Commun. Math. Phys. 95, 161–184 (1984)

Communications in
Mathematical
Physics

Geometry of $N=1$ Supergravity

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Abstract. A new geometrical formalism is suggested for the non-minimal and alternative minimal supergravities. This formalism connects the constrained superspace formulations with the unconstrained ones and is based on the notion of induced geometry. The relevant mathematical technique is that of G-structures. A clear-cut geometrical content of the torsion and curvature constraints is revealed on the basis of a general theorem about the necessary and sufficient properties of induced geometry.

The $N=1$ supergravity can be formulated in superspace in a number of different ways. There is a whole family of supergravity theories, parametrized conveniently by a parameter ζ [1–3]. These superspace formulations correspond in components to different sets of auxiliary fields, minimal ($\zeta=\infty$), non-minimal ($\zeta \neq 1/3, 1, \infty$), and alternative minimal ($\zeta=1$) sets, that were discovered respectively in [4–6]. At the same time various geometrical approaches to superspace are known in all mentioned cases.

In this paper we continue the study of the geometry of $N=1$ supergravities using the framework of induced structures. (The general notion of the induced structure has been introduced in [7]. It describes the internal geometry of a surface inherited from the geometry of the ambient space). The case $\zeta=\infty$ was already investigated by means of these methods [7]. Recently it has been shown [8] that completely analogous constructions can be used in the description of $N=1$ super Yang-Mills theory coupled to supergravity. Our aim in the present paper is to look at the structure of supergravity from a general geometrical point of view. We shall show that the links between prepotentials and constrained supervielbeins in $N=1$ supergravity can be understood on the basis of a theorem about the necessary and sufficient properties of induced geometry. (The formulation and the proof of this theorem are given in [22], which contains also, as a corollary, a derivation of supergravity constraints.) The generality of the methods used allows us to expect further applications. The construction of action functionals is also considered. It is shown, in particular, that the "non-geometrical" action of the alternative minimal supergravity can be obtained from the "geometrical" actions of the non-minimal supergravities by means of a properly defined limiting procedure for $\zeta \to 1$. Some of the main results of the present paper have been published previously in [9].

1. Introduction

There exist at least two basic geometrical approaches to $N=1$ supergravities in superspace. One of these approaches deals with the fields of tangent frames (supervielbeins) defined in superspace up to local transformations of the Lorentz group L or, sometimes, $L \times U(1)$. In order to get the right component content one has to impose certain constraints on the curvature and/or the torsion tensors. The action can be constructed (with some exceptions) as the integral over the volume element defined by the vielbein field. For $N=1$ such superspace formulations have been found in the minimal [1], non-minimal [2, 3] and alternative minimal [10] cases. We will refer further to such formulations as the Wess-Zumino type ones.

The most elegant geometrical approach to $N=1$ supergravity is that of Ogievetsky and Sokatchev [11, 12][1]. In this approach the role of the fields of supergravity is played by the real surface in a complex superspace. Ogievetsky and Sokatchev have shown, for instance, that the real (4|4)-dimensional surface in the superspace $\mathbb{C}^{4/2}$ can play the role of the field of minimal supergravity, provided the symmetry group, $\mathscr{L}$, of the theory is chosen appropriately. Specifically, $\mathscr{L}$ must consist of all complex analytical transformations of $\mathbb{C}^{4/2}$ that preserve the supervolume. The connection of such a group to Poincaré supersymmetry comes from the following observation [11]. Let x_L^a, θ_L^α $(a=1, \dots, 4; \alpha=1,2)$ be the complex coordinates in $\mathbb{C}^{4|2}$. Consider a surface, Q_0, defined in $\mathbb{C}^{4|2}$ by the equations $x_L^a - \bar{x}_R^a = 2i\theta_L^\alpha \sigma^a_{\alpha\dot{\beta}} \bar{\theta}_R^{\dot{\beta}}$. Here $\bar{x}_R$ and $\bar{\theta}_R$ are related to x_L and θ_L by complex conjugation, while $\sigma^a = (\mathbb{1}, \boldsymbol{\sigma})$ with $\boldsymbol{\sigma}$ being the Pauli matrices. This real (4|4)-dimensional surface has a remarkable property that the subgroup of the group $\mathscr{L}$ which leaves it invariant coincides with the super Poincaré group of global supersymmetry. This suggests that the surface Q_0 must play the role of the flat $N=1$ superspace. The curved superspace corresponds then to an arbitrary real (4|4)-dimensional surface in $\mathbb{C}^{4|2}$. The action can be found as a functional on the space of these surfaces invariant under the group $\mathscr{L}$.

A natural generalization would be to consider an action functional on the space of $(p|q)$-dimensional surfaces in the space $\mathbb{R}^{P|Q}$. (In other words, a formalism comes into play, which treats the field and space variables on an equal footing [14].) Note that the space $\mathbb{C}^{4|2}$ can be considered as the real $\mathbb{R}^{8|4}$ with an additional structure. This suggests that, in general, the space $\mathbb{R}^{P|Q}$ must be endowed with some geometry. This geometry may be characterized by its symmetry group, which will play the role of the group $\mathscr{L}$ mentioned above. A sufficiently broad class of such structures is provided by the groups $\Gamma(G)$ consisting of transformations of the space $\mathbb{R}^{P|Q}$ such that the Jacobian matrix of each transformation at every point belongs to a group G [we assume, that $G \subset \mathrm{GL}(P|Q, \mathbb{R})$]. For example, the group $\mathscr{L}$ can be identified with $\Gamma(G)$, where

1 An essentially equivalent formalism has been independently suggested by Siegel and Gates [13, 2, 3]. However, the geometry is rather implicit in their approach

$G=SL(4|2,\mathbb{C})\subset GL(8|4,\mathbb{R})$ is the group of complex linear transformations with unit Berezinian (superdeterminant). The surfaces, connected by a transformation of the group $\Gamma(G)$, will be called (gauge) equivalent. Some surface Q_0 corresponding to flat physics must still obey the property that the subgroup of $\Gamma(G)$ which leaves it invariant contains the transformations of (possibly extended) Poincaré supersymmetry.

It is worth noticing, however, that one may impose some restrictions on the class of surfaces considered. Indeed, it is natural to require that in a small neighborhood of every point these surfaces should be equivalent to Q_0 up to higher order terms. In other words, one may require that every surface should be normalizable by means of allowed gauge transformations up to a fixed order, the normal form being that of Q_0 at some given point. (Of course, it is necessary that the surface Q_0 itself should satisfy this requirement. This is certainly so when the subgroup which leaves Q_0 invariant acts on it transitively.) Such a requirement implies, in particular, that the tangent subspace[2] of every surface at each point can be connected with that of Q_0 at the marked point by means of a transformation of the linear group G. The surfaces satisfying this first order normalizability requirement will be called regular. In minimal supergravity almost all real (4|4)-dimensional surfaces in $\mathbb{C}^{4|2}$ are regular with respect to the group $\mathscr{L}$. Thus the requirement of regularity is not a constraint in this case. The situation will be the same in non-minimal supergravity. It differs, however, in alternative minimal supergravity, as we shall see. This will be the origin of a first order constraint on the "preportentials" H^a, H^α for $\zeta=1$. We shall see also, that for all ζ, except $\zeta=1/3$, almost all regular surfaces are normalizable even up to the second order.

A formalism, which uses the surfaces in an ambient space, can be called the external formalism, as opposed to the internal one. The approach of Wess and Zumino gives an example of the latter type. The external $N=1$ formalism has the advantage of being unconstrained (for $\zeta\neq 1$ at least). On the other hand, the constrained Wess-Zumino type formalism yields immediately the action functional, as we have already noticed. It was not very easy to write down the actions in the external formalism. This task was accomplished [2, 12] essentially by means of a lengthy derivation from the Wess-Zumino action for all ζ (except $\zeta=1/3$, 1). However, it must be pointed out that there exists as well a self-contained procedure for constructing the actions, which invokes the external objects only. In particular, it was shown [7, 15] that the action of minimal supergravity could be found in terms of the so-called Levi form of the real surface in the complex space. Nevertheless, the example of the Wess-Zumino action suggests that it is much more easy to guess the action in the internal framework. Unfortunately, the relation of the Wess-Zumino type formalism to the formalism of surfaces is rather indirect (cf. [2, 12]). Therefore, in order to find an action in an external (presumably unconstrained) formulation of the theory one may proceed as follows. At first one has to construct an internal formalism, which is intimately connected with the external one. Then one can employ it in searching for actions. This

2 Note that the translations always enter the group $\Gamma(G)$. Consequently we may identify the tangent planes at arbitrary points of any surface with corresponding subspaces passing through the origin

formalism must respect the group $\Gamma(G)$, a would-be symmetry of the model. Remember, that $\Gamma(G)$ is defined as the group of automorphisms of a fixed geometry in the external space, in which the surfaces live. Hence we need a proper definition of internal geometry of surfaces, and for each surface this geometry must be determined completely by the structure of the external space. That is to say, we have to define a geometry induced on the surface by the given geometry of the ambient space.

A general notion of induced geometry was introduced in [7]. For the sake of generality it had to involve another general notion, namely that of G-structure [16]. G-structures serve to describe geometry in terms of tangent frames. A large class of geometries can be described in this manner. Let $\mathscr{M}$ be a manifold (or supermanifold) and G be a matrix group, with the size of matrices being equal to the dimension of $\mathscr{M}$ [that is $G \subset \mathrm{GL}(\dim \mathscr{M}, \mathbb{R})$]. Consider two tangent frames at some point of $\mathscr{M}$, generated by the vectors e_a^m and $\tilde{e}_a^m$ respectively. Here the indices run over the dimension of $\mathscr{M}$, with m being a world index, while a is a tangent space one. These frames are called G-equivalent if they can be connected by a tangent space transformation of the group G, that is if $e_a^m = g_a^b \tilde{e}_b^m$ for a matrix (g_a^b) in G. When there are two frame fields $e_a^m(x)$ and $\tilde{e}_a^m(x)$, $x \in \mathscr{M}$, they are called G-equivalent if there is an x-dependent transformation connecting them, that is if $e_a^m(x) = g_a^b(x)\tilde{e}_b^m(x)$ for a G-valued function $(g_a^b(x))$ on $\mathscr{M}$. One says that a G-structure is given in the manifold $\mathscr{M}$, if a G-equivalence class of tangent frame fields on $\mathscr{M}$ is fixed. In this case we have a class of G-equivalent frames fixed at each point. The frames, or the frame fields belonging to the fixed equivalence class are called admissible for this G-structure. The word "geometry," used with an unspecified meaning above, should have been understood as "G-structure" for some group G. Various examples of G-structures and what in more usual terms they correspond to can be found in [16]. Of course, the example of general relativity comes immediately to one's mind: the vierbeins, i.e. the pseudo-orthonormal frame fields in spacetime constitute a Lorentz-group-structure there.

A G-structure is called trivial, if there is a frame field, $e_a^m(x)$, among the admissible ones, that corresponds to a holonomic coordinate frame, i.e. $e_a^m(x) = \delta_a^m$ for some coordinates in the manifold[3]. It can be easily shown that the trivial G-structure has the above group $\Gamma(G)$ as a group of automorphisms (see also Appendix A). Given a space, say $\mathbb{R}^N$, endowed with the trivial G-structure ($G \subset GL(N, \mathbb{R})$) one is able to define the internal geometry of a surface in $\mathbb{R}^N$ by means of frame fields admissible for the G-structure in $\mathbb{R}^N$ and adjusted in a certain way to this surface [7]. Then some M-dimensional surfaces in $\mathbb{R}^N$ receive induced G'-structures, where G' is a subgroup of $GL(M, \mathbb{R})$ related to G in a definite way. The details of the definition of the induced G'-structure are recapitulated in Sect. 3. It turns out that, in general, the induced G'-structure can not be defined on every surface. For this to be possible the surface must satisfy certain regularity conditions coherent with the above regularity up to some conventions.

3 In the mathematical literature such G-structures are commonly called flat, as they correspond to the flat geometry; for example, a holonomic vielbein defines the flat metric. We prefer, however, the term "trivial structure" to avoid confusion with the flat superspace, which is to be endowed with a non-trivial structure

In various mathematical problems as well as in dealing with the relations between external and internal formulations of supergravity the following question arises. Under what conditions can the given geometry (i.e. a G'-structure) on a manifold be realized on some surface in $\mathbb{R}^N$ as one induced by the trivial G-structure in this ambient space? In general such a G'-structure must satisfy some constraints arising as integrability conditions for a certain system of nonlinear partial differential equations. These conditions are imposed as differential constraints on the frame fields corresponding to this G'-structure. In a subsequent paper [22] we prove a theorem that describes in a convenient form these necessary and sufficient constraints. In general, one has a finite chain of integrability conditions of increasing orders. Among them the conditions of the first and second orders can be rewritten as constraints on the torsion and the curvature. It may happen that these torsion and curvature constraints are not only necessary, but also sufficient (that is, no higher order conditions are needed). In [22] we prove that this is just the case in $N=1$ supergravity, and derive the torsion and curvature constraints from the general theorem.

This paper is organized as follows. In Sect. 2 we describe the formalism of real surfaces in the "non-minimal" superspace $\mathbb{C}^{4|4}$, which is used for all members of the family of non-minimal supergravities, parametrized by ζ. The minimal and alternative minimal supergravities are included as special cases, $\zeta=\infty$ and $\zeta=1$, respectively. We point out, in particular, that the surfaces (i.e. the "prepotentials") are to be constrained if $\zeta=1$. The constraint for $\zeta=1$ follows immediately from the requirement of regularity described above. In Sect. 3 we give the definition of the induced structure. As an application we consider in detail induced G'-structures arising in supergravity. This discussion results in certain internal formulations available for all members of the family of $N=1$ models. The role of the field is played now by the internal geometry of the surfaces. These formulations are related to the external ones via the notion of induced structure. It means that considering the internal geometry we deal with G'-structures, which are not arbitrary, but satisfy certain constraints. Establishing these, we make the internal formulations self-contained. In Sect. 4 it is shown that the Wess-Zumino type formulations (for any ζ) can be recovered from the induced structure formulations by means of certain gauge conditions. These reduce the structure group, leaving the Lorentz group L or, if $\zeta=1$, $L\times U(1)$. The action functionals are constructed in Sect. 5. In the gauge of Sect. 4 they assume the form of the Wess-Zumino action for $\zeta \neq 1/3, 1$. It turns out that the action for $\zeta=1$ can be obtained taking $\zeta\to 1$ by means of a properly defined limiting procedure. The discussion of supergravity actions is completed in Appendix B. It will be shown there that the geometrical framework of induced structures yields a simple way to write down the prepotential form of the action.

2. Real Surfaces in a Complex Superspace

Let us consider the complex superspace $\mathbb{C}^{4|4}$ and real (4|4)-dimensional surfaces therein. Let $z^A=(z^a,\theta^\alpha,\bar{\varphi}^{\dot\pi})$ be the complex coordinates in $\mathbb{C}^{4|4}$. Here z^a, $a=1,2,3,4$, are the even (bosonic) coordinates, while θ^α, $\alpha=1,2$, and $\bar{\varphi}^{\dot\pi}$, $\dot\pi=\dot1,\dot2$, are the odd (fermionic) ones. To describe a supergravity model, corresponding to the parameter[4] ζ, we consider the following group of complex analytical transformations of $\mathbb{C}^{4|4}$

$$
\begin{aligned}
&\delta z^a = \lambda^a(z,\theta)\,,\\
&\delta\theta^\alpha = \lambda^\alpha(z,\theta)\,,\\
&\delta\bar\varphi^{\dot\pi} = \bar\varrho^{\dot\pi}(z,\theta,\bar\varphi)\,,\\
&\begin{cases}
\zeta\left(\dfrac{\partial\lambda^a}{\partial z^a} - \dfrac{\partial\lambda^\alpha}{\partial\theta^\alpha}\right) = \dfrac{\partial\bar\varrho^{\dot\pi}}{\partial\bar\varphi^{\dot\pi}} & \text{if} \quad \zeta \neq \infty\,,\\
\dfrac{\partial\lambda^a}{\partial z^a} - \dfrac{\partial\lambda^\alpha}{\partial\theta^\alpha} = 0 & \text{if} \quad \zeta = \infty\,,
\end{cases}
\end{aligned}
\tag{2.1}
$$

where the transformations are written in infinitesimal form. (This group is borrowed from [2, 3, 19, 20].) One can observe immediately that this group (1) coincides, in the notations of Sect. 1, with $\Gamma(G(\zeta))$, where $G(\zeta)$ is the group of complex linear transformations of $\mathbb{C}^{4|4}$, generated by the Jacobian matrices of the transformations (1). Thus the group $G(\zeta)$ consists of the following transformations [we use a notation $z^{\mathscr{A}} = (z^a, \theta^\alpha)$]

$$
\begin{aligned}
&z'^a = z^{\mathscr{B}} U^a_{\mathscr{B}} \equiv z^b A^a_b + \theta^\beta D^a_\beta\,,\\
&\theta'^\alpha = z^{\mathscr{B}} U^\alpha_{\mathscr{B}} \equiv z^b B^\alpha_b + \theta^\beta C^\alpha_\beta\,,\\
&\bar\varphi'^{\dot\pi} = z^b E^{\dot\pi}_b + \theta^\beta F^{\dot\pi}_\beta + \bar\varphi^{\dot\sigma} G^{\dot\pi}_{\dot\sigma}\,,
\end{aligned}
\tag{2.2}
$$

which satisfy a condition corresponding to the finite form of Eq. (1), namely,

$$
\begin{aligned}
&[\mathrm{Ber}(U^{\mathscr{A}}_{\mathscr{B}})]^\zeta = \det(G^{\dot\pi}_{\dot\sigma}) && \text{if} \quad \zeta \neq \infty\,,\\
&\mathrm{Ber}(U^{\mathscr{A}}_{\mathscr{B}}) = 1 && \text{if} \quad \zeta = \infty\,.
\end{aligned}
\tag{2.3}
$$

We intend to assign the meaning of the field of supergravity to the real (4|4)-dimensional surface in $\mathbb{C}^{4|4}$. In doing so, we will consider the space $\mathbb{C}^{4|4}$ as carrying a geometrical structure that corresponds to the symmetry group $\Gamma(G(\zeta))$. According to Sect. 1, first of all we have to specify a surface that corresponds to the flat geometry. This surface can be given as

$$
z^a - z^{\bar a} = 2i\theta^\alpha\sigma^a_{\alpha\dot\beta}\bar\theta^{\dot\beta}\,, \qquad \theta^\alpha = \varphi^\alpha\,.
\tag{2.4}
$$

Here $z^{\bar a}$, $\bar\theta^{\dot\alpha}$, φ^π are related respectively with z^a, θ^α, $\bar\varphi^{\dot\pi}$ by complex conjugation. The surface (4) has the symmetry properties with respect to $\Gamma(G(\zeta))$ required for the flat $N=1$ superspace (similar to the analogous quadric in the "minimal" space $\mathbb{C}^{4|2}$; see Sect. 1).

In agreement with the general considerations of the last section we have to define the regular surfaces. These should be the surfaces that can be transformed at each point to the "flat" form (4) up to the first order by means of the group $\Gamma(G(\zeta))$. The surface (4), considered up to the first order near some point, say $z^A = 0$, is given by the equations of its tangent plane at that point, namely,

$$
z^a = z^{\bar a}\,, \qquad \theta^\alpha = \varphi^\alpha\,.
\tag{2.5}
$$

4 For the sake of simplicity, we take ζ to be a real number throughout

Almost every real (4|4)-dimensional surface in $\mathbb{C}^{4|4}$ can be written in the following form:

$$z^a = x^a + iH^a(x, \theta, \bar{\theta}), \qquad \bar{\varphi}^{\dot{\alpha}} = \bar{\theta}^{\dot{\alpha}} + \bar{H}^{\dot{\alpha}}(x, \theta, \bar{\theta}), \tag{2.6}$$

where $x^a = (z^a + z^{\bar{a}})/2$ and θ^α can be considered as parameters. In Eqs. (6) H^a are some real functions and $\bar{H}^{\dot{\alpha}}$ are some complex ones (we shall use also the notation H^α for the conjugates of $\bar{H}^{\dot{\alpha}}$). To determine the regular surfaces we must consider the tangent plane of a surface (6) at an arbitrary point, say $(x_0, \theta_0, \bar{\theta}_0)$. It is convenient to translate the tangent plane to the origin. The corresponding subspace in $\mathbb{C}^{4|4}$ is given by the following equations

$$\begin{aligned} z^a &= x^b(\delta^a_b + iH^a_{\ ,b}) + \theta^\alpha iH^a_{\ ,\alpha} + \bar{\theta}^{\dot{\alpha}} iH^a_{\ ,\dot{\alpha}}\,, \\ \bar{\varphi}^{\dot{\alpha}} &= x^b \bar{H}^{\dot{\alpha}}_{\ ,b} + \theta^\beta \bar{H}^{\dot{\alpha}}_{\ ,\beta} + \bar{\theta}^{\dot{\beta}}(\delta^{\dot{\alpha}}_{\dot{\beta}} + \bar{H}^{\dot{\alpha}}_{\ ,\dot{\beta}})\,, \end{aligned} \tag{2.7}$$

where the commas mean partial differentiations with respect to the arguments at the point $(x_0, \theta_0, \bar{\theta}_0)$. According to the definition, for a surface (6) to be regular, its tangent subspace (7) must be connected with the fixed subspace (5) by a linear transformation of the group $G(\zeta)$ given in Eqs. (2) and (3). Applying such a transformation to Eq. (5) we obtain an arbitrary subspace in $\mathbb{C}^{4|4}$ that can be recovered from the standard one by means of the group $G(\zeta)$:

$$\begin{aligned} &zA + \theta D - \overline{zA} - \overline{\theta D} = 0\,, \\ &\overline{zB} + \overline{\theta C} - zE - \theta F - \bar{\varphi} G = 0\,, \end{aligned} \tag{2.8}$$

where the indices have been omitted. The subspace (7) corresponds to a regular surface if it coincides with (8) for some matrices A, B, C, D, E, F, G obeying Eq. (3). Let us substitute z and $\bar{\varphi}$ from (7) into (8) and equate the coefficients of $x, \theta, \bar{\theta}$. For the surface (6) to be regular, the resulting equations on $A, \ldots, G$ should be compatible with Eq. (3). It is easy to see that this is always the case if $\zeta \neq 1$ [5]. On the other hand, if $\zeta = 1$ the following condition should be satisfied

$$\operatorname{Im}\{\det(\delta^a_b + iH^a_{\ ,b}) \det(\delta^\alpha_\beta + H^\alpha_{\ ,\beta} + iH^b_{\ ,\beta}(1 - iH)^{-1}_b{}^a H^\alpha_{\ ,a})\} = 0\,, \tag{2.9}$$

where $(1 - iH)^{-1}$ is the inverse of the matrix $(\delta^a_b - iH^a_{\ ,b})$. Thus the requirement of regularity implies a constraint on the surfaces considered for $\zeta = 1$, whereas for $\zeta \neq 1$ no restrictions arise.

It was known that the family of $N = 1$ supergravities can be described in terms of the surfaces (6), or, equivalently, in terms of the superfields $H^a(x, \theta, \bar{\theta})$. $H^\alpha(x, \theta, \bar{\theta})$ called the prepotentials, with the group $\Gamma(G(\zeta))$ of Eq. (1) as a gauge group [2, 3, 11–13, 19–21]. We shall show this later on from a rather general point of view by discussing the internal geometry of the surfaces. Here we notice only that the general regularity condition suggests the constraint (9) on the first derivatives of H^a, H^α for $\zeta = 1$. (The case $\zeta = 1$ corresponds to the alternative minimal supergravity, as we shall see.) The necessity of the constraint (9) was found in [20, 21] by means of completely different methods.

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5 In fact, this is valid only for almost all surfaces. In what follows we will ignore the degenerate surfaces in similar cases as well

3. Supergravity and Induced Geometry

As we have already pointed out, to proceed the study of supergravity it will be useful to have some internal formulations corresponding to the external ones. The latter formulations were outlined above in terms of certain surfaces in the space $\mathbb{C}^{4|4}$. According to the general considerations of Sect. 1 we have to define the internal geometry of the surfaces inherited from the geometry of the ambient space. Now we are going to restore the definition of induced geometry suggested in [7]. At last we shall arrive at the internal formulations for all ζ.

Let us consider the space $\mathbb{R}^N$ provided with the trivial G-structure (see Sect. 1) that corresponds in the standard coordinates, $y^{\hat{n}}$, to the frame field $\delta^{\hat{n}}_{\hat{a}}$, $\hat{a}, \hat{n}=1, \ldots, N$. (In what follows the trivial G-structure in $\mathbb{R}^N$ is always understood to be connected with the standard coordinate frame.) Any tangent frame field $e^{\hat{n}}_{\hat{a}}(y)$ in $\mathbb{R}^N$ admissible for the trivial G-structure satisfies $e^{\hat{n}}_{\hat{a}}(y)=g^{\hat{b}}_{\hat{a}}(y)\delta^{\hat{n}}_{\hat{b}}$ for some $[g^{\hat{b}}_{\hat{a}}(y)]$ in the group G. Suppose an M-dimensional surface $\mathscr{M}$ in $\mathbb{R}^N$ obeys the condition that at each point on $\mathscr{M}$ one can choose such an admissible frame $e^{\hat{n}}_{\hat{a}}$, that the first M of its vectors are tangent to $\mathscr{M}$. Every such frame in $\mathbb{R}^N$, called an adopted frame, then also defines an M-dimensional tangent frame on the surface. For an arbitrary surface the adopted frames may not exist. The surfaces on which the adopted frames do exist at every point are to be called regular, as this condition corresponds to the first order normalizability (cf. Sect. 1). Indeed, let $e^{\hat{n}}_{\hat{a}}$ be an adopted frame on the surface $\mathscr{M}$. The vectors $e^{\hat{n}}_1, \ldots, e^{\hat{n}}_M$ span the tangent space of $\mathscr{M}$, considered as a subspace in $\mathbb{R}^N$. Since the frame $e^{\hat{n}}_{\hat{a}}$ is admissible for the trivial G-structure in $\mathbb{R}^N$, it can be transformed into the standard coordinate frame by means of the group G. Under this transformation the subspace spanned by $e^{\hat{n}}_1, \ldots, e^{\hat{n}}_M$ goes into the subspace V defined in $\mathbb{R}^N$ by the equations

$$y^{M+1}=y^{M+2}= \ldots =y^N=0. \tag{3.1}$$

Therefore the adopted frames exist at every point of a surface if and only if the tangent subspace of this surface at each point can be connected with the fixed subspace V in $\mathbb{R}^N$ by means of a transformation belonging to G. We observe that the adopted frames on a regular surface are defined up to the group $\tilde{G}$, which is a subgroup of G consisting of transformations that leave the subspace V invariant. Consequently, the M-dimensional tangent frames, defined on the surface in terms of different adopted frames, are connected by transformations constituting a certain group G'. This group is a factor group of $\tilde{G}$; it consists of linear transformations in the M-dimensional space V that can be extended to transformations of $\mathbb{R}^N$ belonging to $\tilde{G}$. Thus every regular surface receives a G'-structure. This is the structure induced on a regular surface by the trivial G-structure in $\mathbb{R}^N$ [7].

To be more explicit, if the subspace V is given by (1), the subgroup $\tilde{G}$, which leaves it invariant, consists of linear transformations belonging to G, with matrices $(g^{\hat{b}}_{\hat{a}})$, $\hat{a}, \hat{b}=1, \ldots, N$, satisfying $g^{b'}_a=0$ for $a=1, \ldots, M$; $b'=M+1, \ldots, N$. Then the group G' corresponds to $M \times M$ matrices (g^b_a) that can be extended to $(g^{\hat{b}}_{\hat{a}})$ in G with $g^{b'}_a=0$.

In general a structure that can be obtained as an induced one on some surface is not arbitrary. Let us consider now an M-dimensional manifold $\mathscr{M}$ and a

G'-structure with the above group G' in it. Let x^m, $m=1,\ldots,M$, be some coordinates in $\mathscr{M}$ and $e_a^m(x)$, $a, m=1,\ldots,M$, be an admissible frame field for this G'-structure. If $y^{\hat{n}}=y^{\hat{n}}(x)$ is some embedding of $\mathscr{M}$ into $\mathbb{R}^N$, the vectors $e_a^m(x)$ go into the N-dimensional vectors $\tilde{e}_a^{\hat{n}}=e_a^m(x)\,[\partial y^{\hat{n}}(x)/\partial x^m]$. These vectors are tangent to the surface $y^{\hat{n}}=y^{\hat{n}}(x)$ in $\mathbb{R}^N$ and define a frame representing a G'-structure on this surface, equivalent to the original structure in $\mathscr{M}$. Suppose, the vectors $\tilde{e}_a^{\hat{n}}(x)$ can be supplemented at each x by some vectors $\tilde{e}_{M+1}^{\hat{n}}(x),\ldots,\tilde{e}_N^{\hat{n}}(x)$ so as to obtain a frame $\tilde{e}_{\hat{a}}^{\hat{n}}(x)$ admissible for the trivial G-structure in $\mathbb{R}^N$. In this case the structure on the surface, descended from the given G'-structure in $\mathscr{M}$, coincides with the structure induced on that surface by the trivial G-structure in $\mathbb{R}^N$. Then the functions $y^{\hat{b}}(x)$, $\hat{b}=1,\ldots,N$, giving such an embedding, if they ever exist, must satisfy the equation

$$\frac{\partial y^{\hat{b}}(x)}{\partial x^m}\,e_a^m(x)=g_a^{\hat{b}}(x) \tag{3.2}$$

for an $M\times N$ matrix function $g_a^{\hat{b}}(x)$, which is the rectangular part of some G-valued function $g_{\hat{a}}^{\hat{b}}(x)$ corresponding to $\hat{a}=a=1,\ldots,M$. We conclude, that the given G'-structure in $\mathscr{M}$ is equivalent to the structure induced on some surface by the trivial G-structure in $\mathbb{R}^N$ if and only if the partial differential equation (2) has a solution. Here $e_a^m(x)$ is an arbitrary chosen frame field representing the given G'-structure, while $y^{\hat{b}}(x)$ and $g_a^{\hat{b}}(x)$ are considered as unknown functions, with $g_a^{\hat{b}}(x)$ still obeying the requirement specified above. For Eq. (2) to have a solution the G'-structure in $\mathscr{M}$ must be subjected to certain constraints.

To proceed with supergravity, we notice first that the use of the subspace (1) in the definitions of regular surfaces and induced structures is merely a convention. Instead of this, one may consider an arbitrary fixed subspace V. Then the adopted frames must be defined so that certain linear combinations of their vectors are tangent to the surface, the number of independent combinations being equal to the dimension of V. Their form is determined by the requirement that for the coordinate frame in $\mathbb{R}^N$ these combinations give the vectors belonging to the fixed subspace V.

We are ready now to determine the internal geometry of those surfaces which appeared in the description of supergravity in Sect. 2. We dealt there with the space $\mathbb{C}^{4|4}$ supplied with an additional geometrical structure. This geometry was being specified by its symmetry group $\Gamma(G(\zeta))$. Now we can identify this geometry as the trivial $G(\zeta)$-structure in $\mathbb{R}^{8|8}\simeq\mathbb{C}^{4|4}$. Indeed, as it is known, the group $\Gamma(G)$, for any G, is just the group that preserves the trivial G-structure (see Appendix A). Furthermore, the real (4|4)-dimensional surfaces in $\mathbb{C}^{4|4}$ that we considered in the last section were regular. One sees that the definition of regular surfaces, used there, is consistent with what is required in the definition of induced structure, provided the subspace V is as follows. It is to be the real (4|4)-dimensional subspace of $\mathbb{C}^{4|4}$ given by Eqs. (2.5). The subgroup, $\widetilde{G(\zeta)}$, of the group $G(\zeta)$, that leaves this subspace V invariant, consists of transformations (2.2–3) which satisfy $A_b^a=(A_{\bar{b}}^{\bar{a}})^*$, $B_b^\alpha=(E_{\bar{b}}^{\dot{\alpha}})^*$, $C_\beta^a=(G_{\dot{\beta}}^{\bar{a}})^*$, and $D_\beta^a=F_\beta^{\dot{\alpha}}=0$. Now it is easy to find the group that corresponds to the structure induced on regular surfaces by the trivial $G(\zeta)$-structure in $\mathbb{C}^{4|4}$ [that is the group G', if $G=G(\zeta)$]. We use a special notation for this group, namely SCR(ζ), as we did earlier [7, 9]. Let us take $x^a=(z^a+z^{\bar{a}})/2$,

$a=1,\ldots,4$, and θ^α, $\alpha=1,2$, as the coordinates in the subspace V given in Eq. (2.5). (Note, that x^a are real even coordinates, while θ^α are complex odd ones.) The group SCR(ζ) proves to consist of the following transformations:

$$x'^a = A^a_b x^b\,, \qquad \theta'^\alpha = B^\alpha_b x^b + C^\alpha_\beta \theta^\beta\,, \tag{3.3}$$

where $\bar{A}^a_b = (A^a_b)^*$, and

$$\begin{aligned} &(\det A)^\zeta = (\det C)^\zeta \det \bar{C} && \text{if} \quad \zeta \neq \infty\,, \\ &\det A = \det C && \text{if} \quad \zeta = \infty\,. \end{aligned} \tag{3.4}$$

Thus every real (4|4)-dimensional regular surface in $\mathbb{C}^{4|4}$ has a natural SCR(ζ)-structure induced on it by the trivial $G(\zeta)$-structure in the ambient space. This gives us an internal formulation for every supergravity model in the family parametrized by ζ. The role of the field is played by the internal geometry of surfaces, that is by the induced SCR(ζ)-structure, rather than by the surface itself. We see that the notion of induced geometry yields a natural way to establish internal formulations intimately connected with the external ones. The regular surfaces, used in an external formulation, can be described in terms of the superfields $H^a(x,\theta,\bar{\theta})$, $H^\alpha(x,\theta,\bar{\theta})$ as in Eq. (2.7); these are unconstrained if $\zeta \neq 1$, and satisfy (2.9) if $\zeta=1$. An SCR(ζ)-structure corresponding to the induced geometry on some of these surfaces can be represented by an admissible frame field. Let $x^M=(x^m,\theta^\mu,\bar{\theta}^{\dot{\mu}})$ be some coordinates on a regular surface [for instance, they may be the parameters appearing in Eq. (2.7)]. Then the internal geometry induced on this surface can be described in terms of a frame field $E^M_A(x,\theta,\bar{\theta})$, defined up to local transformations of the group SCR(ζ) acting on the index $A=(a,\alpha,\dot{\alpha})$. [Here $M=(m,\mu,\dot{\mu})$ is the world index.] Given some superfields $H^a(x,\theta,\bar{\theta})$, $H^\alpha(x,\theta,\bar{\theta})$, one can find an explicit expression for a frame field $E^M_A(x,\theta,\bar{\theta})$, admissible for the SCR(ζ)-structure induced on the corresponding surface (see Appendix B). On the other hand, not every arbitrary frame field $E^M_A(x,\theta,\bar{\theta})$ can be expressed in this way in terms of some $H^a(x,\theta,\bar{\theta})$, $H^\alpha(x,\theta,\bar{\theta})$. That is to say, for an SCR(ζ)-structure to be equivalent to an induced one on some surface, it must satisfy certain constraints (in agreement with what we have pointed out above). The constraints corresponding to an induced G'-structure can be imposed on the admissible frame fields. Generally there is always a set of conditions on derivatives of admissible frame fields of increasing order. In some cases it happens, however, that the conditions on the first and second derivatives are not only necessary, but also sufficient.

In order to establish the internal formulations of supergravity in a self-contained manner, the constraints corresponding to the induced SCR(ζ)-structures must be specified explicitly. In the general case the constraints of the first and second order can always be rewritten in terms of torsion and curvature. For the induced SCR(ζ)-structure the resulting conditions on the torsion and curvature tensors are listed below. This result will be derived from a general theorem in a subsequent paper [22]. Moreover, it will be proved there that for SCR(ζ)-structures no higher order conditions are required, that is the torsion and curvature constraints are necessary and sufficient.

Let us consider an arbitrary connection in some SCR(ζ)-structure defined on a real (4|4)-dimensional manifold $\mathcal{M}$. In other words, for each admissible frame field

E_A^M on $\mathscr{M}$ we have the connection coefficients ω_{BA}^{C} corresponding to the Lie algebra scr(ζ) of the group SCR(ζ). That is to say, ω_{BA}^{C} satisfy (apart from the obvious reality conditions):

$$\begin{gathered}\omega_{\beta A}^{c}=0\,, \qquad \omega_{\beta A}^{\dot\gamma}=0\,,\\ \begin{cases}\zeta(\omega_{bA}^{b}-\omega_{\beta A}^{\beta})-\omega_{\dot\beta A}^{\dot\beta}=0 & \text{if} \quad \zeta\neq\infty\,,\\ \omega_{bA}^{b}-\omega_{\beta A}^{\beta}=0 & \text{if} \quad \zeta=\infty\,.\end{cases}\end{gathered} \tag{3.5}$$

Now we may use the torsion, T_{AB}^{C}, and the curvature, R_{BCD}^{A}, of the connection ω_{BA}^{C}, defined as usual.

We assert that if the SCR(ζ)-structure under consideration corresponds to the induced geometry, then the torsion and the curvature of connections in this structure satisfy the following constraints. Firstly, the torsion of an arbitrary connection satisfies

$$\begin{gathered}T_{\alpha\beta}^{c}=0\,, \qquad T_{\alpha\beta}^{\dot\gamma}=0\,,\\ \text{if} \quad \zeta=\infty\,, \quad \text{also} \quad T_{\alpha b}^{b}-T_{\alpha\dot\beta}^{\dot\beta}=0\,.\end{gathered} \tag{3.6}$$

Secondly, if $\zeta\neq\infty$, the curvature satisfies certain constraints as well. [In the case $\zeta=\infty$, that will correspond to the minimal supergravity, no constraints beyond (6) are needed.] To write down the curvature constraints it is convenient to restrict the freedom in the connections. For this purpose, one imposes usually some conditions on the torsion that reduce the choice of connections and imply no additional constraints on the frame fields. It is easy to see that in an arbitrary SCR(ζ)-structure with $\zeta\neq\infty$ one can always choose a connection so that only the following torsion components may be non-zero: $T_{\alpha\dot\beta}^{c}$, $T_{\alpha\beta}^{c}$, $T_{\alpha\beta}^{\dot\gamma}$ and, if $\zeta\neq 1$, also $T_{\alpha b}^{c}=\frac{1}{4}T_{\alpha d}^{d}\delta_b^c$, $T_{\alpha\dot\beta}^{\dot\gamma}=\frac{1}{2}T_{\alpha\dot\delta}^{\dot\delta}\delta_{\dot\beta}^{\dot\gamma}$, $T_{\alpha d}^{d}=T_{\alpha\dot\delta}^{\dot\delta}$.

Now for any connection, subjected to these "conventional" constraints, the curvature in the induced SCR(ζ)-structure satisfies

$$R_{\beta\alpha\gamma}^{\alpha}\varepsilon^{\beta\gamma}=0\,. \tag{3.7}$$

The constraints (6), (7) are not only necessary, but also sufficient: if in an SCR(ζ)-structure there exists a connection obeying Eq. (6), the above conventional constraints and Eq. (7), then this structure coincides certainly with the induced one on some regular surface.

Thus we are led to the following self-contained internal formulation of each $N=1$ supergravity model. The role of the field is played by the frame field defined up to local SCR(ζ)-transformations, provided this frame field satisfies the constraints (6) and (7). The relation with the corresponding external formulation can be readily restored by recalling that these constraints are necessary and sufficient for the existence of a surface (i.e. of prepotentials H^a, H^α), in terms of which the frame fields can be expressed in an explicit way offered by the definition of induced geometry.

4. Lorentz Group Structures

In Sect. 3 we have shown that the description of $N=1$ supergravity in terms of real surfaces in complex superspace leads to the description in terms of internal geometry of these surfaces. In the internal formalism the role of the field of supergravity is played by the frame field $E_A^M(x, \theta, \bar{\theta})$ in a (4|4)-dimensional manifold $\mathcal{M}$ satisfying the constraints, which ensure that the geometry determined by these frames corresponds to an induced structure. (The coframes, E_M^A, may be used, of course, instead of frame fields.) The gauge group consists of general coordinate transformations and arbitrary tangent space transformations

$$E_M^A \to E_M^B U_B^A(x, \theta, \bar{\theta}) . \tag{4.1}$$

with the position-dependent matrix U_B^A, which belongs to the group SCR(ζ), described in Sect. 3[6]. The group SCR(ζ) may be thought of as consisting of matrices

$$(U_B^A) = \begin{pmatrix} A_b^a & 0 & 0 \\ B_b^\alpha & C_\beta^\alpha & 0 \\ \bar{B}_b^{\dot{\alpha}} & 0 & \bar{C}_{\dot{\beta}}^{\dot{\alpha}} \end{pmatrix} \tag{4.2}$$

with A_b^a real and $\bar{B}_b^{\dot{\alpha}}$, $\bar{C}_{\dot{\beta}}^{\dot{\alpha}}$ being conjugates of B_b^α, C_β^α. According to the definition of SCR(ζ), other entries of U_B^A must be zero (i.e. $U_\beta^a = \bar{U}_{\dot{\beta}}^a = 0$), while the matrices A and C in (2) satisfy the ζ-dependent condition

$$\begin{aligned} &(\det A \cdot \det C^{-1})^\zeta = \det \bar{C} \quad \text{if} \quad \zeta \neq \infty , \\ &\det A = \det C \qquad\qquad\quad\; \text{if} \quad \zeta = \infty . \end{aligned} \tag{4.3}$$

The Lie algebra scr(ζ) of the group SCR(ζ) consists of the matrices (2) with the following condition imposed on the traces of blocks

$$\begin{aligned} \zeta A_a^a = (1+\zeta) C_\alpha^\alpha , \quad C_\alpha^\alpha = \bar{C}_{\dot{\alpha}}^{\dot{\alpha}} \quad &\text{if} \quad \zeta \neq \infty , \\ A_a^a = C_\alpha^\alpha = \bar{C}_{\dot{\alpha}}^{\dot{\alpha}} \quad &\text{if} \quad \zeta = \infty , \\ A_a^a = C_\alpha^\alpha + C_{\dot{\alpha}}^{\dot{\alpha}} \quad &\text{if} \quad \zeta = 1 . \end{aligned} \tag{4.4}$$

Each gauge equivalence class of frame fields, related by the transformations (1)–(3), defines an SCR(ζ)-structure. The constraints are to be imposed on torsions and curvatures of connections in this SCR(ζ)-structure. These constraints correspond to the induced geometry [see Sect. 3, in particular, Eqs. (3.6), (3.7)].

Obviously, the approach of the induced SCR(ζ)-structures looks like the Wess-Zumino type formalism in $N=1$ supergravities [1–3], but accounts for the geometrical content of constraints. The only difference is that the group SCR(ζ) is larger than the commonly used Lorentz group L [or $L \times U(1)$] and contains it as a subgroup. Let us show that the equivalence of these two approaches can be established merely by imposing certain gauge conditions on the frame fields. In geometrical terms, we must define a subclass of frames, connected by transformations of the group L at each point, starting from the SCR(ζ) equivalence class of frames in the manifold M. In other words, we must reduce the structure group of the given SCR(ζ)-structure to its subgroup L[7]. There exists sometimes a canonical

6 Until now we considered the kinematics only. We shall see in the next section that these gauge transformations form the symmetry of the action as well

7 When $\zeta = 1$, which corresponds to the alternative minimal supergravity, the reduced group will be $L \times U(1)$

procedure for such a reduction which uses the so-called structure function of the G-structure. (The general case is described in detail in [16].)

Suppose that for a G-structure we can find a function $c(E_A^M)$ of the admissible frames E_A^M at all points of $\mathscr{M}$ such that this function takes values in a space $\mathscr{C}$ of some representation of the group G' and transforms according to this representation under the allowed changes of admissible frames. Thus, if E_A^M runs over the admissible frames at a fixed point $\hat{x}=(x^M)=(x^m, \theta^\mu, \bar{\theta}^{\dot{\mu}})$ of $\mathscr{M}$, the function $c(E_A^M)$ takes values in some orbit of the action of G' in the space $\mathscr{C}$. It may happen that for all points these orbits coincide. In such a case one can use the function c to reduce the structure group G' to the subgroup G'' which is the isotropy subgroup of G' corresponding to a point c_0 in the orbit. Indeed, the frames of the G'-structure that satisfy $c(E_A^M)=c_0$ are connected at each point of $\mathscr{M}$ by the transformations of the group G'' and thus define the reduced G''-structure. The reduction will be canonical, that is it will not depend on the coordinate system, if the function transforms as a scalar under the coordinate changes (or, equivalently, under the diffeomorphisms of $\mathscr{M}$).

The function satisfying these requirements may be constructed for an arbitrary G'-structure from the torsion tensor, T, which is defined, as usual, by the following relation

$$\begin{aligned} T_{BC}^A = & (-)^{MC} E_B^M E_C^N E_{M,N}^A \\ & -(-)^{MB}(-)^{CB} E_C^M E_B^N E_{M,N}^A \\ & -[\omega_{CB}^A - (-)^{BC}\omega_{BC}^A], \end{aligned} \tag{4.5}$$

where $E_A^M(\hat{x})$ is a frame field corresponding to the given G'-structure; E_M^A is the inverse of E_A^M, while $\omega_{BC}^A(\hat{x})$ are the coefficients of a connection in this G'-structure. The torsion components T_{BC}^A at each point depend on the choice of the frame and the connection. The components T_{BC}^A are scalars with respect to the coordinate changes and behave covariantly under the local transformations (1), with $U_B^A(\hat{x})$ in the group G', provided ω_{BC}^A are subjected to the corresponding gauge transformation. If we fix the frame field $E_A^M(\hat{x})$ and alter the connection, $\omega_{BC}^A \to \omega_{BC}^A - \gamma_{BC}^A$, then

$$T_{BC}^A \to T_{BC}^A + \gamma_{CB}^A - (-)^{BC}\gamma_{BC}^A, \tag{4.6}$$

where the coefficients $\gamma_{BC}^A(\hat{x})$ define, of course, a matrix in the Lie algebra $\mathfrak{g}'$ for each fixed value of the index C. Suppressing any possible indices, let us define the function c as the torsion T modulo (6). It is easy to see that such a function does not depend on the connection and, moreover, it satisfies all the requirements specified above. This function c is called the (first) structure function of the G'-structure[8]. If the values of c turn out to lie in a single orbit, it can be used to reduce the structure group.

Let us return to the case of interest, when $G'=\mathrm{SCR}(\zeta)$. The structure function of an $\mathrm{SCR}(\zeta)$-structure can be represented conveniently as follows. For a given

8 See [16]; an exposition suitable for the present purposes can be found in [7,9]

admissible frame field $E_A^M(\hat{x})$ the coefficients ω_{BC}^A in (5) can be chosen to satisfy

$$\begin{aligned}
&T_{ab}^c = T_{ab}^\gamma = T_{\dot{a}b}^\gamma = T_{\dot{a}b}^{\dot{\gamma}} = T_{\alpha\beta}^\gamma = 0\,,\\
&T_{\dot{a}b}^c = \tfrac{1}{4} T_\alpha \delta_b^c \quad (T_\alpha \equiv T_{\alpha d}^d)\,,\\
&T_{\alpha\beta}^{\dot{\gamma}} = \tfrac{1}{2} t_\alpha \delta_\beta^{\dot{\gamma}} \quad (t_\alpha \equiv T_{\alpha\dot{\delta}}^{\dot{\delta}})\,,\\
&\begin{cases} T_\alpha = t_\alpha & \text{if} \quad \zeta \neq 1, \infty\,,\\ T_\alpha = -t_\alpha & \text{if} \quad \zeta = \infty\,,\\ T_\alpha = t_\alpha = 0 & \text{if} \quad \zeta = 1\,.\end{cases}
\end{aligned} \tag{4.7}$$

This is always possible, as it can be seen, e.g. from (6), taking into account that γ_{BC}^A corresponds to the algebra scr(ζ), i.e. for any fixed index C, the matrix γ_{BC}^A satisfies Eqs. (2) and (4). Moreover, the conditions (7) exhaust the freedom (6).

The remaining torsion coefficients, or rather some linear combinations of them, namely

$$\begin{aligned}
&T_{\alpha\dot{\beta}}^c,\ T_{\alpha\beta}^c,\ T_{\alpha\beta}^{\dot{\gamma}}\,,\\
&\begin{cases} \zeta T_{\alpha d}^d - (1+\zeta) T_{\alpha\dot{\delta}}^{\dot{\delta}} & \text{if} \quad \zeta \neq 1, \infty\\ T_{\alpha d}^d - T_{\alpha\dot{\delta}}^{\dot{\delta}} & \text{if} \quad \zeta = \infty\\ 0 & \text{if} \quad \zeta = 1\,,\end{cases}
\end{aligned} \tag{4.8}$$

may be referred to as components of the structure function, since to take the torsion modulo the connection is equivalent to eliminating the freedom (6) by means of a condition like (7). The residual freedom, $\omega_{BC}^A \to \omega_{BC}^A - \gamma_{BC}^A$ with $\gamma_{BC}^A = (-)^{BC} \gamma_{CB}^A$, is irrelevant, as it has no effect on the torsion[9]. In other words, the torsion does not depend on connections, which satisfy (7)[10]. Consequently, we may consider the components (8) as functions of a point and a frame field at this point, and denote these functions together by $c(E_A^M)$. The components (8) of the structure function will be used now to reduce the structure group.

First of all, remember that in $N=1$ supergravity we deal with the induced SCR(ζ)-structures, which satisfy the constraints

$$\begin{aligned}
&T_{\alpha\beta}^c = T_{\alpha\beta}^{\dot{\gamma}} = 0\,,\\
&\begin{cases} T_{\alpha d}^d - T_{\alpha\dot{\delta}}^{\dot{\delta}} = 0 & \text{if} \quad \zeta = \infty\\ R_{\beta\alpha\gamma}^\alpha \varepsilon^{\beta\gamma} = 0 & \text{if} \quad \zeta \neq \infty\,,\end{cases}
\end{aligned} \tag{4.9}$$

where R_{BCD}^A is the curvature of some arbitrary connection obeying (7). It is no surprise that these constraints are imposed on the torsion coefficients which

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9 For the algebra scr(ζ), the condition $\gamma_{BC}^A = (-)^{BC} \gamma_{CB}^A$ does not imply $\gamma_{BC}^A = 0$. Hence any torsion constraints can never fix the connection. It is well known, that for the Lorentz algebra, for instance, the opposite holds

10 It can be seen that an scr(ζ) connection does not enter the explicit expressions for the linear combinations (8) on account of Eqs. (2), (4). Thus the particular linear combinations (8) do not depend on scr(ζ) connections without any reference to condition (7). See [7, 9] for such a treatment of structure functions in general

correspond to the components of the structure function of the SCR(ζ)-structure[11]. The reason is that the structure function depends only on the SCR(ζ)-structure, but not on any additional structures like the connection. Moreover, the structure function is related to the SCR(ζ)-structure in a coordinate independent manner[12].

We shall consider now the torsion components which correspond to non-vanishing components of the structure function of an induced SCR(ζ)-structure. Let us look first at the components $T^c_{\alpha\dot\beta}$, which may survive after imposing the constraints (7), (9). Let us define a 4×4 matrix $\Gamma^c_b=\frac{1}{4i}\sigma_b^{\dot\beta\alpha}T^c_{\alpha\dot\beta}$, which is real. Here $\sigma_b^{\dot\beta\alpha}$ form a basis of Hermitian matricies, which is chosen to consist of the unit matrix and three Pauli matricies. Suppose that

$$\det(\Gamma^c_b)\neq 0\,. \tag{4.10}$$

This requirement is satisfied by generic surfaces (in fact, by almost all surfaces). If the condition (10) holds and $\zeta\neq 1/3$, the function $T^c_{\alpha\dot\beta}(E^M_A)$ takes values in a single orbit of the group SCR(ζ). In other words, any "matrix" $T^c_{\alpha\dot\beta}(E^M_A)$ can be connected with some fixed matrix, say $\mathring{T}^c_{\alpha\dot\beta}=2i\sigma^c_{\alpha\dot\beta}$, by a transformation of the group SCR(ζ). Indeed, under the tangent space transformations (1)–(3) of the group SCR(ζ) this function transforms as follows

$$T^c_{\alpha\dot\beta}\rightarrow A^c_d T^d_{\gamma\dot\delta}C_\alpha^{-1\gamma}\bar C_{\dot\beta}^{-1\dot\delta}\,. \tag{4.11}$$

In terms of $\Gamma^c_b=\frac{1}{4i}\sigma_b^{\dot\beta\alpha}T^c_{\alpha\dot\beta}$, this reads as

$$\Gamma^c_b\rightarrow A^c_d\Gamma^d_e O^e_b|\det C|^{-1}\,, \tag{4.12}$$

where the matrix O^e_b belongs to the group $SO(1,3)$. From condition (3) it follows that

$$\det(A^c_d\cdot|\det C|^{-1})=|\det C|^{-4}\det A=|\det C|^{\frac{1-3\zeta}{\zeta}}\,.$$

If $\zeta\neq 1/3$, the matrix $K^c_d=A^c_d|\det C|^{-1}$ is quite arbitrary. Consequently for all ζ, except $\zeta=1/3$, we can satisfy the gauge condition on the frame fields: $\Gamma^c_b=\delta^c_b$, or equivalently

$$T^c_{\alpha\dot\beta}=2i\sigma^c_{\alpha\dot\beta}\,. \tag{4.13}$$

In the case $\zeta=1/3$, this condition must be imposed as a constraint, which corresponds to a restriction on the second derivatives of the fields H^a, H^α[13]. It can be shown that condition (13) agrees with the requirement of the second order normalizability of surfaces discussed in Sect. 2. This requirement is satisfied by almost all surfaces, if $\zeta\neq 1/3$. In what follows we shall not mention this peculiarity of the $\zeta=1/3$ case.

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11 The constraint $R^\alpha_{\beta\alpha\gamma}\varepsilon^{\beta\gamma}=0$ in (4.9) is, in fact, a condition on the second structure function. The relevant definitions can be found in [16]

12 Analogous statements are valid, of course, for an arbitrary G'-structure

13 This was pointed out also in [21]

Let us impose (13) as a gauge condition on frame fields. The structure group SCR(ζ) is then reduced to a subgroup G'' which consists of transformations (1)–(3) preserving (13). By inspection of (11)–(13) one obtains easily that such transformations (1)–(3) must satisfy

$$\begin{gathered} A_b^a=\tfrac{1}{2}\sigma^a_{\alpha\dot\beta}\sigma_b^{\dot\delta\gamma}C_\gamma^\alpha\bar C_{\dot\delta}^{\dot\beta}\,, \\ \begin{cases} \det C=1 & \text{if} \quad \zeta \neq 1/3, 1 \\ |\det C|=1 & \text{if} \quad \zeta=1 \\ \det C=\det\bar C & \text{if} \quad \zeta=1/3\,. \end{cases} \end{gathered} \tag{4.14}$$

This defines a group G'' and reduces the SCR(ζ)-structure to a G''-structure. The latter has its own structure function which will be used to reduce the group further.

The conditions eliminating the freedom to change the torsion under the shifts of connection may be chosen for a G''-structure to be as follows

$$\begin{gathered} T_{ab}^c=T_{ab}^\gamma=T_{ab}^{\dot\gamma}=T_{\dot\alpha b}^\gamma=T_{\alpha\beta}^\gamma\,, \\ T_{ab}^c(\sigma_c^b)_{\dot\beta}^{\dot\gamma}=0 \quad \left[(\sigma_c^b)_{\dot\beta}^{\dot\gamma}\equiv\frac{i}{2}(\sigma^{b,\dot\gamma\alpha}\sigma_{c,\alpha\dot\beta}-\sigma_c^{\dot\gamma\alpha}\sigma^b_{\alpha\dot\beta})\right], \\ t_\alpha\equiv T_{\alpha\dot\delta}^{\dot\delta}=0\,, \quad \text{if} \quad \zeta=1/3, 1\,. \end{gathered} \tag{4.15}$$

Let us consider the transformations of the unconstrained components under the group G'', which is given by (1)–(3) and (4). If $A_b^a=\delta_b^a$, $C_\beta^\alpha=\delta_\beta^\alpha$, we have, in particular,

$$\begin{gathered} T_{\alpha\dot\beta}^{\dot\gamma}\to T_{\alpha\dot\beta}^{\dot\gamma}+\bar B_c^{\dot\gamma}2i\sigma^c_{\alpha\dot\beta}\,, \\ T_\alpha\to T_\alpha+\bar B_d^{\dot\delta}2i\sigma^d_{\alpha\dot\delta}\,, \end{gathered} \tag{4.16}$$

where $T_\alpha\equiv T_{\alpha d}^d$ as before. Equation (16) shows that we can impose a gauge condition, for instance, as follows

$$\begin{cases} (1-2\zeta)T_\alpha=2(1-\zeta)t_\alpha & \text{if} \quad \zeta\neq\infty \\ T_\alpha=-t_\alpha & \text{if} \quad \zeta=\infty\,, \end{cases} \tag{4.17}$$

which restricts the group G'' to a subgroup G'''. Thus we obtain a G'''-structure, where the group G''' consists of the tangent space transformations $E_M^a\to E_M^bA_b^a$, $E_M^\alpha\to E_M^\beta C_\beta^\alpha$, $E_M^{\dot\alpha}\to E_M^{\dot\beta}\bar C_{\dot\beta}^{\dot\alpha}$, with condition (14) imposed. We see that $G'''=L$, for $\zeta\neq1/3, 1$, and $G'''=L\times U(1)$, for $\zeta=1$, where L is the conventional Lorentz group, while $U(1)$ stands for the transformations of the form: $x^a\to x^a$, $\theta^\alpha\to e^{i\sigma}\theta^\alpha$, $\bar\theta^{\dot\alpha}\to e^{-i\sigma}\bar\theta^{\dot\alpha}$, which correspond to local γ_5-rotations in the tangent space. That is why the case $\zeta=1$ will correspond to the alternative minimal supergravity (cf. [10]). In the case $\zeta=1/3$, according to Eq. (14), the group G''' consists of Lorentz transformations and dilatations. (The relevance of dilatations for $\zeta=1/3$ was pointed out also in [21].)

Let us show that the resulting geometry of G'''-structures correspond to the constrained superspace approach a la Wess-Zumino [1–3, 10] for various ζ. The full list of the constraints on the torsion and the curvature in these G'''-structures is as follows

$$T^c_{ab}=0\,, \qquad T^c_{\dot{a}b}(\sigma^b_c)^{\dot{\gamma}}_{\beta}=0\,, \tag{4.18}$$

$$\begin{cases} T^{\gamma}_{\alpha\beta}=0 & \text{if} \quad \zeta=\infty \\ T^{\gamma}_{\alpha\beta}=\frac{1}{2}\zeta[(T_\alpha-t_\alpha)\delta^{\gamma}_{\beta}+(T_\beta-t_\beta)\delta^{\gamma}_{\alpha}] & \text{if} \quad \zeta\neq\infty\,, \end{cases} \tag{4.19}$$

$$t_\alpha=0 \quad \text{if} \quad \zeta=1/3, 1\,, \tag{4.20}$$

$$T^c_{\alpha\dot{\beta}}=2i\,\sigma^c_{\alpha\dot{\beta}}\,, \tag{4.21}$$

$$\begin{cases} T_\alpha=-t_\alpha & \text{if} \quad \zeta=\infty \\ (1-2\zeta)T_\alpha=2(1-\zeta)t_\alpha & \text{if} \quad \zeta\neq\infty\,, \end{cases} \tag{4.22}$$

$$T^c_{\alpha\beta}=0\,, \qquad T^{\dot{\gamma}}_{\alpha\beta}=0\,, \tag{4.23}$$

$$\begin{cases} T_\alpha-t_\alpha=0 & \text{if} \quad \zeta=\infty \\ R^{\alpha}_{\beta\alpha\gamma}\varepsilon^{\beta\gamma}=0 & \text{if} \quad \zeta\neq 0\,, \end{cases} \tag{4.24}$$

where $T_\alpha\equiv T^d_{\alpha d}$, $t_\alpha\equiv T^{\dot{\delta}}_{\alpha\dot{\delta}}$. One can observe immediatly that these constraints coincide with those of [3]. The constraints (18)–(20) serve to eliminate the freedom in the connection corresponding to the groups G''', while the constraints (21), (22) descend from the gauge conditions imposed under the reductions from $G'=\mathrm{SCR}(\zeta)$ to G'' and from G'' to G'''. Finally, it can be shown by a straightforward calculation that the induced structure constraints for $\mathrm{SCR}(\zeta)$-structures give just the conditions (23), (24) after the described reduction to G'''. Note also, that, in principle, the constraints (18)–(22) may be chosen differently. (These constraints are sometimes called "the conventional constraints"; for a classification of supergravity constraints see, e.g. [3].) Another choice in Eqs. (18)–(22) may change, in particular, the form of the curvature constraint in Eq. (24), which corresponds to an induced structure constraint. In this way one can get other equivalent sets of G'''-constraints, for instance, those of [2] with $\zeta\neq 1/3, 1$ and G''' being the Lorentz group.

Thus we have proved that the Wess-Zumino type formulations of $N=1$ supergravities for all ζ can be recovered from the formulations that use the induced $\mathrm{SCR}(\zeta)$-structures, by imposing certain gauge conditions. On the other hand, the Wess-Zumino type formulations are known to be equivalent to the external formulations, i.e. to the formulations using surfaces in $\mathbb{C}^{4|4}$, or the prepotentials H^a, H^α (see [2, 3, 11–13, 20, 21]). Thus we obtain the equivalence of all three approaches. The internal formulations can be derived from the external ones and vice versa by invoking the notion of induced geometry, as we discussed in Sect. 3.

5. Supergravity Action Functionals

As was already discussed in Sects. 1 and 2, the action of supergravity must be an invariant functional on a space of surfaces. In the particular case of $N=1$ supergravity, we considered real surfaces in complex superspace $\mathbb{C}^{4|4}$ and the group $\Gamma(G(\zeta))$, acting in $\mathbb{C}^{4|4}$ and, hence, on the space of real (4|4)-dimensional regular surfaces[14] in $\mathbb{C}^{4|4}$. Thus the action of $N=1$ supergravity must be a functional on this space of surfaces, invariant with respect to $\Gamma(G(\zeta))$. Such functionals can be defined conveniently in terms of internal geometry of the surfaces. It is worth mentioning that the group $\Gamma(G(\zeta))$ is essentially the group of transformations of $\mathbb{C}^{4|4}$ which preserve the trivial $G(\zeta)$-structure in this space (see Appendix A). In Sect. 3 we have shown that there is a natural internal object on a regular surface, which is determined completely by the surface and the trivial $G(\zeta)$-structure in $\mathbb{C}^{4|4}$. This object is the SCR(ζ)-structure, induced on that surface.

Given a coframe field $E_M^A(\hat{x})$ representing the induced SCR(ζ)-structure on the surface, the action can be constructed as an integral $\int \mathscr{L} dv$ over the surface with the volume element

$$dv = \mathrm{Ber}(E_M^A) d^4x d^2\theta d^2\bar{\theta},$$

where $\hat{x} = (x^M) \equiv (x^m, \theta^\mu, \bar{\theta}^{\dot{\mu}})$ are the coordinates in this (4|4)-dimensional surface and Ber denotes the Berezinian (superdeterminant). Thus we have to look for the functional

$$\int \mathscr{L}(E_M^A(\hat{x}))\, \mathrm{Ber}(E_M^A(\hat{x})) d^4x d^2\theta d^2\bar{\theta}, \tag{5.1}$$

which satisfies the following requirements: (i) It must be invariant under the coordinate changes, (ii) it must be independent of the particular choice of the admissible coframe field corresponding to the SCR(ζ)-structure. These requirements mean that the functional (1) depends only on the SCR(ζ)-structure, induced on the surface by the trivial $G(\zeta)$-structure in $\mathbb{C}^{4|4}$. Consequently, such a functional will be invariant with respect to the group $\Gamma(G(\zeta))$, which preserves that structure in $\mathbb{C}^{4|4}$.

The requirement (i) could be satisfied, if $\mathscr{L}(E_M^A)$ was a world scalar. To satisfy (ii) the function $\mathscr{L}(E_M^A)$ must transform in an obvious way under the tangent space transformations of the group SCR(ζ), acting on E_M^A. Therefore it is natural to try to construct the Lagrangian $\mathscr{L}(E_M^A)$ in terms of the components of the structure function of the induced SCR(ζ)-structure. For those who have not entered into details of the last section, this can be put somewhat differently.

Consider the torsions of the connections in the SCR(ζ)-structure. In particular, consider the components of the torsion tensor represented in the tangent space basis generated by an admissible frame. It may happen that among them there are such components that do not depend on the choice of the connection. Whether or not depends, generally, on the group considered. For the group SCR(ζ) this is the case, in particular, for the components $T^c_{\alpha\dot{\beta}}$. Indeed, $T^c_{\alpha\dot{\beta}}$ involves the components $\omega^c_{\alpha\dot{\beta}}$ and $\omega^c_{\dot{\beta}\alpha}$ of an scr(ζ) connection, but these vanish, according to the definition of the Lie algebra scr(ζ). Consequently, the connection does not enter the expression for $T^c_{\alpha\dot{\beta}}$. These torsion components are the world scalars and transform covariantly under the SCR(ζ)-transformations in the tangent space. Since $T^c_{\alpha\dot{\beta}}$ may be now considered as a function of the point $\hat{x}$ and the admissible coframe E_M^A at that point, it can be used to construct the Lagrangian.

14 Remember (Sect. 2), that the requirement of regularity implies no considerable restrictions on the set of these surfaces for all ζ, except $\zeta = 1$

Thus we are led naturally to the following expression for the action of supergravity

$$\mathscr{A}_\zeta = k \int |\Gamma|^n \operatorname{Ber}(E_M^A) d^4x d^2\theta d^2\bar{\theta}, \tag{5.2}$$

where k is a constant, $n=(1-\zeta)/(3\zeta-1)$ [15], and $\Gamma=\det(\Gamma_b^c)$ with $\Gamma_b^c=\frac{1}{4i}\sigma_b^{\dot\beta\alpha}T_{\alpha\dot\beta}^c$. The functional (2) satisfies the requirements (i) and (ii) specified above. Indeed (i) is obvious, while (ii) is ensured by the choice $n=(1-\zeta)/(3\zeta-1)$. It can be easily verified using Eqs. (4.11) or (4.12), that the integrand of (2) does not change under the tangent space transformations (4.1)–(4.3) of the group SCR(ζ). We remark that the expression (2) makes no sense if $\zeta=1/3$, while other values of ζ are allowed, including $\zeta=\infty$, when $n=-1/3$ (minimal supergravity). It turns out, however, that the integral in Eq. (1) vanishes identically, if $\zeta=1$ $(n=0)$. In this case the integral on the right-hand-side of Eq. (2) is reduced merely to $\int dv$. However, it was proved in [10] that $\int dv \equiv 0$ in alternative minimal supergravity, due to $\zeta=1$ constraints. To apply this result, we have to remind the reader that the constraints on the "$U(1)$-superspace" of [10] differ from the induced structure constraints only by a gauge condition (see Sect. 4).

It is easy to see for an arbitrary ζ as well, how the Wess-Zumino action, $\mathscr{A}=\int dv$ [1], may come out in our approach. For this purpose, let us consider the gauge conditions on the frame fields, which reduce the structure group SCR(ζ) to its subgroup G''', as discussed in Sect. 4. (Remember that G''' is the Lorentz group L, if $\zeta \neq 1, 1/3$, and $G'''=L\times U(1)$, if $\zeta=1$.) In this gauge we have $\Gamma=1$, due to the requirement $T_{\alpha\dot\beta}^c=2i\sigma_{\alpha\dot\beta}^c$, cf. (4.13). This reduces the action (2) to

$$\tilde{\mathscr{A}}_\zeta = k \int \operatorname{Ber}(E_M^A) d^4x d^2\theta d^2\bar{\theta} \equiv k \int dv, \tag{5.3}$$

where the coframe field represents the reduced G'''-structure. This is just the familiar action of Wess and Zumino.

By the way, we have obtained the equivalence of the induced SCR(ζ)-structure approach and the Lorentz group picture on the dynamical level for minimal $(\zeta=\infty)$ and non-minimal $(\zeta \neq 1/3, 1, \infty)$ supergravities. The expression for the action of alternative minimal supergravity $(\zeta=1)$, different from $\int dv \equiv 0$, has been found in [10]. In the rest of the section we show that this "non-geometrical" action for $\zeta=1$ can be obtained as a limit $\zeta \to 1$ of the "geometrical" actions of non-minimal supergravities (2) [or (3), when $\Gamma=1$], if one sets $k=1/n$.

For this purpose, let us take a one-parameter family of coframe fields, corresponding to such a family of structures, that for each value of ζ there is an induced SCR(ζ)-structure. Then we must substitute this family of coframe fields into the expression (2) and consider the behaviour of $\mathscr{A}_\zeta$ near the point $\zeta=1$. It is convenient to use the above parameter n related to ζ via $n=(1-\zeta)/(3\zeta-1)$, or $\zeta=(n+1)/(3n+1)$. Thus we have to consider the family $E_M^A(\hat{x}|n)$, and impose at each value of n the constraints ensuring that the coframe field $E_M^A(\hat{x}|n)$ defines an induced SCR(ζ)-structure with $\zeta=(n+1)/(3n+1)$. Such constraints are given by Eqs. (3.6), (3.7). Moreover, we require that the coframe fields $E_M^A(\hat{x}|n)$ at all values of

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15 One may use n instead of ζ to parametrize the various formulations of supergravity. It coincides with the parameter n of [3]

n define one and the same CR-structure. Here CR denotes the group of linear transformations (4.2) without any further conditions like (4.3)[16]. In other words, $E_M^A(\hat{x}|n_1)$ and $E_M^A(\hat{x}|n_2)$, for arbitrary n_1, n_2 must be connected by a transformation (4.1), (4.2). Using such coframe fields let us pass in expression (2) with $k=1/n$, to the limit $n\to 0$, which corresponds to $\zeta\to 1$. In order to do this we expand the functions $E_M^A(\hat{x}|n)$ up to the first order in the small parameter n:

$$E_M^A(\hat{x}|n)\approx E_M^A(\hat{x}|0)+nE_M^B(\hat{x}|0)V_B^A(\hat{x})\,. \tag{5.4}$$

The above requirement of the CR-equivalence implies that the matrix $(\delta_B^A+nV_B^A)$ must represent an infinitesimal transformation of the group CR [Eq. (4.2)]. Therefore we set

$$V_\alpha^{\dot\beta}=V_{\dot\alpha}^{\beta}=0\,,\qquad V_\alpha^b=V_{\dot\alpha}^b=0\,.$$

After the substitution of the expansion (4), the integrand in Eq. (2) becomes

$$|\Gamma(\hat{x}|n)|^n\,\mathrm{Ber}(E_M^A(\hat{x}|n))\approx(1+n\ln|\Gamma(\hat{x}|0)|+nV(\hat{x}))\,\mathrm{Ber}(E_M^A(\hat{x}|0))\,,$$

where $V=V_A^A\equiv V_\alpha^\alpha-V_{\dot\alpha}^\alpha-V_{\dot\alpha}^{\dot\alpha}$. Since $E_M^A(x|0)$ satisfies the constraints corresponding to alternative minimal supergravity ($\zeta=1$), one has $\int\mathrm{Ber}(E_M^A(\hat{x}|0))d^4xd^2\theta d^2\bar\theta=0$. Consequently the limit $n\to 0$ of the expression $(1/n)\int|\Gamma(n)|^n dv(n)$ makes sense. Thus we obtain

$$\mathscr{A}_{\zeta=1}=\int(\ln|\Gamma|+V)\,\mathrm{Ber}(E_M^A)d^4xd^2\theta d^2\bar\theta\,, \tag{5.5}$$

where Γ is determined as before by the coframe field $E_M^A(\hat{x})$ corresponding to the induced SCR(1)-structure.

The matrix function $V_B^A(\hat{x})$ in the expansion (4) is not fixed completely in terms of $E_M^A(\hat{x})=E_M^A(\hat{x}|0)$. However, it cannot be quite arbitrary, due to the torsion and curvature constraints on the frame fields $E_M^A(\hat{x}|n)$. A straightforward, but still fairly tedious calculation shows that the supertrace V of V_B^A can be represented as $V=U+\bar U$, where the function U may be an arbitrary solution of the following equation

$$\mathscr{D}^\alpha(D_\alpha U-\omega_{\dot\beta\alpha}^{\dot\beta})=0\,. \tag{5.6}$$

Here we have assumed the following notations:

$$D_A=E_A^M(\hat{x})\partial_M\,,\qquad \mathscr{D}^\alpha=\varepsilon^{\alpha\beta}(D_\beta+2\tau_{\beta\gamma}^\gamma)\,,\qquad \omega_{\dot\beta\alpha}^{\dot\beta}=(\tau_{\alpha b}^b+\tau_{\alpha\dot\beta}^{\dot\beta})/3\,,$$

where τ_{AB}^C is determined by the frame field $E_A^M(\hat{x})$ through the relation $[D_A, D_B\}=\tau_{AB}^C D_C$. As it was already mentioned, we have $\Gamma=1$ when reducing to the conventional $L\times U(1)$ superspace. A quick comparison with [10] shows that the functional (5) coincides with the action of alternative minimal supergravity up to

$$\int X\,\mathrm{Ber}(E_M^A)d^4xd^2\theta d^2\bar\theta\,,$$

16 In mathematics the CR-structures (Cauchy-Riemann structures) appear in the study of geometry of real surfaces in complex space. In our case these structures correspond to conformal supergravity

where X is an arbitrary function obeying $\mathscr{D}^{\alpha}D_{\alpha}X=0$. However, it was proved by Frolov (private communication) that such a term does not affect the field equations.

Finally, it is shown in Appendix B that the geometrical framework of this paper makes the task of finding the prepotential form of the actions straightforward.

Appendix A. The Automorphisms of the Trivial G-Structure

Let us find out the transformations that leave the trivial G-structure in the space $\mathbb{R}^N$ invariant (the case of a superspace, say $\mathbb{R}^{P|Q}$, may be included by setting $N=P|Q$). The trivial G-structure ($G\subset \mathrm{GL}(N,\mathbb{R})$) in $\mathbb{R}^N$ can be represented by the coordinate frame $e_a^n=\delta_a^n$, a, $n=1,\ldots,N$, corresponding to the standard coordinates y^n in that space. Other admissible frame fields are connected with the coordinate one by means of y-dependent transformations of the group G; for example, $\tilde{e}_a^n(y)=g_a^b(y)\delta_b^n$ with $g_a^b(y)$ in G. Let us consider a transformation, ϕ, of the space $\mathbb{R}^N$, that is $y^n\to\tilde{y}^n=\phi^n(y)$. This transformation acts on the tangent vectors and, hence, on the tangent frames. Under the action of ϕ an arbitrary frame field $e_a^n(y)$ goes into $\tilde{e}_a^n(\tilde{y})=e_a^{\ell}(y)\partial\phi^n(y)/\partial y^{\ell}$. One says that a transformation leaves the G-structure invariant if it transforms the admissible frames into the admissible ones. It is easy to describe such transformations (i.e. automorphisms) for the trivial G-structure. For the transformation ϕ to be an automorphism, it must give an admissible frame, $\tilde{e}_a^n(y)=g_a^b(y)\delta_b^n$, when applied, for example, to the coordinate frame $e_a^n(y)=\delta_a^n$; thus

$$\delta_a^{\ell}\partial\phi^n(y)/\partial y^{\ell}=g_a^b(\phi(y))\delta_b^n\,.$$

That is to say, at each y the Jacobian matrix $\left[\dfrac{\partial\phi^n(y)}{\partial y^{\ell}}\right]$ must belong to the group G. All the transformations obeying this condition constitute the group $\Gamma(G)$ referred to in Sect. 1 and 3. Thus $\Gamma(G)$ is indeed the group of automorphisms of the trivial G-structure.

Appendix B. Actions in the Prepotential Form

In order to write down the actions directly in terms of the superfields $H^a(x,\theta,\bar{\theta})$, $H^{\alpha}(x,\theta,\bar{\theta})$, we have to find an explicit expression for the frame fields via the prepotentials H^a, H^{α}. Then we shall be able to obtain the desired result by substituting this expression into the action functionals (5.2). Remember, that the frame fields in the internal formulations of supergravity correspond to induced geometry. According to Sects. 2 and 3 we must consider a regular surface of real dimension (4|4) in the space $\mathbb{C}^{4|4}$. Such a surface can be given by Eq. (2.6), where the fields H^a, H^{α} giving a regular surface are constrained by (2.9) if $\zeta=1$. Let us look for the frame fields on the surface (2.6) that correspond to the $\mathrm{SCR}(\zeta)$-structure induced on this surface by the trivial $G(\zeta)$-structure in $\mathbb{C}^{4|4}$. First of all we have the coordinate frame in $\mathbb{C}^{4|4}$, which consists of the vectors $\mathscr{E}_a=\partial/\partial z^a$, $\mathscr{E}_{\alpha}=\partial/\partial\theta^{\alpha}$, $\mathscr{F}_{\dot{\pi}}=\partial/\partial\bar{\varphi}^{\dot{\pi}}$, and their conjugates $\mathscr{E}_{\bar{a}}$, $\mathscr{E}_{\dot{\alpha}}$, and $\mathscr{F}_{\pi}$. An arbitrary frame field, admissible for the trivial $G(\zeta)$-structure in $\mathbb{C}^{4|4}$ is connected with the coordinate one by means of a local transformation $\tilde{\mathscr{E}}_A = U_A^B \mathscr{E}_B$, where $\mathscr{E}_A = (\mathscr{E}_a, \mathscr{E}_\alpha, \mathscr{F}_{\dot{\pi}})$ and similarly for $\tilde{\mathscr{E}}_A$, while $[U_A^B]$ belongs to $G(\zeta)$. In the case under consideration the definition of regular surfaces and SCR(ζ)-structures induced on them uses a subspace V defined in $\mathbb{C}^{4|4}$ by means of the equations $z^a = z^{\bar{a}}$, $\theta^\alpha = \varphi^\alpha$, $\bar{\theta}^{\dot{\alpha}} = \bar{\varphi}^{\dot{\alpha}}$. We see that for the above coordinate frame the vectors $\mathscr{E}_a + \mathscr{E}_{\bar{a}}$, $\mathscr{E}_\alpha + \mathscr{F}_\alpha$, $\mathscr{E}_{\dot{\alpha}} + \mathscr{F}_{\dot{\alpha}}$ belong to V and define a frame in this real (4|4)-dimensional subspace of $\mathbb{C}^{4|4}$. Therefore, according to Sect. 3, the SCR(ζ)-structure induced on a surface can be defined by means of a frame field $(E_A) = (E_a, E_\alpha, E_{\dot{\alpha}})$ if the vectors E_A are tangent to this surface and can be represented as $E_a = \tilde{\mathscr{E}}_a + \tilde{\mathscr{E}}_{\bar{a}}$, $E_\alpha = \tilde{\mathscr{E}}_\alpha + \tilde{\mathscr{F}}_\alpha$, $E_{\dot{\alpha}} = \tilde{\mathscr{E}}_{\dot{\alpha}} + \tilde{\mathscr{F}}_{\dot{\alpha}}$ with $\tilde{\mathscr{E}}_a$, $\tilde{\mathscr{E}}_\alpha$, $\tilde{\mathscr{F}}_\pi$ being the vectors of some frame in $\mathbb{C}^{4|4}$ admissible for the trivial $G(\zeta)$-structure. All such frames (E_A) are connected by means of the group SCR(ζ). For a surface (2.6) it is easy to verify that the following frame field on it does correspond to the induced SCR(ζ)-structure:

$$E_a = \frac{\partial}{\partial x^a},$$
$$E_\alpha = \phi\left[\frac{\partial}{\partial\theta^\alpha} + iH^b_{,\alpha}(1-iH)_b^{-1c}\frac{\partial}{\partial x^c}\right], \tag{B.1}$$

where the vectors tangent to the surface (2.6) are expressed in terms of the coordinates $x^a = (z^a + z^{\bar{a}})/2$, θ^α, $\bar{\theta}^{\dot{\alpha}}$ on it. The field $\phi(x, \theta, \bar{\theta})$ in Eq. (1) is determined by the following equation:

$$[\phi^{-2}\det(\delta_a^b + iH^b_{,a})]^\zeta = (\bar{\phi})^2 \det[\delta_{\dot{\alpha}}^{\dot{\beta}} + H^{\dot{\beta}}_{,\dot{\alpha}} - iH^b_{,\dot{\alpha}}(1+iH)_b^{-1c} H^{\dot{\beta}}_{,c}]. \tag{B.2}$$

Note, that this definition of ϕ is consistent if for $\zeta = 1$ the fields H^a, H^α satisfy (2.9). This agrees with the fact that for the induced structure to be defined on a surface, this surface must be regular.

As we have already discussed, the action of supergravity must be a functional on a space of surfaces. In Sect. 5 we considered the actions written in terms of the internal geometry of surfaces [Eq. (5.2)]. Given a surface, the value of the action functional was determined by the induced SCR(ζ)-structure on that surface and did not depend on the particular choice of an admissible frame field. Substituting the frame field (1) into the functional (5.2) we find the following expression in terms of the fields H^a, H^α corresponding to a regular surface

$$\mathscr{A}_\zeta = k\int |A|^{n+1}|B|^{-(3n+1)}|\tilde{\Gamma}|^n d^4x d^2\theta d^2\bar{\theta}, \tag{B.3}$$

where

$$n = \frac{1-\zeta}{3\zeta - 1},$$
$$A = \det(\delta_b^a + iH^a_{,b}),$$
$$B = \det[\delta_{\dot{\beta}}^{\dot{\alpha}} + H^{\dot{\alpha}}_{,\dot{\beta}} - iH^b_{,\dot{\beta}}(1+iH)_b^{-1a} H^{\dot{\alpha}}_{,a}],$$
$$\tilde{\Gamma} = \det \tilde{\Gamma}_b^c, \qquad \tilde{\Gamma}_b^c = \frac{1}{4i}\Gamma^c_{\alpha\dot{\beta}}\sigma_b^{\dot{\beta}\alpha},$$
$$\Gamma^c_{\alpha\dot{\beta}} = \left(\frac{\partial}{\partial\theta^\alpha} + e_a^b\frac{\partial}{\partial x^b}\right)e^c_{\dot{\beta}} + \left(\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} + e^b_{\dot{\beta}}\frac{\partial}{\partial x^b}\right)e^c_\alpha,$$
$$e^b_\alpha = iH^c_{,\alpha}(1-iH)_c^{-1b}, \qquad e^b_{\dot{\alpha}} = -iH^c_{,\dot{\alpha}}(1+iH)_c^{-1b}.$$

Note, that $\Gamma^c_{\alpha\dot\beta}=|\phi|^{-2}T^c_{\alpha\dot\beta}$, where $T^c_{\alpha\dot\beta}$ are the torsion components entering Eq. (5.2). As a matter of fact, $\Gamma^c_{\alpha\dot\beta}$ are the coefficients of the Levi form of the real surface $z^a=x^a+iH^a(x,\theta,\bar\theta)$ in $\mathbb{C}^{4|2}$ [cf. [15]; one obtains this surface from Eq. (2.6) by a projection of $\mathbb{C}^{4|4}$ onto $\mathbb{C}^{4|2}$, when $(z,\theta,\bar\varphi)\rightarrow(z,\theta)$]. Equation (3) gives the actions for $\zeta \neq 1/3, 1$. According to Sect. 5 the action for $\zeta=1$ can be obtained from (3) by taking $k=1/n$ and $n\rightarrow 0$. For the fields H^a, H^α satisfying the constraint (2.9), we find in this way

$$\mathscr{A}_{\zeta=1}=\int |A\cdot B^{-1}|\cdot\ln|A\cdot B^{-3}\cdot\tilde\Gamma|d^4x d^2\theta d^2\bar\theta. \tag{B.4}$$

Thus we get the prepotential form of the actions for the minimal, non-minimal [Eq. (3) with $\zeta\neq 1/3, 1$] and alternative minimal [Eq. (4)] supergravities; this agrees with [3, 12, 20].

Acknowledgement. The authors are grateful to I. V. Frolov for many helpful discussions.

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Communicated by Ya. G. Sinai

Received September 25, 1983; in revised form May 3, 1984