Commun. Math. Phys. 95, 487–532 (1984)

Communications in Mathematical Physics

Renormalization of Dyson's Hierarchical Vector Valued ϕ^4 Model at Low Temperatures

P. M. Bleher[1] and P. Major[2]

1 Institute of Applied Mathematics of the Soviet Academy of Sciences, Moscow, USSR
2 Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Hungary

Abstract. We investigate Dyson's hierarchical vector valued ϕ^4 model at low temperatures. The case $2 > c > \sqrt{2}$ is considered. The pure phase is constructed, and the existence of its large scale limit is proved. The limit is Gaussian, but an unusual normalization has to be chosen. In the direction of the spontaneous magnetization one has to divide by the square root of the volume, but in the orthogonal direction one has to divide by a different power of the volume for all low temperatures.

1. Introduction

In this paper Dyson's hierarchical vector valued ϕ^4 model is investigated at low temperatures. First we describe the model we are working with (see [2, 7]).

We define the volumes $V_{k,n}$, $V_{k,n} \subset \mathbb{Z}$, $\mathbb{Z} = \{1, 2, \dots\}$ as $V_{k,n} = \{j, j \in \mathbb{Z}, (k-1)2^n < j \leqq k \cdot 2^n\}$, $n = 1, 2, \dots, k = 1, 2, \dots$. Put $V_{1,n} = V_n$. For i, $j \in \mathbb{Z}$ we set

$$n(i,j) = \min\{n, \textit{there exists a } k \text{ such that } i \in V_{k,n}, \ j \in V_{k,n}\}.$$

The hierarchical distance $d(i,j)$, i, $j \in \mathbb{Z}$, is defined as

$$d(i,j) = \begin{cases} 0 & \text{if } \ i = j \\ 2^{n(i,j)-1} & \text{if } \ i \neq j. \end{cases}$$

The spins $\sigma(i)$, $i \in \mathbb{Z}$, take on values in the m-dimensional Euclidean space R^m. The energy of a configuration $\sigma = \{\sigma(i), i \in V_{k,n}\}$ is defined by the formula

$$H_{k,n}(\sigma) = \sum_{\substack{(i,j), i \neq j \\ i,j \in V_{k,n}}} U(i,j)\,(\sigma(i); \sigma(j)), \tag{1.1}$$

where $U(i,j) = -d^{-\alpha}(i,j)$ and $(\cdot\,;\cdot)$ denotes scalar product. In particular

$$H_n(\sigma) = \sum_{\substack{(i,j), i \neq j \\ i,j \in V_n}} U(i,j)\,(\sigma(i); \sigma(j)). \tag{1.1'}$$

The value a, $2>a>1$, is called the parameter of the model. We assume throughout this paper that $3/2>a>1$. We shall often use the quantity $c=2^{2-a}$ instead of a as it is done in Dyson's original paper [8]. Our assumption on a means that $\sqrt{2}<c<2$. Given a configuration σ' on $\mathbb{Z}-V_{k,n}$ we define the Hamiltonian of the configuration $\sigma=\{\sigma(i),\ i\in V_{k,n}\}$ under the boundary condition $\sigma'=\{\sigma(j),\ j\in\mathbb{Z}-V_{k,n}\}$ as

$$H_{k,n}(\sigma|\sigma')=H_{k,n}(\sigma)+\sum_{i\in V_{k,n}}\sum_{j\notin V_{k,n}}U(i,j)(\sigma(i);\sigma(j)).$$

Given a probability measure ν on R^m we define the Gibbs distribution in the volume $V_{k,n}$ at inverse temperature β with boundary conditions σ' by the formula

$$\mu_{k,n}(d\sigma|\beta,\nu,\sigma')=\frac{\exp[-\beta H_{k,n}(\sigma|\sigma')]}{\Xi_{k,n}(\beta,\nu,\sigma')}\prod_{i\in V_{k,n}}\nu(d\sigma(i)), \tag{1.2}$$

$$\Xi_{k,n}(\beta,\nu,\sigma')=\int\exp[-\beta H_{k,n}(\sigma|\sigma')]\,(\sigma)]\prod_{i\in V_n}\nu(d\sigma(i)), \tag{1.2'}$$

In this paper we consider the vector valued ϕ^4 model, where the measure ν is defined as

$$\begin{aligned}\nu(dx)&=L^{-1}\exp\left(-\frac{u}{4}(x;x)^2-\frac{(x;x)}{2}\right)dx,\\ L&=\int_{R^m}\exp\left(-\frac{u}{4}(x;x)^2-\frac{(x;x)}{2}\right)dx.\end{aligned}\tag{1.3}$$

Here $u>0$ is a parameter of the model, and we shall assume that $u<u_0$, where u_0 is a sufficiently small positive constant. A measure μ on $(R^m)^{\mathbb{Z}}$ is called a Gibbs state with potential H at inverse temperature β if a μ distributed sequence $\sigma(i)$, $i\in\mathbb{Z}$, of random variables has the following property: For all volumes $V_{k,n}$ and almost all configurations $\sigma'=\{\sigma(j),\ j\in\mathbb{Z}-V_{k,n}\}$ (with respect to μ) the conditional distribution of the random vector $\sigma=\{\sigma(i),\ i\in V_{k,n}\}$ under the condition σ' is given by the formulas (1.2) and (1.2′). This is the usual definition of Gibbs states (see e.g. [7]).

We are going to investigate the Gibbs states with the above defined potential H at low temperatures. We choose the following approach. We consider an external magnetic field with $\mathbf{h}=he_1$, $e_1=(1,0,\dots,0)$, i.e. we define the potentials

$$H^h_{k,n}(\sigma)=\sum_{\substack{(i,j),\,i\neq j\\ i,j\in V_{k,n}}}U(i,j)(\sigma(i);\sigma(j))-\sum_{i\in V_{k,n}}(\mathbf{h};\sigma(\mathrm{i}))$$

for a configuration $\sigma=\{\sigma(\mathrm{i});\ i\in V_{k,n}\}$, and then the measures

$$\mu^h_n(d\sigma|\beta,\nu)=\frac{\exp[-\beta H^h_n(\sigma)]}{\Xi^h_n(\beta,\nu)}\prod_{i\in V_n}\nu(d\sigma(i)),$$

$$\Xi^h_n(\beta,\nu)=\int\exp[-\beta H^h_n(\sigma)]\prod_{i\in V_n}\nu(d\sigma(i)),$$

over $(R^m)^{V_n}$ for all $n=1,2,\dots,h\geqq 0$ and $\beta>0$. We shall denote by $\mu_n(d\sigma|\beta,\nu)$ the measure $\mu^h_n(d\sigma|\beta,\nu)$ when $h=0$. We shall prove that the limits

$$\mu^h(d\sigma|\beta,\nu)=\lim_{n\to\infty}\mu^h_n(d\sigma|\beta,\nu) \tag{1.4}$$

and

$$\bar{\mu}(d\sigma|\beta, v) = \lim_{h\to 0} \mu^h(d\sigma|\beta, v) \tag{1.5}$$

exist. If these convergences are established, it is not difficult to prove that μ^h and $\bar{\mu}$ are Gibbs states at the inverse temperature β with potentials H^h and H respectively. We are interested in the measure $\bar{\mu}$ constructed in this way.

Our discussion heavily depends on a large deviation result about the distribution $\gamma_n(dx)$ of the average $\frac{1}{2^n}\sum_{i\in V_n} \sigma(i)$ of the $\mu_n(d\sigma|\beta, v)$ distributed spins $\sigma(i)$, which will be formulated in Sect. 2. It states in particular the existence of a critical inverse temperature β_{cr} such that for $\beta > \beta_{cr}$ the measure γ_n is concentrated essentially around a sphere of a positive radius. First we shall formulate the following

Theorem 1. a) *Relation (1.4) holds true for all $\beta > \beta_{cr}$ and $h > 0$.*

b) *Relation (1.5) holds for all $\beta > \beta_{cr}$.*

In both cases the lim *is meant as convergence of the finite dimensional distributions in the variational metrics.*

Actually Theorem 1 holds also for $\beta \leqq \beta_{cr}$ but we shall not prove it. We have considered the double limiting procedure (1.4) and (1.5), because in this way we construct the so-called pure phase, i.e. a Gibbs state which cannot be decomposed into a micture of other Gibbs states. We are interested in the behaviour of the Gibbs state $\bar{\mu}(d\sigma|\beta, v)$ defined by (1.4) and (1.5).

Let $\sigma(j) = (\sigma^{(1)}(j), \ldots, \sigma^{(m)}(j))$, $j \in \mathbb{Z}$, be a sequence of random variables with the distribution $\bar{\mu}(d\sigma|\beta, v)$. Introduce the new random variables $T^n\sigma(j) = (T^n\sigma^{(1)}(j), \ldots, T^n\sigma^{(m)}(j))$, $j \in \mathbb{Z}$,

$$T^n\sigma^{(1)}(j) = 2^{-n/2} \sum_{p\in V_{j,n}} (\sigma^{(1)}(p) - M(\beta)), \quad j\in\mathbb{Z}, \quad n = 1, 2, \ldots$$

$$T^n\sigma^{(i)}(j) = \left(\frac{\sqrt{c}}{2}\right)^n \sum_{p\in V_{j,n}} \sigma^{(i)}(p), \quad i = 2, \ldots, m.$$

The quantity $M(\beta)$ will be defined in Sect. 2. Actually $M(\beta) = E\sigma^{(1)}(p)$. Our main result is the following

Theorem 2. *The multi-dimensional distributions of the random sequence $T^n\sigma(j)$, $j\in\mathbb{Z}$, tend to those of a sequence $\bar{\sigma}(j)$ $j\in\mathbb{Z}$, of Gaussian m-dimensional random variables. For all $k>0$ the random vector $(\bar{\sigma}(1), \ldots, \bar{\sigma}(2^k))$ has the density function*

$$L_k \exp\left\{ -\sum_{i=1}^{2^k} \frac{q}{2} x_1^{(i)2} + \beta \sum_{j=2}^{m} \left(\frac{2-c}{c\cdot 2^{2k}} c^k \left(\sum_{i=1}^{2^k} x_j^{(i)} \right)^2 - \sum_{i=1}^{2^k} \frac{1}{2-c} x_j^{(i)2} + \sum_{i=1}^{2^k} \sum_{l=1}^{i-1} d(i,l)^{-a} x_j^{(i)} x_j^{(l)} \right) \right\},$$

where $L_k = L_k(\beta, u)$ is an appropriate norming constant, and $q = \frac{\partial^2}{\partial s^2} \Phi(s, \beta, u)|_{s=M(\beta)}$, where the function Φ appears in the large deviation result of Sect. 2. The absove result holds for all $\beta > \beta_{cr}$.

Theorem 2 is a central limit theorem for the $\bar{\mu}(d\sigma|\beta, v)$ distributed random variables. In the direction of the first coordinate, i.e. in the direction of the so-called spontaneous magnetization, one has to normalize by the square root of the number of terms. But in the orthogonal direction one has to normalize by a different power of the number of terms. Such an unusual normalization also appears in the scalar model, but only at the critical temperature (see e.g. [5–7]), and not on a whole interval. We emphasize that in our model both the Hamiltonian function and the free measure are invariant under all rotations. It is believed that in models with such a symmetry for all low temperatures one has to normalize in the direction orthogonal to the spontaneous magnetization in the same way as at the critical temperatures. The proof of this conjecture in the general case seems to be very difficult. Our aim in the present paper is to show its validity in a relatively simple case. The unusual normalization in our case is connected with the following

Corollary. *The correlation function of the $\bar{\mu}(d\sigma|\beta, v)$ distributed random variables satisfies the following relation. For $i=2, \ldots, m$*

$$E\sigma^{(i)}(j)\sigma^{(i)}(k) \sim \frac{(2-c)^2 c}{2\beta(c-1)(4-c)} d(j,k)^{\alpha-2} \quad \text{as} \quad d(j,k) \to \infty .$$

The exact order of the correlation function $E[(\sigma^{(1)}(j) - E\sigma^{(1)}(j))\,(\sigma^{(1)}(k) - E\sigma^{(1)}(k)]$ can also be determined as $d(j,k) \to \infty$. Since it requires tedious calculations we omit it. We only remark that

$$\sum_{j \in \mathbb{Z}} E(\sigma^{(i)}(j) - E\sigma^{(i)}(j))\,(\sigma^{(i)}(k) - E\sigma^{(i)}(k))$$

is convergent for $i=1$, and divergent for $i=2, \ldots, m$. This indicates weak dependence in the direction of the magnetization and strong dependence in the orthogonal direction. The unusual normalization in our model is due to this strong dependence.

2. A Large Deviation Result

Let $p_n(s) = p_n(s, \beta, u)$ denote the density function of the distribution of the average spin $2^{-n} \sum_{i \in V_n} \sigma(i)$, where the spins $\sigma(i)$, $i \in V_n$ are $\mu_n(d\sigma|\beta, v)$ distributed. In this section we present an asymptotic formula for $p_n(s)$ which we need later. We deduce it from a more general result.

Introduce the energy function

$$H_{n,\mu,u}(\sigma) = \sum_{i,j \in V_n} U(i,j)\,(\sigma(i); \sigma(j)) + \sum_{i \in V_n} \frac{u}{4}(\sigma(i); \sigma(i))^2 + \frac{\mu}{2}(\sigma(i); \sigma(i)),$$

where $u > 0$ and μ are real numbers, $\sigma = \{\sigma(i);\ i \in V_n\}$, and otherwise we use the notations of the previous section. Put

$$Z_n(s, \mu, u) = \int \delta\Big(2^{-n} \sum_{i \in V_n} \sigma(i) - s\Big) \exp\{-H_{n,\mu,u}(\sigma)\} \cdot \prod_{i \in V_n} d\sigma(i), \qquad s \in R^m,$$

and

$$P_n(s,\mu,u)=\frac{Z_n(s,\mu,u)}{\Xi_n(\mu,u)},$$

where $\delta(\cdot)$ denotes the Dirac delta, and $\Xi_n(\mu,u)=\int Z_n(s,\mu,u)ds$. By substituting $\sigma(i) =\sqrt{\beta}\sigma'(i)$, $u=\frac{u'}{\beta^2}$, $\mu=\frac{1}{\beta}$ we get that

$$p_n(s,\beta,u)=P_n\left(\sqrt{\beta}s,\frac{1}{\beta},\frac{u}{\beta^2}\right)\sqrt{\beta}\,. \tag{2.1}$$

Hence we can deduce a good asymptotic formula for $p_n(s)$ by first proving it for $Z_n(s,\mu,u)$. In order to formulate such a result we introduce some notations.

We say that a function $\Phi(r,\mu)=\Phi(r,\mu,u)$, $r,\mu,u\in R^1$ belongs to the class S_u (we shall omit denoting dependence on u if it causes no ambiguity) if

$$\Phi(r,\mu)=\frac{u}{4}r^4+\frac{\mu-a_0}{2}r^2+R(r,\mu),$$

where $a_0=\frac{2}{2-c}$, the function $R(r,\mu)$ is even in its first coordinate, and it satisfies the following conditions:

i) $\frac{\partial^{i+j}R}{\partial r^i\partial\mu^j}\in C(K)$ if $i+j\leqq 2$, $j\leqq 1$ for all compact sets $K\subset R^2$, and for $i+j=2$ and $j\leqq 1$ $\frac{\partial^{i+j}R}{\partial r^i\partial\mu^j}$ satisfies the Hölder condition of order $\varepsilon_0=\frac{3}{2}-a$ in r and μ with some multiplying factor $C(K)$. Moreover $C(K)\leqq \text{const}\,u(1+\operatorname{diam}K)^2$.

ii) $\left.\frac{\partial^{i+j}R}{\partial r^i\partial\mu^j}\right|_{r=0}$ exists if $i+2j\leqq 4$, $j\leqq 1$ and it is continuous in μ.

$R(r,\mu)=\sum_{i=0}^{4}\frac{1}{i!}\frac{\partial^i R(0,\mu)}{\partial r^i}r^i+o(r^4)$, where $O(\cdot)$ is uniform in μ.

iii) $$R(0,\mu)=0\,,\qquad \left|\frac{\partial R}{\partial r}\right|\leqq Cru\,,\qquad \left|\frac{\partial^2 R}{\partial r^2}\right|\leqq Cu^2\,,$$

$$\left|\frac{\partial^4 R}{\partial r^4}\right|_{r=0}\leqq Cu^{3/2}\,,\qquad \left|\frac{\partial R}{\partial r\partial\mu}\right|\leqq Cu^2\,,\qquad \left|\frac{\partial^3 R}{\partial r^2\partial\mu}\right|_{r=0}\leqq Cu\,,$$

where C does not depend on r and μ.

If $u>0$ is sufficiently small, and $\Phi\in S_u$, then

a) there exists a unique "critical point" $\mu_c=a_0+O(u)$ (depending on Φ) such that $\left.\frac{\partial^2\Phi}{\partial r^2}\right|_{r=0,\,\mu=\mu_c}=0$.

b) Let $M(\mu)\geqq 0$ be the (unique) solution of the equation $\Phi(M(\mu),\mu) =\min_r \Phi(r,\mu)$. Then $M(\mu)=0$ for $\mu\geqq\mu_c$ and $M(\mu)>0$ for $\mu<\mu_c$.

Let us fix a function $\chi(x)\in C^\infty(R^1)$ such that

$$\chi(x)=\begin{cases}0,1 & \text{if}\quad x<0,1\\ x & \text{if}\quad x>0,2\end{cases}$$

and $0,1\leqq\chi(x)\leqq 0,2$ if $0,1\leqq x\leqq 0,2$.

Since the functions $Z_n(s,\mu)$, $P_n(s,\mu)$, $p_n(s,\mu)$ are rotation invariant in their first coordinate, it is enough to define them in the case $s=(r,0,\dots,0)$, $r\in R^1$. We shall denote these functions also by $Z_n(r,\mu)$, $P_n(r,\mu)$ and $p_n(r,\mu)$, $r\in R^1$. Now we formulate the following

Theorem A. *Assume that* $2>c>\sqrt{2}$ *and* $\varepsilon>0$. *There are some constants* $C=C(c)>0$ *and* $u_0>0$ *such that for any* $0<u<u_0$ *there exists a function* $\Phi(r,\mu)=\Phi(r,\mu,u)\in S_u$ *(with the bounding constant C in the definition of the class* S_u*) such that in the domain*

$$U_n=\{(r,\mu),|r|\geqq(1-\varepsilon^{(n)})M(\mu)\}\quad \text{with}\quad \varepsilon^{(n)}=0{,}01\cdot\left(\frac{c}{2}\right)^n$$

the following asymptotic expansion holds true:

$$-\ln Z_n(r,\mu,u)=2^n\Phi(r,\mu)+c^n\frac{a_0}{2}r^2-\frac{1}{2}\ln 2^n-m\ln\left(\frac{4}{\sqrt{\pi}}\right)$$

$$+\psi_n(r,\mu)+O\left(2^{n\left(a-\frac{3}{2}+\varepsilon\right)}\right),$$

where $a_0=\dfrac{2}{2-c}$, $a_1=a_0+1$, *and*

$$\psi_n(r,\mu)=-\frac{1}{4}\sum_{j=0}^{\infty}2^{-j}\left[\ln\left(a_1+\left(\frac{2}{c}\right)^{n+j}\frac{\partial^2\Phi}{\partial r^2}\right)+\ln\left(\chi\left(a_1+\left(\frac{2}{c}\right)^{n+j}\frac{1}{r}\frac{\partial\Phi}{\partial r}\right)\right)\right]. \tag{2.2}$$

and the $O(\cdot)$ *is uniform in r and* μ. *Moreover*

$$Z_n(r,\mu,u)\leqq(1+O(\xi^n))Z_n((1-\varepsilon^{(n)})M(\mu),\mu)\quad \text{if}\quad |r|<(1-\varepsilon^{(n)})M(\mu)$$

with some $\xi<1$.

Theorem A is a multi-dimensional generalization of the result in [3]. We explain the modifications needed in its proof in the Appendix.

Theorem A and formula (2.1) enable us to give a good asymptotic formula for $p_n(s,\beta,u)$. In the sequel we write $\Phi(s,\beta,u)$ instead of $\Phi\left(\sqrt{\beta}s,\frac{1}{\beta},\frac{u}{\beta}\right)$, where Φ is defined in Theorem A.

Let β_{cr} be the (unique) solution of the equation $\left.\dfrac{\partial^2\Phi(s,\beta)}{\partial s^2}\right|_{s=0}=0$, and for $\beta>\beta_{cr}$ $M(\beta)$ the (unique) positive solution of the equation $\dfrac{\partial\Phi(s,\beta)}{\partial s}=0$. Put

$$\Phi_n(r,\beta)=\Phi(r,\beta)+\frac{a_0\beta}{2}\left(\frac{c}{2}\right)^n r^2. \tag{2.3}$$

We write down the estimates on $p_n(s,\beta)$ we need in the sequel. Sometimes we omit the argument β.

Let us fix some $K > M(\beta)$. We claim that

$$p_n(y, \beta) = (1 + O(\xi^n)) p_n(x, \beta) \exp\{-2^n \Phi_n'(x)(y-x) - 2^{n-1}\Phi_n''(x)(y-x)^2\} \quad (2.4)$$

with some $\xi < 1$ if $K \geqq x \geqq M(\beta)$, $|y-x| \leqq 2^{-n/2}\alpha^n$ with $\alpha = \left[\min\left(\frac{2}{c'}, \frac{c^2}{2}\right)\right]^{1/20}$. Here the number $\xi < 1$ and the $O(\cdot)$ are independent of x and y.

The following estimate also holds true:

$$p_n(y, \beta) \leqq (1 + O(\xi^n)) p_n(x, \beta) \exp\{-2^n \Phi_n'(x)(y-x) - 2^n B(y-x)^2\} \quad (2.5)$$

with some $B > 0$ if $M(\beta) < x < K$, $y > M(\beta) - 0{,}01 \cdot \left(\frac{c}{2}\right)^n$. Here again $B > 0$, $\xi < 1$, and the $O(\cdot)$ is independent of x and y. On the other hand

$$p_n(y, \beta) \leqq (1 + O(\xi^n)) p_n\left(M(\beta) - 0{,}01 \cdot \left(\frac{c}{2}\right)^n, \beta\right) \quad (2.5')$$

if $|y| < M(\beta) - 0{,}01 \left(\frac{c}{2}\right)^n$.

To prove (2.4) first we observe that by Theorem A and (2.1)

$$p_n(y) = p_n(x)(1 + O(\xi^n)) \exp\{-2^n(\Phi_n(y) - \Phi_n(x)) - (\psi_n(y) - \psi_n(x))\}.$$

A Taylor expansion shows that

$$2^n[\Phi_n(y) - \Phi_n(x)] = 2^n \Phi_n'(x)(y-x) + 2^{n-1}\Phi_n''(x) \cdot (y-x)^2 + O(\xi^n),$$

because $2^n[\Phi_n''(t) - \Phi_n''(x)](y-x)^2 = O(\xi^n)$ if $|t-x| < 2^{-n/2}\alpha^n$, and $|y-x| < 2^{-n/2}\alpha^n$. It remains to show that

$$\psi_n(y) - \psi_n(x) = O(\xi^n). \quad (2.6)$$

Put $\psi_n(y) - \psi_n(x) = -\frac{1}{4}(I_1 + I_2)$ with

$$I_1 = \sum_{j=0}^{\infty} 2^{-j} \ln \frac{\chi\left(a + \left(\frac{2}{c}\right)^{n+j} \frac{1}{y} \frac{\partial \Phi}{\partial r}(y)\right)}{\chi\left(a_1 + \left(\frac{2}{c}\right)^{n+j} \frac{1}{x} \frac{\partial \Phi}{\partial r}(x)\right)}$$

and

$$I_2 = \sum_{j=0}^{\infty} 2^{-j}\left[\ln\left(\frac{\partial^2 \Phi}{\partial r^2}(y) + \left(\frac{c}{2}\right)^{n+j} a_1\right) - \ln \frac{\partial^2 \Phi}{\partial r^2}(x) + \left(\frac{c}{2}\right)^{n+j} a_1\right)\right].$$

To estimate I_1 let us observe that

$$\sup_{r \in [x, y]} \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial r}\right) \leqq \mathrm{const}(1 + |x-y|) y^{1/2},$$

hence

$$\left|\frac{1}{x}\frac{\partial \Phi(x)}{\partial r} - \frac{1}{y}\frac{\partial \Phi(y)}{\partial r}\right| \leqq \mathrm{const}(|x-y| + |x-y| 2 \cdot y^{1/2}).$$

This relation together with the relations

$$\left|\ln\frac{a}{b}\right| \leqq C|a-b| \quad \text{if} \quad a \geqq 0,1\,, \qquad b \geqq 0,1$$

and

$$C_1|a-b| \leqq |\chi(a)-\chi(b)| \leqq C_2|a-b|/(1+|a-b|^{1/2})$$

imply that

$$\ln\left|\frac{\chi\left(a_1+\left(\frac{2}{c}\right)^{n+j}\frac{1}{y}\frac{\partial\Phi}{\partial r}(y)\right)}{\chi\left(a_1+\left(\frac{2}{c}\right)^{n+j}\frac{1}{x}\frac{\partial\Phi}{\partial r}(x)\right)}\right| \leqq \mathrm{const}\left(\frac{2}{c}\right)^{n+j}(|x-y|+|x-y|^2)\frac{y^{3/2}}{1+|x-y|^{3/2}}\,,$$

hence

$$I_1 \leqq C\cdot\left(\frac{2}{c}\right)^n(|x-y|+|x-y|^2)\frac{y^{3/2}}{1+|x-y|^{3/2}}\,. \tag{2.7}$$

Since $|x-y|<\sqrt{2^n}\alpha^n$, hence (2.7) implies that $I_1=O(\xi^n)$. On the other hand since $\frac{\partial^2\Phi}{\partial r^2}(z) \geqq C>0$ if $z>M(\beta)-0{,}01\cdot\left(\frac{c}{2}\right)^n$, hence

$$I_2=\ln\frac{\partial^2\Phi}{\partial r^2}(y)-\ln\frac{\partial^2}{\partial r^2}(x)+O(\xi^n)=O(\xi^n)\,.$$

The proof of (2.5) is similar but simpler. Since $\frac{\partial^2\Phi}{\partial r^2}(u) \geqq C>0$, we have

$$-2^n[\Phi_n(y)-\Phi_n(x)] \leqq -2^n\Phi_n'(x)\,(y-x)-2^nC(y-x)^2\,.$$

On the other hand $\psi_n(y)-\psi_n(x)=O(\xi^n)\,(2^n(y-x)^2+1)$, which can be proved with the help of (2.7). Formula (2.5) follows from these relations.

3. The Idea of the Proof

The proof contains rather tedious calculations. Hence first we explain the main ideas of the proof in an informal way.

Let $\mu^h_{n,N}(d\sigma|\beta,\nu)$, $n \leqq N$, denote the projection of the measure $\mu^h_N(d\sigma|\beta,\nu)$ to the volume V_n. We are going to give a good asymptotical formula for the Radon–Nikodym derivative $\frac{d\mu^h_{n,N}}{d\mu_n}$. Then we briefly explain how Theorems 1 and 2 can be proved by means of this formula. In this section we state and prove our results in a non-rigorous way. Later we shall prove these statements rigorously.

We need the following result

Theorem B. *Let* $\sigma(i)$, $i \in V_n$, *be* $\mu_n(d\sigma|\beta, v)$ *distributed random variables. Put* $\xi_n = \frac{1}{2^n} \sum_{i \in V_n} \sigma(i)$, *and let* $\gamma_n(dx) = \gamma_{n,\beta,v}(dx)$ *denote the distribution of* ξ_n. *Then*

$$\mu^h_{n,N}(d\sigma|\beta, v) = L^h_{n,N}(\beta) f^h_{n,N}(\xi_n) \mu_n(d\sigma|\beta, v)$$

with

$$f^h_{n,N}(x) = T_{n+1} T_{n+2} \dots T_N f^h_{N,N}(x); \qquad f^h_{N,N}(x) = \exp[2^N \beta(h; x)],$$

where

$$T_m f(x) = \int \exp(\beta c^{m-1}(x; t)) f\left(\frac{x+t}{2}\right) \gamma_{m-1}(dt),$$

and $L^h_{n,N}(\beta)$ *is an appropriate norming constant.*

Theorem B is a slight modification of the main formula in [2]. Its proof goes in the same way, hence we omit it.

Now we are going to find a good asymptotic formula for $f^h_{n,N}(x)$. We omit the subscript N and superscript h if it leads to no ambiguity. For the sake of simplicity we restrict ourselves during the whole proof to the case $m=2$. As we shall see the measure $\mu^h_{n,N}$ is essentially concentrated on such configurations $\sigma(i)$, $i \in V_n$ for which $\frac{1}{2^n} \sum_{i \in V_n} \sigma(i) \sim \mathbf{M} = (M, 0)$, where M is defined through the equation $\Phi'_N(M) = h$. Hence it is enough to give a good asymptotics for $f_n(x)$ in a small neighbourhood of the point $\mathbf{M}$. By Theorem A the measure γ_n has the density function $P_n(x) = p_n(|x|)$, and $p_n(x) \sim L'_n \exp(-2^n \Phi_n(x))$ for $x \sim \mathbf{M}$. Put $f_n(x) = L_n \exp \psi_n(x)$, where L_n is chosen in such a way that $\psi_n(\mathbf{M}) = 0$. We claim that $\psi_n(x) \sim g_n(x_1 - M) + A_n x_2^2$ if $x = (x_1, x_2) \sim \mathbf{M}$, and we give explicit formulas for g_n and A_n. This relation can be proved by induction; namely $\psi_N(x) = 2^N \beta h(x_1 - M) = g_N(x_1 - M) + A_N x_2^2$ with $g_N = 2^N \beta h$ and $A_N = 0$. By Theorems B and A

$$\psi_n(x) \sim C_n + \ln \int \exp\left[\beta c^n(x; t) + \psi_{n+1}\left(\frac{x+t}{2}\right) - 2^n \Phi_n(t)\right] dt. \tag{3.1}$$

The integral in (3.1) is concentrated around its maximum, hence

$$\psi_n(x) \sim C_n + \sup_t \left[\beta c^n(x; t) + \psi_{n+1}\left(\frac{x+t}{2}\right) - 2^n \Phi_n(|t|)\right]. \tag{3.2}$$

This maximum is taken near the point $\mathbf{M}$, but we need a better approximation for the point t where the right-hand side of (3.2) takes its maximum. To get such an approximation we are looking for the maximum of $\psi_n(x)$ on the circle $|t| = M$. The term $2^n \Phi_n(|t|)$ is constant on this circle, and

$$\psi_{n+1}\left(\frac{x+t}{2}\right) \sim g_{n+1}\left(\frac{x_1+t_1}{2} - M\right) + A_n\left(\frac{x_2+t_2}{2}\right)^2.$$

In such a way we give the following approximation $\bar{\psi}_n(x)$ for $\psi_n(x)$:

$$\bar{\psi}_n(x)=C'_n+\sup_{|t|=M}\left[\beta c^n(x_1t_1+x_2t_2)+g_{n+1}\left(\frac{x_1+t_1}{2}-M\right)+A_{n+1}\left(\frac{x_2+t_2}{2}\right)^2\right],$$
$$\bar{\psi}_n(M)=0. \tag{3.3}$$

Let us calculate the place of maximum $t=(t_1,t_2)$ of the expression (3.3) by Lagrange's method of multipliers. We get the equations

$$\beta c^n x_1+\frac{g_{n+1}}{2}=2\lambda t_1,$$

$$\beta c^n x_2+\frac{A_{n+1}}{2}(x_2+t_2)=2\lambda t_2,$$
$$t_1^2+t_2^2=M^2.$$

We shall solve this system of equations only approximately. Since $t_1\sim M$, and $x_1\sim M$, hence $2\lambda\sim\beta c^n+\frac{g_{n+1}}{2M}$, and we get the following approximate solution for t_1 and t_2:

$$t_2=\frac{\beta c^n+\frac{A_{n+1}}{2}}{\beta c^n-\frac{A_{n+1}}{2}+\frac{g_{n+1}}{2M}}x_2,\qquad t_1=M-\frac{t_2^2}{2M}. \tag{3.4}$$

Therefore

$$\bar{\psi}_n(x)=C'_n+\left(\beta c^n M+\frac{g_{n+1}}{2}\right)x_1+\left[\left(-\beta c^n\frac{x_1}{2M}-\frac{g_{n+1}}{4M}\right)\left(\frac{t_2}{x_2}\right)^2\right.$$
$$\left.+\frac{A_{n+1}}{4}\left(1+\frac{t_2}{x_2}\right)^2+\beta c^n\left(\frac{t_2}{x_2}\right)\right]x_2^2-\frac{g_{n+1}}{2M}.$$

If we substitute x_1 by M in the coefficient of x_2^2, then the above formal calculations suggest that

$$\psi_n(x)\sim g_n(x_1-M)+A_n x_2^2,$$

where g_n and A_n are defined by the following recursive formula

$$g_m=\beta c^m M+\frac{g_{m+1}}{2},\qquad g_N=2^M\beta h,\qquad m=N-1,\dots,0. \tag{3.5}$$

$$A_m=-\left(\frac{\beta c^m}{2}+\frac{g_{m+1}}{4M}\right)\left(\frac{\beta c^m+\frac{1}{2}A_{m+1}}{\frac{1}{2}\frac{g_{m+1}}{M}-\frac{A_{m+1}}{2}+\beta c^m}\right)^2$$

$$+\frac{A_{m+1}}{4}\left(1+\frac{\beta c^m+\frac{1}{2}A_{m+1}}{\frac{1}{2}\frac{g_{m+1}}{M}-\frac{A_{m+1}}{2}+\beta c^m}\right)^2+\frac{\left(\beta c^m+\frac{A_{m+1}}{2}\right)\beta c^m}{\frac{g_{m+1}}{2M}-\frac{A_{m+1}}{2}+\beta c^m}$$

$$=\frac{\frac{1}{2}(\beta c^m)^2+\left(\frac{g_{m+1}}{8M}+\frac{3}{4}\beta c^m\right)A_{m+1}}{\frac{g_{m+1}}{2M}-\frac{A_{m+1}}{2}+\beta c^m},\qquad m=N-1,\dots,0,\qquad A_N=0. \tag{3.5'}$$

Later we shall see that these rather rough calculations give an approximation sufficiently good for our purposes.

In the region where the measure $\mu^h_{n,N}$ is concentrated we may write

$$\frac{d\mu^h_{n,N}}{d\mu_n}(x^{(1)},\ldots,x^{(2^n))}) \sim L_n \exp\left\{\frac{g_n}{2^n}\sum_{j=1}^{2^n}(x_1^{(j)}-M)+A_n\left(\frac{1}{2^n}\sum_{j=1}^{2^n}x_2^{(j)}\right)^2\right\}. \tag{3.6}$$

For a fixed $h>0$ the first term is dominating in the exponent of the right-hand side of (3.6), while the second term is negligible. It is easy to see that $g_n=g_n(N,h) \sim \beta h\cdot 2^n+O(c^n)$ as $N\to\infty$. Hence the expression on the right-hand side of (3.6) is almost independent of N. Exploiting this fact, we get, by letting N go to infinity in (3.6), that the restrictions of the measures μ^h_n to V_n have a limit as $N\to\infty$. This implies part a) of Theorem 1. Part b) can be proved by means of a similar but more careful limiting procedure, when $N\to\infty$ and $h\to 0$ simultaneously. In order to carry out this limiting procedure first we have to investigate the behaviour of the sequences g_n and A_n. Put $\bar{g}_n=\frac{g_n}{c^nM}$ and $\bar{A}_n=\frac{A_n}{c^n}$. It is not difficult to see that $\bar{g}_n \to \frac{2\beta}{2-c}$ if $N\to\infty$ and $h\to 0$. On the other hand $\bar{A}_n\to\bar{A}=\beta\frac{2-c}{c}$ in this case. It is natural to expect such a result for the following reason. Because of relation (3.5′) the value $\bar{A}=\lim\limits_{\substack{N\to\infty\\ h\to 0}}\bar{A}_n(N,h)$ has to be the solution of the equation

$$A=c\frac{\frac{1}{2}\left(\frac{\beta}{c}\right)^2+A\left(\frac{\bar{g}}{8}+\frac{3}{4}\frac{\beta}{c}\right)}{\frac{\bar{g}}{2}-\frac{A}{2}+\frac{\beta}{c}}, \qquad \bar{g}=\frac{2\beta}{2-c}.$$

This equation has two solutions: $A^{(1)}=\beta\frac{2-c}{c}$ and $A^{(2)}=\frac{\beta}{2-c}$, $A^{(1)}<A^{(2)}$. Since our iteration starts with $\bar{A}_n=0$, the smaller root of this equation must appear as the limit.

Exploiting the above relations, and carrying out a limiting procedure just as was done with relation (3.6), we get that the limit $\lim\limits_{h\to 0}\mu^h=\bar{\mu}$ exists, and moreover

$$\frac{d\bar{\mu}_{V_n}}{d\mu_n}(x^{(1)},\ldots,x^{(2^n)}) \sim L_n\exp\left\{\bar{g}c^nM\frac{1}{2^n}\sum_{j=1}^{2^n}(x_1^{(j)}-M)+\bar{A}c^n\left(\frac{1}{2^n}\sum_{j=1}^{2^n}x_2^{(j)}\right)^2\right\}, \tag{3.7}$$

where $\bar{\mu}_{V_n}$ is the projection of the measure $\bar{\mu}$ to the volume V_n. Hence Theorem 1 hold true, and formula (3.7) helps us to prove Theorem 2.

Indeed, with the help of formula (3.7) and Theorem A we can get a good approximation for the density function of $\frac{1}{2^n}\sum\limits_{j\in V_n}\sigma(j)$, where the random variables

$\sigma(j)$ are $\bar{\mu}$ distributed. This approximation of the density will be of the form

$$K_n \exp\{\bar{g}c^n M(x_1-M)+\bar{A}c^n x_2^2-2^n\Phi_n(|x|)\}=\exp\{H_n(x)\} \tag{3.8}$$

in the vicinity of the point $\mathbf{M}=(M,0)$. Making a Taylor expansion around the maximum of this density function (the maximum is actually in the point $\mathbf{M}$) we get the exponent of a quadratic form. This implies that $T^n\sigma(j)$ is asymptotically normally distributed. The multi-dimensional distributions in Theorem 2 can be calculated similarly.

Let us finally discuss how the normalization in Theorem 2 must be chosen. To this end we have to determine the variances of the random variables $\bar{\sigma}_n^{(1)} = \frac{1}{2^n}\sum_{j\in V_n}\sigma^{(1)}(j)$ and $\bar{\sigma}_n^{(2)}=\sum_{j\in V_n}\sigma^{(2)}(j)$, where the random variables $\sigma(j)=\{\sigma^{(1)}(j), \sigma^{(2)}(j)\}$ are $\bar{\mu}$ distributed. The vector $\bar{\sigma}_n=(\bar{\sigma}_n^{(1)}, \bar{\sigma}_n^{(2)})$ is asymptotically Gaussian, and its density is given by formula (3.8). Therefore $D\bar{\sigma}_n^{(1)}\sim -\left(\frac{\partial^2 H}{\partial x_1^2}\right)^{-1}$ and $D\bar{\sigma}_n^{(2)} \sim -\left(\frac{\partial^2 H}{\partial x_2^2}\right)^{-1}$, and these partial derivatives are taken in place of the maximum of the function H, i.e. in $\mathbf{M}=(M,0)$. It can be seen that $D\bar{\sigma}_n^{(1)}\sim \text{const}\cdot 2^{-n}$, as it is expected. On the other hand observe that $\left.\frac{\partial\Phi(|x|)}{\partial x_1}\right|_{x=M}=0$, implies that $\left.\frac{\partial^2\Phi}{\partial x_2^2}\right|_{x=M}=0$. Now, since $\Phi_n(x)=\Phi(x)+\frac{a_0\beta}{2}\left(\frac{c}{2}\right)^n x_2^2$, the above relations imply that $\frac{\partial^2 H(\mathbf{M})}{\partial x_2^2}\sim\text{const}\cdot c^n$ and $D\sigma_n^{(2)}\sim\text{const}\cdot c^{-n}$. This fact explains the unusual normalization in Theorem 2.

4. Some Preparatory Remarks

In this section we discuss some technical details needed during the proof. First we give some estimates on the Radon–Nikodym derivative

$$\frac{d\mu_{n,N}^h}{d\mu_n}(x^{(1)},\dots,x^{(2^n)})=f_{n,N}^h\left(\frac{1}{2^n}(x^{(1)}+\dots+x^{(2^n)})\right). \tag{4.1}$$

$\left(\text{By Theorem B the Radon-Nikodym derivative } \frac{d\mu_{n,N}^h}{d\mu_n} \text{ is a function of } x^{(1)}+\dots+x^{(2^n)}.\right)$ We formulate two kinds of estimates. Property A(n) states an estimate on $f_{n,N}^h$ in the typical region, and Property B(n) gives an upper bound everywhere. Later we shall prove that both A(n) and B(n) hold true.

Let $M=M(N,h)$, $h>0$, $N>N_0$ with some N_0, denote the (unique) positive solution of the equation

$$\Phi_N'(s)=h\beta\,, \tag{4.2}$$

where the function Φ_N is defined in (2.3). (Since the argument β is fixed during the whole proof, we shall generally omit it.) Introduce the set

$$D(n)=D(n,N,h)=\left\{x=(x_1,x_2),|x_1-M(N,h)|<2^{-\frac{n}{2}}\alpha^{\frac{n}{2}},x_2^2<c^{-n}\alpha^{\frac{n}{2}}\right\} \tag{4.3}$$

with $\alpha=\left[\min\left(\frac{2}{c},\frac{c^2}{2}\right)\right]^{\frac{1}{20}}$. Now we formulate the following

Property $A(n)$.

$$f^h_{n,N}(x)=L^h_{n,N}\exp\{g_n(N,h)(x_1-M(N,h)) + A_n(N,h)x_2^2+O(\xi^n)\}$$

for $x\in D(n)$, where $f^h_{n,N}(x)$ is defined in (4.1), $0<\xi<1$, $L^h_{n,N}$ is an appropriate norming constant, and the constants $g_n(N,h)$ and $A_n(N,h)$ are defined by the recursive formulas

$$g_N(N,h)=2^N h\beta\,,$$

$$g_m(N,h)=\beta c^m M(N,h)+\frac{g_{m+1}(N,h)}{2}\,,\qquad m=N-1,\ldots,1\,, \tag{4.4}$$

$$A_N(N,h)=0\,,$$

$$A_m(N,h)=\frac{\frac{1}{2}(\beta c^m)^2+A_{m+1}(N,h)\left(\frac{g_{m+1}(N,h)}{8M(N,h)}+\frac{3}{4}\beta c^m\right)}{\frac{g_{m+1}(N,h)}{2M(N,h)}-\frac{A_{m+1}(N,h)}{2}+\beta c^m}\,,\qquad m=N-1,\ldots,1\,. \tag{4.5}$$

Let us introduce the notations $R_n=M(\beta)-0{,}01\cdot\left(\frac{c}{2}\right)^n$ and $\tilde{R}_n=M(\beta)-\frac{0{,}02}{c-1}\left(\frac{c}{2}\right)^n$. We have chosen $\tilde{R}_n$ in such a way that $\frac{1}{2}(R_n+\tilde{R}_n)\geqq\tilde{R}_{n+1}$, $n=1,2,\ldots$.
Now we formulate the following

Property $B(n)$. *Put $M=M(N,h)$, $\mathbf{M}=(M,0)$, $g_n=g_n(N,h)$, $A_n=A_n(N,h)$ and $p=\frac{2}{\alpha}$. There exist some $L>0$, $0<\xi<1$ such that*

a) *For all $y\in R^1$, $x\in R^2$ such that $|x|\leqq y$, $y>\tilde{R}_n$ we have*

$$f^h_{n,N}(x)\leqq f^h_{n,N}(\mathbf{M})\exp\{g_n(y-M)+Lp^n(y-M)^2+O(\xi^n)\}\,.$$

b) *For $|x|\leqq T$, $|T-M|\leqq d_n$ with $d_n=\alpha^n\left(\frac{g_n}{c^n 2^n}\right)^{1/2}$.*

b1) *$f^h_{n,N}(x)\leqq f^h_{n,N}(\mathbf{M})\exp\{g_n(x_1-M)+A_n x_2^2+O(\xi^n)\}$, if $g_n(x_1-T)+A_2x_2^2 \geqq -\frac{g_n}{c^n}\alpha^n$.*

b2) $f^h_{n,N}(x) \leqq f^h_{n,N}(\mathbf{M}) \exp\left\{g_n(T-M) - \frac{g_n}{c^n}\alpha^n + O(\xi^n)\right\}$, *if* $g_n(x_1 - T) + A_n x_2^2$
$\leqq -\frac{g_n}{c^n}\alpha^n$.

[We could have chosen $y = \max(|x|, \tilde{R}_n)$ and $T = |x|$ in Property B(n). The somewhat artificial constants T and y were introduced, because the proofs are simpler with such a formulation.]

Now we formulate the following

Proposition 1. *If* $n \geqq n_0$, $N \geqq n$ *and*

$$h > \frac{2M(N,h)}{2-c}\left(\frac{c}{2}\right)^N, \tag{4.6}$$

then Property A(n) *holds true. The number* ξ, $0 < \xi < 1$, *and the threshold number* n_0 *can be chosen independently of* N *and* h *if* $h < h_0$ *with some* $h_0 > 0$. *The* $O(\cdot)$ *is uniform in* N, h *and* x.

Proposition 2. *Under the conditions of Proposition 1, Property* B(n) *holds true. The constants* $L > 0$, $0 < \xi < 1$, *and the threshold number* n_0 *can be chosen independently of* N *and* h *if* $h < h_0$. *The* $O(\cdot)$ *is uniform in* N, h, x, *and* y.

Later we shall see that under condition (4.6) the constant d_n appearing in part b) of Property B(n) satisfies the inequality $M(N,h) - d_n > R_n$. Property A(n) describes the asymptotic behaviour of $f^h_{n,N}(x)$ in the typical region where the average of the $\mu^h_{n,N}$ distributed spins are concentrated, and Property B(n) gives an upper bound on $f^h_{n,N}$ everywhere. The typical region is around the point $\mathbf{M} = (M(N,h), 0)$ and its size is $2^{-n/2}\alpha(n)$, with some $\alpha(n) \to \infty$ in the direction of the magnetization; In the orthogonal direction its size depends on h, but under condition (4.6) it is always smaller than $c^{-n/2}\alpha(n)$. Condition (4.6) was imposed in order to guarantee that $\mu^h_{n,N}$ is concentrated in a small region. In Property B(n) we have distinguished the cases a) and b) because in the case $|x| \sim M(N,h)$ a sharper bound is needed. The cases b1) and b2) were separated because in the case $x \sim (T,0)$ [this is case b1)] a sharp bound is needed. We shall see that under condition (4.6) $g_n(N,h)(x_1 - T) + A_n(N,h)x_2^2$ is negative for $|x| < T$, and its absolute value is small only if $x \sim (T,0)$.

In the sequel we shall omit the arguments N and h if it leads to no ambiguity. The letter ξ will denote a real number between zero and one. In different formulas it may denote different numbers. What is important for us is that there exists an $\varepsilon > 0$ such that $\xi < 1 - \varepsilon$ for all n, N and h, $0 < h < h_0$ with some $h_0 > 0$, and the $O(\cdot)$ in $O(\xi^n)$ is uniform in all variables of the formulas.

Now we shall investigate the behaviour of the sequences g_n and A_n. Put $\bar{g}_n = \frac{g_n}{c^n M}$ and $\bar{A}_n = \frac{A_n}{c^n}$. By formulas (4.4) and (4.5) the relations

$$\begin{aligned} \bar{g}_n &= \frac{h\beta}{M}\left(\frac{2}{c}\right)^n + \frac{2\beta}{2-c}\left(1 - \left(\frac{c}{2}\right)^{N-n}\right), \qquad n = N, \quad N-1, \ldots, 1, \\ \bar{A}_n &= c\frac{\frac{1}{2}\left(\frac{\beta}{c}\right)^2 + \left(\frac{\bar{g}_{n+1}}{8} + \frac{3}{4}\frac{\beta}{c}\right)\bar{A}_{n+1}}{\frac{\bar{g}_{n+1}}{2} - \frac{\bar{A}_{n+1}}{2} + \frac{\beta}{c}}, \qquad A_N = 0, \quad N = N-1, \ldots, 1 \end{aligned} \tag{4.7}$$

hold true. We shall prove the following

Lemma 1. a) *If relation (4.6) holds then* $\bar{g}_N \geqq \bar{g}_{N-1} \geqq \dots \geqq \bar{g}_1 \geqq \bar{g}$ *with* $\bar{g} = \frac{2\beta}{2-c}$, *and* $0 = \bar{A}_N \leqq \bar{A}_{N-1} \leqq \dots \leqq \bar{A}_1 \leqq \bar{A}$ *with* $\bar{A} = \beta \frac{2-c}{c}$. *The inequality* $\bar{g}_n > \frac{h\beta}{M}\left(\frac{2}{c}\right)^n$ *also holds.*

b) *If moreover* $N \geqq N(n)$, $h < \left(\frac{c}{2}\xi\right)^{2n}$ *with some* $0 < \xi < 1$, *then there exist some* $0 < \xi' < 1$ *and* $C > 0$ *(independently of n and* $N > N(n)$*) such that* $|\bar{g}_n(N,h) - \bar{g}| < C\xi'^n$ *and* $|\bar{A}_n(N,h) - \bar{A}| < C\xi'^n$.

Lemma 1 has the following

Corollary 2. *For all* $0 < \eta < 1$ *and n there exist some constants* $\bar{N} = \bar{N}(n,\eta)$ *and* $p = p(\eta)$, $0 < p < 1$, *such that for* $h < p^n$ *and* $N > \bar{N}(n,\eta)$, $|\bar{A}_n(N,h) - \bar{A}| \leqq \eta^n$ *and* $|\bar{g}_n(N,h) - \bar{g}| \leqq \eta^n$.

Proof of Corollary 2. Choose some integers $K > 0$ and $j > 0$ such that $\eta < \xi'^K$ and $C\xi'^j < 1$. Put $p = \left(\frac{c}{2}\xi\right)^{2(nK+j)}$ and $\bar{N}(n,\eta) = N(Kn+j)$. Then we get, by applying the monotonicity of the sequence A_n together with part b) of Lemma 1 for $\bar{n} = Kn + j$ that

$$\bar{A} \geqq \bar{A}_n \geqq \bar{A}_{\bar{n}} \geqq \bar{A} - C\xi'^{\bar{n}} \geqq \bar{A} - \eta^n .$$

The corresponding statement for $\bar{g}_n$ follows directly from (4.7).

We remark that the condition $h < p^n$ is consistent with (4.6) if N is sufficiently large.

Proof of Lemma 1. It follows immediately from (4.7) that the sequence $\bar{g}_n$ has all properties stated in Lemma 1. Define the function

$$T(\alpha, g) = c \frac{\alpha\left(g + 6\frac{\beta}{c}\right) + 4\left(\frac{\beta}{c}\right)^2}{4g - 4\alpha + 8\frac{\beta}{c}},$$

and the transformation $T_g : R^1 \to R^1$, $T_g(\alpha) = T(\alpha, g)$. Let T_g^n denote the n-fold iteration of the transformation T_g. Clearly, $\bar{A}_m = T(\bar{A}_{m+1}, \bar{g}_{m+1})$. The idea of the proof is the following: We establish some monotonicity properties of the function $T(\alpha, g)$, and we deduce part a) from them. The sequence $T_g^n(0)$ tends exponentially fast to the smaller solution of the equation $T_g(\alpha) = \alpha$. Combining this fact with the monotonicity properties of the sequences $\bar{A}_n$ and $\bar{g}_n$ and the exponentially fast convergence $\bar{g}_n$ to $\bar{g}$ we show that $\bar{A}_n$ tends exponentially fast to the smaller solution of the equation $\alpha = T_{\bar{g}}\alpha$. This equation has two solutions $\alpha_1 = \beta\frac{2-c}{c} = \bar{A}$, $\alpha_2 = \frac{\beta}{2-c}$, and $\alpha_1 < \alpha_2$.

A simple calculation shows that

$$\frac{\partial T(\alpha,g)}{\partial g}=-c\frac{\left(4\frac{\beta}{c}+2\alpha\right)^2}{\left(8\frac{\beta}{c}-4\alpha+4g\right)^2}\leqq 0$$

and

$$\frac{\partial T(\alpha,g)}{\partial \alpha}=c\frac{4\left(g+4\frac{\beta}{c}\right)^2}{\left(8\frac{\beta}{c}+4g+4\alpha\right)^2}\geqq 0\,.$$

Hence in the domain $\left\{(\alpha,g);\ 0\leqq\alpha<g+2\frac{\beta}{c}\right\}$ the function $T(\alpha,g)$ is continuous, and it is monotone decreasing in the argument g and monotone increasing in the argument α. Let α_m, $N\geqq m\geqq 1$ denote the smaller (positive) solution of the equation $\alpha=T(\alpha,\bar{g}_m)$. Since the sequence $\bar{g}_m$ is monotone increasing $\bar{g}_m\geqq\bar{g}$, $T(\bar{A},\bar{g})=\bar{A}$ and $T(0,g)\geqq 0$ for $g\geqq 0$, the monotonicity properties of the function $T(\alpha,g)$ imply that $0\leqq\alpha_N\leqq\ldots\leqq\alpha_1\leqq\bar{A}$. Moreover, a simple induction gives that $0\leqq\bar{A}_m\leqq\alpha_m$, $m=N,\ldots,1$. Indeed, $\bar{A}_m=T(A_{m+1},\bar{g}_{m+1})\leqq T(\alpha_{m+1},\bar{g}_{m+1})=\alpha_{m+1}\leqq\alpha_m$, and $\bar{A}_m=T(\bar{A}_{m+1},\bar{g}_{m+1})\geqq T(0,\bar{g}_{m+1})\geqq 0$. Since $\alpha\leqq T(\alpha,\bar{g}_{m+1})$ for $0\leqq\alpha\leqq\alpha_{m+1}$, the relation $\bar{A}_m=T(\bar{A}_{m+1},\bar{g}_{m+1})\geqq\bar{A}_{m+1}$ holds, i.e. the sequence $\bar{A}_m$ is monotone decreasing. Part a) is proved.

If $h<\left(\frac{c}{2}\xi\right)^{2n}$, then $g_m\leqq\bar{g}+O(\xi'^n)$ for $n\leqq m\leqq 2n$. The relation $T_g^n(0)=\alpha(g)+O(\xi'^n)$ holds true for all $g\geqq\bar{g}$, where $\alpha(g)$ is the smaller solution of the equation $T(\alpha,g)=\alpha$, and the error term $O(\xi'^n)$ is uniformly bounded for $\bar{g}\leqq g\leqq\bar{g}+\varepsilon$. If $g=\bar{g}+O(\xi'^n)$, then $\alpha(g)=\bar{A}+O(\xi'^n)$. Hence the already proved properties of the sequences $\bar{A}_n$ and $\bar{g}_n$ and the function $T(\alpha,g)$ imply that for $h<\left(\frac{c}{2}\xi\right)^{2n}$,

$$\bar{A}\geqq\bar{A}_n\geqq T^n_{g_{2n}}(\bar{A}_{2n})\geqq T^n_{\bar{g}+O(\xi'^n)}(0)\geqq\bar{A}+O(\xi'^n)\,.$$

Lemma 1 is proved.

We shall prove Propositions 1 and 2 by induction, namely we shall prove the following

Lemma 3. *If* $n>n_0$, *and relation (4.6) holds, then Properties* $A(n+1)$ *and* $B(n+1)$ *imply Property* A(n).

Lemma 4. *If* $n>n_0$, *and relation (4.6) holds, then Properties* $A(n+1)$ *and* $B(n+1)$ *imply Property* B(n).

Since $f^h_{N,N}(x)=L_N\exp(\beta h2^Nx_1)$ Properties A(N) and B(N) hold true, and Lemmas 3 and 4 imply Propositions 1 and 2. We postpone their proofs to Sects. 7 and 8. We conclude this section by verifying the relations

$$\Phi_n'(M)=\frac{g_n}{2^n}\quad\text{for}\quad M=M(N,h)\quad\text{and}\quad n\leqq N\,,\tag{4.8}$$

and

$$M(N,h)-2d_n>M(\beta)-2^{-n/2}\alpha^{2n}, \tag{4.9}$$

if relation (4.6) holds.

Relation (4.8) holds for $n=N$ by the definition of g_N and $M(N,h)$. Then we get, by induction from $n+1$ to n, that

$$\Phi_n'(M)=\Phi_{n+1}'(M)+\frac{2-c}{2}a_0\beta\left(\frac{c}{2}\right)^n M=2^{-n}\left(\frac{g_{n+1}}{2}+\beta c^n M\right)=2^{-n}g_n,$$

as we claimed.

To prove (4.9) we write

$$\Phi'(M(N,h))=h\beta-a_0\beta\left(\frac{c}{2}\right)^N M(N,h)=\beta(h-h_0),$$

with $h_0=\frac{2M(N,h)}{2-c}\left(\frac{c}{2}\right)^N$. By (4.6) $h-h_0\geqq 0$, hence the relation $\Phi''(x)\geqq\text{const}>0$, if $x>T_n$ implies that $M(N,h)\geqq M(\beta)$, and moreover

$$M(N,h)\geqq M(\beta)+K(h-h_0)\quad\text{with some}\quad K>0. \tag{4.10}$$

It follows from (4.7) that there exists some $K'>0$ such that $\frac{g_n}{2^n}\leqq\beta(h-h_0)+K'\cdot\left(\frac{c}{2}\right)^n$, and since $\sqrt{x+y}\leqq\sqrt{2x}+\sqrt{2y}$ for $x\leqq 0$, $y\geqq 0$, hence

$$\begin{aligned}
d_n&=\alpha^n\left(\frac{g_n}{2^n c^n}\right)^{1/2}\leqq\alpha^n\left(\beta\frac{h-h_0}{c^n}+K'\cdot 2^{-n}\right)^{1/2}\\
&\leqq\alpha^n\left[\left(\frac{2\beta}{c^n}(h-h_0)\right)^{1/2}+\sqrt{2K'}\cdot 2^{-n}\right]\leqq C\alpha^n\left(\frac{h-h_0}{\alpha^{2n}}+\frac{\alpha^{2n}}{c^n}+2^{-n/2}\right)\\
&\leqq\frac{K}{2}\cdot(h-h_0)+\frac{1}{2}\alpha^{2n}\cdot 2^{-n/2}.
\end{aligned}$$

The last relation with (4.10) imply (4.9).

Since $M(N,h)\geqq M(\beta)$, we can rewrite formulas (2.4) and (2.5) with the choice $x=M(N,h)$ and with the help of (4.8) in the following way:

$$p_n(x)=(1+O(\xi^n))p_n(M)\exp\{-g_n(x-M)-2^{n-1}\Phi_n''(M)(x-M)^2\}, \tag{4.11}$$

if $|x-M|\leqq 2^{-n/2}\alpha^n$,

$$p_n(x)\leqq(1+O(\xi^n))p_n(M)\exp\{-g_n(x-M)-2^nB(x-m)^2\}, \tag{4.12}$$

with some $B>0$ if $x>R_n$,

$$p_n(x)\leqq(1+O(\xi^n))p_n(R_n), \tag{4.12'}$$

if $|x|\leqq R_n$.

We shall need the above estimates on p_n in the sequel.

5. Proof of Theorem 1

We prove Theorem 1 with the help of Propositions 1 and 2 in this section and Theorem 2 in the next one.

Part a). Let us fix some $k \geqq 0$ and choose an $n > k$. We assume that n is sufficiently large, $n > n_0(h)$. Let $\mathscr{F}_k$ denote the natural σ-algebra in $(R^2)^{2^k}$. Given a set $A \in \mathscr{F}_k$, define the cylindrical set

$$A(n) = \{(x^{(1)}, \ldots, x^{(2^n)}),\ x^{(j)} \in R^2,\ j = 1, \ldots, 2^n,\ (x^{(1)}, \ldots, x^{(2^k)}) \in A\}.$$

Sometimes we shall write $\mu_N^h(A)$ instead of $\mu_N^h(A(N))$. We have $\mu_N^h(A) = \mu_{n,N}^h(A(n))$, hence

$$I(A) = \mu_n^h(A) = \int_{A(n)} \frac{d\mu_{n,N}^h}{d\mu_n} d\mu_n.$$

Put

$$\bar{D}(n) = \bar{D}(n, N, h) = \{x = (x_1, x_2),\ |x_1 - M(N, h)| < 2^{-n/2}\alpha^{n/2},\ x_2^2 < 2^{-n}\alpha^{n/2}\},$$
$$\tilde{D}(n) = \tilde{D}(n, h) = \{x = (x_1, x_2),\ |x_1 - M(h)| < \tfrac{1}{2} \cdot 2^{-n/2}\alpha^{n/2},\ x_2^2 < 2^{-n}\alpha^{n/2}\},$$

and

$$\tilde{D}_1(n) = \left\{(x^{(1)}, \ldots, x^{(2^n)}),\ \frac{1}{2^n} \sum_{j=1}^{2^n} x^{(j)} \in \tilde{D}(n)\right\},$$

where $M(h)$ is the solution of the equation $\Phi'(s) = \beta h$. [The difference between the sets $\bar{D}(n)$ and $D(n)$ defined in (4.3) is that x_2^2 is bounded by $2^{-n}\alpha^{n/2}$ in $\bar{D}(n)$ and by $c^{-n}\alpha^{n/2}$ in $D(n)$. We shall work with the set $\tilde{D}(n)$ instead of $\bar{D}(n)$, because it does not depend on N.] We define the integrals

$$I_1(A) = \int_{A(n) \cap \tilde{D}_1(n)} \frac{d\mu_{n,N}^h}{d\mu_n} d\mu_n,$$

and

$$I_2(A) = \int_{A(n) \setminus \tilde{D}_1(n)} \frac{d\mu_{n,N}^h}{d\mu_n} d\mu_n.$$

Then $I(A) = I_1(A) + I_2(A)$. We shall estimate $I_1(A)$ with the help of Proposition 1 and I_2 with the help of Proposition 2. By Proposition 1

$$\left|\frac{d\mu_{n,N}^h}{d\mu_n}(x^{(1)}, \ldots, x^{(2^n)}) \frac{1}{L_{n,N}^h} \exp\left\{-c^n M \bar{g}_n \frac{1}{2^n} \sum_{j=1}^{2^n} (x_1^{(j)} - M)\right\}\right|$$
$$\leqq 1 + O(\xi^n) + O\left(\left(\frac{c}{2}\right)^n \alpha^n\right) = 1 + O(\xi^n) \quad \text{with} \quad M = M(N, h), \tag{5.1}$$

if

$$\frac{1}{2^n} \sum_{j=1}^{2^n} x^{(j)} \in \bar{D}(n), \quad \text{since} \quad A_n \left(\frac{1}{2^n} \sum_{j=1}^{2^n} x_2^{(j)}\right)^2 = O\left(\left(\frac{c}{2}\right)^n \alpha^{n/2}\right)$$

in this case. Moreover (5.1) remains valid if

$$N > N_1(n) = 2\frac{\log(2c)}{\log\left(\frac{2}{c}\right)}n,$$

and the terms $\bar{g}_n$ and $M(N,h)$ are substituted by $\bar{g}'_n$ and $M(h)$, where $\bar{g}'_n = \frac{h\beta}{M(h)}\left(\frac{2}{c}\right)^n + \frac{2\beta}{2-c}$. [Our aim with this substitution is to give a good estimate on $I_1(A)$ whose main term does not depend on N.] Indeed, $M(N,h) - M(h) = O\left(\left(\frac{c}{2}\right)^N\right)$ because of (4.2) and the definition of $M(h)$,

$$\bar{g}_n - \bar{g}'_n = h\beta\left(\frac{2}{c}\right)^n\left(\frac{1}{M} - \frac{1}{M(h)}\right) - \frac{2\beta}{2-c}\left(\frac{c}{2}\right)^{N-n} = O\left(\left(\frac{c}{2}\right)^{N-n}\right).$$

Hence

$$\exp\left\{c^n M\bar{g}_n \frac{1}{2^n}\sum_{j=1}^{2^n}(x_1^{(j)} - M) - c^n M(h)\frac{\bar{g}'_n}{2^n}\sum_{j=1}^{2^n}(x_1^{(j)} - M(h))\right\}$$
$$= O\left(\exp\left(c^n 2^n \left(\frac{c}{2}\right)^N \cdot 2^{-n/2}\alpha^n\right)\right) = O(\xi^n).$$

Moreover for $N > N_1(n)$, $\tilde{D}(n) \subset \bar{D}(n)$. Hence we can write for $N > N_1(n)$,

$$I_1(A) = (1 + O(\xi^n)) \int_{A(n)\cap\tilde{D}_1(n)} L^h_{n,N}\left\{\exp c^n M(h)\bar{g}'_n \frac{1}{2^n}\sum_{j=1}^{2^n}(x_1^{(j)} - M(h))\right\} d\mu_n. \quad (5.2)$$

Put $I'_1 = I_1((R^2)^{2^k})$ and $I'_2 = I_2((R^2)^{2^k})$. We are going to show that

$$I'_2 \leqq I'_1 \exp(-\xi^{-n}). \quad (5.3)$$

First we prove that (5.2) and (5.3) imply part a) of Theorem 1. Since $I'_1 \leqq 1$, (5.3) implies that $I'_2 \leqq \exp(-\xi^{-n})$. Hence we get, by applying (5.2) with the choice of $A = (R^2)^{2^k}$, that

$$1 = I'_1 + I'_2 = (1 + O(\xi^n))L^h_{n,N} \int_{\tilde{D}_1(n)} \exp\left\{\left(\frac{c}{2}\right)^n \bar{g}'_n M(h) \sum_{j=1}^{2^n}(x_1^{(j)} - M(h))\right\} d\mu_n.$$

Hence there exists some constants L^h_n such that $L^h_{n,N} = L^h_n(1 + O(\xi^n))$, and the relations (5.2), (5.3) together with the inequality $I'_2(A) \leqq I'_2$ imply that

$$\mu^h_N(A) = \int_{A(n)\cap D_1(n)} L^h_n \exp\left\{\left(\frac{c}{2}\right)^n \bar{g}'_n M(h) \sum_{j=1}^{2^n}(x_1^{(j)} - M(h))\right\} d\mu_n + O(\xi^n) \quad (5.4)$$

for $N > N_1(n)$. Here the constant $\xi < 1$ does not depend on $A \in \mathscr{F}_k$. By relation (5.4) $|\mu^h_N(A) - \mu^h_{N'}(A)| = O(\xi^n)$ for all $A \in \mathscr{F}_k$ if $N > N_1(n)$ and $N' > N_1(n)$. Letting N tend to infinity we get from this relation that $\mu^h(A) = \lim_{N\to\infty} \mu^h_N(A)$ exists, and $\mu^h_N(A) = \mu^h(A) + O(\xi^n)$. The limit μ^h is a probability measure, hence part a) is proved.

Now we turn to the proof of (5.3). The relations

$$I'_1 = \int_{\tilde{D}(n)} f_n(x)p_n(x)dx\,,$$

$$I'_2 = \int_{R^2-\tilde{D}(n)} f_n(x)p_n(x)dx$$

hold true. It follows from (4.11) and Property A(n) that $f_n(x)p_n(x) \geqq Cf_n(\mathbf{M})p_n(M)$ with some $C>0$ if $|x-\mathbf{M}|<2^{-n/2}$, where $\mathbf{M}=(M(N,h),0)$. Hence

$$I'_1 \geqq C\cdot 2^{-n} f_n(\mathbf{M})p_n(M) \tag{5.5}$$

if $N>N_1(n)$. Obviously

$$I_2 \leqq \int_{F_1} f_n(x)p_n(x)dx + \int_{F_2} f_n(x)p_n(x)dx\,, \tag{5.6}$$

with

$$F_1=\{x\in R^2,\ |x|<M-\tfrac{1}{3}\cdot 2^{-n/2}\alpha^{n/2} \text{ or } |x|>M+\tfrac{1}{3}\cdot 2^{-n/2}\alpha^{n/2}\}\,,$$
$$F_2=\{x\in R^2,\ M-\tfrac{1}{3}\cdot 2^{-n/2}\alpha^{n/2}<|x|<M+\tfrac{1}{3}\cdot 2^{-n/2}\alpha^{n/2},\ x_2^2>2^{-n}\alpha^{n/2}\}\,.$$

Put $t(u)=\max(u,R_n)$. It follows from (4.12), (4.12′) and part a) of Property B(n) with the choice $y=t(x)$ that

$$p_n(x)f_n(x) \leqq p_n(M)f_n(\mathbf{M})\exp\{-2^{n-1}B(t(|x|)-M)^2\}$$

for $x\in F_1$. Therefore

$$\int_{F_1} p_n(x)f_n(x)dx \leqq p_n(M)f_n(\mathbf{M})\exp(-\tfrac{1}{4}\alpha^{n/2})\,. \tag{5.7}$$

If $x\in F_2$ then we can estimate $f_n(x)$ with the help of part b) of Property B(n) with the choice $T=|x|$. We have $x_1-|x| \leqq \frac{-x_2^2}{2|x|} \leqq -\frac{x_2^2}{4M}$, and $g_n \geqq 2^n h\beta$. Therefore, if $n>n_0(h)$, then $g_n>8A_nM$, and $g_n(x_1-T)+A_nx_2^2 \leqq -\frac{g_n}{8M}x_2^2 \leqq -\frac{h\beta}{8M}\alpha^{n/2}$. Hence either b1) or b2) implies that

$$f_n(x) \leqq f_n(\mathbf{M})\exp\{g_n(|x|-M)-\xi^{-n}\}\,.$$

Then by (4.12)

$$\int_{F_2} p_n(x)f_n(x)dx \leqq p_n(M)f_n(\mathbf{M})\exp(-\xi^{-n})\,. \tag{5.8}$$

Relations (5.5)–(5.8) imply (5.3).

Part b). Let us fix again a $k\geqq 0$ and $n\geqq k$. Let $N>N_2(n)$, where $N_2(n)\geqq N_1(n)$, and $N_2(n)$ is so large that Corollary 2 can be applied with $\eta=\frac{1}{2\alpha}$. Let $h<4^{-n}$ and $h<p^n$, where $p=p(\eta)$, and $p(\eta)$ is defined in the corollary of Lemma 1.

We claim that

$$[g_n(x_1-M)+A_nx_2^2]-[c^n\bar{g}M(\beta)(x_1-M(\beta))+c^n\bar{A}x_2^2]=O(\xi^n) \tag{5.9}$$

for $x \in D(n)$ if $N > N_2(n)$, where $\bar{g} = \frac{2\beta}{2-c}$ and $A = \frac{2-c}{c}\beta$. Relation (5.9) follows from the following estimates: By Corollary 2,

$$A_n x_2^2 - c^n \bar{A} x_2^2 \leqq 2^{-n},$$
$$g_n - c^n \bar{g} M(N,h) = O(2^{-n} c^n),$$

and

$$M(\beta) - M(N,h) = O\left(h + \left(\frac{c}{2}\right)^N\right) = O(2^{-n}),$$

if $N > N_2(n)$, $x \in D(n)$. Formula (5.9) follows from the above relations. Put

$$D_2(n) = \{x = (x_1, x_2),\ |x_1 - M(\beta)| < \tfrac{1}{2} \cdot 2^{-n/2} \alpha^{n/2},\ x_2^2 < c^{-n} \alpha^{n/2}\},$$
$$\tilde{D}_2(n) = \left\{(x^{(1)}, \ldots, x^{(2^n)}),\ \frac{1}{2^n} \sum_{j=1}^{2^n} x^{(j)} \in D_2(n)\right\},$$

and define the integrals

$$\mathscr{I}_1(A) = \int_{A(n) \cap \tilde{D}_2(n)} \frac{d\mu_{n,N}^h}{d\mu_n} d\mu_n, \qquad \mathscr{I}_2(A) = \int_{A(n) \setminus \tilde{D}_2(n)} \frac{d\mu_{n,N}^h}{d\mu_n} d\mu_n, \qquad A \in \mathscr{F}_k.$$

Now we argue similarly to part a). Obviously $\mu_n^h(A) = \mathscr{I}_1(A) + \mathscr{I}_2(A)$. Since $D_2(n) \subset D(n)$, it follows from (5.9) and Property A(n) that

$$\mathscr{I}_1(A) = (1 + O(\xi^n)) L_{n,N}^h \int_{A(n) \cap D_2(n)} \cdot \exp\left\{c^n \bar{g} M(\beta) \frac{1}{2^n} \sum_{j=1}^{2^n} (x_1^{(i)} - M(\beta)) + c^n \bar{A} \left(\frac{1}{2^n} \sum_{j=1}^{2^n} x_2^{(j)}\right)^2\right\} d\mu_n. \quad (5.10)$$

Put $\mathscr{I}_1' = \mathscr{I}_1((R^2)^{2^k})$ and $\mathscr{I}_2' = \mathscr{I}_2((R^2)^{2^k})$. Then

$$\mathscr{I}_1' = \int_{\tilde{D}_2(n)} p_n(x) f_n(x) dx, \qquad \mathscr{I}_2' = \int_{R^2 \setminus D_2(n)} p_n(x) f_n(x) dx.$$

It can be proved similarly to (5.3) that

$$\mathscr{I}_2' \leqq \mathscr{I}_1' \exp(-\xi^{-n}). \quad (5.11)$$

The only difference is that now we have to define the set F_2 as

$$F_2 = \{x \in R^2, M - \tfrac{1}{3} 2^{-n/2} \alpha^{n/2} \leqq |x| \leqq M + \tfrac{1}{3} 2^{-n/2} \alpha^{n/2}, x_2^2 > c^{-n} \alpha^{n/2}\},$$

and we have to carry out the estimates in F_2 in the following way: For $x \in F_2$,

$$\begin{aligned} &g_n(x_1 - |x|) + A_n x_2^2 \\ &\leqq c^n[\bar{g} M(N,h)(x_1 - |x|) + \bar{A} x_2^2] \leqq c^n\left(-\frac{\bar{g} x_2^2}{2|x|} M(N,h) + \bar{A} x_2^2\right) \\ &\leqq c^n x_2^2 \left(-(1-\varepsilon)\frac{\bar{g}}{2} + \bar{A}\right) \leqq -B c^n x_2^2 \leqq -B \alpha^{n/2}, \end{aligned}$$

with some appropriately chosen $\varepsilon>0$ and $B>0$. Then (5.11) can be proved similarly to (5.3). Relations (5.10) and (5.11) imply that there exists some L_n such that

$$L^h_{n,N}=(1+O(\xi^n))L_n, \tag{5.12}$$

and

$$\mu^h_N(A)=\int_{A(n)\cap\tilde{D}_2(n)} L_n\exp\left\{c^n\bar{g}M(\beta)\frac{1}{2^n}\sum_{j=1}^{2^n}(x_1^{(j)}-M(\beta))+c^n\bar{A}\left(\sum_{j=1}^{2^n}\frac{1}{2^n}x_2^{(j)}\right)^2\right\}d\mu_n+O(\xi^n), \tag{5.13}$$

if $N>N_2(n)$, $h<p^n$, $h<4^{-n}$ and h satisfies relation (4.6). These relations can be proved similarly to their analogues in part a). Let us emphasize that the main term on the right-hand side of (5.13) depends neither on N nor on h.

Then we can carry out a limiting procedure similarly to part a). Letting first N tend to infinity we get that $|\mu^{h_1}(A)-\mu^{h_2}(A)|=O(\xi^n)$ if $h_1, h_2<4^{-n}$ and $h_1, h_2<p^n$. Then letting h tend to zero we get that the limit $\bar{\mu}(A)=\lim_{h\to 0}\mu^h(A)$ exists, and $\mu^h(A)=\bar{\mu}(A)+O(\xi^\delta)$ with $\delta=\log\frac{1}{h}\cdot\left(\max\left(\log 4,\log\frac{1}{p}\right)\right)^{-1}$. The number $\xi<1$ and the $O(\cdot)$ in the last relation does not depend on the set A. Hence it implies part b) of Theorem 1.

6. Proof of Theorem 2 and the Corollary

Let $\mu^h_{V_n}$ and $\bar{\mu}_{V_n}$ denote the projection of the measures μ^h and $\bar{\mu}$ to V_n. We claim that the Radon-Nikodym derivatives

$$\frac{d\mu^h_{V_n}}{d\mu_n}(x^{(1)},\dots,x^{(2^n)})=F^h_n\left(\frac{1}{2^n}\sum_{j=1}^{2^n}x^{(j)}\right), \tag{6.1}$$

and

$$\frac{d\bar{\mu}_{V_n}}{d\mu_n}(x^{(1)},\dots,x^{(2^n)})=F_n\left(\frac{1}{2^n}\sum_{j=1}^{2^n}x^{(j)}\right), \tag{6.2}$$

exist.

Indeed, the convergence of the measures, $\mu^h_{n,N}$ to $\mu^h_{V_n}$ in variational metric, established in Theorem 1, is equivalent to the $L^1_{\mu_n}$ convergence of $f^h_{n,N}$ to (the existing) F^h_n. Then the convergence of the measures $\mu^h_{V_n}$ to $\bar{\mu}_{V_n}$ in variational metric implies the convergence of F^h_n to F^h. Moreover, these convergences help us to show that Properties A(n) and B(n) remain valid for $F^h_n(x)$ and $F_n(x)$.

To be more precise, let us first define

$$g_n(h)=\beta h 2^n+\frac{2\beta}{2-c}c^n\cdot M(h).$$

It is easy to see from (4.7) that

$$g_n(h)-g_n(N,h)=O(c^N\cdot 2^{n-N}).$$

By Corollary 2 there exists a $\frac{1}{4}>p_1>0$ such that if $h<p_1^n$ and $N>N(p_1,n)$, then $A(N,p)-\bar{A}c^n=O(\xi^n)$. Clearly $\left(M(N,h)-M(h)=O\left(\frac{c}{2}\right)^N\right)$. It can be seen with the help of these facts and formula (5.12) that properties A(n) and B(n) hold for $F_n^h(x)$ if $h<p_1^n$, $g_n(N,h)$ is substituted by $g_n(h)$, $A_n(N,h)$ by $\bar{A}c^n$, $M(N,h)$ by $M(h)$ and $L_{n,N}^h$ by L_n. Then letting h go to zero we get that Properties A(n) and B(n) hold also for $F_n(x)$ if $M(h)$ is substituted by $M(\beta)$ and $g_1(h)$ by $c^n\bar{g}M(\beta)$. Thus we get with the notation of Properties A(n) and B(n), the following relations:

$$F_n(x)=L_n\exp\{c^n\bar{g}M(\beta)(x_1-M(\beta))+c^n\bar{A}x_2^2+O(\xi^n)\} \quad \text{for} \quad x\in D_0(n),$$
$$D_0(n)=\{x=(x_1,x_2), |x_1-M(\beta)|<2^{-n/2}\alpha^{n/2}, x_2^2<c^{-n}\alpha^{n/2}\}, \tag{6.3}$$

$$\frac{F_n(x)}{F_n(\mathbf{M})}\leqq\exp\{\bar{g}c^n(y-M(\beta))+Lp^n(y-M(\beta))^2+O(\xi^n)\} \quad \text{for} \quad x\in R^2,$$
$$y\in R^1, \quad |x|<y, \quad y>\tilde{R}_n \quad \text{and} \quad \mathbf{M}=(M(\beta),0). \tag{6.4}$$

For $|T-M(\beta)|<\sqrt{\bar{g}}\cdot 2^{-n/2}\alpha^n$, $|x|>T$, $x\in R^2$,

$$\frac{F_n(x)}{F_n(\mathbf{M})}\leqq\exp\{c^n\bar{g}(x_1-M)+\bar{A}c^nx_2^2+O(\xi^n)\} \quad \text{if} \quad \bar{g}(x_1-T)+\bar{A}x_2^2\geqq-\frac{\alpha^n}{c^n}\bar{g}, \tag{6.5}$$

and

$$\frac{F_n(x)}{F_n(\mathbf{M})}\leqq\exp\{c^n\bar{g}(T-M)-\alpha^n+O(\xi^n)\} \quad \text{if} \quad \bar{g}(x_1-T)+\bar{A}x_2^2\leqq-\frac{\alpha^n}{c^n}\bar{g}. \tag{6.5'}$$

To prove Theorem 2 still we need a theorem to be formulated below. Let us fix some integer $k\geqq 0$, and define the transformation $Q_n=Q_n(k):(R^2)^{2^{n+k}}\to(R^2)^{2^k}$:

$$Q_n(x^{(1)},\ldots,x^{(2^{n+k})})=(y^{(1)},\ldots,y^{(2^k)}), \quad y_j=\frac{1}{2^n}\sum_{l=(j-1)2^n+1}^{j\cdot 2^n}x^{(l)},$$
$$j=1,\ldots,2^k; \quad x^{(p)}\in R^2, \quad p=1,\ldots,2^{n+k}.$$

Given a probability measure ν over $(R^2)^{2^{n+k}}$, we denote by $Q_n\nu$ the measure over $(R^2)^{2^k}$ induced by the above transformation Q_n; i.e. $Q_n\nu$ is the distribution of the random vector $Q_n(\eta(1),\ldots,\eta(2^{n+k}))$, where $(\eta(1),\ldots,(2^{n+k}))$ is ν-distributed. Now we formulate the following

Theorem C. *The measure $Q_n\mu_{n+k}=Q_n\mu_{n+k}(d\sigma|\beta,\nu)$ has a density function of the form*

$$L_{n,k}^{\beta}\exp\{-\beta H_k(c^{n/2}x^{(1)},\ldots,c^{n/2}x^{(2^k)})\}$$
$$\cdot\prod_{j=1}^{2^k}p_n(|x^{(j)}|,\beta), \quad x^{(j)}\in R^2, \quad j=1,\ldots,2^k,$$

where H_k is defined in (1.1)', p_n is defined at the beginning of Sect. 2 and $L_{k,n}^{\beta}$ is an appropriate norming constant.

Theorem C follows from Theorem 1 of [2].

It follows easily from (6.2) and the definition of the transformation Q_n that

$$\frac{dQ_n\bar{\mu}_{V_{n+k}}}{dQ_n\mu_{n+k}}(x^{(1)},\dots,x^{(2^k)})=F_{n+k}\left(\frac{1}{2^k}\sum_{j=1}^{2^k}x^{(j)}\right).$$

This relation together with Theorem C imply that the measure $Q_n\bar{\mu}_{V_{n+k}}$ has the density function

$$G_{k,n}(x^{(1)},\dots,x^{(2^k)})$$
$$=L^{\beta}_{k,n}F_{n+k}\left(\frac{1}{2^k}\sum_{j=1}^{2^k}x^{(j)}\right)\exp\{-\beta H_k(c^{n/2}x^{(1)},\dots,c^{n/2}x^{(2^k)})\}\prod_{j=1}^{2^k}p_n(|x^{(j)}|,\beta)\,. \tag{6.6}$$

First we give a good asymptotic formula for $G_{k,n}$ in the case when $x^{(j)}\in D_0(n)$, $j=1,\dots,2^k$. [The set $D_0(n)$ is defined in (6.3).]

We claim that for $x\in D_0(n)$,

$$p_n(|x|)=C_n\exp\left\{-\frac{a_0\beta}{2}c^n(x_1^2+x_2^2)-2^{n-1}q(x_1-M)^2+O(\xi^n)\right\}, \tag{6.7}$$

where $M=M(\beta)$, and $q=\Phi''(M)$. Indeed, we get from (2.4) with $x=M(\beta)$ that

$$p_n(|x|)=(1+O(\xi^n))p_n(M)$$
$$\cdot\exp\left\{-a_0\beta c^nM(|x|-M)-2^{n-1}q(|x|-M)^2-\frac{a_0\beta}{2}c^n(|x|-M)^2\right\},$$

and the expression in the exponent can be written as $\frac{a_0\beta}{2}c^nM^2-\frac{a_0\beta}{2}c^n|x|^2$ $-2^{n-1}q(|x|-M)^2$, and since $|x|-M\sim x_1+\frac{x_2^2}{2x_1}-M=x_1-M+O(c^{-n}\alpha^{n/2})$ for $x\in D_0(n)$, these relations imply (6.7).

By (6.3), (6.6) and (6.7)

$$G_n(x^{(1)},\dots,x^{(2^k)})=L'_{k,n}\{\exp P_{k,n}(x^{(1)},\dots,x^{(2^k)})+O(\xi^n)\}, \tag{6.8}$$

with

$$P_{k,n}(x^{(1)},\dots,x^{(2^k)})$$
$$=c^{k+n}\bar{g}M\frac{1}{2^k}\sum_{j=1}^{2^k}(x_1^{(j)}-M)+c^{k+n}\bar{A}\frac{1}{2^{2k}}\left(\sum_{j=1}^{2^k}x^{(2)j}\right)^2$$
$$+\beta c^n\sum_{i=1}^{2^k}\sum_{j=1}^{i-1}d^{-a}(i,j)\,(x_1^{(i)}x_1^{(j)}+x_2^{(i)}x_2^{(j)})$$
$$-\sum_{j=1}^{2^k}\frac{a_0\beta}{2}c^n(x_1^{(j)2}+x_2^{(j)2})+2^{n-1}q(x_1^{(j)}-M)^2\,, \tag{6.8'}$$

if $x_j\in D_0(n), j=1,\dots,2^k$. We rewrite the quadratic polynomial $P_{k,n}$ as a polynomial of $x_1^{(j)}-M$ and $x_2^{(j)}$. To this end let us first observe that for all $i=1,\dots,2^k$,

$$\sum_{j=1}^{2^k}d^{-a}(i,j)=\sum_{j=1}^{2^k}d^{-a}(1,j)=\sum_{j=0}^{k-1}2^{j(1-a)}=\frac{1-\left(\frac{c}{2}\right)^k}{1-\frac{c}{2}}.$$

Then a simple calculation shows that

$$\left.\frac{\partial P(x^{(1)},\dots,x^{(2^k)})}{\partial x_1^{(j)}}\right|_{(\mathbf{M},\dots,\mathbf{M})}=0 \quad \text{for all} \quad j=1,\dots,2^k,$$

where $\mathbf{M}=(M,0)$. This means that the linear terms $x_1^{(j)}-M$ disappear when we rewrite $P_{k,n}$ as a polynomial of $x_1^{(j)}-M$ and $x_2^{(j)}$. Moreover, since $x_1-M=O(2^{-n/2}\alpha^{n/2})$ if $x\in D_0(n)$, the terms $\beta c^n d(i,j)\,(x_1^{(i)}-M)\,(x_1^{(j)}-M)$ and $\frac{a_0\beta}{2}c^n(x_1^{(j)}-M)^2$ are negligible in $P_{k,n}$, and we get

$$\begin{aligned}&P_{k,n}(x^{(1)},\dots,x^{(2^k)})\\&=c^{n+k}\bar{A}\frac{1}{2^{2k}}\left(\sum_{j=1}^{2^k}x_2^{(j)}\right)^2+\beta c^n\sum_{i=1}^{2^k}\sum_{j=1}^{i-1}d^{-a}(i,j)x_2^{(i)}x_2^{(j)}\\&-\sum_{j=1}^{2^k}\left(\frac{a_0\beta}{2}c^n x_2^{(j)2}+2^{n-1}q(x_1^{(j)}-M)^2\right)+O(\xi^n),\\&\text{if}\quad x_j\in D_0(n),\quad j=1,\dots,2^k.\end{aligned}\tag{6.9}$$

A calculation similar to that in Theorem 1 shows with the help of relations (6.4), (6.5), and (6.5') that

$$\int_{x\in R^2\setminus D_0(n)}F_{0,n}(x)dx=O(\exp(-\xi^n))\int_{x\in D_0(n)}F_{0,n}(x)dx.$$

This means that the measure $Q_n\bar{\mu}_{V_{n+k}}$ is essentially concentrated in the set

$$\{x=(x^{(1)},\dots,x^{(2^k)}),x^{(j)}\in D_2(n),j=1,\dots,2^k\}.$$

Hence, after an appropriate rescaling, formulas (6.8) and (6.9) imply Theorem 2.

Proof of the Corollary. It follows from the symmetry properties of the model that $E\sigma^{(2)}(i)\sigma^{(2)}(j)$ depends on i and j only through $d(i,j)$. Hence, if $d(i,j)=n$, then

$$E\sigma^{(2)}(i)\sigma^{(2)}(j)=c^{-n}ET^{(n)}\sigma^{(2)}(1)T^{(n)}\sigma^{(2)}(2).$$

On the other hand

$$\lim_{n\to\infty}ET^{(n)}\sigma^{(2)}(1)T^{(n)}\sigma^{(2)}(2)=E\bar{\sigma}^{(2)}(1)\bar{\sigma}^{(2)}(2)=\frac{(2-c)^2}{2\beta(c-1)(4-c)},$$

where $\bar{\sigma}(i)$ is the same as in Theorem 2. These relations imply the Corollary.

7. The Proof of Lemma 3

In this section we prove Lemma 3 together with some formulas which will be useful in the proof of Lemma 4. By Theorem B,

$$f_n(x)=T_{n+1}f_{n+1}(x)=\int\exp(\beta c^n(x;t))f_{n+1}\left(\frac{x+t}{2}\right)p_n(t)dt.\tag{7.1}$$

Put

$$s_2=s_2(x)=\frac{\beta c^n+\frac{A_{n+1}}{2}}{\frac{g_{n+1}}{2M}-\frac{A_{n+1}}{2}+\beta c^n}x_2\,, \qquad s_1=s_1(x)=M-\frac{s_2^2}{2M}\,,$$

where $M=M(N,h)$. We have chosen the point $s=(s_1,s_2)$ as a good approximation of the maximum point in the integrand in (7.1). Put

$$D_3(x)=\left\{t=(t_1,t_2);\,|t_1-s_1|<(\sqrt{2}-1)2^{-n/2}\alpha^{n/2},\,|t_2-s_2|\leqq 2\frac{\sqrt{c}-1}{c}c^{-n/2}\alpha^{n/4}\right\},$$

$$I_1=\int_{t\in D_3(x)}\exp(\beta c^n(x;t))f_{n+1}\left(\frac{x+t}{2}\right)p_n(t)dt\,,$$

and

$$I_2=\int_{t\in R^2\setminus D_3(x)}\exp(\beta c^n(x;t))f_{n+1}\left(\frac{x+t}{2}\right)p_n(t)dt\,.$$

We shall estimate I_1 with the help of Property $A(n+1)$ and I_2 with the help of Property $B(n+1)$. We claim that if $x\in D(n)$, then

$$I_1=L_n\exp\{g_n(x_1-M)+A_nx_2^2+O(\xi^n)\}\,, \tag{7.2}$$

with

$$L_n=(2^{n-1}\omega_nq_n)^{-1/2}\pi f_{n+1}(\mathbf{M})p_n(M)\exp(\beta c^nM^2)\,, \tag{7.3}$$

where

$$M=M(N,h)\,,\qquad q_n=\Phi_n''(M)\,,$$

$$\mathbf{M}=(M,0)\,,\qquad \omega_n=\frac{g_n}{2M}-\frac{A_{n+1}}{4}\,,$$

and

$$I_2=\exp(-O(\xi^{-n}))I_1\,. \tag{7.4}$$

Formulas (7.2), (7.3), and (7.4) imply Lemma 3 together with the relation

$$f_n(\mathbf{M})=L_n=(2^{n-1}m_n\omega_n)^{-1/2}\pi f_{n+1}(\mathbf{M})p_n(M)\exp(\beta c^nM^2)\,. \tag{7.5}$$

To prove (7.2) let us first observe that if $t\in D_3(x)$ and $x\in D(n)$, then $\frac{x+t}{2}\in D(n+1)$. Indeed, in this case

$$\begin{aligned}\left|\frac{x_1+t_1}{2}-M\right|&\leqq\frac{1}{2}(|x_1-M|+|s_1-M|+|t_1-s_1|)\\&\leqq\frac{1}{2}\left(2^{-n/2}\alpha^{n/2}+\frac{s_2^2}{2M}+(\sqrt{2}-1)2^{-n/2}\alpha^{n/2}\right)\\&=2^{-\frac{n+1}{2}}\alpha^{n/2}+\frac{1}{2}\frac{s_2^2}{4M}\leqq 2^{-\frac{n+1}{2}}\alpha^{\frac{n+1}{2}}\,,\end{aligned}$$

since $s_2^2 \leqq x_2^2 \leqq c^{-n}\alpha^{n/2}$, and

$$\left|\frac{x_2+t_2}{2}\right| \leqq \frac{1}{2}\left(1+\frac{s_2}{x_2}\right)|x_2| + \frac{\sqrt{c}-1}{c} c^{-n/2}\alpha^{n/4} \leqq c^{-\frac{n+1}{2}}\alpha^{n/4} \leqq c^{-\frac{n+1}{2}}\alpha^{\frac{n+1}{4}},$$

since by Lemma 1 and the monotonicity properties of the function $\frac{(ax+b)}{(cx+d)}$,

$$1+\frac{s_2}{x_2} \leqq 1+\frac{\beta+\frac{c}{2}\bar{A}}{\frac{\bar{g}c}{2}-\frac{\bar{A}c}{2}+\beta} = \frac{2}{c}.$$

Therefore, we can estimate I_1 by means of Property $A(n+1)$ and Theorem A. (We put slightly more general estimates which we will need in the proof of Lemma 4). We get with the substitution $\tau = \tau(x) = t' - s$ that

$$I_1 = (1+O(\xi^n))L'_n \exp(I_{1,1}(s)) \cdot \int_{\tau \in D_3(x)-s} \exp(I_{1,2}(\tau))d\tau, \tag{7.6}$$

with

$$I_{1,1}(s) = \beta c^n(x;s) - g_{n+1}\left(\frac{x_1+s_1}{2}-M\right) + A_{n+1}\left(\frac{x_2+s_2}{2}\right)^2,$$

$$I_{1,2}(\tau) = \beta c^n(x;\tau) + g_{n+1}\frac{\tau_1}{2} + A_{n+1}\frac{\tau_2^2}{4} + \frac{A_{n+1}}{2}\tau_2(x_2+s_2)$$
$$-(\log p_n(|s+\tau|) - \log p_n(M)),$$

$$L'_n = f_{n+1}(\mathbf{M})p_n(M). \tag{7.7}$$

Now we give a good asymptotic formula for $I_{1,1}(s)$ and $I_{1,2}(\tau)$. We recall that

$$|x_1-M| < 2^{-n/2}\alpha^{n/2}, \quad x_2^2 < c^{-n}\alpha^{n/2}, \quad s_2^2 \leqq c^{-n}\alpha^{n/2},$$
$$|s_1-M| = \frac{s_2^2}{2M} \leqq \frac{c^{-n}\alpha^{n/2}}{M}, \quad |\tau_1| \leqq 2\cdot 2^{-n/2}\alpha^{n/2}, \quad |\tau_2| \leqq 2c^{-n/2}\alpha^{n/2}$$

and $||s|-M| \leqq \frac{s_2^4}{8M}$ if $x \in D(n)$ and $t \in D_3(x)$. We have

$$I_{1,1}(s) = \beta c^n\left(x_1\left(M-\frac{s_2^2}{2M}\right)+x_2s_2\right) + g_{n+1}\left(\frac{x_1+M}{2}-\frac{s_2^2}{4M}-M\right)$$
$$+A_{n+1}\left(\frac{x_2+s_2}{2}\right)^2 = x_1\left(\beta c^n M + \frac{g_{n+1}}{2}\right) - \frac{g_{n+1}}{2}M$$
$$+x_2^2\left(\left(-\frac{\beta c^n}{2}-\frac{g_{n+1}}{4M}\right)\left(\frac{s_2}{x_2}\right)^2 + \beta c^n\frac{x_2}{s_2}+\right.$$
$$\left.+\frac{A_{n+1}}{4}\left(1+\frac{s_2}{x_2}\right)^2\right) + \beta c^n\frac{M-x_1}{2M}s_2^2.$$

The coefficient of x_1 in the last expression is $\beta c^n M + \frac{g_{n+1}}{2} = g_n$, and that of x_2^2 is

$$-\frac{1}{2}\left(\frac{g_{n+1}}{2M} - \frac{A_{n+1}}{2} + \beta c^n\right)\left(\frac{s_2}{x_2}\right)^2 + \left(\beta c^n + \frac{A_{n+1}}{2}\right)\frac{s_2}{x_2} + \frac{A_{n+1}}{4}$$

$$= \frac{1}{2}\left(\beta c^n + \frac{A_{n+1}}{2}\right)\frac{s_2}{x_2} + \frac{A_{n+1}}{4} = \frac{\beta^2 c^{2n} + A_{n+1}\left(\frac{g_{n+1}}{4M} + \frac{3}{2}\beta c^n\right)}{\frac{g_{n+1}}{M} - A_{n+1} + 2\beta c^n} = A_n\,.$$

Since $\beta c^n \frac{M-x_1}{2M} s_2^2 = O(\xi^n)$ if $x_2^2 = O(c^{-n}\alpha^n)$ and $|x_1 - M| \leqq 2d_n$ [the quantity d_n was defined in Property B(n)], hence

$$I_{1,1}(s) = g_n(x_1 - M) + A_n x_2^2 + \beta c^n M^2 + O(\xi^n \quad \text{if} \quad x \in D(n)\,, \tag{7.8}$$

or, more generally, if $x_2^2 = O(c^{-n}\alpha^n)$ and $|x_1 - M| = O(d_n)$. Now we turn to the estimation of $I_{1,2}(\tau)$. We estimate $\log p_n(|s+\tau|) - \log p_n(M)$ by mens of (4.11). We have

$$\begin{aligned}|s+\tau| - M &= M\left\{\left[\left(\frac{1}{M^2}(2s_1\tau_1 + 2s_2\tau_2 + \tau_1^2 + \tau_2^2 + (|s|^2 - M^2)\right) + 1\right]^{1/2} - 1\right\}\\
&= \frac{1}{2M}\left(2s_1\tau_1 + 2s_2\tau_2 + \tau_1^2 + \tau_2^2 + \frac{s_2^4}{4M}\right)\\
&\quad - \frac{1}{8M^3}\left(2s_2\tau_1 + 2s_2\tau_2 + \tau_1^2\tau_2^2 + \frac{s_1^4}{4M^2}\right)^2\\
&\quad + O(\tau_1^3 + s_2^3\tau_2^3 + \tau_2^6 + s_2^{12})\,.\end{aligned}$$

Hence (4.11) implies that

$$\begin{aligned}&\log p_n(|s+\tau|) - \log p_n(M)\\
&\quad = -g_n\left(2s_1\tau_1 + 2s_2\tau_2 + \tau_1^2 + \tau_2^2 - \frac{s_1^2\tau_1^2}{2M^3}\right)\\
&\quad\quad - 2^{n-1} q_n \frac{s_1^2}{M^2}\tau_1^2 + 2^n O(\tau_1^3 + \tau_2^4 + s_2^4 + \tau_1\tau_2^2 + \tau_1\tau_2 s_2)\,.\end{aligned}$$

Therefore

$$\begin{aligned}I_{1,2}(\tau) &= \left(\beta c^n x_1 + \frac{g_{n+1}}{2} - g_n\frac{s_1}{M}\right)\tau_1 + \left(\beta c^n x_2 + \frac{A_{n+1}}{2}(s_2 + x_2) - \frac{g_n s_2}{M}\right)\tau_2\\
&\quad - \left(g_n\left(\frac{1}{2M} - \frac{s_1^2}{2M^3}\right) + 2^{n-1} q_n \frac{s_1^2}{M^2}\right)\tau_1^2 - \left(\frac{g_n}{2M} - \frac{A_{n+1}}{4}\right)\tau_2^2\\
&\quad + 2^n O(\tau_1^3 + \tau_2^4 + s_2^4 + \tau_1\tau_2^2 + \tau_1\tau_2 s_2)\,.\end{aligned} \tag{7.9}$$

Let us consider the coefficients of τ_1, τ_2, and τ_1^2 in (7.9).

$$\left(\beta c^n x_1 + \frac{g_{n+1}}{2} - \frac{s_1}{M} g_n\right)\tau_1$$
$$= \left(\beta c^n(x_1 - M) + g_n\left(1 - \frac{s_1}{M}\right)\right)\tau_1 = O(c^n(x_1-M)\tau_1 + 2^n x_2^2 \tau_1)\,, \qquad (7.10)$$

$$\beta c^n x_2 + \frac{A_{n+1}}{2}(x_2+s_2) - \frac{s_2}{M} g_n$$
$$= \left(\frac{g_{n+1}}{2M} - \frac{A_{n+1}}{2} + \beta c^n\right)s_2 + \left(\frac{A_{n+1}}{2} - \frac{g_n}{M}\right)s_2 = 0\,, \qquad (7.10)$$

and

$$g_n\left(\frac{1}{2M} - \frac{s_1^2}{2M^3}\right) + 2^{n-1} q_n\left(\frac{s_1^2}{M^2}\right)\tau_1^2$$
$$= 2^{n-1} q_n \tau_1^2 + \left(\frac{s_2^2}{2M^3} g_n - 2^{n+1} q_n \frac{s_2^2}{M^2} - \frac{s_2^4}{8M^5} g_n + 2^{n-3} q_n \frac{s_2^4}{M^4}\right)$$
$$= 2^{n-1} q_n \tau_1^2 + O(2^n x_2^2 \tau_1^2)\,. \qquad (7.10'')$$

The above estimates imply that

$$I_{1,2}(\tau) = -2^{n-1} q_n \tau_1^2 - \left(\frac{g_n}{2M} - \frac{A_{n+1}}{4}\right)\tau_2^2 + O(\xi^n) \qquad (7.11)$$

if $x \in D(n)$ and $\tau \in D_3(x)$, or more generally if $x_2^2 = O(c^{-n}\alpha^n)$, $\tau_2^2 = O(c^{-n}\alpha^n)$, $\tau_1 = O(2^{-n/2}\alpha^n)$ and $|x_1 - M| \leqq 2d_n$. Observe that the coefficients of τ_1^2 and τ_2^2 in (7.11) are positive. We get, by integrating (7.11), that

$$\int_{\tau \in D_3(x)-s} \exp I_{1,2}(\tau)d\tau = (1 + O(\xi^n))\,(2^{n-1} q_n \omega_n)^{-1/2}\,. \qquad (7.12)$$

Relations (7.2) and (7.3) follow from (7.6), (7.7), (7.8), and (7.12).

Now we turn to the proof of (7.4). We need a good upper bound on

$$\mathscr{I}_n(x,t) = \exp\{\beta c^n(x;t)\} f_{n+1}\left(\frac{x+t}{2}\right) p_n(t)\,.$$

Given an $x \in R^2$, $||x| - M| \leqq d_n$, we define the sets

$$F_1(x) = \left\{t = (t_1, t_2);\ g_{n+1}\left(\frac{x_1+t_1}{2} - \frac{|x|+|t|}{2}\right)\right.$$
$$\left. + A_{n+1}\left(\frac{x_2+t_2}{2}\right)^2 > -\frac{g_{n+1}}{c^{n+1}}\alpha^{n+1},\ ||t| - M| \leqq (\sqrt{2}-1)d_n\right\},$$

and

$$D_4(n) = \{t \in R^2,\ ||t| - M| \leqq (\sqrt{2}-1)d_n\}\,.$$

First we estimate $\mathcal{J}_n(x,t)$ in the case $||x|-M|\leqq d_n$ and $t\in F_1(x)$. In this case $f_{n+1}\left(\frac{x+t}{2}\right)$ can be estimated by means of part b1) of Property $B(n+1)$ with the choice $T=\frac{|t|+|x|}{2}$. $\Big[$Observe that $|T-M|\leqq d_{n+1}$, because $|T-M|\leqq\frac{1}{2}[(|t|-M|+(|x|-M)]\leqq\frac{1}{\sqrt{2}}d_n\leqq d_{n+1}$ by Lemma 1.$\Big]$ We get that

$$\mathcal{J}_n(x,t)\leqq p_n(M)f_{n+1}(\mathbf{M})\cdot\exp\{I_{1,1}(s)+I_{1,2}(\tau)+O(\xi^n)\}\quad\text{if}\quad t\in F_1(x). \tag{7.13}$$

In order to estimate the expression in (7.13), we prove that

$$x_2^2=O(c^{-n}\alpha^n),\qquad |x|-x_1=O(c^{-n}\alpha^n)\quad\text{if}\quad ||x|-M|\leqq d_n,\quad\text{and}\quad F_1(x)\quad\text{is non-empty}. \tag{7.14}$$

$$t_2^2=O(c^{-n}\alpha^n),\qquad |t|-t_1=O(c^{-n}\alpha^n)\quad\text{if}\quad t\in F_1(x). \tag{7.14'}$$

Indeed, if there exists some $u\in R^2$, $T\in R^1$, $|u|\leqq T\leqq M+d_{n+1}$ such that

$$g_{n+1}(u_1-T)+A_{n+1}u_2^2\geqq-\frac{g_{n+1}}{c^{n+1}}\alpha^{n+1},$$

then

$$\frac{g_{n+1}}{c^{n+1}}\alpha^{n+1}\geqq g_{n+1}(T-u_1)-A_{n+1}u_2^2\geqq(g_{n+1}-2TA_{n+1})(T-u_1).$$

We have by Lemma 1, $g_{n+1}-2TA_{n+1}>Kg_{n+1}$ with some $K>0$. Hence $T-u_1=O(c^{-n}\alpha^n)$. By applying this result to the case $u=\frac{t+x}{2}$, $t\in F_1(x)$, $T=\frac{|t|+|x|}{2}$, we get that $x_1-t_1-(|x|+|t|)=O(c^{-n}\alpha^n)$. Hence $|x|-x_1=O(c^{-n}\alpha^n)$, and $|t|-t_1=O(c^{-n}\alpha^n)$. Then the inequalities $x_2^2\leqq 2|x|(|x|-|x_1|)$, $t_2^2\leqq 2|t|(|t|-t_1)$ imply (7.14) and (7.14').

We claim that

$$I_{1,2}(\tau)=-2^nB_1\tau_1^2-c^nB_2\tau_2^2+O(\xi^n), \tag{7.15}$$

with some $B_1>0$, and $B_2>0$ if $x_2^2=O(c^{-n}\alpha^n)$, $t_2^2=O(c^{-n}\alpha^n)$, $|t_1-M|<2d_n$, $|x_1-M|<2d_n$.

Indeed, we can verify (7.15) similarly to (7.11), only we have to estimate $\log p_n(|s+t|)-\log p_n(M)$ by formula (4.12) instead of (4.11). We get a relation similar to (7.9), only we have to write inequality instead of equaity, and q_n must be substituted by B in the coefficient of τ_1^2. Then it remains to bound the error terms in (7.9), (7.10), and (7.12). The term $2^n\tau_1\tau_2s_1$, e.g. can be estimated in the following way: Since $\tau_2=O(c^{-n/2}\alpha^{n/2})$ and $\tau_1=O(d_n)=O(c^{-n/2}\alpha^n)$, $(\tau_j=t_j-s_j(x))$; hence $2^n\tau_1\tau_2s_2\leqq 2^n\tau_1^2\alpha^{-n}+2^n\alpha^n\tau_2^2s_2^2=(2^n\tau_1^2+1)\cdot O(\xi^n)$.

By relations (7.14) and (7.14') the relations (7.9) and (7.15) hold if $||x|-M|<d_n$ and $t\in F_1(x)$. [Observe that in this case $|x_1-M|\leqq|x_1-|x||+||x|-M|=O(d_n)$ by (7.14) and $|t_1-M|=O(d_n)$ similarly.] Hence by (7.13)

$$\mathcal{J}_n(x,t)\leqq p_n(M)f_{n+1}(\mathbf{M})\cdot\exp\{\beta c^nM^2+g_n(x_1-M)+A_nx_2^2-2^nB_1\tau_1^2-c^nB_2\tau_2^2\}, \tag{7.16}$$

if $||x|-M|<d_n$, and $t\in F_1(x)$. Moreover by (7.11),

$$\mathscr{I}_n(x,t)=p_n(M)f_{n+1}(\mathbf{M})\cdot\exp\{\beta c^n M^2+g_n(x_1-M)+A_n x_2^2-2^{n-1}q_n\tau_1^2-\omega_n\tau_2^2+O(\xi^n)\}\,, \quad (7.16')$$

if $||x|-M|<d_n$, $t\in F_1(x)$ and $\tau_1=O(2^{-n/2}\alpha^n)$. Put $F_2(x)=D_4(n)-F_1(x)$.

Now we estimate $\mathscr{I}_n(x,t)$ in the case $||x|-M|\leqq d_n$, $t\in F_2(x)$. In this case the function $f_{n+1}\left(\frac{x+t}{2}\right)$ can be estimated by part b2) of Property $B(n+1)$ and $p_n(t)$ by (4.12). Moreover by using the inequality $(x;t)\leqq|x|\cdot|t|$, we get that

$$\mathscr{I}_n(x,t)\leqq p_n(M)f_{n+1}(\mathbf{M})\exp\{G_n(x,t)\}\,,$$

with

$$G_n(x,t)=\beta c^n|x|\cdot|t|+g_{n+1}\left(\frac{|x|+|t|}{2}-M\right)-g_n(|t|-M)-2^nB(|t|-M)^2-\frac{g_{n+1}}{c^{n+1}}\alpha^{n+1}\,.$$

Clearly

$$G_n(x,t)=\beta c^n M^2+g_n(|x|-M)+\beta c^n(|x|-M)(|t|-M)-2^nB(|t|-M)^2-\frac{g_{n+1}}{c^{n+1}}\alpha^{n+1}\,.$$

We have

$$\beta c^n(|x|-M)(|t|-M)=O\left(\frac{g_n}{2^n}\alpha^{2n}\right)=O\left(\xi^n\frac{g_n}{c^n}\alpha^n\right)\quad\text{if}\quad t\in D_4(n)\,,$$

and $||x|-M|\leqq d_n$. On the other hand by Lemma 1, $\frac{g_{n+1}}{c^{n+1}}\alpha^{n+1}\geqq\frac{g_n}{c^n}\alpha^{n+1}=\frac{g_n}{c^n}\alpha^n(1+2K)$ with $2K=\alpha-1>0$. Hence

$$\mathscr{I}_n(x,t)\leqq L_n\exp\left\{g_n(|x|-M)-\frac{g_n}{c^n}\alpha^n-K\frac{g_n}{c^n}\alpha^n\right\}, \quad (7.17)$$

with some $K>0$ if $||x|-M|\leqq d_n$ and $t\in F_2(x)$. [The constant L_n is defined in (7.3).] If $x_2^2=O(c^{-n}\alpha^{n/2})$, and this relation holds if $x\in D(n)$, then

$$g_n(|x|-M)=g_n(x_1-M)+O(g_nx_2^2)=g_n(x_1-M)+O\left(\frac{g_n}{c^n}\alpha^{n/2}\right).$$

Hence

$$\mathscr{I}_n(x,t)\leqq L_n\exp\{g_n(x_1-M)+A_nx_2^2-K\alpha^n\}\,, \quad (7.18)$$

with some $K>0$ if $x\in D(n)$ and $t\in F_2(x)$. Since $2^nB_1\tau_1^2+c^nB_2\tau_2^2\geqq K\alpha^{n/2}$ if $t\notin D_3(x)$, hence (7.16) and (7.18) imply that

$$\int\limits_{t\in D_4(n)-D_3(x)}\mathscr{I}_n(x,t)dt\leqq L_n\exp\{g_n(x_1-M)+A_nx_2^2-O(\xi^{-n})\}\,, \quad (7.19)$$

if $x\in D(n)$.

Now we turn to the estimation of $\mathscr{I}_n(x,t)$ in the case $||x|-M|\leqq d_n$, $t\in R^2-D_4(n)$. In this case we get, by estimating $f_{n+1}\left(\frac{x+t}{2}\right)$ by means of part a) of Property $B(n+1)$ with $y=\frac{|x|+\max(|t|,R_n)}{2}$, and by using (4.12) with (4.12'), that

$$\mathscr{I}_n(x,t)\leqq(1+O(\xi^n))p_n(M)f_{n+1}(\mathbf{M})\exp\bar{G}_n(|x|,u(t)),$$

with

$$\begin{aligned}\bar{G}_n(v,z)&=\beta c^n vz-g_n(z-M)-2^nB(z-M)^2+g_{n+1}\left(\frac{z+v}{2}-M\right)\\&+Lp^{n+1}\left(\frac{z+v}{2}-M\right)^2=\beta c^nM^2+g_n(v-M)+\beta c^n(v-M)(z-M)\\&+Lp^{n+1}\left(\frac{z+v}{2}-M\right)^2-2^nB(z-M)^2,\end{aligned}\tag{7.20}$$

where $u(t)=\max(|t|,R_n)$.

Clearly, $(u(t)-M)^2\geqq(\sqrt{2}-1)^2d_n^2$, hence

$$\begin{aligned}&-2^nB(u(t)-M)^2+\beta c^n(|x|-M)(u(t)-M)+Lp^{n+1}\left(\frac{u(t)+x}{2}-M\right)^2\\&<-2^{n-1}B(u(t)-M)^2-\frac{g_n}{c^n}\alpha^n.\end{aligned}$$

As a consequence

$$\mathscr{I}_n(x,t)\leqq L_n\exp\left\{g_n(|x|-M)-\frac{g_n}{c^n}\alpha^n-2^{n-2}B\ Y(\max(|t|,R_n)-M)^2-\alpha^n\right\},\tag{7.21}$$

if $||x|-M|\leqq d_n$ and $t\in D_4(n)$.

If $x\in D(n)$, then $x_2^2=O(c^{-n}\alpha^{n/2})$; hence $g_n(|x|-M)=g_n(x_1-M)+O\left(\frac{g_n}{c^n}\alpha^{n/2}\right)$, and

$$\mathscr{I}_n(x,t)\leqq L_n\exp\{g_n(x_1-M)+A_nx_2^2-2^{n-2}B(\max(t,R_n)-M)^2\},\tag{7.22}$$

if $x\in D(n)$, $t\notin D_4(n)$. Relation (7.4) follows from (7.19), (7.22), and (7.2). Lemma 3 is proved.

8. The Proof of Lemma 4

We have to give a good upper bound for the function $f_n(x)$ defined by the integral (7.1). First we prove part b) of Property B(n). We shall use the estimates of the previous section, and we also preserve its notations. We can write $f_n(x)=I_1+I_2+I_3$, with

$$I_1=\int_{t\in F_1(x)}\mathscr{I}_n(x,t)dt,\qquad I_2=\int_{t\in F_2(x)}\mathscr{I}_n(x,t)dt,$$

and

$$I_3=\int_{t\in R^2-D_4(n)}\mathscr{I}_n(x,t)dt.$$

We get, with the help of (7.16), (7.16′), and (7.5) that

$$I_1 = f_n(\mathbf{M}) \exp\{g_n(x_1 - M) + A_n x_2^2 + O(\xi^n)\}, \tag{8.1}$$

if $||\mathrm{x}| - \mathrm{M}| \leqq \mathrm{d}_n$. On the other hand, by (7.17), (7.21), and (7.5)

$$I_2 + I_3 \leqq f_n(\mathbf{M}) \exp\left\{ g_n(|x| - M) - \frac{g_n}{c^n} \alpha^n - \xi^{-n} \right\}. \tag{8.2}$$

If the conditions of part b1) are satisfied then, since $T \geqq |x|$,

$$\begin{aligned} g_n(|x| - M) - \frac{g_n}{c^n} \alpha^n &\leqq g_n(|x| - M) + g_n(x_1 - |x|) + A_n x_2^2 \\ &= g_n(x_1 - M) + A_n x_2^2 . \end{aligned}$$

Hence (8.2) implies that

$$I_2 + I_3 \leqq f_n(\mathbf{M}) \exp\{g_n(x_1 - M) + A_n x_2^2 - \xi^{-n}\},$$

and the last relation together with (8.1) imply Part b1). If the conditions of part b2) of Property B(n) are satisfied, then relation (8.1) has the consequence

$$I_1 \leqq f_n(\mathbf{M}) \exp\left\{ g_n(T - M) - \frac{g_n}{c^n} \alpha^n + O(\xi^n) \right\}.$$

The last relation together with (8.2) imply part b2).

Now we turn to the proof of part a) of Property B(n). Let $|x| \leqq y$, $y \geqq \tilde{R}_n$. Since part b) is already proved we may restrict ourselves to the case $|y - M| \geqq d_n$.

We get, by estimating $f_{n+1}\left(\frac{x+t}{2}\right)$ by means of part a) of Property $B(n+1)$ with $\frac{y+u(t)}{2}$ as upper bound for $\frac{x+t}{2}$, ($u(t) = \max(|t|, R_n)$), and by using (4.12) and (4.12′) for the estimation of $p_n(t)$ that

$$\mathscr{I}_n(x, t) \leqq (1 + O(\xi^n)) p_n(M) f_{n+1}(\mathbf{M}) \exp \bar{G}_n(y, u(t)), \tag{8.3}$$

where the function $\bar{G}_n(v, z)$ is defined in (7.20). We have to bound the integral $\int \mathscr{I}_n(x, t)dt$. By integrating the right-hand side of (8.3) first on the concentric circles $|t| = z$, and then by integrating with respect to z, we get that

$$f_n(x) \leqq (1 + O(\xi^n)) p_n(M) f_{n+1}(\mathbf{M}) \int_0^\infty 2\pi z \exp(\bar{G}_n(y, u(z)))dz . \tag{8.4}$$

Given a fixed y, let $z_0 = z_0(y)$ denote the maximum point of the polynomial $\bar{G}_n(y, z)$. Then

$$\bar{G}_n(y, z) = \bar{G}_n(y, z_0) - \left(2^n B - \frac{L}{4} p^{n+1}\right)(z - z_0)^2;$$

hence

$$\int_0^\infty 2\pi z \exp\{\bar{G}_n(y, u(z))\}dz = O(1)\,(z_0 + 1) \exp \bar{G}_n(y, z_0),$$

and by (8.4)

$$f_n(x) \leqq C p_n(M) f_{n+1}(\mathbf{M})(z_0+1) \exp \bar{G}_n(y, z_0). \tag{8.5}$$

We claim that

$$\bar{G}_n(y, z_0) \leqq \beta c^n M^2 + g_n(y-M) + L p^n \frac{p}{2}(y-M)^2 \tag{8.6}$$

if L and n_0 are chosen sufficiently large. (First we chose L and then the threshold n_0 depends on L in Propositions 1 and 2.)

Indeed, a simple calculation yields that

$$z_0 - M = \frac{\beta c^n + \frac{L}{2} p^{n+1}}{2^{n+1} B - \frac{L}{2} p^{n+1}} (y-M).$$

Hence

$$|z_0 - M| \leqq \frac{L}{B} \left(\frac{p}{2}\right)^n |y-M|,$$

$$\beta c^n (y-M)(z_0-M) \leqq \beta c^n \cdot \left(\frac{p}{2}\right)^n \cdot \frac{L}{B}(y-M)^2 \leqq L \cdot p^n \frac{p}{6}(y-M)^2,$$

$$L p^{n+1} \left(\frac{z_0+y}{2} - M\right)^2 \leqq L \frac{p}{3} p^n (y-M)^2,$$

and

$$\bar{G}_n(y, z_0) = \beta c^n M^2 + g_n(y-M) + \beta c^n (y-M)(z_0-M) + L p^{n+1} \left(\frac{z_0+y}{2} - M\right)^2$$

$$- 2^n B (z_0 - M)^2 \leqq \beta c^n M^2 + g_n(y-M) + L p \frac{p}{2}(y-M)^2.$$

Since $p^n(y-M)^2 \geqq \alpha^n$, $\alpha > 1$, if $|y-M| \geqq d_n$ and $p<2$, hence relations (8.5), (8.6), and (7.5) imply that part a) of Property B(n) holds true. Lemma 4 is proved.

Appendix: The Proof of Theorem A

Theorem A is the multi-dimensional generalization of the result in [3]. The proof goes on the same line with slight modifications. We outline the proof briefly. Our main goal is to explain the necessary modifications. For the sake of simpler notations we restrict ourselves to the case $m=2$.

The following recursive relation holds for Z_n:

$$Z_n(s, \mu) = 4 \int \exp\{2^{n(2-\alpha)}[(s; s) - (v; v)]\} Z_n(s-v, \mu) Z_n(s+v, \mu) dv,$$

$$Z_0(s, \mu) = \exp\left\{ -\frac{u}{4}(s; s)^2 - \frac{\mu}{2}(s; s) \right\}.$$

This is a simple multi-dimensional generalization of formula (3.1) in [3]. Put

$$Z_n(s,\mu)=\exp\left\{-\frac{a_0}{2}\cdot 2^{n(2-a)}(s;s)\right\}S_n(s,\mu),\qquad a_0=\frac{2}{2-c}\cdot \tag{A.1}$$

Then $S_0(s,\mu)=\exp\left\{-\frac{u}{4}(s;s)^2-\frac{\mu-a_0}{2}(s;s)\right\}$ and

$$S_{n+1}(s,\mu)=4\int\exp\{-2^{n(2-a)}a_1(v;v)\}S_n(s-v,\mu)\cdot S_n(s+v,\mu)dv\,,\qquad a_1=a_0+1\,. \tag{A.2}$$

This formula differs from the convolution because of the kernel $\exp\{-2^{n(2-a)}a_1(v;v)\}$. This kernel term however has a very strong influence. We shall see it by considering two special cases. Introduce the notation

$$Q_nf(s)=4\int\exp\{-2^{n(2-a)}a_1(v;v)\}f(s-v)f(s+v)dv\,,\qquad s,v\in R\,.$$

Then $S_{n+1}=Q_n(S_n)$, and we are interested in $Q_nQ_{n-1},\ldots,Q_1(S_0)$. We consider the asymptotic behaviour of the functions $Q_n,\ldots,Q_1(f)$ for two types of functions f.

Type 1. The Gaussian Density

If $f=G(s,\gamma)=\frac{1}{2\pi\gamma}\exp\left(-\frac{|s|^2}{2\gamma}\right)$ is a Gaussian density, then $f_n=Q_n,\ldots,Q_1(f)$ is also Gaussian, $f_n=\text{const}\cdot G(s,\gamma_n)$ with $\gamma_n=2^{-n}\gamma$. This behaviour is stable in the following sense. If f is a small perturbation of the Gaussian density $G(s,\gamma)$, then f_n is asymptotically Gaussian when $n\to\infty$.

Type 2. The "Craters"

We call a function $f(s)$, $s\in R^2$ a "crater" if it is rotation invariant and δ-shaped near some $|s|=m>0$ along the radius. The crater is concentrated near the sphere $|s|=m$, and has a width

$$\delta=\langle(|s|-m)^2\rangle^{1/2}=[\int(|s|-m)^2f(s)ds/\int(s)ds]^{1/2}\,.$$

The Gaussian craters are defined by the formula

$$B(s,m,\chi)=\exp\left(-\frac{(|s|-m)^2}{2\chi}\right).$$

Choose $\chi=\chi_n=2^{-n}\chi_0$, $\frac{m^2}{\chi_0}\gg 1$, and compute $Q_n(B)$ with $B=B(s,m,\chi_n)$. We have for $s=(r,0)$, $r>0$,

$$Q_n(B)(r)=4\iint\exp\left\{-[(\sqrt{(r+v_1)^2+v_2^2}-m)^2+\sqrt{(r-v_1)^2+v_2^2}-m)^2]\frac{1}{2\chi_n}-2^{n(2-a)}a_1(v_1^2+v_2^2)\right\}dv_1dv_2\,. \tag{A.3}$$

This integral is sharply localized around the minima of the function

$$\phi(v_1, v_2) = \frac{1}{2\chi_n}[(\sqrt{(r+v_1)^2+v_2^2}-m)^2+(\sqrt{(r-v_1)^2+v_2^2}-m)^2] + 2^{n(2-a)}(v_1^2+v_2^2).$$

If we substitute the maximum of the integrand, we get with the help of some calculation the approximate equation,

$$\frac{Q_n(B)(r)}{Q_n(B)(m)} \approx \begin{cases} B(r, m, \chi_{n+1}) & \text{if } r \geqq r^* \\ \varepsilon_{n+1}\exp(-2^{n(2-a)}(r^*2-r^2)) & \text{if } r \leqq r^*, \end{cases} \qquad r^* = \frac{m}{1+2^{n(2-a)}\chi_n a_1}, \tag{A.4}$$

with $\chi_n = 2^{-n}\chi_0$, $\varepsilon_{n+1} = B(r^*, m, \chi_{n+1})$. We shall call the function appearing at the right-hand side of (A.4) a "special crater". Estimations similar to the one above show that Q_n maps a special crater to another special crater with a small perturbation.

Thus there are two different types of asymptotic behaviour for the iterations $f_{n+1} = Q_n(f_n)$; the Gaussian and special "crater" behaviour. They are stable with respect to small rotation invariant perturbations. The Gaussian asymptotic behaviour corresponds to the single phase region of the hierarchical model, the crater one to the multiphase region. For large μ the function $S_0(s, \mu) = \exp\left\{-\frac{u}{4}s^4 - \frac{\mu - a_0}{2}s^2\right\}$ is close to the Gaussian density, and so is the function $S_n(s, \mu)$. For small μ the function $S_0(s, \mu)$ is close to the Gaussian "crater" with $m \approx \sqrt{\frac{\tau}{u}}$, $\chi_0 = (2\tau)^{-1}$, where $\tau = -(\mu - a_0)$. In this case $S_n(s, \mu)$ is close to a special "crater." The critical point μ_c separates these two cases.

Now we turn to the rigorous considerations. Put $S_n(s, \mu) = \exp(-2^n\phi_n(s, \mu))$. By (A.2),

$$\phi_{n+1}(s, \mu) = -2^{-(n-1)}\ln\left\{4\int_{R^2}\exp\{-2^{n(2-a)}a_1(v, v) - 2^n\phi_n(s+v, \mu) - 2^n\phi_n(s-v, \mu)\}dv\right\}, \tag{A.5}$$

$$\phi_0(s, \mu) = \frac{u}{4}(s; s)^2 + \frac{\mu - a_0}{2}(s; s).$$

Let us emphasize that the functions p_n and ϕ_n are rotation in variant, i.e. they depend on s only through $|s|$. First we are going to give a local expansion for $\phi_n(s)$ around the point s in the coordinate system whose coordinates are either parallel with or orthogonal to the vector s. Because of the rotational invariance of the function ϕ_n, we can restrict ourselves to the case $s = (r, 0)$, $r = |s|$. Put

$$\phi_n(r+h, k, \mu) = \phi_n^{(0,0)}(r, \mu) + h\phi_n^{(1,0)}(r, \mu) + \frac{h^2}{2}\phi_n^{(2,0)}(r, \mu) + \frac{k^2}{2}\phi_n^{(0,2)}(r, \mu) + \varrho_n(h, k, r, \mu). \tag{A.6}$$

We remark that, because of the symmetry property $\phi_n(r+h, k, \mu)=\phi_n(r+h, -k, \mu)$, in the above formula and also in the subsequent expansions of the function $\phi_n(r+h, k, \mu)$ all polynomials are even in their coordinate k. The following formulas are a straightforward adaptation of those in [3]. It follows from (A.5) and (A.6) that

$$\phi_{n+1}(r+h, k, \mu)=\phi_{n+1}^{(0,0)}(r, \mu)+h\phi_{n+1}^{(1,0)}(r, \mu)+\frac{h^2}{2}\phi_{n+1}^{(2,0)}(r, \mu)$$
$$+\frac{k^2}{2}\phi_{n+1}^{(0,2)}(r, \mu)+\varrho_{n+1}(h, k, r, \mu),$$

with

$$\phi_{n+1}^{(1,0)}(r, \mu)=\phi_n^{(1,0)}(r, \mu), \qquad \phi_{n+1}^{(2,0)}(r, \mu)=\phi_n^{(2,0)}(r, \mu),$$
$$\phi_{n+1}^{(0,2)}(r, \mu)=\phi_n^{(0,2)}(r, \mu),$$
$$\phi_{n+1}^{(0,0)}(r, \mu)=\phi_n^{(0,0)}(r, \mu)+2^{-n-1}\ln\frac{\sqrt{\lambda_n^{(1)}(r, \mu)\lambda_n^{(2)}(r, \mu)}}{\pi}, \tag{A.7}$$

$$\varrho_{n+1}(h, k, r, \mu)=-2^{-n-1}\ln\left\{\frac{\sqrt{\lambda_n^{(1)}(r, \mu)\lambda_n^{(2)}(r, \mu)}}{\pi}\cdot\int_{R^2}\exp\{-\lambda_n^{(1)}(r, \mu)v_1^2\right.$$
$$-\lambda_n^{(2)}(r, \mu)v_2^2-2^n\varrho_n(h+v_1, k+v_2, r, \mu)$$
$$\left.-2^n\varrho_n(h-v_1, k-v_2, r, \mu)\}dv_1dv_2\right\}, \tag{A.8}$$

where

$$\lambda_n^{(1)}(r, \mu)=a_1c^n+\phi_n^{(2,0)}(r, \mu)\cdot 2^n,$$

and

$$\lambda_n^{(2)}(r, \mu)=a_1c^n+\phi_n^{(0,2)}(r, \mu)\cdot 2^n.$$

In linear approximation for ϱ_{n+1}, $\varrho_n\approx\bar{\varrho}_n$, and

$$\bar{\varrho}_{n+1}(h, k, x, \mu)\approx\frac{\sqrt{\lambda_n^{(1)}(x, \mu)\lambda_n^{(2)}(x, \mu)}}{\pi}\int_{R^2}\exp[-\lambda_n^{(1)}(x, \mu)v_1^2$$
$$-\lambda_n^{(2)}(x, \mu)v_2^2]\bar{\varrho}_n(h-v_1, k-v_2, x, \mu)dv_1dv_2. \tag{A.9}$$

For fixed x and μ we have in (A.9) a multi-dimensional Gaussian operator. For a fixed $\sigma=(\sigma^{(1)}, \sigma^{(2)})$, $\sigma^{(1)}>0$, $\sigma^{(2)}>0$ define the Gaussian operator A_σ

$$A_\sigma f(h, k)=\frac{1}{\pi\sqrt{\sigma^{(1)}\sigma^{(2)}}}\int_{R^2}\exp\left\{-\frac{v_1^2}{\sigma^{(1)}}-\frac{v_2^2}{\sigma^{(2)}}\right\}f(h-v_1, k-v_2)dv_1dv_2.$$

For fixed $m=(m_1, m_2)$ and $\gamma=(\gamma_1, \gamma_2)$ define the function $G_m(h, k, \gamma)=G_{m_1}(h, \gamma_1)G_{m_2}(k, \gamma_2)$ with $G_m(x, \gamma)=\left(\frac{\sqrt{\gamma}}{2}\right)^m H_m\left(\frac{x}{\sqrt{\gamma}}\right)$, where $H_m(x)$, $m=0, 1, \ldots$ are the Hermite polynomials with weight function $\exp(-x^2)$ and leading coefficient 1. We have (see e.g. [1])

Proposition A.1. (i) $A_{\sigma_1}A_{\sigma_2}=A_{\sigma_1+\sigma_2}$,
(ii) $A_\sigma G_m(h,k,\gamma)=G_m(h,k,\gamma-\sigma)$.

This proposition motivates the following expansion of the remaining term ϱ_n by Hermite polynomials:

$$\varrho_n(h,k,r,\mu)=\sum_{3\leqq i+j\leqq 4}\Phi_n^{(i,j)}(r,\mu)\frac{G_i(h,\gamma_n^{(1)})}{i!}\frac{G_j(k,\gamma_n^{(2)})}{j!}+R_n(h,k,r,\mu), \tag{A.10}$$

and the term R_n satisfies the following orthogonality relations,

$$\iint \exp(-\gamma_n^{(1)}h^2-\gamma_n^{(2)}k^2)G_i(h,\gamma_n^{(1)})G_j(k,\gamma_n^{(2)})R_n(h,k,r,\mu)dhdk=0 \tag{A.10'}$$

for $i+j\leqq 4$.

By the evenness in k, $\phi_n^{(i,j)}(r,\mu)=0$ when j is odd. The choice of the parameters $\gamma_n^{(1)}$, $\gamma_n^{(2)}$ is motivated by Proposition A.1. Namely by this proposition

$$\gamma_{n+1}^{(i)}=\gamma_n^{(i)}-\frac{1}{\lambda_n^{(i)}(r,\mu)}$$

in the linear approximation (A.9). Thus it is natural to define

$$\gamma_n^{(i)}=\gamma_n^{(i)}(r,\mu)=\sum_{m=n}^{\infty}\frac{1}{\lambda_{n,m}^{(i)}(r,m)},\qquad i=1,2, \tag{A.11}$$

with

$$\lambda_{n,m}^{(1)}(r,\mu)=2^{m(2-a)}a_1+2^m\phi_n^{(2,0)}(r,\mu),$$

and

$$\lambda_{n,m}^{(2)}(r,\mu)=2^{m(2-a)}a_1+2^m\phi_n^{(0,2)}(r,\mu).$$

The definition (A.11) is correct if $\phi_n^{(2,0)}(r,\mu)\geqq 0$, $\phi_n^{(0,2)}(r,\mu)\geqq 0$. The second condition is violated in the narrow strip $\{M_n>r>M_n-\varepsilon_n\}$, where M_n is the minimum point of the function ϕ_n. To overcome this difficulty we introduce a regularization of the quantity $\gamma_n^{(2)}(r,\mu)$ in this strip. Define the functions

$$\lambda_n(t)=2^{n(2-a)}a_1+2^n t, \tag{A.12}$$

$$\gamma_n(t)=\sum_{m=n}^{\infty}\left[s^{m(2-a)}a_1+2^m t\chi\left(\frac{2^{m(a-1)}}{a_1}\right)\right]^{-1}, \tag{A.13}$$

where $\chi(t)\in C^\infty(R^1)$ is an arbitrary function such that $\chi(t)=0$ if $t\leqq -1$, $\chi(t)=1$ if $t\geqq -1+(4a_1)^{-1}$, $0\leqq\chi(t)\leqq 1$ otherwise. We shall write $\gamma_n^{(2)}=\gamma_n(\phi_n^{(0,2)}(r,\mu))$ as the continuation of the relation (A.11) into the domain $\{r<M_n\}$ (cf. [3]).

We need an expansion of the starting function

$$\phi_0(x,y,\mu)=\frac{u}{4}(x^2+y^2)^2+\frac{\mu-a_0}{2}(x^2+y^2)$$

in the form (A.6), (A.10). We shall choose

$$R_0(h,k,r,\mu)\equiv 0,\qquad \gamma_0^{(1)}=\gamma(\phi_0^{(2,0)}(x,\mu)$$

and

$$\gamma_0^{(2)} = \gamma(\phi_0^{(0,2)}(x,\mu)).$$

Then the coefficients $\phi_0^{(i,j)}(x,\mu)$ satisfy the equations

$$\phi_0^{(4,0)}(x,\mu) = 6u, \quad \phi_0^{(0,4)}(s,\mu) = 6u, \quad \phi_0^{(2,2)}(x,\mu) = 2u, \tag{A.14}$$

$$\phi_0^{(3,0)}(x,\mu) = 6ux, \quad \phi_0^{(1,2)}(x,\mu) = 2ux, \tag{A.15}$$

$$\phi_0^{(2,0)}(x,\mu) = \mu - a_0 + \frac{3}{2}u\gamma(\phi_0^{(2,0)}(x,\mu)) + \frac{u}{2}\gamma(\phi_0^{(0,2)}(x,\mu)) + 3ux^2, \tag{A.16}$$

$$\phi_0^{(0,2)}(x,\mu) = \mu - a_0 + \frac{u}{2}\gamma(\phi_0^{(2,0)}(x,\mu)) + \frac{3}{2}u\gamma(\phi_0^{(2,0)}(x,\mu)) + ux^2, \tag{A.17}$$

$$\phi_0^{(1,0)}(x,\mu) = (\mu - a_0)x + ux^3 + \frac{3}{2}ux\gamma(\phi_0^{(2,0)}(x,\mu)) + \frac{u}{2}x\gamma(\phi_0^{(0,2)}(x,\mu)) \tag{A.18}$$

$$\phi_0^{(0,0)}(x,\mu) = \frac{\mu_0 - a_0}{2}x^2 + \frac{u}{4}x^4 - \frac{3}{16}u\gamma(\phi_0^{(2,0)}(x,\mu)) - \frac{3}{16}u\gamma(\phi_0^{(0,2)}(x,\mu))^2 - \frac{1}{8}u\gamma(\phi_0^{(2,0)}(x,\mu))\gamma(\phi_0^{(0,2)}(x,\mu)). \tag{A.19}$$

In (A.16) and (A.17) we have a set of equations for $\phi_0^{(2,0)}$ and $\phi_0^{(0,2)}$. After solving them we find the remaining coefficients $\phi_0^{(i,j)}$ by a simple substitution to the other relation. The following result holds true.

Proposition A.2. *There exists a $u_0 > 0$ such that for all $0 < u < u_0$, $x \in R^1$ $\mu \in R^1$ there is a unique solution $\phi_0^{(2,0)}$, $\phi^{(0,2)}$ of the equations (A.16) and (A.17). This solution satisfies the estimates*

$$|\phi_0^{(2,0)}(x,\mu) - (\mu - a_0) - 3ux^2| < Cu, \quad |\phi_0^{(0,2)}(x,\mu) - (\mu - a_0) - ux^3| < Cu,$$

$$\left|\frac{\partial}{\partial\mu}\phi_0^{(2,0)}(x,\mu) - 1\right| < Cu, \quad \left|\frac{\partial}{\partial\mu}\phi_0^{(0,2)}(x,\mu) - 1\right| < Cu,$$

$$\left|\frac{\partial}{\partial x}\phi^{(2,0)}(x,\mu) - 6ux\right| < Cu^2x, \quad \left|\frac{\partial}{\partial x}\phi_0^{(0,2)}(x,\mu) - 2ux\right| < Cu^2x.$$

The proof of Proposition A.2 is based on the contractive properties of the mapping

$$(\phi(x),\psi(x))\to\left(\mu-a_0+3ux^2+\frac{3}{2}u\gamma(\phi(x))+\frac{u}{2}\gamma(\psi(x)),\right.$$
$$\left.\mu-a_0+ux^2+\frac{u}{2}\gamma(\phi(x))+\frac{3}{2}u\gamma(\psi(x))\right).$$

It goes on the same line as the proof of Proposition 4.2 in [3]. We omit the details.

Now we formulate the inductive hypotheses we need for ϕ_n. For a set $\Omega\subset R^2$ let $S_u(\Omega)$ denote the set of functions which have the form $\Phi(r,\mu)=\frac{u}{4}r^4+\frac{\mu-a_0}{2}r^2+R(r,\mu)$, where $R(r,\mu)$ is the restriction to Ω of a function $R\in S_u$. The class S_u was defined in Sect. 2.

Inductive Assumption $I^{(N)}$

1) For any function $\phi_n(r,\mu)$, $n=0,1,\dots,N$ there is a critical point $\mu_c^{(n)}$ and a continuous monotone function $M_n(\mu)$ (the spontaneous magnetization) on the half-line $\{\mu<\mu_c^{(n)}\}$ such that

$$\left.\frac{\partial^2\phi_n(r,\mu_c^{(n)})}{\partial r^2}\right|_{r=0}=0\,,\qquad \frac{\partial\Phi_n}{\partial r}(r,\mu)|_{r=r=M_n(\mu)}=0\,,\qquad \mu\leqq\mu_c^{(n)}\,.$$

The function $\mu=\mu_n(r)$ which is the inverse of the function $r=\pm M_n(\mu)$ belongs to $C^1(R)$, and satisfies the inequality

$$\|\mu_n(r)-\mu_c^n-(\mu_{n-1}(r)-\mu_c^{(n-1)})\|_{C^1(R)}\leqq Cu^{3/2}|r|\cdot 2^{n(1-a)}q^n$$

with $q=2^{a-3/2}<1$, $n=0,\dots,N$.

2) Define the domains

$$U^{(n)}=\{(r,\mu),\mu\geqq\mu_c-A\cdot 2^{n(1-a)}\text{ or }r\geqq M_n(\mu)\,(1-B\cdot 2^{n(1-a)})\}\,,$$

with

$$A=a_1-\tfrac{1}{4},\qquad B=2(\mu_c^{(n)}-\mu)a_1(1-u^{1/2})\,,\qquad n=0,\dots,N\,,$$

and $U^{(-1)}=R^2$. Then $U^{(-1)}\supset U^{(0)}\supset\dots\supset U^{(N)}$, and $\phi_n(s,\mu)\in S_u(U^{(n-1)})$.

3) For $(r,\mu)\in U^{(n-1)}$, $n=0,1,\dots,N$ the expansions (A.6), (A.10) hold true together with the orthogonality relation (A.10′), where

$$\gamma_n^{(1)}=\gamma_n(\phi_{n-1}^{(2,0)}(r,\mu))\,,\qquad \gamma_n^{(2)}=\gamma_n(\phi_{n-1}^{(0,2)}(r,\mu))\,.$$

The coefficients $\phi_n^{(i,j)}$ and the error term R_n satisfy the inequalities:

$i_1^{(n)}$) For $(r,\mu)\in U^{(n-1)}$, $n\geqq 1$,

$$\left|\frac{\partial^{i+j+l}}{\partial h^i\partial k^j\partial\mu^l}R_n^{(+)}\right|\leqq C\cdot 2^{-n}\theta_{n-1}^{3-\varepsilon}(\lambda_{n-1}^{(1)})^{i/2}(\lambda_{n-1}^{(2)})^{j/2}\cdot\left(\frac{2^n}{\lambda_{n-1}^{(2)}}\right)^l,$$
$$\left|\frac{\partial^{i+j+l}}{\partial h^i\partial k^j\partial\mu^l}R_n^{(-)}\right|\leqq C\cdot 2^{-n}\eta_{n-1}\theta_{n-1}^{2-\varepsilon}(\lambda_{n-1}^{(2)})^{i/2}(\lambda_{n-1}^{(2)})^{j/2}\left(\frac{2^n}{\lambda_{n-1}^{(2)}}\right)^l,$$

where

$$q = 2^{a-3/2+\varepsilon} < 1\,, \qquad \xi_n^{(i,j)} = \frac{2^n u}{(\lambda_n^{(1)})^i (\lambda_n^{(2)})^j}\,,$$

$$\eta_n = \frac{2^n u r}{(\lambda_n^{(1)})^{1/2} \lambda_n^{(2)}}\,, \qquad \theta_n = \xi_n^{(1,1)} + \eta_n^2\,,$$

$R_n(h,k,r,\mu) = R_n^+(h,k,r,\mu) + R_n^-(h,k,r,\mu)$ is the decomposition of R_n into its even and odd parts in h. The above estimates hold for $i,j \leqq 5$, $l=0,1$ and $|h| \leqq \delta_{n-1}^{(1)}$, $|k| \leqq \delta_{n-1}^{(2)}$, where $\delta_{n-1}^{(i)} = \frac{\mathscr{D}_{n-1}}{\lambda_{n-1}^{(i)}}$, and $\mathscr{D}_{n-1} = \sqrt{\left|\ln\left(\frac{u}{\lambda_{n-1}^{(1)}}\right)\right|}$.

$i_2^{(n)}$) For $(r,\mu) \in U^{(n-1)}$, $n \geqq 1$, $l \leqq 1$,

$$\left|\frac{\partial^l}{\partial \mu^l}(\phi_n^{(i,j)} - \phi_{n-1}^{(i,j)})\right| \leqq C \cdot 2^{-n} \cdot \left(\frac{2^n}{\lambda_{n-1}^{(2)}}\right)^l \cdot \begin{cases} (\omega_{n-1} + \tau_{n-1}^2)^2, & i=4, \; j=0 \\ \omega_{n-1}(\omega_{n-1} + \tau_{n-1}^2), & i=2, \; j=2 \\ \omega_{n-1}^2, & i=0, \; j=4 \\ \tau_{n-1}(\omega_{n-1} + \tau_{n-1}^2), & i=3, \; j=0 \\ \tau_{n-1}\omega_{n-1}, & i=1, \; j=2 \\ \tau_{n-1}^2 + \dfrac{\omega_{n-1}^2}{\lambda_{n-1}^{(2)}}, & i=2, \; j=0 \\ \omega_{n-1}, & i=0, \; j=2 \\ \dfrac{\tau_{n-1}\omega_{n-1}}{\lambda_{n-1}^{(2)}}, & i=1, \; j=0, \end{cases}$$

$$\left|\frac{\partial}{\partial \mu^l}\left\{(\phi_n^{(0,0)} - \phi_{n-1}^{(0,0)}) - 2^{-n}\ln\left[\frac{4}{\pi}(\lambda_{n-1}^{(1)}\lambda_{n-1}^{(2)})^{1/2}\right]\right\}\right|$$
$$\leqq C \cdot \left(\frac{2^n}{\lambda_{n-1}^{(2)}}\right)^l u^2 \cdot \frac{2^n}{(\lambda_{n-1}^{(2)})^4}\,,$$

where $\omega_{n-1} = \frac{2^n u}{\lambda_n^{(2)}}$, $\tau_{n-1} = \omega_{n-1} r$.

$i_3^{(n)}$). For $(r,\mu) \in U^{(n-1)}$, $n \geqq 1$

$$|\phi_n^{(0,4)} - \phi_0^{(0,4)}| \leqq C u^2\,, \qquad |\phi_n^{(1,2)} - \phi_0^{(1,2)}| \leqq C r u^2\,,$$

$$|\phi_n^{(2,0)} - \phi_0^{(2,0)}| \leqq C u^2\,, \qquad \left|\frac{\partial^j}{\partial \mu^j}(\phi_n^{(0,2)} - \phi_0^{(0,2)})\right| \leqq C u\,, \qquad j=0,1\,,$$

$$\left|\frac{\partial^j}{\partial \mu^j}(\phi_n^{(1,0)} - \phi_0^{(1,0)})\right| \leqq C r u^2\,, \qquad j=0,1\,,$$

$$\left|\frac{\partial}{\partial \mu}(\phi_n^{(2,0)} - \phi_0^{(2,0)})\right|_{r=0} \leqq C u^2\,.$$

The inductive assumptions $I^{(N)}$ are similar to those of paper [3]. Properties $i_1^{(n)}$ $-i_3^{(n)}$ enable us to control the function $\phi_n(r,\mu)$ in the domain $U^{(n-1)}$.

Since the integration in formula (A.5) goes in the space $v \in R^2$, we also need an estimate on $\phi_n(r,\mu)$ in the domain $B^{(n)} = R^2 - V^{(n)}$. We call it external estimate. Its formulation is motivated by the analysis of the "crater" densities.

External Estimate $J^{(n)}$. For $\mu \leqq \mu_c^{(n)}$ the estimate

$$\exp\{-2^n(\phi_n(r,\mu) - \phi_n(M_n(\mu),\mu)\} \leqq B_s(r, M_n(\mu), \chi_n(\mu))$$

holds true, where $\chi_n(\mu) = 2|\mu - \mu_c^{(n)}| 2^{-n}(1-u^{0,2})$, and B_s is the special crater defined at the right-hand side of (A.4).

In the proof of the assumptions $I^{(N)}$ and $J^{(n)}$ we shall need the following three types of estimates $A^{(n)}$, $B^{(n)}$, $C^{(n)}$. Put $q = 2^{a-3/2} < 1$ and

$$\Lambda_n^{(1)}(r,\mu) = 2^{n(2-a)}a_1 + (|\mu - \mu_c^{(n)}| + ur^2)2^n,$$
$$\Lambda_n^{(2)}(r,\mu) = 2^{n(2-a)}a_1 + u(r^2 - M_n^2(\mu)) \cdot 2^n.$$

In all subsequent estimates the quantity $C > 1$ is an absolute constant.

$A^{(n)}a_1^{(n)}$). For $(r,\mu) \in U^{(n-1)}$ and $\mu < \mu_c^{(n-1)} \phi_n^{(2,0)}(r,\mu) > 0$.
$a_2^{(n)}$). For $(r,\mu) \in U^{(n)}$ $4\Lambda_n^{(i)}(r,\mu) \geqq \lambda_n^{(i)}(r,\mu) \geqq \frac{1}{4}\Lambda_n^{(i)}(r,\mu)$, $i = 1, 2$.

$B^{(n)}$). For $(r,\mu) \in U^{(n-1)}$,

$$\left|\frac{\partial^j}{\partial\mu^j}\left(\frac{\partial^2}{\partial r^2}\phi_n - \phi_n^{(2,0)}\right)\right| \leqq C\frac{u}{\Lambda_n^{(2)}} \cdot \left(\frac{2^n}{\Lambda_n^{(2)}}\right)^j, \quad r = 0, \quad j = 0, 1,$$

$$\left|\frac{\partial^j}{\partial\mu^j}\left(\frac{\partial}{\partial r}\phi_n - \phi_n^{(1,0)}\right)\right| \leqq C\frac{u^2 r 2^n}{(\Lambda_n^{(2)})^3}\left(\frac{2^n}{\Lambda_n^{(2)}}\right)^j, \quad j = 0, 1,$$

$$\left|\frac{\partial^j}{\partial\mu^j}\left(\phi_n^{(0,2)} - \frac{1}{r}\phi_n^{(1,0)}\right)\right| \leqq Cu\frac{1}{\Lambda_n^{(2)}}\left(\frac{2^n}{\Lambda_n^{(2)}}\right)^j, \quad j = 0, 1.$$

$C^{(n)}$). For $(r,\mu) \in U^{(n-1)}$.

$c_1^{(n)}$). $\dfrac{ur^2 2^n}{\Lambda_n^{(1)}} \leqq 1$,

$c_2^{(n)}$). $\dfrac{2^{n/2}}{\Lambda_n^{(2)}} \leqq q^n$.

The central part in the proof of Theorem A is the following lemma. We say that the inductive assumptions $I^{(N)}$ and $I^{(M)}$, $N \leqq M$ are consistent if the functions $\phi_n^{(i,j)}$, $R_n^{(i)}$, R_n coincide in $I^{(N)}$ and $I^{(M)}$ if $n \leqq N$, and the conditions $i_1^{(n)} - i_3^{(n)}$ hold with the same constant C.

Lemma A.3. *There exists some $u_0 > 0$ such that for any $0 < u < u_0$ and $n = 0, 1, \ldots$ the assumptions $I^{(n)}$ together with the estimates $A^{(n)}$, $B^{(n)}$, $C^{(n)}$, $J^{(n)}$ hold true, and the assumptions $I^{(n)}$ for different n are consistent.*

The proof of this lemma is rather tedious, and we omit it. It applies the same method as the papers [3, 4]. Now we prove Theorem A with the help of Lemma A.3. Put $\mathscr{U} = \bigcap_{n=0}^{\infty} U^{(n)}$.

Lemma A.4. *In the domain $(s,u)\in\mathscr{U}$ the functions $\phi_n^{(i,0)}(s,\mu)$ tend to the limits $\phi_*^{(i,0)}(s,\mu)$, $i=1,2$, as $n\to\infty$, and the estimates*

$$|\phi_n^{(i,0)}-\phi_*^{(i,0)}|\leqq C\cdot\frac{2^n u^2}{(\Lambda_n^{(2)})^2}\cdot\begin{cases} r^2+\dfrac{1}{\Lambda_n^{(2)}}, & i=1\\ \dfrac{r}{\Lambda_n^{(2)}}, & i=1,\end{cases}$$

$$\left|\frac{\partial}{\partial\mu}(\phi_n^{(1,0)}-\phi_*^{(1,0)})\right|\leqq C\left(\frac{2^n}{(\Lambda_n^{(2)})^2}\right)^2 u^2 r,$$

$$\left|\phi_n^{(0,2)}-\frac{1}{r}\phi_*^{(1,0)}\right|<\frac{Cu}{\Lambda_n^{(2)}}.$$

Proof. According to condition $i_2^{(n)}$,

$$|\phi_n^{(2,0)}-\phi_{n-1}^{(2,0)}|\leqq C\cdot\frac{2^n u^2}{(\Lambda_n^{(2)})^2}\left(r^2+\frac{1}{\Lambda_n^{(2)}}\right).$$

Since $\dfrac{2^n}{(\Lambda_n^{(2)})^2}$ tends to zero exponentially fast, $\lim\limits_{n\to\infty}\phi_n^{(2,0)}=\phi_*^{(2,0)}$ exists, and

$$|\phi_n^{(2,0)}-\phi_*^{(2,0)}|\leqq C\cdot\frac{2^n u}{(\Lambda_n^{(2)})^2}\left(r^2+\frac{1}{\Lambda_n^{(2)}}\right).$$

The other relations can be proved similarly.

Lemma A.5. *In the domain $(r,\mu)\in\mathscr{U}$ the limit $\lim\limits_{n\to\infty}\phi_n(r,\mu)=\Phi(r,\mu)$ exists, and for $j=0,1,2$ the estimates*

$$\left|\frac{\partial}{\partial\mu^j}\left\{\phi_n(r,\mu)-\Phi(r,\mu)+\frac{1}{2}\sum_{m=n}^{\infty}2^{-m}\ln\frac{4}{\pi}(\lambda_n^{(1)}\lambda_n^{(2)})^{1/2}\right\}\right|$$
$$\leqq Cu^2\frac{2^n}{(\Lambda_{n-1}^{(2)})^4}\left(\frac{2^n}{\Lambda_n^{(2)}}\right)^j$$

hold true. Moreover

$$\frac{\partial^i}{\partial r^i}\Phi(r,\mu)=\phi_*^{(i)}(r,\mu),\qquad i=1,2.$$

Proof. By the relations $i_3^{(n)}$,

$$\left|\frac{\partial^j}{\partial\mu^j}\left\{\phi_n^{(0,0)}-\phi_{n-1}^{(0,0)}-2^{-n}\ln\left[\frac{4}{\pi}(\lambda_{n-1}^{(1)}\lambda_{n-1}^{(2)})^{1/2}\right]\right\}\right|$$
$$\leqq Cu^2\frac{2^n}{(\Lambda_{n-1}^{(2)})^4}\cdot\left(\frac{2^n}{\Lambda_{n-1}^{(2)}}\right)^j.$$

Relations (A.6) and (A.10) imply with the choice $h=k=0$ that

$$\begin{aligned}\phi_n(r,\mu)&=\phi_n^{(0,0)}(r,\mu)+\tfrac{1}{32}[\gamma_n^{(1)2}\phi_n^{(4,0)}(r,\mu)+2\gamma_n^{(1)}\gamma_n^{(2)}\phi_n^{(2,2)}(r,\mu)\\&\quad+(\gamma_n^{(2)})^2\phi_n^{(0,4)}(r,\mu)]+R_n^{(0)}(0,r,\mu)-\tfrac{1}{4}\gamma_n^{(1)}R_n^{(2)}(0,r,\mu)\\&\quad+\tfrac{1}{32}(\gamma_n^{(1)})^2R_n^{(4)}(0,r,\mu)+R_n^{+}(0,0,r,\mu).\end{aligned}$$

By applying relations $i_1^{(n)}-i_3^{(n)}$, we get from this relation that

$$\left|\frac{\partial}{\partial\mu^j}(\phi_n^{(0,0)}-\phi_n)\right|\leqq Cu^2\frac{2^n}{(\Lambda_{n-1}^{(2)})^4}\cdot\left(\frac{2^n}{\Lambda_{n-1}^{(2)}}\right)^j.$$

Hence

$$\left|\frac{\partial^j}{\partial\mu^j}\left(\phi_n-\phi_{n-1}-2^{-n}\ \ln\left[\frac{4}{\pi}(\lambda_{n-1}^{(1)}\lambda_{n-1}^{(2)})^{1/2}\right]\right)\right|$$
$$\leqq Cu^2\frac{2^m}{(\Lambda_{n-1}^{(2)})^2}\cdot\left(\frac{2^n}{\Lambda_{n-1}^{(2)}}\right)^j.$$

Summing up the last relation in n we get the convergence of $\phi_n(r,\mu)$ to a limit $\Phi(r,\mu)$ together with the desired estimate on $\phi_n-\Phi$. The formulas $\frac{\partial^i}{\partial r^i}\Phi(r,\mu)=\phi_*^{(i)}(r,\mu)$ follows from the estimates $B^{(n)}$.

Lemma A.6.

$$\left|\phi_n(r,\mu)-\Phi(r,\mu)+\frac{1}{2}\sum_{m=n}^{\infty}2^{-m}\ln\left[\frac{4}{\pi}(\lambda_{m*}(r,\mu)\lambda_{m*}^{(2)}(r,\mu))^{1/2}\right]\right|$$
$$\leqq C\cdot 2^{-n}u\cdot 2^{n(2a-3)},$$

where

$$\lambda_{m*}^{(1)}(r,\mu)=2^{m(2-a)}a_1+2^m\frac{\partial^2\Phi(r,\mu)}{\partial r^2},$$

$$\lambda_{m*}^{(2)}(r,\mu)=2^{m(2-a)}a_1+2^m\frac{1}{r}\frac{\partial\Phi(r,\mu)}{\partial r}.$$

Proof. By Lemmas A.4, A.5

$$|\ln\lambda_{m*}^{(1)}-\ln\lambda_m^{(1)}|=\left|\ln\left(1+\frac{2^m(\phi_*^{(2,0)}-\phi_m^{(2,0)})}{\lambda_m^{(1)}}\right)\right|\leqq C\cdot\frac{2^m}{\Lambda_m^{(1)}}\cdot\frac{2^m u^2}{(\Lambda_m^{(2)})^2}$$

$$\cdot\left(r^2+\frac{1}{\Lambda_m^{(2)}}\right)\leqq C\frac{2^m u}{(\Lambda_m^{(2)})^2},$$

hence

$$\left|\sum_{m=n}^{\infty}2^{-m}(\ln\lambda_{m*}^{(1)}-\ln\lambda_m^{(1)})\right|\leqq C\frac{u}{(\Lambda_n^{(2)})^2}\leqq C\cdot 2^{-n}u\cdot 2^{n(2a-3)}.$$

Similarly

$$\left|\sum_{m=n}^{\infty}2^{-m}(\ln\lambda_{m*}^{(2)}-\ln\lambda_m^{(2)})\right|\leqq C\cdot 2^{-n}u\cdot 2^{n(2a-3)}.$$

These estimates together with Lemma A.5 (with the choice $j=0$) imply Lemma A.6.

Lemma A.7. $\Phi(r,\mu)\in S_u$.

Proof. Let $|h|\leqq 5(\Lambda_n^{(1)})^{-1/2}$. By differentiating the identity (A.6) and (A.10) twice by the variable h, and exploiting $I^{(n)}$ we get that

$$\left|\frac{\partial^2\phi_n}{\partial r^2}(r+h,\mu)-\Phi_n^{(2,0)}(r,\mu)\right|$$

$$\leqq C\left(\frac{|\phi_n^{(3,0)}|}{(\Lambda_n^{(1)})^{1/2}}+\frac{|\phi_n^{(4,0)}|}{\Lambda_n^{(1)}}+|\phi_n^{(2,2)}|\cdot\gamma_n^{(2)}\right.$$

$$\left.+\sum_{i=2}^{4}|R_n^{(i)}|(\Lambda_n^{(1)})^{1/2(2-i)}+\left|\frac{\partial^2 R}{\partial h^2}\right|\right)\leqq Cu\left(\frac{1}{\Lambda_n^{(1)}}\right)^{\varepsilon_0},$$

with $\varepsilon_0=\frac{3}{2}-a$. On the other hand by Lemmas A.4 and A.5

$$\left|\phi_n^{(2,0)}-\frac{\partial^2\Phi}{\partial r^2}\right|\leqq C\frac{2^n u^2}{(\Lambda_n^{(2)})^2}\left(r^2+\frac{1}{\Lambda_n^{(2)}}\right)\leqq C_1 u(\Lambda_n^{(1)})^{-\varepsilon_0},$$

hence

$$\left|\frac{\partial^2\Phi(r+h,\mu)}{\partial r^2}-\frac{\partial^2\Phi(r,\mu)}{\partial r^2}\right|\leqq C_2 u(\Lambda_n^{(1)})^{-\varepsilon_0}.$$

Apply this result for $(\Lambda_n^{(1)})^{-1/2}\leqq|h|\leqq 5(\Lambda_n^{(1)})^{-1/2}$. Then we obtain

$$\left|\frac{\partial^2\Phi(r+h,\mu)}{\partial r^2}-\frac{\partial^2\Phi(r,\mu)}{\partial r^2}\right|\leqq C_2 u|h|^{\varepsilon_0}.$$

Since the above estimate holds for all n (the length h depends on n), we get by covering the interval $[r-h_0, r+h_0]$, $h_0=(\Lambda_0^{(1)})^{-1/2}$ appropriately that the last estimate holds for any $|h|\leqq h_0$. Thus the Hölder condition is satisfied for the function R. The Hölder conditions for $\frac{\partial^{i+j}R}{\partial r^i\partial\mu^j}$ can be proved similarly. Part ii) and iii) of the definition of the class S_u can be proved with the help of $i_3^{(n)}$.

Lemma A.6. implies the asymptotic formula for $\ln Z_n(r,\mu,u)$ in Theorem A. On the other hand $\Phi\in S_u$.

This completes the proof of the asymptotic formula in the domain $U=\bigcap_{n=0}^{\infty}U^{(n)}$. Now we briefly explain how it can be extended to the domains $U^{(n)}$, $n=0,1,2,\dots$. Let us observe that in the domains $U^{(n)}\backslash U^{(n+1)}$, $n=0,1,2,\dots$, we have to consider the asymptotic formula for the functions $\ln Z_k(r,\mu)$ only for $k\leqq n$, hence the function $\Phi(r,\mu)$ is not uniquely defined in these domains. We defined $\Phi(r,\mu)$ in the domain U as $\Phi(r,\mu)=\lim_{n\to\infty}\phi_n(r,\mu)$. Due to Lemma A.5 the expression

$$\phi_n(r,\mu)+(1/2)\sum_{m=n}^{\infty}2^{-m}\ln\left[\frac{4}{\pi}(\lambda_m^{(1)}\lambda_m^{(2)})^{1/2}\right]$$

yields a good approximation for $\Phi(r,\mu)$. Thus it is natural to define

$$\Phi(r,\mu)=\phi_n(r,\mu)+(1/2)\sum_{m=n}^{\infty}2^{-m}\ln\left[\frac{4}{\pi}(\lambda_{mn}^{(1)}\lambda_{mn}^{(2)})^{1/2}\right]$$

in the domain $U^{(n)}\backslash U^{(n+1)}$, where

$$\lambda_{mn}^{(1)}=a_1c^m+\phi_n^{(2,0)}\cdot 2^m\,,\qquad \lambda_{mn}^{(2)}=a_1c^m+\phi_n^{(0,2)}\cdot 2^m\,.$$

In this case the asymptotic formula for $\ln Z_k(r,\mu)$ holds in the domain $U^{(n)}\backslash U^{(n+1)}$ for $k\leqq n$. However this definition of the function $\Phi(r,\mu)$ has a defect: $\Phi(r,\mu)$ has jumps on the boundary of the domains $U^{(n)}$, $n=1,2,\ldots$. To get rid of this defect we smooth the function $\Phi(r,\mu)$. For this purpose we use a decomposition of the unit connected with the domains $U^{(n)}\backslash U^{(n+1)}$, $n=0,1,2,\ldots$. Namely, let us consider a set of non-negative functions $\chi_n(r,\mu)\geqq 0$ such that

$$\sum_{n=0}^{\infty}\chi_n(r,\mu)=1\,,\qquad \chi_n(r,\mu)=0\quad\text{if}\quad (r,\mu)\notin U^{(n-1)}\backslash U^{(n+2)}$$

and

$$\left|\frac{\partial^{i+j}}{\partial r^i\partial\mu^j}\chi_n(r,\mu)\right|<C\cdot d^{-(i+j)}\,,\qquad i+j\leqq 4\,,$$

where C is an absolute constant and $d=\varrho(x,U^{(n+2)})+\varrho(x,U^{(n)c})$, $x=(r,\mu)$, $\varrho(x,V)$ is the distance between x and V, and $U^{(n)c}$ is the component to $U^{(n)}$. The set of functions $\chi_n(r,\mu)$, $n=0,1,\ldots$, is called a decomposition of the unit connected with the domains $U^{(n)}\backslash U^{(n+1)}$. Using these functions we put

$$\Phi(r,\mu)=\sum_{n=0}^{\infty}\chi_n(v,\mu)\cdot\left\{\phi_n(r,\mu)+(1/2)\sum_{m=n}^{\infty}2^{-m}\ln\left[\frac{4}{\pi}(\lambda_{mn}^{(1)}\lambda_{mn}^{(2)})^{1/2}\right]\right\}.$$

A slight modification of the proof given above shows that this definition of the function $\Phi(r,\mu)$ satisfies all statements of Theorem A. This completes the proof of Theorem A.

References

1. Bateman, G., Erdélyi, A.: Higher transcendental functions, Vol. 2. New York, Toronto, London: McGraw-Hill 1955
2. Bleher, P.M.: Construction of non-Gaussian self-similar random fields with hierarchical structure. Commun. Math. Phys. **84**, 557–578 (1982)
3. Bleher, P.M.: Large deviations results in Dyson's hierarchical model (in Russian). Teor. Verojatnost. Primenen (to appear)
4. Bleher, P.M., Sinai, Ya.G.: Investigation of the critical point in models of the type of Dyson's hierarchical models. Commun. Math. Phys. **33**, 23–42 (1973)
5. Bleher, P.M., Sinai, Ya.G.: Critical indices for Dyson's asymptotically hierarchical models. Commun. Math. Phys. **45**, 247–278 (1975)
6. Collet, P., Eckmann, J.P.: A Renormalization group analysis of the hierarchical model in statistical mechanics. In: Lecture Notes in Physics, No. 74. Berlin, Heidelberg, New York: Springer 1978
7. Sinai, Ya.G.: Theory of phase transitions: rigorous results. London: Pergamon Press 1982
8. Dyson, F.: An Ising ferromagnet with distribution soflong-range order. Commun. Math. Phys. **21**, 269–283 (1971)

Communicated by Ya. G. Sinai

Received April 5, 1984