefficient*, whose best estimate is given in the following

**Theorem 4.2.**
(i) $C_{\beta,0}=\beta$,
(ii) $\beta-\int_0^\infty\lambda^{-1}\mu(d\lambda)\leqq C_{\beta,\gamma}\leqq\beta$,
(iii) $\gamma=0$ *if and only if either of two inequalities in* (ii) *becomes equal.*

*Proof.* (i) is easy to be checked. By Theorems 2.4 (ii) and 4.1 (ii), we have

$$C_{\beta,\gamma}=\beta+\int_{-\infty}^{0}\frac{R(y)}{R(0)}\gamma(y)dy=\beta-\int_0^\infty\left(\int_{-\infty}^{0}e^{y\lambda}\frac{R(y)}{R(0)}dy\right)\mu(d\lambda)\,. \tag{4.5}$$

Since $0<R(y)\leqq R(0)$ for any $y\in\mathbb{R}$, we have (ii). (iii) can be proved as follows; we assume that $\beta=C_{\beta,\gamma}$. It then follows from (4.5) that

$$\int_0^\infty\left(\int_{-\infty}^{0}e^{y\lambda}\frac{R(y)}{R(0)}dy\right)\mu(d\lambda)=0\,,$$

which implies $\mu=0$, because $\dfrac{R(y)}{R(0)}>0$. Next we assume that $C_{\beta,\gamma}=\beta-\int_0^\infty\lambda^{-1}\mu(d\lambda)$.

By using (4.5) again,

$$\int_0^\infty \left(\lambda^{-1} - \int_{-\infty}^0 e^{y\lambda} \frac{R(y)}{R(0)} dy\right) \mu(d\lambda) = 0,$$

which implies that $\mu = 0$, because $0 < \frac{R(y)}{R(0)} < 1$. Q.E.D.

## 5. Kubo's Problem

For the purpose of a proposal for solutions of Kubo's problem stated in Sect. 1, we shall reformulate the results in Sects. 2–4 to a stationary curve on a Hilbert space $\mathscr{H}$. Let $L$ be any self-adjoint operator on $\mathscr{H}$ and $(U(t); t \in \mathbb{R})$ be a one-parameter unitary group with infinitesimal generator $iL$. For a given initial vector $A \in \mathscr{H}$, we define a curve $A = (A(t); t \in \mathbb{R})$ in $\mathscr{H}$ by

$$A(t) = U(t)A, \tag{5.1}$$

and then its covariance function $R_A$ on $\mathbb{R}$ by

$$R_A(t) = (A(t), A)_{\mathscr{H}}. \tag{5.2}$$

We assume that the covariance function $R_A$ can be represented in the form (2.1) with a Borel measure $\sigma$ satisfying (2.2). Since there exists a linear isometric operator $\Phi$ from the Hilbert space $\mathbb{M}$ generated by $\{X_A(t); t \in \mathbb{R}\}$ in $L^2(\Omega, \mathscr{B}, P)$ into such that $\Phi(X_A(t)) = A(t)$ $(t \in \mathbb{R})$, where $X_A = (X_A(t); t \in \mathbb{R})$ is a real stationary Gaussian process on a probability space $(\Omega, \mathscr{B}, P)$ having the covariance function $R_A$ in (5.2), we see from Theorems 2.1, 2.2, and 4.1 that

**Theorem 5.1.** (i) *([α, β, γ]-Langevin equation.) There exists a unique quadruplet* $[\alpha, \beta, \gamma, I]$ *such that* $A(t)$ *satisfies the following* $[\alpha, \beta, \gamma]$*-Langevin equation:*

$$A(t) - A(s) = -\int_s^t \left(\beta A(u) + \int_{-\infty}^0 A(u+v)\gamma(v)dv\right) du + \alpha(I(t) - I(s)) \quad (s < t). \tag{5.3}$$

*Here the triple* $[\alpha, \beta, \gamma]$ *satisfies conditions* (2.16) *and* (2.17) *and the curve* $I = (I(t); t \in \mathbb{R})$ *in* $\mathscr{H}$ *is an innovation curve with causal condition associated with* $A$, *that is, for any* $s, t \in \mathbb{R}$, $s < t$,

$$\|I(t) - I(s)\|^2_{\mathscr{H}} = t - s, \tag{5.4}$$

the closed linear hull of $\{A(u); u \leqq t\}$ equals the closed linear hull of $\{I(u) - I(v); u, v \leqq t\}$. (5.5)

(ii) *(A generalized second fluctuation-dissipation theorem)*

$$\frac{\alpha^2}{2} = R_A(0) C_{\beta,\gamma}. \tag{5.6}$$

*Here* $C_{\beta,\gamma}$ *is a generalized friction coefficient given by* (4.4).

*Remark 5.1.* The triple $[\alpha, \beta, \gamma]$ can be calculated from the formulae (i), (ii), and (iii) in Theorem 3.2.

We shall give an example of the curve $A$ satisfying conditions (2.1) and (2.2).

*Example 5.1. (A homogeneous random field with T-positivity.)* For a given measurable space $(S, \mathscr{F})$, we define the path space $\Omega_S$ and the coordinate mapping $N(t)$ from $\Omega_S$ into $S$ by

$$\Omega_S = S^{\mathbb{R}}, \tag{5.7}$$

$$N(t)\omega = \omega(t), \tag{5.8}$$

and then four $\sigma$-fields $\mathscr{B}$, $\mathscr{B}^+$, $\mathscr{B}^-$, and $\mathscr{B}^\cdot$ by

$$\mathscr{B} = \sigma(N(t); t \in \mathbb{R}), \quad \mathscr{B}^+ = \sigma(N(t); t \geqq 0), \quad \mathscr{B}^- = \sigma(N(t); t \leqq 0),$$

and

$$\mathscr{B}^\cdot = \sigma(N(0)). \tag{5.9}$$

Moreover we denote by $\theta_t$ and $\tau$ the shift operator and the time reflection operator on $\Omega_S$, respectively;

$$(\theta_t\omega)(s) = \omega(s+t) \quad \text{and} \quad (\tau\omega)(t) = \omega(-t). \tag{5.10}$$

Let $P$ be a given symmetric and stationary measure on $(\Omega_S, \mathscr{B})$;

$$\tau(P) = P \quad \text{and} \quad (\theta_t)(P) = P \quad \text{for any} \quad t \in \mathbb{R}. \tag{5.11}$$

Then we define five Hilbert spaces $\mathscr{H}$, $\mathscr{H}^+$, $\mathscr{H}^-$, $\mathscr{H}^\cdot$, and $\mathscr{H}^{-/+}$ by

$$\mathscr{H} = L^2(\Omega_S, \mathscr{B}, P), \quad \mathscr{H}^+ = L^2(\Omega_S, \mathscr{B}^+, P), \quad \mathscr{H}^- = L^2(\Omega_S, \mathscr{B}^-, P),$$
$$\mathscr{H}^\cdot = L^2(\Omega_S, \mathscr{B}^\cdot, P), \text{ and } \mathscr{H}^{-/+} = \text{the closed linear hull of } \{P_{\mathscr{H}^+}B; B \in \mathscr{H}^-\}. \tag{5.12}$$

Here $P_{\mathscr{H}^+}$ denotes the projection operator from $\mathscr{H}$ onto $\mathscr{H}^+$. Furthermore we define a unitary group $(U(t); t \in \mathbb{R})$ and the time reflection operator $T$ on $\mathscr{H}$ by

$$U(t)B = B(\theta_t) \quad \text{and} \quad T(B) = B(\tau). \tag{5.13}$$

*Definition 5.1.* We say that $P$ has *T-positivity* if $S \equiv P_{\mathscr{H}^+}TP_{\mathscr{H}^+} \geqq 0$.

We assume that $P$ has $T$-positivity. By taking the same consideration as in [3], we have (cf. [10]),

**Proposition 5.1.** *There exists a unique non-negative self-adjoint operator $H$ on $\mathscr{H}^{-/+}$ such that*

$$S^{1/2}P_{\mathscr{H}^{-/+}}U(t) = e^{-tH}S^{1/2} \quad on \quad \mathscr{H}^{-/+} \quad (t > 0), \tag{5.14}$$

$$(U(t)B, B)_{\mathscr{H}} = (e^{-|t|H}B, B)_{\mathscr{H}} \quad for\ any \quad B \in \mathscr{H}^\cdot \quad (t \in \mathbb{R}). \tag{5.15}$$

*Definition 5.2.* We call the operator $H$ in Theorem 6.1 the *Hamiltonian operator* associated with $P$.

For a given vector $A \in \mathscr{H}^\cdot$, we define a covariance function $R_A$ by

$$R_A(t) = (U(t)A, A)_{\mathscr{H}} \quad (t \in \mathbb{R}). \tag{5.16}$$

By using the spectral resolution of $H$, we obtain

**Proposition 5.2.** (i) *The covariance function $R_A$ can be represented in the form* (2.1) *with a Borel measure $\sigma_A$ on $[0, \infty)$.*

(ii) *The Borel measure $\sigma_A$ satisfies condition* (2.2) *if and only if*

$$A \in \mathscr{D}(H), \tag{5.17}$$

$$\lim_{t \to \infty} e^{-tH} A = 0. \tag{5.18}$$

Next, we shall give an example of $P$ satisfying $T$-positivity and conditions (5.11), (5.17), and (5.18).

*Example 5.2. (A stationary symmetric Markov process.)* Let $S$ be a complete separable metric space or a locally compact topological space with the second axiom of countability and $\mathscr{F}$ a topological Borel field in $S$. Let us be given a $\sigma$-finite Borel measure $m_\infty$ on $(S, \mathscr{F})$ and a symmetric conservative Markov semi-group $(P_t; t \in [0, \infty))$ on $L^2(S, \mathscr{F}, m_\infty)$ with infinitesimal generator $\mathscr{G}$. By Kolmogorov's extension theorem, we can get a symmetric and stationary measure $P$ on $(\Omega_S, \mathscr{B})$ such that for $n \in \mathbb{N}$, $t_1, t_2, \ldots, t_n \in \mathbb{R}$, $t_1, < t_2 < \ldots < t_n$,

$$\begin{aligned} &P((N(t_1), N(t_2), \ldots, N(t_n)) \in dx_1 dx_2 \ldots dx_n) \\ &\quad = m_\infty(dx_1) p(t_2 - t_1, x_1, dx_2) \ldots p(t_n - t_{n-1}, x_{n-1}, dx_n). \end{aligned} \tag{5.19}$$

Here $p(t, x, dy)$ is a transition probability of Markov semi-group $(P_t; t \in [0, \infty))$. By a standard argument we can show

**Proposition 5.3.**

(i) *$P$ has Markovian property, that is, $\mathscr{H}^{-/+} = \mathscr{H}^{\cdot}$.*

(ii) *$P$ has T-positivity.*

(iii) *The Hamiltonian $H$ associated with $P$ is unitarily equivalent to the infinitesimal generator $\mathscr{G}$ of $(P_t; t \in [0, \infty))$ through the following correspondence,*

$$A \in \mathscr{H}^{\cdot} \underset{A = f(N(0))}{\longleftrightarrow} f \in L^2(S, \mathscr{F}, m_\infty). \tag{5.20}$$

*In particular, for a vector $A = f(N(0))$ in $\mathscr{H}$ ($f \in L^2(S, \mathscr{F}, m_\infty)$), $A$ satisfies conditions* (5.17) *and* (5.18) *if and only if $f$ satisfies the following conditions*

$$f \in \mathscr{D}(\mathscr{G}), \tag{5.21}$$

$$\lim_{t \to \infty} P_t f = 0. \tag{5.22}$$

Finally we shall consider two concrete examples of symmetric Markov processes in order to see what kinds of realization Theorem 5.1 gives for these examples.

*Example 5.3.* Let $N = (N(t); t \in \mathbb{R})$ be an Ornstein-Uhlenbeck process with covariance function $R_N$ given by

$$R_N(t) = \frac{\alpha^2}{2\beta} e^{-\beta|t|}. \tag{5.23}$$

Here $\alpha$ and $\beta$ are positive constants. Then we consider a stochastic process $Y = (Y(t); t \in \mathbb{R})$ defined by $Y(t) = N(t)^3$. It can be proved that $Y$ is a symmetric and stationary Markov process and the covariance function $R_Y$ has the following form:

$$R_Y(t)=9v^3e^{-\beta|t|}+6v^3e^{-3\beta|t|}, \tag{5.24}$$

where $v=\alpha^2/2\beta$. Therefore we can apply Theorem 5.1, Propositions 5.2 and 5.3 to the process $Y$ to find that the time evolution of $Y$ is described by $[\alpha_Y, \beta_Y, \gamma_Y]$-Langevin equation (5.3) for $f(x)=x^3$. Here the coefficients $\{\alpha_Y, \beta_Y, \gamma_Y\}$ are given by

$$\left.\begin{aligned} &\alpha_Y=\sqrt{54\beta v^3}, \quad \beta_Y=\left(4-\sqrt{\frac{11}{3}}\right)\beta \\ &\text{and} \\ &\gamma_Y(t)=-\chi_{(-\infty,0)}(t)\frac{4(\sqrt{33}-5)}{3}\beta^2 e^{\sqrt{11/3}\beta t}. \end{aligned}\right\} \tag{5.25}$$

Therefore, by a generalized second fluctuation-dissipation theorem (5.6), we note that

$$\frac{\alpha_Y^2}{2}=27\beta v^3, \quad R_Y(0)=15v^3, \quad \text{and} \quad C_{\beta_Y,\gamma_Y}=\frac{9}{5}\beta. \tag{5.26}$$

*Example 5.4.* Let $\mathcal{X}=(\mathcal{X}_t, P_x; t\in[0,\infty), x\in\mathbb{R})$ be a one-dimensional diffusion process with infinitesimal generator $\mathcal{G}$ given by

$$\mathcal{G}=a(x)\frac{d^2}{dx^2}+b(x)\frac{d}{dx}. \tag{5.27}$$

Here $a$ and $b$ are continuous functions on $\mathbb{R}$ and $a(x)>0$ for any $x\in\mathbb{R}$. We define a Borel measure $m$ on $\mathbb{R}$ by

$$m(dx)=\frac{1}{a(x)}\exp(C(x))dx, \tag{5.28}$$

where $C$ is given by

$$C(x)=\int_0^x \frac{b(y)}{a(y)}dy. \tag{5.29}$$

We suppose the following conditions:

$$m(\mathbb{R})<\infty, \tag{5.30}$$

$$\int_{-\infty}^0\left(\int_y^0 m(dx)\right)e^{-C(y)}dy=\int_0^\infty\left(\int_0^y m(dx)\right)e^{-C(y)}dy=\infty. \tag{5.31}$$

We define a probability Borel measure $m_\infty$ on $\mathbb{R}$ by

$$m_\infty(dx)=\frac{1}{m(\mathbb{R})}m(dx). \tag{5.32}$$

Since we know from [9] that $m_\infty$ is an invariant probability measure of the diffusion process $\mathcal{X}$ and the differential operator $\mathcal{G}$ generates a symmetric and conservative Markov semi-group $(P_t=e^{t\mathcal{G}}; t\in[0,\infty))$, we can get a symmetric and stationary Markov process $N=(N(t));t\in\mathbb{R})$ describing the stationary state of the diffusion process $\mathscr{X}$ with initial distribution $m_\infty$, by using Proposition 5.3.

Now we are in a position to state the following main theorem.

**Theorem 5.2.** *Besides conditions* (5.30) *and* (5.31), *we suppose the following conditions:*

$$\int_{\mathbb{R}} (x^2+b(x)^2)m(dx)<\infty\,, \tag{5.33}$$

$$\int_{\mathbb{R}} xm(dx)=0\,. \tag{5.34}$$

*Then there exists a unique quadruplet* $[\alpha,\beta,\gamma,I]$ *such that the time evolution of* $N=(N(t);t\in\mathbb{R})$ *is governed by the following* $[\alpha,\beta,\gamma]$*-Langevin equation:*

$$dN(t)=-\left(\beta N(t)+\int_{-\infty}^{0} N(t+s)\gamma(s)ds\right)dt+\alpha dI(t)\,. \tag{5.35}$$

*Here the triple* $[\alpha,\beta,\gamma]$ *satisfies conditions* (2.16) *and* (2.17) *and the stochastic process* $I=(I(t);t\in\mathbb{R})$ *is an innovation process satisfying conditions* (5.4) *and* (5.5) *for* $f(x)=x$.

By noting that conditions (5.33) and (5.34) imply conditions (5.21) and (5.22) for $f(x)=x$, we find that Theorem 5.2 follows from Theorem 5.1 (i), Propositions 5.2 and 5.3.

*Remark 5.2.* We define a reference family $(\mathscr{B}_t;t\in\mathbb{R})$ by

$$\mathscr{B}_t=\sigma(N(s);s\leqq t)\,. \tag{5.36}$$

Then it follows from (5.5), (5.21), and (5.35) that there exists a $(\mathscr{B}_t;t\in\mathbb{R})$-martingale $(M(t);t\in\mathbb{R})$ such that for any $s,t\in\mathbb{R}$, $s<t$,

$$\alpha(I(t)-I(s))=\int_s^t\left(b(N(u))+\beta N(u)+\int_{-\infty}^{0} N(u+v)\gamma(v)dv\right)du$$
$$+M(t)-M(s)\,. \tag{5.37}$$

*Remark 5.3.* We suppose that the functions $a$ and $b$ in (5.27) satisfy the following relation

$$b(x)=a'(x)-\frac{x}{v}a(x)\,, \tag{5.38}$$

where $v$ is a positive constant. By (5.28) and (5.29), we note

$$m(dx)=\frac{1}{a(0)}\exp\left(-\frac{x^2}{2v}\right)dx\,. \tag{5.39}$$

Then it can be proved that conditions (5.31), (5.33), and (5.34) are satisfied if the function $a$ satisfies the following conditions:

$$\int_{-\infty}^{0}\frac{-x}{a(x)}dx=\int_0^{\infty}\frac{x}{a(x)}dx=\infty\,, \tag{5.40}$$

$$\int_{-\infty}^{\infty}(a(x)^2+(a'(x))^2)\exp\left(-\frac{x^2}{2v}\right)dx<\infty\,. \tag{5.41}$$

Finally, by virtue of Theorem 5.1 (ii), we can obtain a generalized second fluctuation-dissipation theorem for $N$ which gives a proposal for solutions of Kubo's open problem [7].

**Theorem 5.3.** *(A generalized second fluctuation-dissipation theorem.)*

*In the generalized second fluctuation-dissipation theorem* (5.6) *for* $[\alpha,\beta,\gamma]$-*Langevin Eq.* (5.35), *we have*

$$\left.\begin{aligned}
\frac{\alpha^2}{2} &= \left(-\int_{\mathbb{R}} \frac{xb(x)}{a(x)}\exp(C(x))dx\right)\left(\int_{\mathbb{R}} \frac{1}{a(x)}\exp(C(x))dx\right)^{-1}\\
R(0) &= \left(\int_{\mathbb{R}} \frac{x^2}{a(x)}\exp(C(x))dx\right)\left(\int_{\mathbb{R}} \frac{1}{a(x)}\exp(C(x))dx\right)^{-1}\\
C_{\beta,\gamma} &= \left(-\int_{\mathbb{R}} \frac{xb(x)}{a(x)}\exp(C(x))dx\right)\left(\int_{\mathbb{R}} \frac{x^2}{a(x)}\exp(C(x))dx\right)^{-1}.
\end{aligned}\right\} \tag{5.42}$$

*Remark 5.4.* For the Ornstein-Uhrenbeck process with its covariance function given by (5.23), Theorem 5.3 is nothing but the classical fluctuation-dissipation theorem (1.5) stated in Sect. 1 [6–8, 11].

More generally, we shall give the following

*Remark 5.5.* We suppose that the functions $a$ and $b$ in (5.27) satisfy relation (5.38) with conditions (5.40) and (5.41). Then we note that (5.42) in Theorem 5.3 is reduced to the following

$$\left.\begin{aligned}
\frac{\alpha^2}{2} &= \int_{\mathbb{R}} a(x)\frac{1}{\sqrt{2\pi v}}\exp\left[-\frac{x^2}{2v}\right]dx\\
R(0) &= v\\
C_{\beta,\gamma} &= -\frac{1}{v}\int_{\mathbb{R}} xb(x)\frac{1}{\sqrt{2\pi v}}\exp\left[-\frac{x^2}{2v}\right]dx.
\end{aligned}\right\} \tag{5.43}$$

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