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On Absence of Diffusion near the Bottom of the Spectrum for a Random Schrödinger Operator on $L^2(\mathbb{R}^\nu)^+$

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Abstract. We consider a random Schrödinger operator on $L^2(\mathbb{R}^\nu)$ of the form $H_\omega = -\Delta + V_\omega$, $V_\omega(x) = \Sigma\chi_{C_i}(x)q_i(\omega)$, $\{C_i\}$ being a covering of $\mathbb{R}^\nu$ with unit cubes around the sites of $\mathbb{Z}^\nu$ and $\{q_i\}$ i.i.d. random variables with values in $[0,1]$. We assume that the q_i's are continuously distributed with bounded density $f(q)$ and that $0 < P(q_0 < \frac{1}{2}) = \alpha < 1$. Then we show that an ergodic mean of the quantity $\langle \int dx|x|^2|(\exp(itH_\omega)\Phi)(x)|^2\rangle t^{-1}$ vanishes provided $\Phi = g_E(H_\omega)\Psi$, where Ψ is well-localized around the origin and g_E is a positive C^∞-function with support in $(0,E)$, $E \leqq E^*(\alpha, |f|_\infty)$. Our estimate of $E^*(\alpha, |f|_\infty)$ is such that the set $\{x\in\mathbb{R}^\nu | V_\omega(x) \leqq E^*(\alpha, |f|_\infty)\}$ may contain with probability one an infinite cluster of cubes $\{C_i\}$ which are nearest neighbours. The proof is based on the technique introduced by Fröhlich and Spencer for the analysis of the Anderson model.

Section 1. Introduction

Let us consider a quantum mechanical particle moving in $\mathbb{R}^\nu$ and interacting with a random potential V_ω given by

$$V_\omega(x) = \sum_{i\in\mathbb{Z}^\nu} \chi_{C_i}(x)q_i(\omega). \tag{1.1}$$

Here $C_i = \{x\in\mathbb{R}^\nu | -\frac{1}{2} < x_j \leqq \frac{1}{2}; j = 1,\dots,\nu\} + i$ and $\{q_i(\omega)\}_{i\in\mathbb{Z}^\nu}$ are independent identically distributed (i.i.d.) random variables with values in $[0,1]$ such that $P(q_0(\omega)\in[a,b]) = \int_a^b dq f(q)$, $|f|_\infty \equiv \operatorname{ess\,sup}|f| < \infty$ and $0 < P(q_0 \leqq \frac{1}{2}) = \alpha < 1$. We are interested in the spectral properties of the corresponding random Hamiltonian H_ω on $L^2(\mathbb{R}^\nu)$

$$H_\omega = -\Delta + V_\omega, \tag{1.2}$$

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and mainly in the asymptotic behaviour in t as $t \to \infty$ of the quantity

$$r_E^2(t) = \langle \int dx |x|^2 |(\exp(itH_\omega)\Psi_E)(x)|^2 \rangle, \tag{1.3}$$

where $\langle \dots \rangle$ denotes the average over the random variables $\{q_i(\omega)\}$ and Ψ_E is a function well localized in space belonging to the spectral subspace of H_ω with energy less than or equal to $E, E > 0$. This quantity measures the mean square distance from the origin of the time-evolution of the particle whose wave-function at time 0 is given by Ψ_E.

The asymptotic properties of $r_E^2(t)$ for large t is a measure of the diffusion of the particle. More precisely, when one has a finite diffusion one expects that $r_E^2(t)$ behaves as

$$r_E^2(t) \sim D(E)t \quad \text{as} \quad t \to \infty,$$

where $D(E)$ is the diffusion constant.

Here we prove that in our model a particle will not diffuse in the sense that an ergodic mean of $r_E(t)^2/t$ will vanish as $t \to \infty$, provided it initially is well localized in space and has sufficiently small energy.

Let now $S_E(\omega) = \{x \in \mathbb{R}^\nu | V_\omega(x) \leqq E\}$ be the classically allowed region. Then depending on the distribution f of the random variables $\{q_i(\omega)\}$, $S_E(\omega)$ will consist with probability one of *only* lakes or of lakes *and a sea* (possibly more than one sea) and it is expected that the behaviour in t of $r_E^2(t)$ will be different in the two situations. It is not difficult to show (see Sect. 2) that the study of $r_E^2(t)$ as $t \to \infty$ can be reduced to the analysis of $\varepsilon^2 \int dx |x|^2 \langle |G(\omega, E + i\varepsilon, 0, x)|^2 \rangle$ as $\varepsilon \to 0$, with E in the spectrum $\sigma(H_\omega)$ of H_ω and $G(\omega, E + i\varepsilon, 0, x) \equiv (H_\omega - E - i\varepsilon)^{-1}(0, x)$. Using the ergodic theorem and Weyl's criterium it is easy to convince oneself that $\sigma(H_\omega) = [0, \infty)$ with probability one (see e.g. [8]). For such a quantity in the Anderson model, i.e. in the discrete version of (1.2) where $\mathbb{R}^\nu$ is replaced by $\mathbb{Z}^\nu$ and $-\Delta$ is the finite difference analogue to the Laplacian, Fröhlich and Spencer recently developed [2, 3] a very powerful technique to prove that, for the energy in a suitable range, $\langle |G(E + i\varepsilon, 0, x)|^2 \rangle$, $\varepsilon \neq 0$, decays exponentially in $|x|$ with mass $m = m(E)$ bounded away form zero uniformly in ε, with probability greater than $1 - \text{const}|x|^{-p}$. Here p can be made arbitrarily large by choosing E in a suitable way. This in turn implies that if the initial state of the particle is well localized near zero and has energy in a suitable range, then the corresponding $r_E^2(t)$ satisfies

$$\lim_{T \to \infty} \frac{1}{T} \int_1^T dt \frac{r_E^2(t)}{t} = 0.$$

Their method is based on perturbation theory about an infinite sequence of "block" Hamiltonians. The "blocks" correspond to regions where the potential V_ω is singular in the sense that the eigenvalues of the corresponding "block" Hamiltonian are close to the given energy E. Distinct singular regions are decoupled by introducing Dirichlet boundary conditions at the respective boundaries. As the size of the blocks increases the eigenvalues of the corresponding Hamiltonian are permitted to get closer to E; the corresponding divergent terms in $(H_{\text{block}} - E - i\varepsilon)^{-1}$ (small divisors problem) are killed by the exponential decay of $(H_{\text{block}} - E - i\varepsilon)^{-1}$ on scale of order of the size of the block. However in order to remove perturbatively the Dirichlet boundary conditions it is necessary to control the tunneling between singular regions, and for this it is important that the "blocks" are well separated from one another.

A natural candidate for the set of singular sites in our model (1.2) would be the set $S_E(\omega)=\{j\in\mathbb{Z}^\nu | q_j(\omega)\leqq 2E\}$. In fact if $\Lambda\cap S_E(\omega)=\emptyset$, then $V_\omega(x)>2E$ for all $x\in\Lambda$ so that $\mathrm{dist}(E,\sigma(H_\Lambda(\omega))>E$. Here $H_\Lambda(\omega)=-\Delta_\Lambda^D+V_\omega$, $-\Delta_\Lambda^D$ being the Dirichlet Laplacian on $L^2(\Lambda)$, and $\sigma(H_\Lambda(\omega))$ denotes its spectrum. Hence no divergent terms arise in $(H_{\omega,\Lambda}-E-i\varepsilon)^{-1}$ as $\varepsilon\to 0$.

Actually using an argument due to Combes–Thomas it is possible to prove that the integral kernel of $(H_\Lambda(\omega)-E-i\varepsilon)^{-1}$ decays exponentially with mass of the order of $E^{1/2}$. Thus with this choice of the set of singular sites one could apply the Fröhlich–Spencer method for energy E so small that $S_W(\omega)$ has no infinite cluster with probability one. Typically the energy threshold would be of the form $E\ll c/|f|_\infty$ with $c\ll 1$. We remark here that Fröhlich–Spencer technique does not apply immediately to the continuum case; however at least for our special model the extension can be made without too much trouble. In what follows we propose another choice of the set of singular sites which shows that the above result holds even if the set $S_E(\omega)$ *has an infinite cluster.* More precisely we get an estimate on the energy threshold of the form

$$E\leqq\min\left\{E_0(\alpha),\left[\ln\left(\frac{|f|_\infty}{E_1(\alpha)}\right)\right]^{-2/\nu}\right\}\equiv E^*(\alpha,|f|_\infty),\tag{1.4}$$

where $E_0(\alpha)$ and $E_1(\alpha)$ are small constants depending *only* on α and not on $|f|_\infty$. It is clear from the logarithmic dependence of $|f|_\infty$ in the expression (1.4) for E^* that if we keep α fixed and change the probability distribution f suitably, we can still have $P(q_0\leqq E^*(\alpha,|f|_\infty))=\alpha$, when we let $|f|_\infty$ increase. (Let e.g. $f_a=(\alpha/a)\chi_{[0,a]}+2(1-\alpha)\chi_{[1/2,1]}$. By increasing $|f|_\infty=\alpha/a$ (when $a\ll 1$) we can have $P(q_0\leqq E^*)=\alpha$.) Thus if α was chosen greater than the the percolation probability for the site-percolation model in $\mathbb{Z}^\nu$ (see e.g. [5]), then the set $S_E(\omega)$ may contain with probability one, an infinite cluster $\{C_i\}$ of cubes which are nearest neighbours.

The idea is the following:

Let $C_E(0)$ be a cube centered at $x=0$ of size $L(E)\sim E^{-1/2}$ and let $\lambda_1(H^N_{C_E(0)}(\omega))$ be the lowest eigenvalue of $H(\omega)$ on $L^2(C_E(0))$ with Neumann boundary conditions. It was shown in [6] (see also Sect. 4) that:

$$P(\lambda_1(H^N_{C_E(0)}(\omega))\leqq 2E)\leqq\exp(-cE^{-\nu/2})\tag{1.5}$$

for some $c>0$ and all $E<E_0(\alpha)$, where $E_0(\alpha)$ is *independent* of $|f|_\infty$.

Let now $\mathbb{Z}^\nu(E)=L(E)\mathbb{Z}^\nu$, and let $\{C_E(j)\}_{j\in\mathbb{Z}^\nu(E)}$ be a covering of $\mathbb{R}^\nu$ of cubes with size $L(E)$ around the sites of $\mathbb{Z}^\nu(E)$. The above estimate shows that although the set $S_E(\omega)$ may contain an infinite cluster $\{C_i\}$, $i\in\mathbb{Z}^\nu$, of cubes which are nearest neighbours, it *does not* contain an infinite cluster of cubes $C_E(j)$, $j\in\mathbb{Z}^\nu(E)$, where the condition $\lambda_1(H^N_{C_E(j)}(\omega))\leqq 2E$ is violated, provided the energy E is sufficiently small but *uniformly* in $|f|_\infty$. Furthermore using Neumann–Dirichlet bracketing (see Sect. 3) it is easy to see that if $\Lambda\subseteq\mathbb{R}^\nu$ is such that $\lambda_1(H^N_{C_E(j)}(\omega))\geqq 2E$ for all $j\in\mathbb{Z}^\nu(E)$ with $C_E(j)\cap\Lambda\neq\emptyset$, then $-\Delta_\Lambda^D+V_\omega\geqq 2E$ so that $(-\Delta_\Lambda^D+V_\omega-E-i\varepsilon)^{-1}$

(x, y) decays exponentially in $|x-y|$ with "mass" $m(E) \geqq E^{1/2}$ uniformly in ε. It is then natural to define the set of singular sites *on scale* $L(E)$ as

$$S_0 = \{j \in \mathbb{Z}^v(E) | \lambda_1(H^N_{C_E(j)}(\omega)) \geqq 2E\},$$

and to perform the Fröhlich–Spencer perturbation argument on clusters of cubes of the form $C_E(j)$.

As in the papers by Fröhlich–Spencer [2, 3] our probabilistic estimates rely on the following result valid for any $\Lambda \subset \mathbb{R}^v$:

$$P(\mathrm{dist}(\sigma(H_\Lambda(\omega), E) < k) \leqq k^{1/2} |\Lambda|^{3/2} \rho(E, |f|_\infty)^{1/2}, \tag{1.6}$$

where $\rho(E, |f|_\infty) \sim |f|_\infty N(E)$, if $k \sim E$, $N(E)$ being the integrated density of states of the system. Since $N(E)$ has a singular behaviour near $E=0$ of the form $N(E) \sim \exp(-c_2 E^{-v/2})$ (see [6]) we see that in order to have $\rho(E, |f|_\infty)$ small, it is enough to take $E \leqq E^*(\alpha, |f|_\infty)$ with $E^*(\alpha, |f|_\infty)$ of the form (1.4).

We remark here that a sharp estimate of the left-hand side of (1.6) in the case of discrete distributed $\{q_i(\omega)\}$, say $P(q_0 = 0) = p$, $P(q_0 = 1) = 1 - p$, is still missing even in the discrete case, i.e. for the Anderson model.

Needless to say, all our proofs rely heavily on the Fröhlich–Spencer paper and most of the time they are just the translation into our context of their proofs. Therefore we do not give here all the details of the proofs, but we only discuss the main steps where the differences between the two situations appear clearly. For simplicity we also discuss only the case $v = 3$, but the only place in the proof where the dimension enters crucially is in the exact localization rate of the initial wave-function Ψ.

Notations. We fix here some notations which will be used in the rest of the paper. For any measurable $\Lambda \subset \mathbb{R}^v$, let $H_\Lambda(\omega) = -\Delta^D_\Lambda + V_\omega$ and $H^N_\Lambda(\omega) = -\Delta^N_\Lambda + V_\omega$ be operators on $L^2(\Lambda)$, where $-\Delta^D_\Lambda$ and $-\Delta^N_\Lambda$ are the Laplacian with Dirichlet and Neumann boundary conditions respectively. (For the precise definition of these operators, see e.g. [9].) We also denote by $\sigma(H_\Lambda(\omega))$, $\sigma(H^N_\Lambda(\omega))$ their spectra and by $N_\Lambda(E, \omega) = \#\{k | \lambda_k(H_\Lambda(\omega)) < E\}$, $E > 0$, where $\lambda_1(H_\Lambda(\omega)) \leqq \lambda_2(H_\Lambda(\omega)) \leqq \dots$ are the eigenvalues of $H_\Lambda(\omega)$ in nondecreasing order. It follows from the ergodic theorem (see e.g. [7]) that $\lim_{\Lambda \uparrow \mathbb{R}^v} (1/|\Lambda|) N_\Lambda(E, \omega) = N(E)$ exists almost surely and is independent of ω. Here $|\Lambda|$ denotes the Lebesgue measure of Λ. The quantity $N(E)$ is called the integrated density of states.

In order to describe precisely the degree of localization in space of the initial state Ψ of the particle at time $t = 0$ (see (1.3)) we also need to introduce the weighted L^p spaces (see [10]), L^p_δ, defined as follows:

$$L^p_\delta = \{f \in L^p | (1 + |x|^2)^{\delta/2} f \in L^p\}, \quad \delta \in \mathbb{R}.$$

Throughout the estimates several constants *independent* of E, $|f|_\infty$, ω will occur and they will always be denoted by c, c_1, c_2, although their values may change from estimate to estimate.

Green's Identities. Let $\Lambda \subset R^3$ be given and let Λ_1, Λ_2 be such that $\Lambda_1 \cap \Lambda_2 = \emptyset$ $\Lambda_1 \cup \Lambda_2 = \Lambda$. Let also $G_\Lambda(\omega, E + i\varepsilon, x, y)$ be the Green's function of $H_\Lambda(\omega) -$

$E - i\varepsilon$, i.e.

$$G_\Lambda(\omega, E + i\varepsilon, x, y) = (H_\Lambda(\omega) - E - i\varepsilon)^{-1}(x, y),$$

and let $\partial_{n_z} G_\Lambda(\omega, E + i\varepsilon, x, z)$, $x \in \Lambda$, $z \in \partial\Lambda$ denote the outward normal derivative at z of $G_\Lambda(\omega, E + i\varepsilon, x, y)$. Then we have the following two identities which follow immediately from Green's first and second formula:

$$G_\Lambda(\omega, E + i\varepsilon, x, y) = G_{\Lambda_1|\Lambda_2}(\omega, E + i\varepsilon, x, y) + \int_{\partial\Lambda_1} dz (\partial n_z G_{\Lambda_1|\Lambda_2}(\omega, E + i\varepsilon, x, z)) G_\Lambda(\omega, E + i\varepsilon, z, y), \quad (1.7)$$

when $x \in \Lambda_1$, $x \neq y$, and

$$G_\Lambda(\omega, E + i\varepsilon, x, y) = G_{\Lambda_1|\Lambda_2}(x, y) + \int_{\partial\Lambda_1} dz\, G_\Lambda(\omega, E + i\varepsilon, x, z)(\partial_{n_z} G_{\Lambda_1|\Lambda_2}(\omega, E + i\varepsilon, z, y)), \quad (1.8)$$

when $x \in \Lambda_1$, $x \neq y$, where $G_{\Lambda_1|\Lambda_2}(\omega, E + i\varepsilon, z, y)$ is the Green's function for the operator $H(\omega) = -\Delta + V(\omega)$ on $L^2(\Lambda) \simeq L^2(\Lambda_1) \oplus L^2(\Lambda_2)$ with Dirichlet boundary conditions on $\partial\Lambda \cup \partial\Lambda_1$. We note that for $x, y \in \Lambda_1(\Lambda_2) G_{\Lambda_1|\Lambda_2}(\omega, E + i\varepsilon, z, y)$ coincides with $G_{\Lambda_1}(\omega, E + i\varepsilon, z, y)(G_{\Lambda_1}(\omega, E + i\varepsilon, z, y))$ and if

$$x \in \Lambda_1, \quad y \in \Lambda_2\ G_{\Lambda_1|\Lambda_2}(\omega, E + i\varepsilon, x, y) = 0.$$

Section 2. The Main Result

In this section we state our main result and show how to derive it from an estimate on the decay of the Green's function $G(\omega, E + i\varepsilon, x, y)$ for E near the bottom of the spectrum of H_ω and as $\varepsilon \to 0$.

Theorem 2.1. *Let* $g_E \in C_0^\infty(\mathbb{R})$, $g_E \geqq 0$, *and* $\operatorname{supp} g_E \subset (0, E)$, *and let for any* $\Psi \in L_2^2(\mathbb{R}^3)$, $\Phi_\omega \equiv g_E(H_\omega)\Psi$. *Then the quantity*

$$r_E^2(t) = \langle \int dx\, |x|^2 |(e^{itH_\omega} \Phi_\omega)(x)|^2 \rangle$$

is well defined for any $0 \leqq t < \infty$, *and there exist two constants* $E_0(\alpha)$, $E_1(\alpha)$ *depending only on* α *and not on* $|f|_\infty$ *such that if*

$$E \leqq \min\left\{ E_0(\alpha), \left[\ln\left(\frac{|f|_\infty}{E_1(\alpha)} \right) \right]^{-2/3} \right\} \equiv E^*(\alpha, |f|_\infty) \equiv E^*,$$

then

$$\lim_{T \to \infty} \frac{1}{T} \int_1^T dt \frac{r_E^2(t)}{t} = 0.$$

Actually Theorem 2.1 follows from the next more general result which will be proved in Sect. 3.

Theorem 2.2. *Let* $E^*(\alpha, |f|_\infty)$ *be as above. Then*

$$\lim_{\varepsilon \to 0} \varepsilon \int dx (|x| + 1) [\langle |G(w, E + i\varepsilon, 0, x)|^4 \rangle]^{1/4} = 0$$

uniformly in E *for* E *in a compact subset of* $(0, E^*(\alpha, |f|_\infty))$.

Next we show how to derive Theorem 2.1 from Theorem 2.2. We fix $0<E<E^*(\alpha,|f|_\infty)$, and first prove that for any $0<\eta<1$

$$\lim_{T\to\infty}\frac{1}{T}\int_{\eta T}^{T} dt\frac{r_E^2(t)}{t}=0. \tag{2.1}$$

Set $\varepsilon=1/T$. Then we get:

$$\begin{aligned}\frac{1}{T}\int_{\eta T}^{T} dt\frac{r_E^2(t)}{t} &\leqq \frac{\varepsilon^2 e}{\eta}\int_0^\infty dt\, e^{-\varepsilon t} r_E^2(t)\\ &=\frac{e}{\eta}\varepsilon^2\int_{-\infty}^{\infty} dE'\langle\int dx|x|^2|(G(\omega,E'+i\varepsilon)\Phi_\omega)(x)|^2\rangle,\end{aligned} \tag{2.2}$$

where for the equality we have used a vector-valued version of the Plancherel theorem (see [9], Lemma 1, p. 142). We divide the integration in (2.2) in three parts:

$$\begin{aligned}&\frac{e}{\eta}\varepsilon^2\int_{-\infty}^{\bar E} dE'\langle\int dx|x|^2|(G(\omega,E'+i\varepsilon)\Phi_\omega)(x)|^2\rangle\\ &+\frac{e}{\eta}\varepsilon^2\int_{E}^{E^*} dE'\langle\int dx|x|^2|(G(\omega,E'+i\varepsilon)\Phi_\omega)(x)|^2\rangle\\ &+\frac{e}{\eta}\varepsilon^2\int_{E^*}^{\infty} dE'\langle\int dx|x|^2|(G(\omega,E'+i\varepsilon)\Phi_\omega)(x)|^2\rangle,\end{aligned} \tag{2.3}$$

where $\bar E>0$ is such that $\operatorname{supp} g_E\subseteq(\bar E,E)$.

We start by discussing the first and the last term of (2.3). Since $G(\omega,E'+i\varepsilon)\Phi_\omega=f_{\varepsilon,E'}(H_\omega)\Psi$, where

$$f_{\varepsilon,E'}(x)=\frac{g_E(x)}{x-E'-i\varepsilon}\in C_0^\infty(\mathbb{R})\quad\text{for}\quad E'\in(-\infty,\bar E)\cup[E^*(\alpha,|f|_\infty),\infty)$$

and ε sufficiently small, we have

$$\int dx|x|^2|G(\omega,E'+i\varepsilon)\Phi_\omega|^2\leqq\|f_{\varepsilon,E'}(H_\omega)\|^2_{L_2^2\to L_2^2}\|\Psi\|^2_{L_2^2}. \tag{2.4}$$

To estimate $\|f_{\varepsilon,E'}(H_\omega)\|^2_{L_2^2\to L_2^2}$, we use a lemma whose proof is given in Appendix B.

Lemma 2.1. *Let $f\in C_0^\infty(\mathbb{R})$. Then*

$$\|f(H_\omega)\|_{L_2^2\to L_2^2}\leqq c_1|f|_\infty+c_2|\operatorname{supp} f|\left\{\left|\frac{d^4}{dx^4}f\right|_\infty+\left|\frac{d^4}{dx^4}h\right|_\infty\right\},\quad\textit{where}\quad h(x)=xf(x)$$

and $|\operatorname{supp} f|$ is the Lebesgue measure of $\operatorname{supp} f$.

Using the lemma we get that for ε sufficiently small $\|f_{\varepsilon,L'}(H_\omega)\|^2_{L_2^2\to L_2^2}$ is bounded uniformly in ω,ε,E', $E'\in(-\infty,0)\cup[E^*(\alpha,|f|_\infty),\infty)$, and that for large $|E'|$ it is bounded by $(c/|E'|^2)$. Thus the first and the last integral in (2.3) are uniformly bounded in ε for ε small enough and when multiplied by ε^2 they vanish in the limit $\varepsilon\to 0$.

We now discuss the second term in (2.3). It clearly suffices to show, uniformly in $E'\in[\bar{E}, E^*(\alpha, |f|_\infty)]$, that

$$\lim_{\varepsilon\to 0} \varepsilon^2 \int dx |x|^2 \langle (\int dy |G(\omega, E'+i\varepsilon; x, y)| |\Phi_\omega(y)|)^2 \rangle = 0. \tag{2.5}$$

By "expanding" the square and using the Hölder inequality with respect to the ω-integration $\langle \ldots \rangle$, we get that (2.5) is bounded by:

$$\lim_{\varepsilon\to 0} \varepsilon^2 \int dx |x|^2 (\int dy \{\langle |G(\omega, E'+i\varepsilon, x, y)|\rangle^4\}^{1/4} \cdot \{\langle |\Phi_\omega(y)|^4 \rangle\}^{1/4})^2. \tag{2.6}$$

We now observe that using the stationarity of V_ω

$$(\langle |G(\omega, E'+i\varepsilon, x, y)|^4 \rangle)^{1/4} = (\langle |G(\omega, E'+i\varepsilon, 0, y-x)|^4 \rangle)^{1/4} \equiv K(y-x).$$

Using $|x| \leqq |x-y| + |y|$ and Young's inequality, we bound (2.6) by:

$$2 \lim_{\varepsilon\to 0} \varepsilon^2 \{(\int dx |x| K(x))^2 \cdot \int dy (\langle |\Phi_\omega(y)|^4 \rangle)^{1/2} + (\int dx K(x))^2 \int dy (\langle |y\Phi_\omega(y)|^4 \rangle)^{1/2}\}. \tag{2.7}$$

To estimate $\int dy (\langle |y\Phi_\omega(y)|^4 \rangle)^{1/2}$ we need the following lemma which is proved in Appendix B.

Lemma 2.2. *Let $f \in C_0^\infty(\mathbb{R})$. Then $\| f(H_\omega) \|_{L_2^2 \to L_2^\infty} \leqq C$.*

From the lemma we get that $|\Phi_\omega(y)| \leqq c_1(1+|y|^2)^{-1}$ uniformly in ω and from Lemma 2.1 that $\| \Phi_\omega \|_{L_2^2} \leqq c_2$. Hence:

$$\int dy (\langle |y\Phi_\omega(y)|^4 \rangle)^{1/2} \leqq (\int dy (1+|y|^2)^{-2})^{1/2} \cdot (\langle dy (1+y^2)^2 y^4 |\Phi_\omega(y)|^4 \rangle)^{1/2} < \infty,$$

and analogously for $\int dy (\langle |\Phi_\omega(y)|^4 \rangle)^{1/2}$. Thus, using now Theorem 2.2, the limit as $\varepsilon \to 0$ in (2.7) vanishes uniformly in $E' \in [0, E^*(\alpha, |f|_\infty)]$, and (2.7) is proved.

We now complete the proof of Theorem 2.1.

Let $T_n = \sup\left\{ t \in [1, n] \,\middle|\, \frac{1}{t} \int_1^t ds \frac{r_E^2(s)}{s} = \sup_{1 \leqq T \leqq n} \frac{1}{T} \int_1^T ds \frac{r_E^2(s)}{s} \right\}$.

Clearly $T_{n+1} \geqq T_n$. Suppose first that $\{T_n\}$ is bounded. Then in this case $\overline{\lim}_{T\to\infty} \frac{1}{T} \int_1^T dt \frac{r_E^2(t)}{t} < \infty$, and

$$\overline{\lim}_{T\to\infty} \frac{1}{T} \int_1^T dt \frac{r_E^2(t)}{t} = \overline{\lim}_{n\to\infty} \frac{1}{T} \int_1^\eta dt \frac{r_E^2(t)}{t} \tag{2.8}$$

for all $\eta \in (0,1)$, using (2.1). Since the quantity $\lim_{T\to\infty} \frac{1}{T} \int_1^T dt (r_E^2(t)/t)$ by assumption is finite, (2.8) implies that it is actually zero. If $T_n \to \infty$ as $n \to \infty$ it follows from the definition of T_n that

$$\overline{\lim}_{T\to\infty} \frac{1}{T} \int_1^T dt \frac{r_E^2(t)}{t} = \overline{\lim}_{n\to\infty} \frac{1}{T_n} \int_1^{T_n} ds \frac{r_E^2(s)}{s}. \tag{2.9}$$

Now

$$\frac{1}{T_n}\int_1^{T_n} dt\frac{r_E^2(t)}{t}=\frac{\eta}{\eta T_n}\int_1^{\eta T_n} dt\frac{r_E^2(t)}{t}+\frac{1}{T_n}\int_{\eta T_n}^{T_n} dt\frac{r_E^2(t)}{t}$$
$$\leqq\frac{\eta}{T_n}\int_1^{T_n} dt\frac{r_E^2(t)}{t}+\frac{1}{T_n}\int_{\eta T_n}^{T_n} dt\frac{r_E^2(t)}{t},$$

i.e.

$$\frac{1}{T_n}\int_1^{T_n} dt\frac{r_E^2(t)}{t}\leqq(1-\eta)^{-1}\frac{1}{T_n}\int_{\eta T_n}^{T_n} dt\frac{r_E^2(t)}{t}, \tag{2.10}$$

which again implies using (2.1) that

$$\overline{\lim_{T\to\infty}}\frac{1}{T}\int_1^T dt\frac{r_E^2(t)}{t}=0.$$

Section 3. Exponential Decay of the Green's Function

In this section we prove Theorem 2.2. As in Fröhlich–Spencer [2], see also [3], it is sufficient to prove the following lower bound on the probability that the Green's function $G(\omega, E+i\varepsilon, x, y)$ decays exponentially for large distances $|x-y|$:

Theorem 3.1. *Given any $p>0$, there exist two constants $E_0(\alpha, p)$, $E_1(\alpha, p)$ independent of $|f|_\infty$ such that if*

$$E\leqq\min\left\{E_0(\alpha,p),\left[\ln\left(\frac{|f|_\infty}{E_1(\alpha,p)}\right)\right]^{-2/3}\right\}\equiv E^*(\alpha,|f|_\infty,p),$$

then

$$P\left(|G(\omega,E+i\varepsilon,0,x)|\leqq\max\left\{e^{m(E)(NL(E)^2-|x|)},\frac{e^{m(E)(NL(E)^2-|x|)}}{|x|}\right\};\varepsilon\neq 0\right)\geqq 1-\frac{K_p}{N^p}$$

for any $N\in\mathbb{Z}$, $N>0$, and some constant K_p independent of E. Here the "mass" $m(E)$ satisfies $m(E)\geqq cE^{1/2}$ and $L(E)=c(\alpha)E^{-1/2}$ for some constant $c(\alpha)$ independent of E.

Let now $x\in\mathbb{R}^3$ be fixed and let Λ be a large cube around 0 containing x. Applying the first Green's identity (1.7) to $G(\omega, E+i\varepsilon, 0, x)$ and $G_{\Lambda|\mathbb{R}^3\sim\Lambda}(\omega, E+i\varepsilon, 0, x)$, we get

$$G(\omega,E+i\varepsilon,0,x)=G_{\Lambda|\mathbb{R}^3\sim\Lambda}(\omega,E+i\varepsilon,0,x)$$
$$+\int_{\partial\Lambda}dz\{\partial_{n_z}G_{\Lambda|\mathbb{R}^3\sim\Lambda}(\omega,E+i\varepsilon,0,z)\}G(\omega,E+i\varepsilon,z,x). \tag{3.1}$$

Using the Combes–Thomas argument (see e.g. Simon [9]), one shows that both $G(\omega, E+i\varepsilon, x, y)$ and $G_{\Lambda|\mathbb{R}^3\sim\Lambda}(\omega, E+i\varepsilon, x, y)$ decay exponentially in $|x-y|$ as long as $\varepsilon\neq 0$. To control the normal derivative of $G_{\Lambda|\mathbb{R}^3\sim\Lambda}(\omega, E+i\varepsilon, x, y)$ at the boundary $\partial\Lambda$ we use the following lemma which will be proven in Appendix A.

Lemma 3.1. *Assume $y \in \partial \Lambda$ is not one of the corners. Then*

$$|\partial_{n_y} G_\Lambda(\omega, E + i\varepsilon, x, y)| \leqq c \sup_{|y'-y| \leqq 1} |G_\Lambda(\omega, E + i\varepsilon, x, y')|$$

for some $c > 0$ and any x with $|x - y| > 1$.

From the lemma it follows that the second term in the right-hand side of (3.1) is bounded by $c\exp(-m(E)\operatorname{dist}(x, \partial\Lambda))$. Hence $G_{\Lambda|\mathbb{R}^3 \sim \Lambda}(\omega, E + i\varepsilon, 0, x)$ converges to $G(\omega, E + i\varepsilon, 0, x)$ as $\Lambda \uparrow \mathbb{R}^3$ uniformly on compact sets and it is really enough to prove the statement of Theorem 3.1 for $G_\Lambda(\omega, E + i\varepsilon, 0, x)$ provided the constants $E^*(\alpha, |f|_\infty, p)$, K_p, $m(E)$ are independent of Λ for large Λ.

The Singular Sets. We specify here our choice of the set of singular sites which allows us to perform the Fröhlich–Spencer induction argument also in the case when the set $S_E(\omega) = \{x \in \mathbb{R}^3 | V_\omega(x) \leqq E\}$ contains an infinite cluster of cubes $\{C_i\}$ which are nearest neighbours. First we fix the energy $E > 0$ and define our *basic length scale* $L(E) \equiv \pi((1 + \alpha)/(1 - \alpha))^{-1/2} E^{-1/2}$. Let now $\mathbb{Z}^3(E) \equiv L(E)\mathbb{Z}^3$, and let $C_E(j) = C_E(0) + j, j \in \mathbb{Z}^3(E)$, with $C_E(0) = \{x \in \mathbb{R}^3 | -L(E)/2 \leqq x_i < L(E)/2; i = 1, 2, 3\}$. On $\mathbb{Z}^3(E)$ we will consider the norm

$$|j|_E = \max_{i=1,2,3} |j_1| \cdot L(E)^{-1}.$$

Definition. A site $j \in \mathbb{Z}^3(E)$ is said to be singular iff

$$\lambda_1(H^N_{C_E(j)}(\omega)) \leqq 2E$$

We will then denote by S_0 the set of all singular sites. We are now in a position to given an inductive definition of the singular sets S_i of strength i. Assume that $S_0 \supseteq S_1 \supseteq \ldots \supseteq S_i$, $S_k \subset \mathbb{Z}^3(E)$ for all k with $0 \leqq k \leqq i$ have been defined. Then we set

$$S_{i+1} \equiv S_i \sim S_i^g, \tag{3.2}$$

where $S_i^g = \cup_\beta C_i^\beta$ is the *maximal union* of components C_i^β such that

Condition $A(i)$.

a) $C_i^\beta \subseteq S_i$ for all β, (3.3)

b) $\operatorname{diam}_E C_i^\beta \leqq d_i$, (3.4)

c) $\operatorname{dist}_E(C_i^\beta, S_i \sim C_i^\beta) \geqq 2d_{i+1}$, (3.5)

d) $\operatorname{dist}\left(\sigma\left(H_{(\overline{C_i^\beta})^0}\right), E\right) \geqq \exp(-d_i^{1/2})$, (3.6)

where $d_0 = d_0(E) = [L(E)]$ ([ ] denotes integer part), $d_i = d_0^{(5/4)^i}$ and

$$\overline{C_i^\beta} = \{j \in \mathbb{Z}^3(E) | \operatorname{dist}_E(j, C_i^\beta) \leqq 4d_i\}, \tag{3.7}$$

and for any set $D \subset \mathbb{Z}^3(E)$, $D^0 = \bigcup_{j \in D} C_E(j)$. Here diam_E and dist_E are measured using the norm $|\cdot|_E$. Recall that H_Λ has Dirichlet boundary condition on $\partial\Lambda$ for any set Λ (see Sect. 1).

Definition. A set $A \subset \mathbb{Z}^3(E)$ is k-admissible iff $\partial A \cap \overline{C_i^\beta} = \emptyset$, $i = 0, 1, \ldots, k$. A is admissible iff it is k-admissible for all k.

Following Fröhlich–Spencer we will prove in the next paragraph the following exponential decay estimate on the Green's function by using iteratively the Green's identities (1.7) and (1.8).

Theorem 3.4. *There exists a constant $E_1(\alpha) > 0$ such that if $E \leqq E_1(\alpha)$ and A is a k-admissible subset of $\mathbb{Z}^3(E)$ with $A \cap S_{K+1} = \emptyset$, then for arbitrary real ε*

$$|G_{A^0}(\omega, E + i\varepsilon, x, y)| \leqq \exp[-m(E)|x - y|],$$

provided $|x - y| \geqq \frac{1}{5}L(E)d_{k+1}$. The "mass" $m(E)$ satisfies: $m(E) \geqq c_1 E^{1/2}$ with $c_1 \equiv c_1(\alpha)$ independent of A and k.

We postpone the proof of Theorem 3.4 till the end of this section.

Our next step is to estimate the probability that a given site $i \in \mathbb{Z}^3(E)$ belongs to S_j^q. In the next section we will prove the following:

Theorem 3.5. *For any $p > 0$ there exist two constants $E_2(\alpha, p)$, $E_3(\alpha, p)$ independent of $|f|_\infty$ such that if $E < \min\{E_2(\alpha, p), [\ln(|f|_\infty / E_3(\alpha, p))]^{-2/3}\}$, then for all $i \in \mathbb{Z}^3(E)$*

$$P(i \in S_j^q) \leqq d_j^{-p}.$$

The proof of Theorem 3.5 is deferred to Sect. 4. Combining now Theorem 3.4 and Theorem 3.5 (see Fröhlich–Spencer [2], Sect. 6 for details) we get the basic probabilistic estimate

Theorem 3.6. Given any $p > 0$, there exist two constants $E_0(\alpha, p)$, $E_1(\alpha, p)$ independent of $|f|_\infty$ such that if $E \leqq \min\{E_0(\alpha, p), [\ln(|f|_\infty / E_1(\alpha, p))]^{-2/3}\} \equiv E^*(\alpha, |f|_\infty, p)$, then the following event holds with probability at least $1 - l^{-p}$:

$$F_l = \{\omega \mid \exists A \subset \mathbb{Z}^3(E), 0 \in A, A \text{ admissible},$$

$$\frac{l}{2} \leqq \min_{b \in \partial A} |b|_E \leqq \max_{b \in \partial A} |b|_E \leqq l,$$

and

$$|G_{A^0}(\omega, E + i\varepsilon, x, y)| \leqq e^{-m(E)|x-y|}$$

for $|x - y| \geqq L(E) l^{3/4}\}$.

The "mass" $m(E)$ is as in Theorem 3.4 and $l \gg L(E)$.

We now turn to the proof of Theorem 3.3 but with $G(\omega, E + i\varepsilon, 0, x)$ replaced by $G_\Lambda(\omega, E + i\varepsilon, 0, x)$, Λ being a cube of size $L \cdot L(E)$ centered at the origin, $L \gg 1$. As we already pointed out this is enough to obtain the result for $G_\Lambda(\omega, E + i\varepsilon, 0, x)$.

Fix $N \geqq 10$, N integer, and assume first $|x| \geqq L(E)(N/2)$. Let $l_j = [2|x| 4^j / L(E)]$, $j = 0, 1, \ldots$ ([ ] here denotes integer part), and let F_{l_j} be the event in the probability space described in Theorem 3.6 with $l = l_j$. Using Theorem 3.6 we have

$$P\left(\bigcap_{j=0}^{\infty} F_{l_j}\right) \geqq 1 - \frac{K_p}{N^p}, \tag{3.8}$$

if $E \leqq E^*(\alpha, |f|_\infty, P)$.

Let now $A_j \subset \mathbb{Z}^\nu(E)$, $j = 0, 1, 2, \ldots$, be a sequence of sets associated to the events F_{l_j}. Let $\gamma_j \equiv \partial(A_j^0)$. We set $G(x, y) \equiv G_\Lambda(\omega, E + i\varepsilon, x, y)$ and $G_j(x, y) \equiv$

$G_{A_j^0}(\omega, E+i\varepsilon, x, y)$. Using the Green's identity (1.7) we have

$$G(0,x) = G_0(0,x) + \int_{\gamma_0} dz (\partial_{n_z} G_0(0,z)) G_1(z,x)$$

$$+ \int_{\gamma_0} dz \int_{\gamma_1} dz' (\partial_{n_z} G_0(0,z))(\partial_{n_{z'}} G_1(z,z')) G_2(z',x) + \dots \quad (3.9)$$

Using the definition of the sets A_j, A_j^0 we see that in the above expansion both $\partial_{n_z} G_j$ and G_j are evaluated at sites with distance at least

$$L(E)\left(\frac{|x|4^j}{2L(E)} - 2\right) \geqq L(E) l^{3/4}.$$

Hence from Theorem 3.6 and Lemma 3.1 they are estimated from above by $c \exp[-m(E)(\frac{1}{2}|x|4^j - 2L(E))]$, where c is independent of E, N, ω, and $m(E) \geqq c_1 E^{1/2}$. Inserting this estimate in (3.9) we get that

$$|G(0,x)| \leqq c \exp(-m(E)|x|), \; m(E) \geqq c_1 E^{1/2} \quad (3.10)$$

For all $\omega \in \bigcap_j F_{l_j}$, i.e. with probability greater than $1 - K(p)/N^p$ provided $E \leqq E^*(\alpha, |f|_\infty, p)$.

Suppose now $|x| \leqq L(E)(N/2)$. Let $l_j = N4^{j+1}, j = 0, 1, \dots$ and let $F = \{\omega \in \Omega | \operatorname{dist}(\sigma(H_{A_0^0}(\omega)), E) \geqq \exp(-m(E)(N/4)L^2(E))\}$.

To estimate $P(F)$ we use the following result due essentially to Wegner [10] (see also Fröhlich–Spencer [2, 3] which will be proven in Appendix C.

Lemma 3.2. *Let $\Lambda \subset \mathbb{R}^3$ be a bounded measurable set. Then*

$$P(\operatorname{dist}(\sigma(H_\Lambda(\omega), E) \leqq k) \leqq c_1 |\bar{\Lambda}|^{3/2} k^{1/2} N(E+k)^{1/2} |f|_\infty^{1/2},$$

where $N(E)$ is the integrated density of states for H_ω at the point E and $\bar{\Lambda} \supseteq \Lambda$ is the smallest cube containing Λ.

Thus using the above lemma we get

$$P(F) \geqq 1 - (4NL(E))^{9/2} \cdot e^{-m(E)(N/8)L^2(E)} |f|_\infty^{1/2} N(E + e^{-m(E)(N/4)L^2(E)}). \quad (3.11)$$

Now we observe that from Theorem 4 in [6] $N(E) \leqq \exp(-CE^{-3/2})$ for $E \leqq E_0(\alpha, 1)$. Hence, using the estimates $L(E) \geqq c_2 E^{-1/2}$, $m(E) \geqq c_1 E^{1/2}$ we get that for $E \leqq E^*(\alpha, |f|_\infty, p)$ the right-hand side of (3.11) is greater than

$$1 - \frac{K'(p)}{N^p} \quad (3.12)$$

for some constant $K'(p) > 0$.

We now estimate the terms in the expansion (3.9) for $\omega \in F \cap \bigcap_{j>1} F_j$ (where $F_j \equiv F_{l_j}$). Terms involving G_j or $\partial_{n_z} G_j$ when $j > 0$ can be estimated as in the previous case, $|x| \geqq L(E)(N/2)$, since they are evaluated at sites with distance greater than or equal to $N4^j L(E)$. The zeroth order term $G_0(0,x)$ is estimated by the next lemma (see Appendix A for a proof).

Lemma 3.3. *Let $\Lambda \subseteq \mathbb{R}^3$ be a bounded, measurable set. Then*

$$|G_\Lambda(\omega, E+i\varepsilon, x, y)| \leqq \frac{c}{|x-y|} \frac{1}{\operatorname{dist}(\sigma(H_\Lambda(\omega)), E)}.$$

Furthermore from Lemma 3.1 and Lemma 3.3 we get

$$|\partial_{n_z} G_0(0,z)| \leqq \frac{c}{|z|} \frac{1}{\operatorname{dist}(\sigma(H_\Lambda(\omega)), E)}.$$

Hence for $\omega \in F \cap \bigcap_{j>1} F_j$, the expansion (3.8) is bounded from above by

$$\begin{aligned} &c|x|^{-1}\exp\left(m(E)\frac{N}{4}L^2(E)\right)\left[1+\sum_{j=1}^{\infty} e^{-m(E)N4^jL(E)}(N4^{j+1}L(E))^3\right] \leqq \\ &\leqq |x|^{-1}\exp\left(m(E)\frac{N}{4}L^2(E)\right) \leqq |x|^{-1}\exp(m(E)(NL^2(E)-|x|)) \end{aligned} \tag{3.13}$$

for $E \leqq E^*(\alpha, |f|_\infty, P)$. Combining now Theorem 3.6 and (3.12), we get

$$P\left(F \cap \bigcap_{j>1} F_j\right) \geqq 1 - \frac{K''(p)}{N^p} \tag{3.14}$$

for some constant $K''(p)$.

Proof of Theorem 3.4. We give here the proof of Theorem 3.4. Since however the argument is the same as the one given by Fröhlich and Spencer [2, 3] we limit ourselves to prove the result for special sets $\Lambda \subseteq \mathbb{Z}^3(E)$. The extension to general $\Lambda \subseteq \mathbb{Z}^3(E)$ can be done as in [2, 3]. The proof is based on an induction argument. Let θ_k denote the following statement

$$|G_{\Lambda^0}(\omega, E+i\varepsilon, x, y)| \leqq \exp\{-m_k(E)|x-y|\},$$

if $|x-y| \geqq \frac{1}{5}d_k L(E)$, assuming Λ is $(k-1)$-admissible and $\Lambda \cap S_k = \emptyset$. Here $m_k(E) = m_0(E)\left(1 - 90\sum_{i=0}^{k} d_i^{-1/4}\right)$, $m_0(E) = E^{1/2}$. We observe that since $d_0 \sim E^{-1/2}$, if E is sufficiently small, $m_k(E) \geqq cE^{1/2}$ uniformly in k.

Proof of θ_0. Let $\Lambda \subset \mathbb{Z}^3(E)$ be such that $\Lambda \cap S_0 = \emptyset$. Then, using Neumann–Dirichlet bracketing (see [9]) we have

$$-\Delta^D_{\Lambda^0} + V_\omega \geqq -\Delta^N_{\Lambda^0} + V_\omega \geqq \bigoplus_{j\in\Lambda} (-\Delta^N_{C_E(j)} + V_\omega) \geqq 2E, \tag{3.15}$$

since Λ is non-singular. Hence $\operatorname{dist}(\sigma(H_{\Lambda^0}(\omega)), E) \geqq E$, and using the Combes–Thomas argument we infer that

$$|G_{\Lambda^0}(\omega, E+i\varepsilon, x, y)| \leqq \exp(-E^{1/2}|x-y|) \tag{3.16}$$

for all $|x-y| \geqq \frac{1}{5}d_0$.

Thus we have proved that θ_0 holds. Next we assume θ_k to be true and prove θ_{k+1} for special $\Lambda \subset \mathbb{Z}^3(E)$.

Lemma 3.4. *Let $R_k^\beta \subset \mathbb{Z}^3(E)$ be a k-admissible region containing a component C_k^β of S_k^g and such that*

(i) $\frac{1}{5}d_{k+1} \leqq \operatorname{diam}_E R_k^\beta \leqq \frac{3}{2}d_{k+1}$,

(ii) $\operatorname{dist}_E(\partial R_k^\beta, \overline{C_k^\beta}) > 0$, $\overline{C_k^\beta}$ as in (3.7).

Then there exists a constant $\bar{E}(\alpha)$ such that if $E \leqq \bar{E}(\alpha)$,

$$|G_{(R_k^\beta)^0}(\omega, E + i\varepsilon, x, y)| \leqq \exp[-(m_k(E) - \mu_k(E))|x - y|].$$

Remark. It is worthwhile to observe that, using condition $A(k)$, i.e. Eqs. (3.3)–(3.6), one has $\operatorname{dist}_E(C_k^\beta, S_k \sim C_k^\beta) \geqq 2d_{k+1}$ so that $R_k^\beta \cap (S_k \sim C_k^\beta) = \emptyset$. In particular $R_k^\beta \cap S_{k+1} = \emptyset$.

Proof of the lemma. Let $R \equiv R_k^\beta$, $C \equiv C_k^\beta$, $\bar{C} \equiv \overline{C_k^\beta}$, and let $B \subset \mathbb{Z}^3(E)$ be a $(k-1)$-admissible set such that $R \sim B$ is $(k-1)$-admissible and

$$\begin{aligned} &\bar{C} \supset B \supset C, \\ &\operatorname{dist}_E(\partial B, \{j(x), j(y)\}) \geqq d_k, \\ &\operatorname{dist}_E(\sim B, C) \leqq 3d_k, \end{aligned} \tag{3.17}$$

where $j(x) \in \mathbb{Z}^3(E)$ is such that $x \in C_E(j(x))$. From the definition of the cubes $C_E(j)$ we see that $j(x)$ is uniquely defined.

The existence of the sets R and B has been shown in [2], Appendix D.

We also set $Q = R \sim B$, $\gamma = \partial(B^0)$, $\bar{\gamma} = \partial(\bar{C}^0)$, and assume for simplicity that $x, y \in Q^0$. The cases $x \in Q^0$, $y \in B^0$ or vice versa can be treated analogously (see [2] for details).

Using the Green's identities (1.7) and (1.8), we write

$$\begin{aligned} G_{R^0}(\omega, E + i\varepsilon, x, y) &\equiv G_{R^0}(x, y) = G_{Q^0|B^0}(x, y) \\ &\quad + \int_\gamma dz (\partial_{n_z} G_{Q^0|B^0}(x, z)) G_{R^0}(z, y) \\ &= G_{Q^0}(x, y) + \int_\gamma dz (\partial_{n_z} G_{Q^0}(x, z)) \\ &\quad \cdot \{G_{Q^0|B^0}(z, y) + \int_\gamma dz' G_{R^0}(z, z')(\partial_{n_{z'}} G_{Q^0|B^0}(z', y))\} \\ &= G_{Q^0}(x, y) + \int_\gamma dz (\partial_{n_z} G_{Q^0}(x, y)) \\ &\quad \cdot \int_\gamma dz' G_{R^0}(z, z')(\partial_{n_{z'}} G_{Q^0}(z', y)). \end{aligned} \tag{3.18}$$

Since $Q \cap S_k = \emptyset$ and Q is $(k-1)$-admissible, we can apply θ_k to G_{Q^0} to get

$$|G_{Q^0}(x, y)| \leqq \exp(-m_k(e)|x - y|). \tag{3.19}$$

Furthermore since $\inf_{z \in \gamma} \operatorname{dist}(z, \{x, y\}) > \frac{1}{5}d_k L(E)$ for E sufficiently small, the exponential decay of $G_{Q^0}(x, y)$ for $|x - y| \geqq \frac{1}{5}d_k L(E)$ and Lemma 3.1 imply that

$$\begin{aligned} &z \in \gamma \text{ implies that } |\partial_{n_z} G_{Q^0}(x, y)| \leqq c e^{-m_k|y - z|}. \\ &z' \in \gamma \text{ implies that } |\partial_{n_{z'}} G_{Q^0}(z', y)| \leqq c e^{-m_k|y - z'|}, \end{aligned} \tag{3.20}$$

To estimate $G_{R^0}(z, z')$ when $z, z' \in \gamma$, we use the following result.

Lemma 3.5. *Let $u, w \in B^0$. Then*

$$|G_{R^0}(\omega, E+i\varepsilon, u, w)| \leqq c\frac{\exp(d_k^{1/2})}{|u-w|}.$$

Proof. From the Green's identities we obtain the following two expressions for $G_{R^0}(\omega, E+i\varepsilon, u, w) \equiv G_{R^0}(u, w)$:

$$G_{R^0}(u,w) = G_{B^0|R^0}(u,w) + \int_\gamma dz'(\partial_{n_{z'}} G_{B^0|R^0}(u,z'))G_{R^0}(z',w),$$

$$G_{R^0}(u,w) = G_{C^0|R^0}(u,w) + \int_{\bar\gamma} dz'(\partial_{n_{z'}} G_{\bar C^0|R^0}(u,z'))G_{R^0}(z',w).$$

Alternating the above expression we get an expansion for $G_{R^0}(u,w)$.

$$\begin{aligned} G_{R^0}(u,w) = G_{\bar C^0}(u,w) &+ \int_{\bar\gamma} dz(\partial_{n_z} G_{\bar C^0|R^0}(u,z))G_{B^0|R^0}(z,w) \\ &+ \int_{\bar\gamma} dz(\partial_{n_z} G_{\bar C^0}(u,z))\int_\gamma dz'(\partial_{n_z} G_{B^0|R^0}(z,z'))G_{\bar C^0}(z',w) + \dots . \end{aligned} \tag{3.21}$$

In (3.21) terms like $G_{\bar C^0}(u', w')$ are estimated using Lemma 3.3 and condition $A(k)$ by

$$|G_{\bar C^0}(u',w')| \leqq \frac{c\exp(d_k^{1/2})}{|u'-w'|}. \tag{3.22}$$

Terms $\partial_{n_z} G_{\bar C^0}(z', z)$ always appear with $|z-z'| \geqq \frac{1}{5}L(E)d_k$. Hence using Lemma 3.1 and (3.22) they are bounded by $c_1\exp(d_k^{1/2})$. Terms like $G_{B^0|R^0}$, or $\partial_{n_z} G_{B^0|R^0}, z\in\gamma$, always coincide with G_{Q^0} and $\partial_{n_z} G_{Q^0}, z\in\gamma$, and they are evaluated at sites z, z' where $z'\in\bar\gamma$ and $z\in\gamma$ such that $|z-z'| \geqq L(E)d_k \geqq \frac{1}{5}d_k L(E)$. Thus we can use θ_k to estimate G_{Q^0} and Lemma 3.1 to estimate $\partial_{n_z} G_{Q^0}$, and to obtain for both of them a bound of the form $c_2\exp(-m_k(E)|z-z'|)$. As a result the right-hand side of (3.21) can be bounded by:

$$\frac{c\exp(d_k^{1/2})}{|u-w|}\sum_{n=0}^{\infty}[c_1\exp(-m_k(E)d_kL(E) + d_k^{1/2})|\bar\gamma||\gamma|]^n, \tag{3.23}$$

where $|\bar\gamma|(|\gamma|)$ is the surface measure of $\bar\gamma(\gamma)$. Such factors arise when we estimate in (3.21) integrals of the form $\int_\gamma dz'(1/|z'-z|)$, $z\in\bar\gamma$.

Using now the estimates $m_k(E) \geqq c_2E^{1/2}$, $L(E) \sim E^{-1/2}$ and $d_0(E) \sim E^{-1/2}$, we easily obtain that for E sufficiently small (3.23) is bounded by $c\exp(d_k^{1/2})/|u-w|$, which proves Lemma 3.5.

We now return to the proof of Lemma 3.4. Inserting the estimate on $G_{R^0}(u,w)$ given by Lemma 3.5 in (3.18), and using (3.19) and (3.20) we obtain

$$\begin{aligned} |G_{R^0}(x,y)| &\leqq \exp(-m_k(E)|x-y|) \\ &+ C\int_\gamma dz\int_\gamma dz'\frac{\exp(-m_k(E)|x-z| + d_k^{1/2} - m_k|z'-y|)}{|z-z'|}. \end{aligned} \tag{3.24}$$

Hence

$$|G_{R^0}(x,y)| \leqq \exp(-m_k(E)|x-y|)[1 + c|\gamma|^2 \exp(d_k^{1/2} + m_k(7d_k L(E) + L(E)))]$$
$$\leqq \exp(-(m_k(E) - \mu_k(E))|x-y|),$$

with $\mu_k(E) = 45E^{1/2}d_k^{1/2}$, provided that E is so small that

$$1 + c|\gamma|^2 \exp(d_k + m_k(7d_k L(E) + L(E)) \leqq \exp(8m_k L(E) d_k).$$

Let now the set $A \subset \mathbb{Z}^3(E)$ appearing in Theorem 3.4 be such that $\operatorname{diam}_E A \leqq \frac{3}{2}d_{k+1}$. If $A \cap S_k^g = \emptyset$, then θ_{k+1} follows immediately from θ_k. If $A \cap S_k^g \neq \emptyset$, then we can take the set R of Lemma 3.4 to be equal to A, and θ_{k+1} follows from Lemma 3.4. If now $\operatorname{diam}_E A > \frac{3}{2}d_{k+1}$, we can repeat in our context the proof by Fröhlich–Spencer of Lemma 3.3 in [2] without any problem and get θ_{k+1} for general A.

Section 4. Probabilistic Estimates

We prove here the basic probabilistic estimate given in Theorem 3.5. Define for a fixed site $i \in \mathbb{Z}^3(E)$, $p_j \equiv P(i \in S_j^g)$. Then clearly

$$p_j \leqq \sum_{\substack{D \subset \mathbb{Z}^3(E) \\ i \in D}} P(D \text{ is a component of } S_j^g).$$

Let now $\mathbb{Z}^3(E,n) \equiv 2^n \mathbb{Z}^3(E)$, $n \geqq 0$. With each site $x \in \mathbb{Z}^3(E, n-1)$ we associate a cube $c_n(x)$ centered at x with sides of length (measured with $|\ |_E$) 2^n parallel to the lattice axes. Then $c_n(x)$ will be called an n-cube. For $n=0$ we set $\mathbb{Z}^3(E,-1) = \mathbb{Z}^3(E)$ and $c_n(x) = x$. Given $D \subset \mathbb{Z}^3(E)$, let $C_n(D)$ be the minimal family of n-cubes which cover D, let $|C_n(D)|$ be its cardinality and let $V(D) = \sum_{n=0}^{n_0(D)} |C_n(D)|$, where $n_0(D)$ is the smallest integer such that $2^{n_0(D)} \geqq 2 \operatorname{diam}_E D$. Let also $C_n'(D) \equiv \{c_n \in C_n(D)| \cdot \operatorname{dist}_E(c_n, c_n') \geqq 2.2^{5n/3}$ for all $c_n' \in C_n(D)$, $c_n' \neq c_n\}$, and let $V'(D) = \sum_{n=1}^{n_0(D)} |C_n'(D)|$. In order to prove Theorem 3.5 it is sufficient to prove, following Fröhlich–Spencer ([2], Sect. 6), the following two estimates:

(a) Let $\bar{D}_j = \{i \in \mathbb{Z}^3(E) | \operatorname{dist}_E(i, D) \leqq 4d_{j-1}\}$. Then $P(\operatorname{dist}(\sigma(H_{(\bar{D}_j)^0}(\omega)), E) \leqq e^{-d_{j-1}^{1/2}}) \leqq \phi(E, |f|_\infty)(d_{j-1})^{9\ 2} e^{-(d_{j-1}^{1/2})/2}$, with $0 < \phi(E, |f|_\infty) \leqq \eta$ for an arbitrary $\eta > 0$ provided $E \leqq [\ln(|f|_\infty / E(\eta, \alpha))]^{-2/3}$ for some constant $E(\eta, \alpha)$ *independent* of $|f_\infty|$.

(b) $P_D \equiv \langle \chi_D \rangle \equiv P$ ($\exists i$ such that D is a component of $S_i^g) \leqq \exp[-K_0(E)|D| - K'(E)V'(D)]$, where $K_0(E)$ and $K'(E)$ are both independent of $|f|_\infty$ and arbitrary large for E sufficiently small.

Estimate (a) can be settled using Lemma 3.2 with $k = e^{-d_{j-1}^{1/2}}$. One finds $\phi(E, |f|_\infty) = C|f|_\infty^{1/2} N(2E)^{1/2}$. We have here used the fact that for E small enough $\exp(-d_{j-1}^{1/2}) \leqq E$. Using now the bound $N(E) \leqq \exp(-cE^{-3/2})$ for E small we see that (a) is satisfied. To estimate P_D we make use of the following result, the proof of which is just a copy of the proof of Lemma 5.2 in [2] and is therefore omitted:

Lemma 4.1. *Let $I=\{0\}\cup\{n\in\mathbb{Z}|n\geqq\ln(d_0(E))/\ln(2)\}$, and let for any $n\in I, n>0, j(n)$ be the smallest integer such that $d_{j(n)}\geqq 2^n$. Let furthermore for any $c\in C_n(D), n>0$,*

$$\chi_{n,c}=\chi_{\{\omega|\mathrm{dist}(\sigma(H(\overline{C\cap D})\circ(\omega)),E)\leqq\exp(-d_{j(n)}^{1/2})\}},$$

where $\overline{C\cap D}=\{i\in\mathbb{Z}^3(E)|\mathrm{dist}_E(i,C\cap D)\leqq 4d_{j(n)}\}$. For $n=0$ define $\chi_{0,c}=\chi_{\{\omega|c\in S_0\}}$. Then

$$P_D\leqq\left\langle\prod_{n\in I}\prod_{c\in C'_n(D)}\chi_{n,c}\right\rangle.$$

It is easy to see that if c_1, c_2 belong to $C'_n(D)$, then if $d_0(E)$ is sufficiently large, i.e. E is sufficiently small, then $\overline{c_1\cap D}\cap\overline{c_2\cap D}=\emptyset$. This implies the independence of $\chi_{n,c}$, when $c\in C'_n$ and $n\in I$. Thus, from the lemma and the Hölder inequality we infer

$$P_D\leqq\prod_{n\in I}\prod_{c\in C'_n(D)}\langle\chi_{n,c}\rangle^{r^n(1-r)},\quad 0<r<1. \tag{4.1}$$

For $n>0$ we can estimate $\langle\chi_{n,c}\rangle$ using Lemma 3.2 with $k=\exp(-2^{n/2})$, and obtain

$$\langle\chi_{n,c}\rangle^{r^n}\leqq[|f|_\infty^{1/2}(c_1 d_{j(n)}L(E)^{9/2}\exp(-\tfrac{1}{2}2^{n/2})N(E+e^{-2^{n/2}})^{1/2}]^{r^n}. \tag{4.2}$$

Since $n\geqq\ln d_0(E)/\ln 2$, we can take E so small that $\exp(-2^{n/2})\leqq E$. Using again the bound $N(E)\leqq e^{-CE^{-3/2}}$, the right-hand side of (4.2) is bounded by

$$(d_{j(n)}^{9/2}\exp(-\tfrac{1}{2}2^{n/2}))r^n, \tag{4.3}$$

provided $E\leqq[\ln(|f|_\infty/E_1(\alpha))]^{-2/3}$ for some constant $E_1(\alpha)$.

Next, using $d_{j(n)}\leqq 2^{n5/4}$, we see that (4.3) is bounded by a decreasing function of n if e.g. $r=0.8$. In conclusion if $E\leqq[\ln(|f|_\infty/E_1(\alpha))]^{-2/3}$ for some constant $E_1(\alpha)$, the left-hand side of (4.2) is bounded by:

$$[d_1(E)^{9/2}\exp(-\tfrac{1}{2}d_0(E)^{1/2})]r^{n_0(E)}(1-r)\equiv e^{-JK'(E)}, \tag{4.4}$$

with $r=0.8$, and $K'(E)\to+\infty$ if $E\to 0$.

To estimate $\langle\chi_{0,c}\rangle$ we first observe that using translation invariance it is enough to estimate the probability that the origin of $\mathbb{Z}^3(E)$ belongs to S_0. To do this let us define new random variables $\xi_i(\omega)$ as follows

$$\xi_i(\omega)=\begin{cases}0 & \text{if } q_i(\omega)\leqq 1/2,\\ 1/2 & \textit{if } q_i(\omega)>1/2.\end{cases}$$

Using our assumption on the distribution of the q_i's, we get that $P(\xi_i=0)=\alpha$. Furthermore since $\xi_i(\omega)\leqq q_i(\omega)$ for all $i\in\mathbb{Z}^3$, we have

$$P(0\in S_0)=P(\lambda_1(H^N_{C_E(0)}(\omega))\leqq 2E)\leqq P\left(\lambda_1\left(-\Delta^N_{C_E(0)}+\sum_i\xi_i(\omega)\chi_{C_i}\right)\leqq 2E\right). \tag{4.5}$$

This last probability has been estimated in [6] and it is bounded by:

$$\exp(-cE^{-3/2}),\quad c>0 \tag{4.6}$$

for all $E<\bar{E}(\alpha)$. Combining now (4.6) and (4.4) we see that if $E\leqq\min\{\bar{E}(\alpha),[\ln(|f|_\infty/E_1(\alpha))]^{-2/3}\}$, where $\bar{E}(\alpha)$ and $E_1(\alpha)$ are two positive constants *independent* of $|f|_\infty$, then

$$P_D \leqq \exp(-K_0(E)|D| - K'(E)V'(D)), \tag{4.7}$$

with $K_0(E) = C(2E)^{-3/2}$ and $K'(E) \to +\infty$ as $E \to 0$. We emphasize here that in the definition of $K'(E)$ (see (4.4)) $|f|_\infty$ *does not appear*. This proves (b).

Appendix A. Some Estimates on $G_\Lambda(\omega, E+i\varepsilon, x, y)$

We prove her Lemma 3.1 and 3.2.

Proof of Lemma 3.1. Let A be one of the corners nearest to y and assume first $\operatorname{dist}(A, y) \leqq 1/2\sqrt{3}$. Let $\Lambda_0 \subset \Lambda$ be a cube of size $1/\sqrt{3}$ such that $y \in \partial\Lambda_0$ and $\operatorname{dist}(y, B) \geqq \operatorname{dist}(y, A)$, where B is any corner of Λ_0. Clearly such a cube always exists and $\partial\Lambda_0 \cap \partial\Lambda \neq \emptyset$. Let us fix $x \in \Lambda$, with $|x-y| \geqq 1$, and let $g(z) \equiv G_\Lambda(\omega, E+i\varepsilon, x, z)$ when $z \in \Lambda_0$. Since $x \notin \Lambda_0$, the continuous function g is the weak solution of the Dirichlet problem

$$\begin{aligned}
\Delta v &= (V_\varphi - E - i\varepsilon)g && \text{in } \Lambda_0^0,\\
v &= g && \text{on } \partial\Lambda_0 \sim \partial\Lambda,\\
v &= 0 && \text{on } \partial\Lambda_0 \cap \partial\Lambda,
\end{aligned} \tag{A.1}$$

where for any set $A \subseteq \mathbb{R}^\nu$, A^0 denotes the interior of A. We write $g = w + u$, where w and u are the unique weak solutions of the problems

$$\begin{aligned}
\Delta w &= (V_\omega - E - i\varepsilon)g && \text{in } \Lambda_0^0,\\
w &= 0 && \text{on } \partial\Lambda_0,
\end{aligned} \tag{A.2}$$

and

$$\begin{aligned}
\Delta u &= 0 && \text{in } \Lambda_0^0,\\
u &= g && \text{in } \partial\Lambda_0 \sim \partial\Lambda,\\
u &= 0 && \text{on } \partial\Lambda_0 \cap \partial\Lambda.
\end{aligned} \tag{A.3}$$

Using the method of strong barrier functions (see Courant–Hilbert, Vol. II [1], p. 343), we get that

$$|\partial_{n_y} W| \leqq c_1 |V_\omega - E - i\varepsilon|_\infty |g|_\infty. \tag{A.4}$$

To estimate $|\partial_{n_y} u|$ we use the Schwartz reflection principle (see [4], p. 28). For simplicity we assume that Λ_0 has three faces F_1, F_2, F_3 on $\partial\Lambda$; the case when the faces on $\partial\Lambda$ are only two can be discussed along the same lines. We then construct a new cube $\tilde{\Lambda}_0$ and a new harmonic function $\tilde{u}$ from Λ_0 and u as follows. We reflect Λ_0 with respect to the hyperplane through F_1 to get a new set Λ_1; then we reflect $\Lambda_0 \cup \Lambda_1$ with respect to F_2 to get a set Λ_2 and finally we reflect $\Lambda_0 \cup \Lambda_1 \cup \Lambda_2$ with respect to F_3 to get Λ_3 and define $\tilde{\Lambda}_0 = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$. Clearly $y \in \tilde{\Lambda}_0$ and $\operatorname{dist}(y, \partial\tilde{\Lambda}_0) \geqq 1/2\sqrt{3}$. Let now $u_1(x) = u(x)$, if $x \in \Lambda_0$, and $u_1(x) = -u(x^*)$, if $x \in \Lambda_1$, where x^* is the reflection of x with respect to F_1; $u_2(x)$ and $\tilde{u}(x)$ are defined on $\Lambda_0 \cup \Lambda_1 \cup \Lambda_2$ and on $\tilde{\Lambda}_0$ in the same way, but with $u(x)$ replaced by $u_1(x)$ and $u_2(x)$ respectively. Since $u = 0$ on $F_1 \cup F_2 \cup F_3$, according to the Schwartz reflection principle, $\tilde{u}$ is again harmonic and $|\tilde{u}|_\infty = |u|_\infty$ by construction. Since y has positive distance from the boundary of $\tilde{\Lambda}_0$ we can now use interior gradient estimates for harmonic functions ([4], Th. 2.10) to get

$$|\partial_{n_y} u| = |\partial_{n_y} \tilde{u}| \leqq \frac{3}{\left(1 - \frac{1}{2\sqrt{3}}\right)} |u|_\infty . \tag{1.5}$$

Thus we have proved:

$$|\partial_{n_y} g| \leqq c_1 |V_\omega - E - i\varepsilon|_\infty |g|_\infty + c_2 |u|_\infty . \tag{A.6}$$

Since it is easy to see that $|u|_\infty \leqq c_3 |g|_\infty$, we get:

$$|\partial_{n_y} g| \leqq c_4 |g|_\infty \leqq c_5 \exp(-m|x-y|). \tag{A.7}$$

In the case $\mathrm{dist}(y, \Lambda) > 1/2\sqrt{3}$ we can use a similar argument if we construct the cube Λ_0 in such a way that $y \in \partial\Lambda_0$ and y is the middle point of the face of Λ_0 which contains it.

Proof of Lemma 3.3. Following Simon ([10], p. 479) we write, using the resolvent identity

$$\begin{aligned}(H_\Lambda(\omega) - E - i\varepsilon)^{-1} = {} & (H_\Lambda(\omega) + 1)^{-1} + (1 + E + i\varepsilon)(H_\Lambda(\omega) + 1)^{-2} \\ & + (1 + E + i\varepsilon)^2 (H_\Lambda(\omega) + 1)^{-2} (H_\Lambda(\omega) - E - i\varepsilon)^{-1}. \end{aligned} \tag{A.8}$$

Using now a result of Simon ([10], Th. B.7.2), the first term in (A.8) has kernel bounded by $c_1/|x-y|$, while the second one has bounded kernel. To estimate the third term we use the following inequality valid for any $\psi, \phi \in L^1(\Lambda)$

$$\begin{aligned} & |\langle \psi, (H_\Lambda(\omega)+1)^{-2} (H_\Lambda(\omega) - E - i\varepsilon)^{-1} (H_\Lambda(\omega)+1)^{-1} \phi \rangle| \\ & \leqq \|\psi\|_1 \|\phi\|_1 \|(H_\Lambda(\omega)+1)^{-1}\|_{2,\infty} \|(H_\Lambda(\omega) - E - i\varepsilon)^{-1}\|_{2,2} \|(H_\Lambda(\omega)+1)^{-1}\|_{1,2}, \end{aligned} \tag{A.9}$$

where $\|\cdot\|_{p,q}$ denotes the operator norm from L^p to L^q. Again using the results of [10], $\|(H_\Lambda(\omega)+1)^{-1}\|_{2,\infty}$ and $\|(H_\Lambda(\omega)+1)^{-1}\|_{1,2}$ are bounded uniformly in ω. Hence the left-hand side of (A.9) is bounded by

$$c_2 (\mathrm{dist}(E, \sigma(H_\Lambda(\omega))))^{-1} \|\psi\|_1 \|\phi\|_1 . \tag{A.10}$$

Taking now ϕ and ψ to be smooth approximations of Dirac's delta-function at the points x and y respectively, we get from (A.10) and from the continuity of the kernel of $(H_\Lambda(\omega)+1)^{-2}(H_\Lambda(\omega) - E - i\varepsilon)^{-1}$ (see [10]):

$$|\{(H_\Lambda(\omega)+1)^{-2}(H_\Lambda(\omega) - E - i\varepsilon)^{-1}\}(x,y)| \leqq c_2 (\mathrm{dist}(E, \sigma(H_\Lambda(\omega))))^{-1}. \tag{A.11}$$

Appendix B. Proof of Lemma 2.1 and 2.2

We prove here Lemma 2.1 and 2.2. It is clearly enough to show that

$$\|(1+|x|^2)^{-1} f(H_\omega)(1+|x|^2)\|_{2,2} \tag{B.1}$$

or

$$\|(1+|x|^2)^{-1} [f(H_\omega), x^2]\|_{2,2} \tag{B.2}$$

satisfy the stated bound.

Following Simon [10] we write $f(H_\omega) = (1/2\pi)\int d\lambda \hat{f}(\lambda) e^{i\lambda H_\omega}$. Hence we have to compute $[e^{i\lambda H_\omega}, x^2] = (e^{i\lambda H_\omega} x^2 e^{-i\lambda H_\omega} - x^2) e^{i\lambda H_\omega}$. Now

$$e^{i\lambda H_\omega} x^2 e^{-i\lambda H_\omega} = x^2 - 2i\lambda + 4\int_0^\lambda d\mu e^{i\mu H_\omega} \mathbf{x}\cdot \mathbf{p} e^{-i\mu H_\omega}$$

$$= x^2 - 2i\lambda + 8\int_0^\lambda d\mu \int_0^\mu d\mu' e^{i\mu' H_\omega} p^2 e^{-i\mu' H_\omega}. \tag{B.3}$$

From (B.3) we see that

$$\|(1+|x|^2)^{-1}[e^{i\lambda H_\omega}, x^2](H_\omega + 1)^{-1}\|_{2,2} \leqq c(1+|\lambda|^2). \tag{B.4}$$

from which it follows:

$$\|(1+|x|^2)^{-1}[f(H_\omega), x^2](H_\omega + 1)^{-1}\|_{2,2} \leqq c\int d\lambda |\hat{f}(\lambda)|(1+|\lambda|^2). \tag{B.5}$$

To prove the result for $(1+|x|^2)^{-1}[f(H_\omega), x^2]$, we write it as:

$$\begin{aligned}(1+|x|^2)^{-1}x^2 f(H_\omega) &+ (1+|x|^2)^{-1} f(H_\omega) H_\omega x^2 (H_\omega + 1)^{-1}\\ &+ (1+|x|^2)^{-1} f(H_\omega) x^2 (H_\omega + 1)^{-1}\\ &+ (1+|x|^2)^{-1} f(H_\omega)[x^2, H_\omega](H_\omega + 1)^{-1}.\end{aligned} \tag{B.6}$$

The first term is obviously bounded while the second and third term are bounded using (B.4) for the functions $f(x)$, $h(x) \equiv f(x) x \in C_0^\infty(\mathbb{R})$. The last term is equal to $(1+|x|^2)^{-1} f(H_\omega)(2i - 4\mathbf{x}\cdot\mathbf{p})(H_\omega + 1)^{-1}$. Since $p(H_\omega + 1)^{-1}$ is bounded, it is sufficient to show that $(1+|x|^2)^{-1} f(H_\omega) x$ is bounded and for this it suffices to repeat the steps (B.3) to (B.6) with x^2 replaced by x. Collecting all the estimates together we finally get:

$$\begin{aligned}\|(1+|x|^2)[f(H_\omega), x^2]\|_{2,2} &\leqq c_1 |f|_\infty + c_2 \int d\lambda |\hat{h}(\lambda)|(1+|\lambda|^2)\\ &\quad + c_3 \int d\lambda |\hat{f}(\lambda)|(1+|\lambda|^2)\\ &\leqq c_1 |f|_\infty + c_2 |\mathrm{supp}\, f| \left(\left|\frac{d^4}{dx^4} f\right|_\infty + \left|\frac{d^4}{dx^4} h\right|_\infty \right),\end{aligned} \tag{B.7}$$

where $|\mathrm{supp}\cdot f|$ denotes the Lebesgue measure of supp f.

Proof of Lemma 2.2. Write $f(H_\omega) = (H_\omega + 1)^{-1} g(H_\omega)$, where $g(x) = (x+1) f(x) \in C_0^\infty(\mathbb{R})$. Using Lemma 2.1, $\|g(H_\omega)\|_{L_2^2 \to L_2^2}$ is bounded uniformly in ω. Since, using a result of Simon [10], $\|H_\omega + 1\|_{L_2^2 \to L_2^\infty}$ is bounded uniformly in ω, we get the result.

Appendix C. Proof of Lemma 3.2

We prove here Lemma 3.2. Following Wegner [11] and Fröhlich–Spencer [2] we write:

$$\begin{aligned}P(\mathrm{dist}(\sigma(H_\Lambda(\omega)), E) < k) &= P(N(E+k, H_\Lambda(\omega)) - N(E-k, H_\Lambda(\omega)) \geqq 1)\\ &\leqq \left\langle \int_{E-k}^{E+k} dE' \frac{d}{dE'} N(E', H_\Lambda(\omega)) \right\rangle\end{aligned}$$

$$= - \sum_{\substack{i\in\mathbb{Z}^3 \\ c_i\cap\Lambda\neq\emptyset}} \int_{E-k}^{E+k} dE' \int \prod_{\substack{j \\ j\neq i, c_j\cap\Lambda\neq\emptyset}} (dq_j f(q_j))$$
$$\cdot \int dq_i f(q_i) \partial_{q_i} N(0, H_\Lambda(\omega) - E') \tag{C.1}$$
$$\leqq |f|_\infty \sum_{\substack{i\in\mathbb{Z}^3 \\ c_i\cap\Lambda\neq\emptyset}} \int \prod_{\substack{j \\ j\neq i, c_j\cap\Lambda\neq\emptyset}} (dq_j f(q_j))$$
$$\cdot [N(E, H_\Lambda(\omega))|_{q_i=0} - N(E, H_\Lambda(\omega))|_{q_i=+1}],$$

where $N(E, H_\Lambda(\omega))|_{q_i=0} = \#\left\{k\in\mathbb{N} \mid \lambda_k\left(-\Delta_\Lambda^D + \sum_{\substack{j\neq i \\ c_j\cap\Lambda\neq\emptyset}} \chi_{C_j} q_j\right) \leqq E\right\}$,

and $N(E, H_\Lambda(\omega))|_{q_i=+1} = \#\left\{k\in\mathbb{N} \mid \lambda_k\left(-\Delta_\Lambda^D + \sum_{\substack{j\neq i \\ c_j\cap\Lambda\neq\emptyset}} \chi_{C_j} q_j + \chi_{C_i}\right) \leqq E\right\}$.

Since $N(E, H_\Lambda(\omega))|_{q_i=0} \leqq N(E, -\Delta_\Lambda^D) \leqq \text{const}\,|\Lambda|$, the right-hand side of (C.1) is bounded by

$$C|f|_\infty k|\Lambda|^2. \tag{C.2}$$

On the other hand:

$$P(\operatorname{dist}(\sigma(H_\Lambda(\omega)), E) \leqq k) \leqq \langle N(E+k, H_\Lambda(\omega))\rangle \leqq \langle N(E+k, H_{\bar\Lambda}(\omega))\rangle$$
$$\leqq |\bar\Lambda| \sup_{\substack{\Lambda\subset\mathbb{R}^3 \\ \Lambda\,\text{cube}}} \frac{1}{|\Lambda|} \langle N(E+k, H_\Lambda(\omega))\rangle, \tag{C.3}$$

where $\bar\Lambda \supseteq \Lambda$ is the smallest cube containing Λ. Here we have used the monotonicity of the eigenvalues $\lambda_k(H_\Lambda(\omega))$ with respect to Λ (see [7]). Observing that from the results of [7] we have that

$$\sup_{\substack{\Lambda\subset\mathbb{R}^3 \\ \Lambda\,\text{cube}}} \frac{1}{|\Lambda|} \langle N(E+k, H_\Lambda(\omega))\rangle = N(E+k), \tag{C.4}$$

where $N(E)$ is the integrated density of states of the system, we conclude the proof by taking the geometric mean of the two estimates.

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