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The Broadwell Model for Initial Values in $L^1_+(\mathbb{R})$

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Abstract. The Cauchy problem for the Broadwell model is shown to have a global mild solution for initial data in $L^1_+(\mathbb{R})$ with small L^1-norm, and a local solution for arbitrary initial data in $L^1_+(\mathbb{R})$. For data which are small in $L^1(\mathbb{R})$, the asymptotic behaviour of the solutions as $t \to \infty$ is determined. Moreover, it is shown that a global solution exists for all initial values in $L^1_+(\mathbb{R})$ with finite entropy if the H-Theorem holds.

Introduction

The Broadwell model is one of the simplest non-trivial discrete velocity models of the Boltzmann equation and has found a lot of attention as a model problem in kinetic theory. Nishida and Mimura [5] and Crandall and Tartar [6] have shown that the Cauchy problem has always a global unique solution for bounded initial data. Inoue and Nishida have studied the asymptotic stability of equilibrium solutions [3], and Caflisch and Papanicolaou have investigated the fluid dynamical limit as the mean free path between collisions tends to 0 [1]. Open questions are the asymptotic behaviour of solutions as $t \to \infty$ and the largest possible class of admissible initial values.

The last question is of relevance because the mass conservation law suggests L^1-spaces as natural spaces in which to look for solutions of the Boltzmann equation. However, the quadratic terms in the collision operator are a priori not defined in L^1, and this is one major reason why existence theorems for the Boltzmann equation have only been proven for smaller sets of initial data and in general only locally in time.

In this paper I investigate the question of solvability of the Broadwell model for initial values in $L^1_+(\mathbb{R})$. The basic idea is to obtain an a priori upper bound of the solution by solving a model with suitably truncated collision terms, for which I give monotone approximations – this is done in Sect. 2 for nonnegative initial data

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with small L^1-norm. The proofs use some auxiliary results which I give in Sect. 1. In Sect. 3, I use the a priori-bound and a modification of the Kaniel-Shinbrot iteration scheme [4] to obtain a global solution to the Broadwell model when the initial data have small L^1_+-norm. This result implies local existence for any initial values in L^1_+. If the initial data have finite entropy and if an H-Theorem holds, then the method of Crandall and Tartar [6] applies and global existence follows. This is done in Sect. 6. I have no proof of the H-Theorem for the solutions constructed in Sect. 3, but I give a semi-formal discussion in Sect. 5.

In Sect. 4, the asymptotic behaviour as $t \to \infty$ of the solutions constructed in Sect. 3 is determined. The methods I use combine techniques introduced by Tartar [7], Kaniel and Shinbrot [4], and myself [2].

Notation. Let $\alpha \in \mathbb{R}$. D_α will denote the differential operator $\partial_t + \alpha\partial_x$. For $\alpha = +1$, -1 and 0, I will write D_+, D_- and D, respectively. λ denotes the Lebesgue measure on $\mathbb{R}$, $\mathscr{B}(\mathbb{R})$ the class of all Borel sets, and L^1_+ the set of all nonnegative functions in L^1. The letter C will denote different constants in different formulas.

1. Some Auxiliary Results

Let $T > 0$ be arbitrary but fixed, and let $\Omega = [0, T] \times \mathbb{R}$.

Lemma 1. *Let* $h \in L^1_+(\mathbb{R})$, $f \in L^1_+(\Omega)$. *Then the initial value problem*

$$\begin{aligned} &\text{a) } D_\alpha w = f - w^2 \\ &\text{b) } w(0, \cdot) = h \end{aligned} \tag{1}$$

has a unique nonnegative mild solution $w \in C([0, T]; L^1(\mathbb{R}))$ *in the sense that* $w^2 \in L^1(\Omega)$, $D_\alpha w \in L^1(\Omega)$ *and* a) *holds in* $L^1(\Omega)$, b) *in* $L^1(\mathbb{R})$.

Proof. Let $f_N(x) := \min\{f(x), N\}$, $N = 0, 1, 2, \ldots$, and let w_N be the solution of

$$D_\alpha w = f_N - w^2, \qquad w(0, \cdot) = h. \tag{2}$$

All the w_N exist, are uniquely determined and satisfy

$$0 \leqq w_N(t, x) \leqq w_{N+1}(t, x) \quad \text{a.e. in } \Omega.$$

From (2), it follows that

$$w_N(t, x + \alpha t) + \int_0^t w_N^2(\tau, x + \alpha\tau)d\tau = h(x) + \int_0^t f_N(\tau, x + \alpha\tau)dr. \tag{3}$$

Thus $w_N(t, \cdot + \alpha t)$ is bounded by the integrable function

$$h + \int_0^t f(\tau, \cdot + \alpha\tau)d\tau$$

and monotone increasing in N. Hence there is a limit $w(t, \cdot + \alpha t)$ in $L^1_+(\mathbb{R})$. Clearly, $w_N^2(t, x + \alpha t)$ converges monotonically to $w^2(t, x + \alpha t)$ for almost all x. From (3), for all $t \in [0, T]$ and almost all x

$$\int_0^t w_N^2(\tau, x + \alpha\tau)d\tau \leqq h(x) + \int_0^t f(\tau, x + \alpha\tau)d\tau.$$

By Beppo Levi's theorem,

$$\lim_{N\to\infty} \int_0^t w_N^2(\tau, x+\alpha\tau)d\tau = \int_0^t w^2(\tau, x+\alpha\tau)d\tau .$$

Passing to the limit $N\to\infty$ in (3), w is seen to be a mild solution of (1) in the sense of the definition. The properties $w^2 \in L^1(\Omega)$, $D_\alpha w \in L^1(\Omega)$ are easily read off from (3), and from

$$w(t,x+\alpha t) + \int_0^t w^2(\tau, x+\alpha\tau)d\tau = h(x) + \int_0^t f(\tau, x+\alpha\tau)d\tau ,$$

it follows that $w \in C([0, T]; L^1(\mathbb{R}))$.

To show uniqueness, assume that $\tilde{w}$ is a second solution. Then

$$D(w-\tilde{w}) = \tilde{w}^2 - w^2 = -(w+\tilde{w})(w-\tilde{w}) ,$$

which implies $w=\tilde{w}$ a.e. in Ω. □

Now let $f_1, f_2 \in L^1_+(\Omega)$ such that $f_1 \leqq f_2$ a.e., and let h_1, $h_2 \in L^1_+(\mathbb{R})$ such that $h_1 \leqq h_2$ a.e.

Lemma 2. *If w_i denotes the mild solution of*

$$D_\alpha w = f_i - w^2 , \qquad w(0, \cdot) = h_i, \qquad i = 1, 2 ,$$

then $w_1 \leqq w_2$ a.e. *in Ω.*

Proof. Left to the reader. □

Now suppose that $f_1, f_2 \in L^1_+(\Omega)$ such that $f_1 \leqq f_2$ a.e., and $g_1, g_2 \in L^1_+(\Omega)$ such that $g_1 \geqq g_2$ a.e., and let $h \in L^1_+(\mathbb{R})$.

Lemma 3. *The unique solutions w_i of the linear Cauchy problems*

$$D_\alpha w = f_i - g_i w, \qquad w(0, \cdot) = h, \qquad i = 1, 2 ,$$

satisfy $w_1 \leqq w_2$ a.e. *in Ω.*

Proof. By inspection (the statement is read off from the explicit solutions).

2. A Truncated System

Let $f, g, h \in L^1_+(\mathbb{R})$, and let $\gamma = \max\{\|f\|_{L^1}, \|g\|_{L^1}, \|h\|_{L^1}\}$. On Ω, consider the Cauchy problem

$$\begin{aligned} D_+ u &= w^2 , & u(0, \cdot) &= f \\ D_- v &= w^2 , & v(0, \cdot) &= g \\ Dw &= 2(uv - w^2), & w(0, \cdot) &= h . \end{aligned} \tag{4}$$

Definition. (u, v, w) is called a mild solution of (4), if

a) $u, v, w \in C([0, T]; L^1_+(\mathbb{R}))$
b) $D_+ u, D_- v, Dw \in L^1(\Omega)$
c) $w^2, uv \in L^1(\Omega)$
d) (4) holds (a.e. in Ω) in the sense of $L^1(\Omega)$, and the initial conditions hold in $L^1(\mathbb{R})$.

Theorem 1. *If* $\gamma \leqq 1/4$, *then* (4) *has a mild solution* $(\bar{u}, \bar{v}, \bar{w})$, *which is the limit of the iteratively defined functions*

$$u_0(t,x) = f(x-t), \qquad v_0(t,x) = g(x+t),$$

$$\begin{aligned} &\text{a)} \quad Dw_n = 2(u_n v_n - w_n^2), \qquad w_n(0,\cdot) = h \\ &\text{b)} \quad D_+ u_{n+1} = w_n^2, \qquad u_{n+1}(0,\cdot) = f, \\ &\text{c)} \quad D_- v_{n+1} = w_n^2, \qquad v_{n+1}(0,\cdot) = g, \end{aligned} \tag{5}$$

$n = 0, 1, 2, \ldots$. *The iterates* u_n, v_n *and* w_n *are well-defined and form a non-decreasing sequence. Furthermore, there is a non-decreasing sequence of functions* $\tilde{u}_n$, $\tilde{v}_n$, $\tilde{w}_n \in L^1_+(\mathbb{R})$ *and there are* $\tilde{u}$, $\tilde{v}$, $\tilde{w} \in L^1_+(\mathbb{R})$, *such that*

$$\|\tilde{u}\|_{L^1}, \|\tilde{v}\|_{L^1}, \|\tilde{w}\|_{L^1} \leqq 2\gamma,$$

and

$$\|\tilde{u}_n\|_{L^1}, \|\tilde{v}_n\|_{L^1}, \|\tilde{w}_n\|_{L^1} \leqq 2\gamma, \qquad n = 0, 1, 2, \ldots, \tag{6}$$

and

$$\begin{aligned} &u_n(t,x) \leqq \tilde{u}_n(x-t) \leqq \tilde{u}(x-t) \\ &\bar{u}(t,x) \leqq \tilde{u}(x-t) \\ &v_n(t,x) \leqq \tilde{v}_n(x+t) \leqq \tilde{v}(x+t) \\ &\bar{v}(t,x) \leqq \tilde{v}(x+t) \\ &w_n(t,x) \leqq \tilde{w}_n(x) \leqq \tilde{w}(x) \\ &\bar{w}(t,x) \leqq \tilde{w}(x) \end{aligned} \tag{7}$$

a.e. *in* Ω.

Proof. The plan of the proof is as follows: I first prove the existence of all the iterates; on the way, (6), (7) and the monotonicity are obtained. The existence of $(\bar{u}, \bar{v}, \bar{w})$ and $(\tilde{u}, \tilde{v}, \tilde{w})$ then follows from Beppo Levi's theorem.

Let $\tilde{u}_0 = f$, $\tilde{v}_0 = g$, $\tilde{w}_0(x) = h(x) + 2\int_0^T u_0 v_0(\tau, x)\,d\tau$.

Notice that

$$\begin{aligned} \int_{\mathbb{R}} \int_0^T u_0 v_0(\tau,x)\,d\tau\,dx &= \int_{\mathbb{R}} \int_0^T f(x-\tau) g(x+\tau)\,d\tau\,dx \\ &\leqq \int_{\mathbb{R}} \int_0^T f(z) g(z+2\tau)\,d\tau\,dz \\ &\leqq (1/2) \cdot \|f\|_{L^1} \|g\|_{L^1} \leqq (1/2)\gamma^2. \end{aligned}$$

Hence $u_0 v_0 \in L^1_+(\Omega)$, and by Lemma 1 w_0 is well defined. Clearly $w_0(t,x) \leqq \tilde{w}_0(x)$, and $\|\tilde{w}_0\|_{L^1} \leqq \gamma + \gamma^2 \leqq 2\gamma$. This proves (6) and (7) for $n = 0$. As w_0 is nonnegative and

$$w_0(t,x) + 2\int_0^t w_0^2(\tau,x)\,d\tau = h(x) + 2\int_0^t u_0 v_0(\tau,x)\,d\tau,$$

it follows that $2\int_{\mathbb{R}} \int_0^T w_0^2(\tau,x)\,d\tau\,dx \leqq \gamma + \gamma^2$. In particular, $w_0^2 \in L^1_+(\Omega)$.

Now assume that u_n, v_n and $\tilde{u}_n$, $\tilde{v}_n$ have been constructed such that (6) and (7) are true. As $u_n v_n \in L^1_+(\Omega)$, it follows from Lemma 1 that $w_n(t,x)$ exists a.e. in Ω and is nonnegative. By (5) a),

$$\begin{aligned} w_n(t,x) &= h(x) + 2\int_0^t [u_n v_n - w_n^2](\tau,x)d\tau \\ &\leqq h(x) + 2\int_0^t \tilde{u}_n(x-\tau)\tilde{v}_n(x+\tau)d\tau - 2\int_0^t w_n^2(\tau,x)d\tau\,, \end{aligned}$$

and therefore

$$\begin{aligned} 2\int_{\mathbb{R}}\int_0^T w_n^2(\tau,x)d\tau dx &\leqq \|h\|_{L^1} + \|\tilde{u}_n\|_{L^1}\|\tilde{v}_n\|_{L^1} \\ &\leqq \gamma + 4\gamma^2 \leqq 2\gamma\,. \end{aligned}$$

Consequently, $u_{n+1}(t,x+t) = f(x) + \int_0^t w_n^2(\tau,x+\tau)d\tau$, and

$$v_{n+1}(t,x-t) = g(x) + \int_0^t w_n^2(\tau,x-\tau)d\tau$$

are well defined, because $\int_0^t w_n^2(\tau,x+\tau)d\tau$ and $\int_0^t w_n^2(\tau,x-\tau)d\tau$ exist for all $t \leqq T$ and almost all x. Let

$$\tilde{u}_{n+1}(x) := f(x) + \int_0^T w_n^2(\tau,x+\tau)d\tau\,,$$

$$\tilde{v}_{n+1}(x) := g(x) + \int_0^T w_n^2(\tau,x-\tau)d\tau\,,$$

$$\tilde{w}_n(x) := h(x) + 2\int_0^T u_n v_n(\tau,x)d\tau\,.$$

Clearly

$$\|\tilde{u}_{n+1}\|_{L^1} \leqq \gamma + (1/2)\gamma + 2\gamma^2 \leqq 2\gamma\,,$$

$$\|\tilde{v}_{n+1}\|_{L^1} \leqq 2\gamma\,,$$

$$\|\tilde{w}_n\|_{L^1} \leqq \gamma + 4\gamma^2 \leqq 2\gamma\,.$$

Now note that $u_1 \geqq u_0$ and $v_1 \geqq v_0$ a.e. and that therefore, by Lemma 2, $w_1 \geqq w_0$ a.e. Inductively one shows that a.e. $u_{n+1} \geqq u_n$, $v_{n+1} \geqq v_n$ and $w_{n+1} \geqq w_n$, and from this the corresponding inequalities for the $\tilde{u}_n$, $\tilde{v}_n$ and $\tilde{w}_n$ follow.

Beppo Levi's theorem implies that the u_n, v_n and w_n have limits $\bar{u}$, $\bar{v}$ and $\bar{w}$ in $L^1_+(\Omega)$, and that the $\tilde{u}_n$, $\tilde{v}_n$ and $\tilde{w}_n$ have limits $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ in $L^1_+(\mathbb{R})$. Also, the relations $\bar{u}(t,x) \leqq \tilde{u}(x-t)$, $\bar{v}(t,x) \leqq \tilde{v}(x+t)$ and $\bar{w}(t,x) \leqq \tilde{w}(x)$ hold a.e. in Ω.

From (5), I know that for all $t \in [0,T]$ and almost all x

$$u_{n+1}(t,x+t) = f(x) + \int_0^t w_n^2(\tau,x+\tau)d\tau\,. \tag{8}$$

Notice that w_n^2 is non-decreasing as $n\to\infty$, that $w_n^2(t,x)\to\bar{w}^2(t,x)$ a.e. in Ω, and that

$$2\int_{\mathbb{R}}\int_0^T \bar{w}^2(\tau,x+\tau)d\tau dx \leqq \gamma+(2\gamma)^2 \leqq 1/2\,.$$

Thus

$$\lim_{n\to\infty}\int_{\mathbb{R}}\int_0^T (\bar{w}^2-w_n^2)(\tau,x+\tau)d\tau dx=0\,,$$

which implies that for almost all $x\in\mathbb{R}$

$$\int_0^T (\bar{w}^2-w_n^2)(\tau,x+\tau)d\tau\to 0 \quad \text{as} \quad n\to\infty\,.$$

Let $n\to\infty$ in (8). It follows that

$$\bar{u}(t,x+t)=f(x)+\int_0^t \bar{w}^2(\tau,x+\tau)d\tau$$

holds a.e. in Ω. The corresponding equations for $\bar{v}$ and $\bar{w}$ are obtained similarly. This completes the proof. □

Remark. For $\gamma>1/4$, a standard domain of dependence argument shows that (4) has a local mild solution in the sense of the definition. There is, however, in general no mild solution for arbitrary T, as the following example shows:

Let $f\equiv g\equiv h\equiv 1$. These initial data are in $L^1_{\mathrm{loc},+}(\mathbb{R})$, and (4) reduces to the system of ordinary differential equations

$$\dot{u}=w^2\,,\qquad u(0)=1\,,\qquad \dot{w}=2(u^2-w^2)\,,\qquad w(0)=1 \tag{9}$$

(note that necessarily $v=u$).

Equation (9) has a local solution satisfying $u(t)\geqq w(t)$. Thus

$$\frac{d}{dt}(2u+w)=2u^2\geqq 2[(1/3)(2u+w)]^2=(2/9)(2u+w)^2\,.$$

This differential inequality implies that $2u+w$ becomes infinite after a finite time, and there is no global solution. To treat the case where the initial data are in L^1_+, it is sufficient to multiply the initial values given above by 0 outside of a sufficiently large interval.

3. The Broadwell Model for Initial Data with Small L^1-Norm

As before, let $\Omega=[0,T]\times\mathbb{R}$, $f,g,h\in L^1_+(\mathbb{R})$, $\gamma=\max\{\|f\|_{L^1},\|g\|_{L^1},\|h\|_{L^1}\}$. I consider the following Cauchy problem for the Broadwell model:

$$\begin{aligned} D_+u&=w^2-uv\,, & u(0,\cdot)&=f\,,\\ D_-v&=w^2-uv\,, & v(0,\cdot)&=g\,,\\ Dw&=2(uv-w^2)\,, & w(0,\cdot)&=h\,. \end{aligned} \tag{10}$$

Let $\gamma\leqq 1/4$ and let $(\bar{u},\bar{v},\bar{w})$ be the solution of (4) given by Theorem 1. I will construct a solution of (10) as limit of the following coupled system of iterations, which is modeled after the iteration scheme introduced by Kaniel and Shinbrot [4]:

$$u_0^l = v_0^l = w_0^l = 0\,,$$
$$u_0^h = \bar{u}\,, \qquad v_0^h = \bar{v}\,, \qquad w_0^h = \bar{w}\,,$$

and

$$\begin{aligned} D_+ u_{n+1}^h &= (w_n^h)^2 - u_{n+1}^h v_n^l\,, & u_{n+1}^h(0,\cdot) &= f\,,\\ D_- v_{n+1}^h &= (w_n^h)^2 - v_{n+1}^h u_n^l\,, & v_{n+1}^h(0,\cdot) &= g\,,\\ D w_{n+1}^h &= u_n^h v_n^h - (w_{n+1}^h)^2\,, & w_{n+1}^h(0,\cdot) &= h\,, \end{aligned} \tag{11a}$$

$$\begin{aligned} D_+ u_{n+1}^l &= (w_n^l)^2 - u_{n+1}^l v_n^h\,, & u_{n+1}^l(0,\cdot) &= f\,,\\ D_- v_{n+1}^l &= (w_n^l)^2 - v_{n+1}^l u_n^h\,, & v_{n+1}^l(0,\cdot) &= g\,,\\ D w_{n+1}^l &= u_n^l v_n^l - (w_{n+1}^l)^2\,, & w_{n+1}^l(0,\cdot) &= h\,. \end{aligned} \tag{11b}$$

Lemma 4. *The functions u_n^l, u_n^h, v_n^l, v_n^h, w_n^l and w_n^h are well defined by* (11), *are in $L^1(\Omega)$ and satisfy* a.e. *in Ω,*

$$\begin{aligned} 0 \leqq u_n^l \leqq u_{n+1}^l \leqq u_{n+1}^h \leqq u_n^h \leqq \bar{u}\\ 0 \leqq v_n^l \leqq v_{n+1}^l \leqq v_{n+1}^h \leqq v_n^h \leqq \bar{v}\\ 0 \leqq w_n^l \leqq w_{n+1}^l \leqq w_{n+1}^h \leqq w_n^h \leqq \bar{w} \end{aligned} \tag{12}$$

for all $n \geqq 0$.

Proof. The proof is by induction. The existence and monotonicity properties for $n=0, 1$ follow from Lemmas 1, 2, and 3 in Sect. 1. Assume the existence of all the iterates in (11) and (12) have been shown for $n \leqq m$. Then $w_m^l \leqq w_m^h$, $v_m^l \leqq v_m^h$ a.e., and Lemma 3 implies that a.e. $u_{m+1}^h \geqq u_{m+1}^l$, and similarly it follows from $w_{m+1}^h \geqq w_m^h$ and $v_m^l \geqq v_{m-1}^l$ that $u_{m+1}^h \leqq u_m^h$. All other inequalities follow likewise. □

Relation (12) implies that there are u^h, u^l, v^h, v^l, w^h, $w^l \in L_+^1(\Omega)$, such that $u_n^l \nearrow u^l$, $u_n^h \searrow u^h$, $v_n^l \nearrow v^l$, $v_n^h \searrow v^h$, $w_n^l \nearrow w^l$, $w_n^h \searrow w^h$ a.e. in Ω. Since $u^l \leqq u^h \leqq \bar{u}$ and $v^l \leqq v^h \leqq \bar{v}$ a.e. in Ω and because $\bar{u} \cdot \bar{v}$ is integrable over Ω, it follows that $u_{n+1}^h \cdot v_n^l \to u^h v^l$ in $L^1(\Omega)$ as $n \to \infty$ by the dominated convergence theorem. Similar convergence statements hold for all other terms on the right of (11).

Using convergence in $L^1(\Omega)$ and the monotonicity properties, one shows

$$\begin{aligned} &\left.\begin{aligned} D_+ u^h &= (w^h)^2 - u^h v^l\\ D_+ u^l &= (w^l)^2 - u^l v^h\,, \end{aligned}\right\} \quad u^h(0,\cdot) = u^l(0,\cdot) = f\,,\\ &\left.\begin{aligned} D_- v^h &= (w^h)^2 - v^h u^l\\ D_- v^l &= (w^l)^2 - v^l u^h\,, \end{aligned}\right\} \quad v^h(0,\cdot) = v^l(0,\cdot) = g\,,\\ &\left.\begin{aligned} D w^h &= 2(u^h v^h - (w^h)^2)\\ D w^l &= 2(u^l v^l - (w^l)^2)\,, \end{aligned}\right\} \quad w^h(0,\cdot) = w^l(0,\cdot) = h\,, \end{aligned} \tag{13}$$

in the mild sense.

Theorem 2. *If γ is sufficiently small, then necessarily $u^h = u^l$, $v^h = v^l$, $w^h = w^l$* a.e. *in Ω, and $(u, v, w) = (u^h, v^h, w^h)$ is a mild solution of* (10).

Proof. Note that $u^l(t,x+t)\leqq u^h(t,x+t)\leqq \bar{u}(t,x+t)\leqq \tilde{u}(x)$ and $v^l(t,x-t)\leqq v^h(t,x-t)\leqq \bar{v}(t,x-t)\leqq \tilde{v}(x)$ for all $t\in[0,T]$ and almost all $x\in\mathbb{R}$. Also,

$$u^h(t,x-t)=f(x-2t)+\int_0^t((w^h)^2-u^hv^l)(\tau,x-2t+\tau)d\tau$$

a.e. in Ω. Therefore, for almost all $x\in\mathbb{R}$,

$$\int_0^T(u^h-u^l)(t,x-t)dt=\int_0^T\int_0^t((w^h)^2-(w^l)^2)(\tau,x-2t+\tau)d\tau dt + \int_0^T\int_0^t(u^lv^h-u^hv^l)(\tau,x-2t+\tau)d\tau dt\,. \tag{14}$$

Almost everywhere in Ω, $w^l\leqq w^h$, and a.e. in $\mathbb{R}$

$$w^h(T,x)+2\int_0^T(w^h)^2(\tau,x)d\tau=h(x)+2\int_0^T u^hv^h(\tau,x)d\tau\,,$$

$$w^l(T,x)+2\int_0^T(w^l)^2(\tau,x)d\tau=h(x)+2\int_0^T u^lv^l(\tau,x)d\tau\,.$$

Substracting and integrating, *I* get

$$\int_{\mathbb{R}}\int_0^T((w^h)^2-(w^l)^2)(\tau,x)d\tau dx\leqq\int_{\mathbb{R}}\int_0^T(u^hv^h-u^lv^l)(\tau,x)d\tau dx\,.$$

I use this inequality to estimate the first term on the right of (14). The result is

$$\int_0^T(u^h-u^l)(t,x-t)dt\leqq(1/2)\int_{\mathbb{R}}\int_0^T(u^hv^h-u^lv^l)(\tau,x)d\tau dx + (1/2)\int_{\mathbb{R}}\int_0^T(u^lv^h-u^hv^l)(\tau,x)d\tau dx\,.$$

By the triangle inequality

$$\int_0^T(u^h-u^l)(t,x-t)dt\leqq(1/2)\int_{\mathbb{R}}\int_0^T[u^h(v^h-v^l) + v^l(u^h-u^l)+u^l(v^h-v^l) + v^l(u^h-u^l)](\tau,x)d\tau dx\,. \tag{15}$$

Moreover, *I* have the estimate

$$\int_{\mathbb{R}}\int_0^T u^h(v^h-v^l)(\tau,x)d\tau dx\leqq\int_{\mathbb{R}}\int_0^T\tilde{u}(x-\tau)(v^h-v^l)(\tau,x)d\tau dx$$

$$=\int_{\mathbb{R}}\int_0^T\tilde{u}(z)(v^h-v^l)(\tau,z+\tau)d\tau dz$$

$$\leqq\|\tilde{u}\|_{L^1}\cdot\sup_{z\in\mathbb{R}}\int_0(v^h-v^l)(\tau,z+\tau)d\tau\,. \tag{16}$$

Let $|||(u,v)|||:=\sup_x \int_0^T u(t,x-t)dt+\sup_x \int_0^T v(t,x+t)dt$, then $|||(u^h,v^h)|||$ and $|||(u^l,v^l)|||$ are finite, and (15) and (16) imply an estimate

$$|||(u^h,v^h)-(u^l,v^l)|||\leqq C\cdot\gamma|||(u^h,v^h)-(u^l,v^l)|||\,, \tag{17}$$

where C is a fixed constant. If $\gamma<1/C$, it follows that $u^h=u^l$ and $v^h=v^l$ a.e. in Ω, and then necessarily also $w^h=w^l$. The proof is complete. □

4. Behaviour as $t\to\infty$

For initial data which are admissible in Theorem 2, the asymptotic behaviour of the solutions as $t\to\infty$ is easily described. The following lemma is helpful.

Lemma 5. *Let $f\in L^1_+[0,\infty)$. Then the Cauchy problem*

$$\frac{d}{dt}u=f-u^2\,,\qquad u(0)=a>0\,. \tag{18}$$

has a unique, absolutely continuous and nonnegative solution u, and $\lim_{t\to\infty}u(t)=0$.

Proof. One easily verifies that (18) has a unique, absolutely continuous and nonnegative solution. For $t_0>0$ and $t\geqq t_0$, it follows that

$$u(t)-u(t_0)=\int_{t_0}^t f(\tau)d\tau-\int_{t_0}^t u^2(\tau)d\tau\,, \tag{19}$$

hence

$$u(t)\leqq u(t_0)+\int_{t_0}^\infty f(\tau)d\tau\,.$$

Let $\varepsilon>0$. There is a t_0 such that $u(t_0)<\varepsilon/2$ and $\int_{t_0}^\infty f(\tau)d\tau<\varepsilon/2$ [otherwise a contradiction to (19) results], and therefore $u(t)\leqq\varepsilon$ for all $t\geqq t_0$. □

Theorem 3. *For the initial data which are admissible in Theorem 2, the solution (u,v,w) of* (10) *satisfies*

a) $\lim_{t\to\infty} w(t,x)=0$ *for almost all* $x\in\mathbb{R}$

b) $u(t,\cdot+t)$ *and* $v(t,\cdot-t)$ *have limits in* $L^1(\mathbb{R})$ *as* $t\to\infty$.

c) $\lim_{t\to\infty} w(t,\cdot)=0$ *in* $L^1(\mathbb{R})$.

Proof. a) is an immediate consequence of Lemma 5 and the fact that $u\cdot v\in L^1([0,\infty)\times\mathbb{R})$. [So far, I have only shown $u\cdot v\in L^1(\Omega)$; as $T>0$ was arbitrary, and as $\|u\cdot v\|_{L^1(\Omega)}$ is bounded independently of T, it actually follows that $u\cdot v\in L^1([0,\infty)\times\mathbb{R})$.] For b), the argument given in [7] is applicable. I repeat it for the convenience of the reader. Let $u_+(t,x)=u(t,x+t)$, $v_-(t,x)=v(t,x-t)$. Then $\frac{\partial}{\partial t}u_+$

and $\frac{\partial}{\partial t} v_-$ are in $L^1([0,\infty)\times\mathbb{R})$. Thus

$$\int_{\mathbb{R}} |u_+(t_2,x)-u_+(t_1,x)|dx = \int_{\mathbb{R}} \left|\int_{t_1}^{t_2} \frac{\partial}{\partial t} u_+(t,x)dt\right| dx$$
$$\leqq \int_{\mathbb{R}} \int_{t_1}^{t_2} \left|\frac{\partial}{\partial t} u_+(t,x)\right| dtdx,$$

and it follows that $u_+(t_i, \cdot)$ is Cauchy in $L^1(\mathbb{R})$ for every sequence $\{t_i\}_{i\in\mathbb{N}}$ such that $t_i \to \infty$. Hence there is a function $u_+ \in L^1(\mathbb{R})$ such that $\lim_{t\to\infty} \|u_+(t,\cdot)-u_+\|_{L^1}=0$. Similarly, there is a $v_- \in L^1(\mathbb{R})$ such that

$$\lim_{t\to\infty} \|v_-(t,\cdot)-v_-\|_{L^1}=0.$$

c) As in b) one shows that there is a $w_\infty \in L^1(\mathbb{R})$ such that $\lim_{t\to\infty} \|w(t,\cdot)-w_\infty\|_{L^1}=0$. a) implies that $w_\infty(x)=0$ a.e., i.e. $w_\infty=0$ in $L^1(\mathbb{R})$. □

Roughly speaking, Theorem 3 depicts the following asymptotic behaviour as $t\to\infty$: w decays to 0, and the long-time behaviour of u and v is approximately given by free flow.

5. The *H*-Theorem: A Semi-Formal Discussion

The weakness of the global existence theorem proved in Sect. 3 is the smallness condition on the initial values. In the next section I will show how local existence follows from Theorem 2 for any initial data in L^1_+. If the initial values have in addition finite entropy and if the H-Theorem is true, then the argument of Crandall and Tartar [6] is applicable, and one actually gets global existence.

Unfortunately I have no rigorous proof of the H-Theorem for the solutions given by Theorem 2. Also, I do not know whether the solutions depend smoothly on the initial values (in the sense of $L^1(\mathbb{R})$), even though this seems reasonable to expect.

A formal proof of the H-Theorem can be given following the classical lines: Let $\varepsilon>0$, consider the solution (u,v,w) of (10) given by Theorem 2 on a set $\Omega_0:=[t_0,t_1]\times\mathbb{R}$, where $t_0\geqq 0$, and let

$$\Omega_\varepsilon:=\Omega_0\cap\{(t,x);\ \varepsilon\leqq u(t,x),\ v(t,x),\ w(t,x)\leqq 1/\varepsilon\}.$$

As D_+u, D_-v and Dw exist a.e. in Ω_ε, so do $D_+(u\cdot\log u)$, $D_-(v\cdot\log v)$ and $D(w\cdot\log w)$, and I have

$$\begin{aligned} &D_+(u\cdot\log u)+D_-(v\cdot\log v)+D(w\cdot\log w)\\ &=(1+\log u)D_+u+(1+\log v)D_-v+(1+\log w)Dw\\ &=(w^2-uv)[\log(uv)-\log w^2]\leqq 0.\end{aligned}$$

After integration over Ω_3 I get

$$\int_{\Omega_\varepsilon} D_+(u\cdot\log u)dtdx+\int_{\Omega_\varepsilon} D_-(v\cdot\log v)dtdx+\int_{\Omega_\varepsilon} D(w\cdot\log w)dtdx\leqq 0.$$

Note that all integrals exist. As $\varepsilon \searrow 0$, Ω_ε blows up to the set

$$\Omega_0' = \{(t,x) \in \Omega_0 ; u(t,x), v(t,x), w(t,x) > 0\}.$$

Now suppose that

$$\lim_{\varepsilon \to \infty} \int_{\Omega_\varepsilon} D_+(u \cdot \log u)(t,x) dt dx = \int_{\mathbb{R}} u \cdot \log u(t_1, x) dx - \int_{\mathbb{R}} u \cdot \log u(t_0, x) dx, \tag{20}$$

that the integrals on the right exist, and that the corresponding identities hold for v and w. Then, if $U = (u, v, w)$ is a shorthand for the solution of (10), and if

$$H(U)(t) := \int_{\mathbb{R}} (u \cdot \log u + v \cdot \log v + w \cdot \log w)(t,x) dx,$$

it follows that

$$H(U)(t_1) \leqq H(U)(t_0). \tag{21}$$

This is the H-Theorem. The gap in this "proof" is clearly Eq. (20). For bounded solutions, considered in [6], (20) is easy to verify, and (21) follows. For the L^1-solutions given by Theorem 2, (20) remains to be shown.

6. Large Initial Data

Theorem 4. *The Broadwell model* (10) *has a local nonnegative mild solution for all initial data in* $L^1_+(\mathbb{R})$.

Proof. This follows from Theorem 2 by a standard domain of dependence argument. At time $t=0$, split the real axis into finitely many intervals I_j such that

$$\int_{I_j} (f+g+h)(x) dx \leqq \gamma/3,$$

with γ as in Theorem 2. Let $J_j = I_{j-1} \cup I_j \cup I_{j+1}$, and let $f_j = \chi_{J_j} \cdot f$, $g_j = \chi_{J_j} \cdot g$ and $h_j = \chi_{J_j} \cdot h$. By Theorem 2, (10) has a global mild solution for the initial values f_j, g_j and h_j. For each $x \in I_j$ there is a time $t_j > 0$ such that the domain of dependence of (t,x), where $0 \leqq t \leqq t_j$, is contained in J_j. In other words, the solution at (t,x) does not know what the initial data look like outside of J_j. Therefore all the solutions constructed above can locally be glued together to a solution of the original problem. □

Corollary. *If* $f, g, h \in L^1_{+,\mathrm{loc}}(\mathbb{R})$ *and if there is a* $\delta > 0$ *such that* $\int_I (f+g+h)(x) dx \leqq \gamma/3$ *for each interval* I *of length* $\leqq \delta$, *then the Broadwell model* (10) *has a local nonnegative mild solution for the initial data* f, g, *and* h.

Proof. The reasoning from the proof of Theorem 4 can be repeated literally.

Global Existence. Let $U_0 = (f, g, h) \in (L^1_+(\mathbb{R}))^3$ and assume that $H(U_0) < \infty$. By Theoem 3, (10) has a local solution belonging to the initial values (f, g, h), and the semi-formal discussion from Sect. 5 suggests that this solution satisfies the H-Theorem

Theorem 5. *If for every U_0 as above the local solution to* (10) *satisfies the H-Theorem, then the solution to* (10) *exists globally.*

Proof. This follows with an argument first used by Crandall and Tartar [6]. I need to show that the local solution does not leave $L^1_+(\mathbb{R})$, or, more specifically, that no point measures develop. To do this, I have to prove that a solution on an interval $[0, T)$ consists of uniformly absolutely continuous measures in the sense that for any $\varepsilon > 0$, there is a $\delta > 0$ such that for all $t \in [0, T)$

$$\lambda(M) < \delta \Rightarrow \int_M (u+v+w)(t,x)dx < \varepsilon . \tag{22}$$

If (22) holds, then the solution can be continued to a larger time interval $[0, T+\eta)$. Global existence follows if (22) holds indepentently of T.

I show (22) by contradiction. Assume that $[0, T)$ is the maximal existence interval for the local solution. Then (22) must be violated, i.e. there is an $\varepsilon > 0$ such that for all $\delta > 0$ there is a $t \in [0, T)$ and a Borel set $M \subset \mathbb{R}$ with $\lambda(M) < \delta$, but $\int_M (u+v+w)(t,x)dx \geqq \varepsilon$.

By the H-Theorem, we have a uniform bound for $H(U)(t)$:

$$\int_{\mathbb{R}} (u \cdot \log u + v \cdot \log v + w \cdot \log w)(t,x)dx \leqq C . \tag{23}$$

Because of the finite propagation speed, it is no restriction of generality to assume that the solution has compact support K for all $t \in [0, T)$. From $x \cdot \log_+ x \leqq x \cdot \log x + 2/e$, it follows that

$$\int_{\mathbb{R}} (u \cdot \log_+ u + v \cdot \log_+ v + w \cdot \log_+ w)(t,x)dx \leqq H(U)(t) + C , \tag{24}$$

where the constant C depends on K.

By assumption, there are sequences $\delta_i \to 0$, $\{t_i\} \subset [0, T)$, and there are $M_i \in \mathscr{B}(\mathbb{R})$ such that $\lambda(M_i) < \delta_i$, but

$$\int_{M_i} (u+v+w)(t_i, x)dx \geqq \varepsilon .$$

By passing to subsequences, it is no restriction of generality to assume

$$\int_{M_i} u(t_i, x)dx \geqq \varepsilon/3 .$$

From (23) and (24), I have $\int_{M_i} (u \cdot \log_+ u)(t_i, x)dx \leqq C$.

Now let $m \geqq 1$ be arbitrary, and let $M_{i,1} := \{x \in M_i; u(t_i, x) \geqq e^m\}$, $M_{i,2} := \{x \in M_i; u(t_i, x) < e^m\}$. Then

$$\begin{aligned} \varepsilon/3 &\leqq \int_{M_i} u(t_i, x)dx = \int_{M_{i,1}} u(t_i, x)dx + \int_{M_{i,2}} u(t_i, x)dx \\ &\leqq (1/m) \int_{M_{i,1}} u(t_i, x) \log_+ u(t_i, x)dx + \delta_i \cdot e^m \\ &\leqq C \cdot (1/m) + \delta_i \cdot e^m . \end{aligned}$$

Choose m such that $C/m<\varepsilon/6$. Then choose i such that $e^m \cdot \delta_i<\varepsilon/6$, and the contradiction $\varepsilon/3 \leqq e^m \cdot \delta_i + C/m < \varepsilon/3$ results. The proof is complete. □

Corollary. *If the assumptions of Theorem 4 hold, then the Broadwell model (10) has a global solution for all initial data in $L^1_{+,\mathrm{loc}}(\mathbb{R})$ which satisfy*

$$\int_M (f \cdot \log f + g \cdot \log g + h \cdot \log h)(x)dx < \infty$$

for each bounded measurable set M.

Proof. This follows from the local existence theorem (Theorem 3) and the domain of dependence argument used above.

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