Commun. Math. Phys. 93, 19–31 (1984)

Communications in Mathematical Physics

Uniform Boundedness of Conditional Gauge and Schrödinger Equations

Zhongxin Zhao*

Institute of Systems Science and Mathematical Sciences, Academica Sinica, Beijing, People's Republic of China

Abstract. We prove that for a bounded domain $D \subset R^n$ with C^2 boundary and $q \in K_n^{\mathrm{loc}}$ $(n \geqq 3)$ if $E^x \exp \int_0^{\tau_D} q(x_t)dt \not\equiv \infty$ in D, then

$$\sup_{\substack{x\in D\\ z\in\partial D}} E_z^x \exp \int_0^{\tau_D} q(x_t)dt < +\infty \quad (\{x_t\}: \text{Brownian motion}).$$

The important corollary of this result is that if the Schrödinger equation $\frac{\Delta}{2}u + qu = 0$ has a strictly positive solution on D, then for any $D_0 \subset\subset D$, there exists a constant $C = C(n, q, D, D_0)$ such that for any $f \in L^1(\partial D, \sigma)$, ($\sigma$: area measure on ∂D) we have

$$\sup_{x\in D_0} |u_f(x)| \leqq C \int_{\partial D} |f(y)|\sigma(dy),$$

where u_f is the solution of the Schrödinger equation corresponding to the boundary value f.

To prove the main result we set up the following estimate inequalities on the Poisson kernel $K(x,z)$ corresponding to the Laplace operator:

$$C_1 \frac{d(x,\partial D)}{|x-z|^n} \leqq K(x,z) \leqq C_2 \frac{d(x,\partial D)}{|x-z|^n}, \qquad x\in D, \qquad z \in \partial D,$$

where C_1 and C_2 are constants depending on n and D.

Let D be a bounded domain in R^n $(n \geqq 3)$ with C^2 boundary, $(x_t, t>0)$ be the Brownian motion and $\tau_D = \inf(t>0 : x_t \notin D)$. According to Doob [3], for any positive harmonic function h on D, h-conditioned Brownian motion in D is

---

* Visiting Department of Mathematics, Stanford University, Stanford, CA 94305, USA

determined by the following transition probability density:

$$P_h(t,x,y)=\frac{1}{h(x)}P^D(t,x,y)h(y)\,,\qquad t>0\,,\qquad x,y\in D\,, \tag{1}$$

where $P^D(t,x,y)$ is the density of the Brownian motion killed outside D (see [6]).

In this paper, we only consider $h(x)$ as the Poisson kernel of $D: K(x,z)$, $(x\in D, z\in\partial D)$. For any $x\in D$, $K(x,\cdot)$ is defined as the density of the harmonic measure on ∂D:

$$K(x,z)\sigma(dz)=P^x(x(\tau_D)\in dz)\,. \tag{2}$$

According to the Green formula and smoothness of the boundary, $K(x,z)$ can also be defined as follows:

$$K(x,z)=\frac{\partial G}{\partial n_z}(x,z)\,, \tag{3}$$

where $G(\cdot,\cdot)$ is the Green function of D and $\frac{\partial G}{\partial n_z}$ is the internal normal derivative. G also has the following probabilistic meaning: (see [6])

$$G(x,y)=\int_0^\infty P^D(t,x,y)dt\,. \tag{4}$$

For any $z\in\partial D$, if we let $h(\cdot)$ be $K(\cdot,z)$ in (1), then the corresponding process is called z-conditioned Brownian motion in D. Let P_z^x and E_z^x denote respectively the probability and expectation determined by z-conditioned Brownian motion starting at x.

By (1) and (2), it is easy to check the following properties:

For any positive and F_{τ_D}-measurable function Φ, we have

$$E^x[\Phi(\omega)|x(\tau_D)]=E^x_{x(\tau_D)}\Phi(\omega)\,. \tag{5}$$

For any stopping time $T<\tau_D$ and any positive, F_T-measurable function Φ,

$$E_z^x\Phi(\omega)=\frac{1}{K(x,z)}E^x[\Phi(\omega)K(x_T,z)]\,. \tag{6}$$

Let q be a Borel function belonging to the class K_n^{loc} (see [1, 7]), i.e. q satisfies the condition: for each $R>0$,

$$\lim_{\alpha\downarrow 0}\left[\sup_{|x|\leqq R}\int_{|y-x|\leqq\alpha}\frac{|q(y)|}{|y-x|^{n-2}}dy\right]=0\,. \tag{7}$$

Set $e_q(t)=\exp\int_0^t q(x_s)ds$, $(t\geqq 0)$.

The main result in this paper is the following:

Theorem 1. *If $E^x e_q(\tau_D)\not\equiv\infty$ in D, then*

$$\sup_{\substack{x\in D\\ z\in\partial D}} E_z^x[e_q(\tau_D)]<+\infty\,.$$

Remark 1. This result improves the main theorems in [1] by Aizenman and Simon, in [4] by Falkner, and in [9] by Zhao.

Remark 2. By Theorem A.4.1 and Theorem A.4.9 in [1], the condition $E^x e_q(\tau_D) \not\equiv \infty$ can be replaced by condition (A):

$$\sup\left[\operatorname{spec}\left(\frac{\Delta}{2}+q\right)\right] = \sup_{\substack{v\in C_0^\infty(D)\\ \|v\|_{L^2}=1}}\left[-\int_D |\nabla v|^2 + \int_D qv^2\right] < 0,$$

or condition (B): there exists a solution u of $\left(\frac{\Delta}{2}+q\right)u=0$ with a positive lower bound on D.

Theorem 1 has the following important corollary:

Theorem 2. *If* (D, q) *satisfies* (A) *or* (B), *then for any domain* $D_0, \bar{D}_0 \subset D$, *there exists a constant* $C = C(n, q, D, D_0)$ *such that for any* $f \in L^1(\partial D, \sigma)$, *we have*

$$\sup_{x\in D_0} |u_f(x)| \leqq C \int_{\partial D} |f(z)|\sigma(dz),$$

where $u_f(x) = E^x[e_q(\tau_D) f(x_{\tau_D})]$. *($u_f$ is the solution of the Schrödinger boundary problem corresponding to f.)*

Proof. By the Harnack inequality, for $D_0 \subset\subset D$,

$$J_1 = \sup_{\substack{x\in D_0\\ z\in\partial D}} K(x,z) < +\infty.$$

According to Theorem 1, $J_2 = \sup_{\substack{x\in D\\ z\in\partial D}} E_z^x e_q(\tau_D) < +\infty$. By definition and (5), we have

$$u_f(x) = \int_{\partial D} f(z) E_z^x e_q(\tau_D) K(x,z)\sigma(dz).$$

Hence

$$\sup_{x\in D_0} |u_f(x)| \leqq J_1 J_2 \int_{\partial D} |f(z)|\sigma(dz). \quad \square$$

To prove Theorem 1 we need some lemmas. The following lemma has independent interest:

Lemma 1. *There exist two constants* C_1 *and* C_2, *which only depend on* n *and* D, *such that*

$$C_1 \frac{d(x)}{|x-z|^n} \leqq K(x,z) \leqq C_2 \frac{d(x)}{|x-z|^n}, \qquad x\in D, \qquad z\in\partial D, \tag{8}$$

where $d(x) = d(x, \partial D)$.

Proof. By Theorem 2.3 in [8], there exists some $C > 0$ such that

$$\left|\frac{\partial}{\partial y_i} G(x,y)\right| \leqq C \frac{d(x)}{|x-y|^n}, \qquad x, y\in D, \qquad i=1,\dots,n.$$

Considering the continuity of $\frac{\partial}{\partial y_i} G(x, y)$ on $\bar{D}$, we have

$$K(x, z) = \frac{\partial}{\partial n_z} G(x, z) \leqq C\sqrt{n}\frac{d(x)}{|x-z|^n}.$$

The inequality on the right hand of (8) is proved.

Since D is of class C^2, there exists $r>0$ such that for any $z \in \partial D$, a ball of radius r, $B_z \subset D$ and $z \in \partial B_z$.

Set

$$D_1 = \left\{x \in D : d(x, \partial D) < \frac{r}{2}\right\}.$$

By the Harnack inequality, there exists a constant $C>0$ such that for all $x \in D \backslash D_1$, $z \in \partial D$,

$$K(x, z) \geqq C. \tag{9}$$

Then we have

$$K(x, z) \geqq \frac{Cr^n}{d(D)2^n} \cdot \frac{d(x)}{|x-z|^n}, \tag{10}$$

where $d(D)$ is the diameter of D.

For $x \in D_1$, $\exists w \in \partial D$ such that $d(x) = |w-x|$. Let u_0 be the center of ball B_w and

$$\bar{B}\left(u_0; \frac{r}{2}\right) = \left(u : |u-u_0| \leqq \frac{r}{2}\right).$$

We consider the domain $R = B_w \backslash \bar{B}\left(u_0; \frac{r}{2}\right)$ and introduce a function v as in Lemma 3.4 in [5]:

$$v(u) = \exp\left(-\frac{2n}{r^2}|u-u_0|^2\right) - \exp(-2n), \qquad u \in R,$$

$$\Delta v(u) = \exp\left(-\frac{2n}{r^2}|u-u_0|^2\right)\left(\frac{16n^2}{r^4}|u-u_0|^2 - \frac{4n^2}{r^2}\right) \geqq 0, \qquad u \in R.$$

Set $\varepsilon = \dfrac{C}{\exp(-n/2) - \exp(-2n)}$. Since $\bar{B}(u_0; r/2) \subset D \backslash D_1$, if $u \in \partial \bar{B}(u_0; r/2)$, $K(u, z) \geqq C \geqq \varepsilon v(u)$, $(z \in \partial D)$. If $u \in \partial B_w$, $v(u) = 0$. So for all $u \in \partial R$,

$$F_z(u) \equiv \varepsilon v(u) - K(u, z) \leqq 0.$$

On the other hand we have

$$\Delta F_z(u) = \varepsilon \Delta v(u) \geqq 0, \qquad u \in R.$$

Now the maximum principle implies that

$$\begin{aligned} K(x, z) &\geqq \varepsilon v(x) = \varepsilon\left\{\exp\left[-\frac{2n}{r^2}(r-d(x))^2\right] - \exp(-2n)\right\} \\ &\geqq \frac{2n\varepsilon}{r}\exp(-2n)d(x), \qquad x \in R. \end{aligned}$$

So we have

$$K(x,z) \geqq \left[\frac{2nC}{r[\exp(-n/2)-\exp(-2n)]}\cdot \exp(-2n)d(x)\right], \qquad x\in D_1\,. \tag{11}$$

Since the normal unit vectors n_z are uniformly continuous with respect to z on ∂D, there exists some $b>0$ such that if $z_1, z_2 \in \partial D$, $|z_1 - z_2| < b$, then

$$\sin\frac{\angle(n_{z_1}, n_{z_2})}{2} < 1/8\,, \tag{12}$$

where $\angle(\cdot,\cdot)$ denotes the angle between two vectors.

Take $a=\min(b/2, r/8)$. For any $z\in\partial D$, set $D_z=(u\in D: |u-z|<a)$. From (11), for any $x\in D_1\backslash D_z$, we have

$$K(x,z) \geqq \frac{2nCa^n\exp(-2n)}{r[\exp(-n/2)-\exp(-2n)]}\cdot\frac{d(x)}{|x-z|^n}\,. \tag{13}$$

For $x\in D_z$, $\exists w\in\partial D$ such that $|w-x|=d(x)$. Set $D_0=B_z\cup B_w$. Since $D_0\subset D$, D, and D_0 have the same normal direction at z, we have

$$K(x,z)\geqq K_{D_0}(x,z)\,. \tag{14}$$

Let o_z and o_w be the centers of B_z and B_w respectively. Then we have

$$|o_z - o_w| \leqq |z-w| + 2r\sin\frac{\angle(n_z, n_w)}{2}\,.$$

Take a point $u\in\partial B_z\cap\partial B_w$. Set $\theta=\angle(o_w u, o_w o_z)$. Since

$$|z-w|\leqq|z-x|+|x-w|\leqq 2|x-z|\leqq 2a\leqq b$$

and (12),

$$\sin\frac{\angle(n_z, n_w)}{2} < 1/8\,.$$

So we have

$$\cos\theta = \frac{|o_z - o_w|}{2r} \leqq a/r + \sin\frac{\angle(n_z, n_w)}{2} \leqq 1/8 + 1/8 = 1/4\,. \tag{15}$$

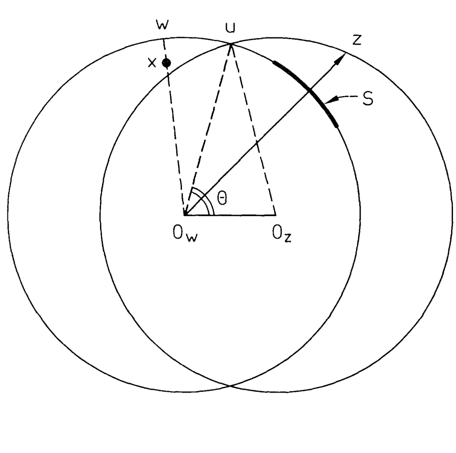

Fig. 1

Take $\alpha=\theta-\angle(o_w z, o_w o_z)$. We set up a polar coordinate system with principal axis $o_w z$. Set

$$S=\{y=(\varrho,\varphi_1,\dots,\varphi_{n-1}):\varrho=r,\varphi_1<\alpha/2\},\qquad S_1=\partial B_w\cap B_z.$$

Obviously $S\subset S_1$.

According to the harmonic property of the Poisson kernel and the Green formula, we have

$$\begin{aligned}K_{D_0}(x,z)&\geqq[1/\omega_{n-1}(r)]\int_{S_1}K_{B_w}(x,y)K_{B_z}(y,z)\sigma(dy)\\&\geqq\frac{r^{2n-4}}{\omega_{n-1}(r)}(r^2-|x-o_w|^2)\int_S\frac{r^2-|y-o_z|^2}{|x-y|^n|y-z|^n}\sigma(dy),\end{aligned}\tag{16}$$

where $\omega_{n-1}(r)=\sigma[\partial B(0;r)]$.

When $y\in S$,

$$\begin{aligned}&|x-y|\leqq|x-z|+|y-z|\leqq 2|x-z|,\\&|y-o_z|^2=r^2+|o_w-o_z|^2-2r|o_w-o_z|\cos\angle(o_w y,o_w o_z),\\&|o_w-o_z|=2r\cos\theta,\qquad\angle(o_w y,o_w o_z)\leqq\theta-\alpha/2.\end{aligned}\tag{17}$$

Then

$$\begin{aligned}r^2-|y-o_z|^2&\geqq 4r^2\cos\theta\cos(\theta-\alpha/2)-4r^2\cos^2\theta\\&=8r^2\cos\theta\sin(\theta-\alpha/4)\sin\alpha/4.\end{aligned}\tag{18}$$

Set $L=|z-o_w|$, $A=\cos\theta$, $B=\cos(\theta-\alpha)$. Then we have

$$\begin{aligned}r^2&=|z-o_z|^2=|z-o_w|^2+|o_w-o_z|^2-2|z-o_w||o_w-o_z|\cos(\theta-\alpha)\\&=L^2-4r^2\cos^2\theta-4Lr\cos\theta\cos(\theta-\alpha);\\&L^2-4rABL-(4r^2A^2-r^2)=0;\\&L=2rAB+(4r^2A^2B^2-4r^2A^2+r^2)^{1/2};\end{aligned}\tag{19}$$

$$\begin{aligned}L-r&=\frac{(4r^2A^2B^2-4r^2A^2+r^2)-(r-2rAB)^2}{(4r^2A^2B^2-4r^2A^2+r^2)^{1/2}+r-2rAB}\\&=\frac{4rA(B-A)}{(1-4A^2+4A^2B^2)^{1/2}+1-2AB}\\&\leqq\frac{8r\cos\theta\sin(\theta-\alpha/2)\sin\alpha/2}{2(1-2A)}\\&\leqq 8r\cos\theta\sin(\theta-\alpha/2)\sin\alpha/2,\quad(A=\cos\theta\leqq 1/4);\end{aligned}$$

$$\begin{aligned}|y-z|^2&=L^2+r^2-2Lr\cos\varphi_1(y)\\&=(L-r)^2+2Lr[1-\cos\varphi_1(y)]\quad(L\leqq 2r)\\&\leqq(L-r)^2+4r^2\sin^2\varphi_1(y).\end{aligned}\tag{20}$$

By (17), (18) and (20), we have

$$\int_S \frac{r^2-|y-o_z|^2}{|x-y|^n|y-z|^n}\sigma(dy) \geqq \frac{8r^2\cos\theta\sin(\theta-\alpha/4)\sin(\alpha/4)}{2^n|x-z|^n}\int_S \frac{\sigma(dy)}{[(L-r)^2+4r^2\sin^2\varphi_1(y)]^{n/2}}. \tag{21}$$

$$\begin{aligned}\int_S \frac{\sigma(dy)}{[(L-r)^2+4r^2\sin^2\varphi_1(y)]^{n/2}} &= \omega_{n-2}(1)\int_0^{r\sin(\alpha/2)} \frac{u^{n-2}du}{[(L-r)^2+4u^2]^{n/2}(r^2-u^2)^{1/2}/r}\\ &\geqq \frac{\omega_{n-2}(1)}{L-r}\int_0^{\frac{r\sin(\alpha/2)}{L-r}} \frac{v^{n-2}dv}{(1+4v^2)^{n/2}}\\ &\underset{(19)}{\geqq} \frac{\omega_{n-2}(1)}{8r\cos\theta\sin(\theta-\alpha/2)\sin(\alpha/2)}\int_0^{\frac{1}{8\cos\theta\sin(\theta-\alpha/2)}} \frac{v^{n-2}dv}{(1+4v^2)^{n/2}};\end{aligned} \tag{22}$$

$$r^2-|x-o_w|^2 \geqq rd(x); \tag{23}$$

$$\frac{\sin(\alpha/4)}{\sin(\alpha/2)} \geqq 1/\pi. \tag{24}$$

By (14), (16), and (21)–(24), we have

$$\begin{aligned}K(x,z) &\geqq \frac{r^{2n-2}d(x)\sin(\theta-\alpha/4)\sin(\alpha/4)\omega_{n-2}(1)}{\omega_{n-1}(r)2^n|x-z|^n\sin(\theta-\alpha/2)\sin(\alpha/2)}\int_0^{1/2}\frac{v^{n-2}}{(1+4v^2)^{n/2}}\\ &\geqq \frac{r^{n-1}\omega_{n-2}(1)}{2^n\pi\omega_{n-1}(1)}\int_0^{1/2}\frac{v^{n-2}dv}{(1+4v^2)^{n/2}}\cdot\frac{d(x)}{|x-z|^n}.\end{aligned} \tag{25}$$

So by (10), (13), and (25), if we take

$$C_1 = \min\left(\frac{Cr^n}{d(D)2^n}, \frac{2nCa^n\exp(-2n)}{r[\exp(-n/2)-\exp(-2n)]}, \frac{r^{n-1}\omega_{n-1}(1)}{2^n\pi\omega_{n-1}(1)}\int_0^{1/2}\frac{v^{n-2}dv}{(1+4v^2)^{n/2}}\right),$$

we have

$$K(x,z) \geqq C_1\frac{d(x)}{|x-z|^n}, \qquad \text{for all } x\in D,\ z\in\partial D. \quad \square$$

Lemma 2. *There exists a constant* C_3 *such that for all* $x, y \in D$,

$$G(x,y) \leqq C_3\frac{d(x)}{d(y)}\cdot\frac{1}{|x-y|^{n-2}}, \tag{26}$$

and

$$G(x,y) \leqq C_3\frac{d(x)d(y)}{|x-y|^n}. \tag{27}$$

Proof. It is known that

$$G(x,y) \leqq A_n\frac{1}{|x-y|^{n-2}}, \qquad \left(A_n = \frac{\Gamma(n/2-1)}{(2\pi)^{n/2}}\right).$$

By Theorem 2.3 in [8], we have

$$G(x,y) \leqq C \frac{d(x)}{|x-y|^{n-1}}.$$

Since $d(y) \leqq d(x) + |x-y|$,

$$d(y)G(x,y) \leqq d(x)G(x,y) + |x-y|G(x,y) \leqq A_n \frac{d(x)}{|x-y|^{n-2}} + C \frac{d(x)}{|x-y|^{n-2}}.$$

$$G(x,y) \leqq (A_n + C) \frac{d(x)}{d(y)} \cdot \frac{1}{|x-y|^{n-2}}.$$

Inequality (26) is proved.

For any $x, y \in D$, take a point $z \in D$ such that $|y-z| = d(y)$. Set

$$\begin{aligned} f(t) &= G(x, z+t(y-z)), \quad 0 \leqq t \leqq 1. \\ G(x,y) &= f(1) = f(1) - f(0) = f'(\theta), \quad (0 \leqq \theta \leqq 1) \\ &= \sum_{i=1}^{n} \left[\frac{\partial}{\partial y_i} G(x, z+\theta(y-z)) \right] (y_i - z_i) \\ &\leqq |y-z| \left\{ \sum_{i=1}^{n} \left[\frac{\partial}{\partial y_i} G(x, z+\theta(y-z)) \right]^2 \right\}^{1/2} \\ &\leqq n^{1/2} C d(y) \frac{d(x)}{|x-(z+\theta(y-z))|^n}, \quad \text{(since Theorem 2.3 in [5]).} \end{aligned} \tag{28}$$

If $|x-y| > 2d(y)$,

$$\begin{aligned} |x-(z+\theta(y-z))| &\geqq |x-y| - |(y-z)(1-\theta)| \\ &\geqq |x-y| - d(y) \geqq \tfrac{1}{2}|x-y|. \end{aligned}$$

From (28),

$$G(x,y) \leqq n^{1/2} 2^n C \frac{d(x)d(y)}{|x-y|^n}.$$

If $|x-y| \leqq 2d(y)$, also by Theorem 2.3 in [8],

$$G(x,y) \leqq C \frac{d(x)}{|x-y|^{n-1}} \leqq 2C \frac{d(x)d(y)}{|x-y|^n}.$$

Inequality (27) is proved. □

Similar to (4), we define the Green function corresponding the z-conditioned Brownian motion as follows:

$$G_z(x,y) = \int_0^\infty P_z(t,x,y)\,dt. \tag{29}$$

So by (1) and (4), we have

$$G_z(x,y) = \frac{1}{K(x,z)} G(x,y) K(y,z). \tag{30}$$

Lemma 3. *For any sub-domain $D_0 \subset D$ such that $\partial D \subset \partial D_0$, we have*

$$\sup_{\substack{x\in D_0\\ z\in\partial D}} \int_{D_0} G_z(x,y)|q(y)|dy \leqq C \sup_{x\in\bar{D}_0} \int_{D_0} \frac{|q(y)|}{|x-y|^{n-2}}dy,$$

where the constant C only depends on n and D.

Proof. For any given $x\in D_0$, $z\in\partial D$, set

$$D_1=(y\in D_0:|y-x|<|z-x|/2),$$
$$D_2=(y\in D_0:|y-x|\geqq|z-x|/2).$$

By Lemmas 1 and 2 and (30), we obtain

$$\begin{aligned}\int_{D_0} G_z(x,y)|q(y)|dy &\leqq \frac{C_2C_3}{C_1}\frac{|x-z|^n}{d(x)}\int_{D_1}\frac{d(x)}{d(y)|x-y|^{n-2}}\cdot\frac{d(y)}{|y-z|^n}|q(y)|dy\\ &\quad+\frac{C_2C_3}{C_1}\frac{|x-z|^n}{d(x)}\int_{D_2}\frac{d(x)d(y)}{|x-y|^n}\frac{d(y)}{|y-z|^n}|q(y)|dy\\ &\leqq \frac{C_2C_3}{C_1}2^n\left(\int_{D_1}\frac{|q(y)|}{|x-y|^{n-2}}dy+\int_{D_2}\frac{d(y)^2}{|z-y|^n}|q(y)|dy\right)\\ &\leqq \frac{C_2C_3}{C_1}2^n\left(\int_{D_0}\frac{|q(y)|}{|x-y|^{n-2}}dy+\int_{D_0}\frac{|q(y)|}{|z-y|^{n-2}}dy\right)\\ &\leqq \frac{C_2C_3}{C_1}2^{n+1}\sup_{x\in\bar{D}_0}\int_{D_0}\frac{|q(y)|}{|x-y|^{n-2}}dy. \quad \square\end{aligned}$$

Now for any $\delta>0$, set

$$D(\delta)=(x\in D:d(x,\partial D)<\delta),$$
$$S(\delta)=(x\in D:d(x,\partial D)=\delta),$$
$$B(\delta)=(x\in D:d(x,\partial D)>\delta).$$

Lemma 4. *There exists some $\delta_1>0$ such that for any $0<\delta\leqq\delta_1$,*

$$\sup_{\substack{x\in D(\delta)\\ z\in\partial D}} E_z^x e_{|q|}(\tau_{D(\delta)})\leqq 4/3, \tag{31}$$

and

$$\sup_{x\in D(\delta)} E^x e_{|q|}(\tau_{D(\delta)})\leqq 4/3. \tag{32}$$

Proof. For any $x\in D$, measurable set $D_0\subset D$ and $\alpha>0$, we have

$$\int_{D_0}\frac{|q(y)|}{|x-y|^{n-2}}dy\leqq \int_{|x-y|\leqq\alpha}\frac{|q(y)|}{|x-y|^{n-2}}dy+\frac{1}{\alpha^{n-2}}\int_{D_0}|q(y)|dy. \tag{33}$$

By definition (7), we can take some $\alpha>0$ such that

$$\sup_{x\in\bar{D}}\int_{|x-y|\leqq\alpha}\frac{|q(y)|}{|x-y|^{n-2}}dy\leqq 1/(8C), \tag{34}$$

where C is given in Lemma 3.

It is easy to see from (7) that $\int_D |q(y)|dy < +\infty$. Then there exists some $\delta_1 > 0$ such that for any $0 < \delta \leqq \delta_1$.

$$\int_{D(\delta)} |q(y)|dy \leqq \alpha^{n-2}/(8C). \tag{35}$$

It follows from (33), (34), and (35) that

$$\sup_{x \in \bar{D}(\delta)} \int_{D(\delta)} \frac{|q(y)|}{|x-y|^{n-2}} dy \leqq 1/(4C). \tag{36}$$

By (29), Lemma 3, and (36), we have

$$\begin{aligned} \sup_{\substack{x \in D(\delta) \\ z \in \partial D}} E_z^x \int_0^{\tau_{D(\delta)}} |q(x_t)|dt &\leqq \sup_{\substack{x \in D(\delta) \\ z \in \partial D}} E_z^x \int_0^{\tau_D} |1_{D(\delta)} q(x_t)|dt \\ &= \sup_{\substack{x \in D(\delta) \\ z \in \partial D}} \int_{D(\delta)} G_z(x, y)|q(y)|dy \\ &\leqq C \sup_{x \in \bar{D}(\delta)} \int_{D(\delta)} \frac{|q(y)|}{|x-y|^{n-2}} dy \leqq 1/4. \end{aligned} \tag{37}$$

For any $x \in D(\delta)$, $z \in \partial D$, by the Markov property of z-Brownian motion and (37), we have

$$\begin{aligned} E_z^x e_{|q|}(\tau_{D(\delta)}) &= 1 + \sum_{k=1}^{\infty} E_z^x \left(\int_{0 < t_1 < \ldots < t_k < \tau_{D(\delta)}} |q(x_{t_1})| \ldots |q(x_{t_k})| dt_1 \ldots dt_k \right) \\ &\leqq \sum_{k=0}^{\infty} (1/4)^k = 4/3. \end{aligned}$$

Similarly, since

$$\begin{aligned} \sup_{x \in D(\delta)} E^x \left(\int_0^{\tau_{D(\delta)}} |q(x_t)|dt \right) &\leqq \sup_{x \in D(\delta)} \int_{D(\delta)} G(x, y)|q(y)|dy \\ &\leqq \sup_{x \in D(\delta)} A_n \int_{D(\delta)} \frac{|q(y)|}{|x-y|^{n-2}} dy \leqq 1/4, \end{aligned}$$

for any $x \in D(\delta)$, we have

$$E^x e_{|q|}(\tau_{D(\delta)}) \leqq 4/3. \quad \square$$

Lemma 5. *If $E^x e_q(\tau_D) \not\equiv \infty$ in D, then*

$$u(x) = E^x e_q(\tau_D), \qquad x \in \bar{D},$$

is a continuous function on $\bar{D}$.

Proof. By Theorem 7 in [9], we have $M = \sup_{x \in D} u(x) < +\infty$. Set

$$G(qu)(x) = \int_D G(x, y) q(y) u(y) dy, \qquad x \in \bar{D}.$$

Since for any measurable set $A \subset D$,

$$\int_A G(x, y)|q(y)u(y)|dy \leqq M \int_A G(x, y)|q(y)|dy,$$

it follows from (7) and (33) that the integrals $\left\{\int_A G(x,y)q(y)u(y)dy : A\subset D\right\}$ are uniformly absolutely continuous with respect to $x\in D$. Hence it is easy to see that $G(qu)(\cdot)$ is a continuous function on $\bar{D}$.

Simulating the proof of Theorem 2.1 in [2], we have $u=1+G(qu)$. This shows that u is continuous on $\bar{D}$. □

Lemma 6. *If* $E^x e_q(\tau_D)\not\equiv\infty$, *then there exists some* $\delta_2>0$ *such that for any* $0\leqq\delta<\delta_2$, $x\in B(\delta)\backslash B(\delta_2)$, *we have*

$$2/3\leqq E^x e_q(\tau_{B(\delta)})\leqq 4/3\,.$$

Proof. Take an $\varepsilon>0$ such that $2/3\leqq(1-\varepsilon)/(1+\varepsilon)\leqq(1+\varepsilon)/(1-\varepsilon)\leqq 4/3$. By Lemma 5, there exists some $\delta_2>0$ such that if $x\in D\backslash B(\delta_2)$, then

$$1-\varepsilon\leqq E^x e_q(\tau_D)\leqq 1+\varepsilon\,.$$

For any $0\leqq\delta<\delta_2$, $x\in B(\delta)\backslash B(\delta_2)$,

$$1+\varepsilon\geqq E^x e_q(\tau_D)=E^x\{e_q(\tau_{B(\delta)})E^{x(\tau_{B(\delta)})}[e_q(\tau_D)]\}\geqq(1-\varepsilon)E^x e_q(\tau_{B(\delta)})\,,$$

$$1-\varepsilon\leqq E^x e_q(\tau_D)=E^x\{e_q(\tau_{B(\delta)})E^{x(\tau_{B(\delta)})}[e_q(\tau_D)]\}\leqq(1+\varepsilon)E^x e_q(\tau_{B(\delta)})\,.$$

Hence

$$2/3\leqq(1-\varepsilon)/(1+\varepsilon)\leqq E^x e_q(\tau_{B(\delta)})\leqq(1+\varepsilon)/(1-\varepsilon)\leqq 4/3\,. \quad \square$$

Proof of Theorem 1. Set $b=\min(\delta_1,\delta_2)$, where δ_1 and δ_2 are given by Lemmas 4` and 6, respectively.

Since $\phi(x)=P^x[x(\tau_{D(b)})\in S(b)]$ is a continuous function on $\overline{D(b)}$ and $\phi(z)=0$ for $z\in\partial D$, there exists a number $0<r<b$ such that for all $x\in D\backslash B(r)$,

$$P^x[x(\tau_{D(b)})\in S(b)]<1/3\,. \tag{38}$$

Set $T_0=0$,

$$T_1=\inf(t>0: x_t\notin B(r))\,,$$

$$\left.\begin{aligned} T_{2k}&=\inf(t>T_{2k-1}: x_t\notin D(b))\,,\\ T_{2k+1}&=\inf(t>T_{2k}: x_t\notin B(r),\ T_{2k}<\tau_D)\,.\end{aligned}\right\}\quad(k\geqq 1)$$

We want to prove inductively that for any $k\geqq 1$,

$$\sup_{y\in S(r)} E^y[e_q(T_{2k+1}),\ T_{2k}<\tau_D]\leqq(8/9)^k\,. \tag{39}$$

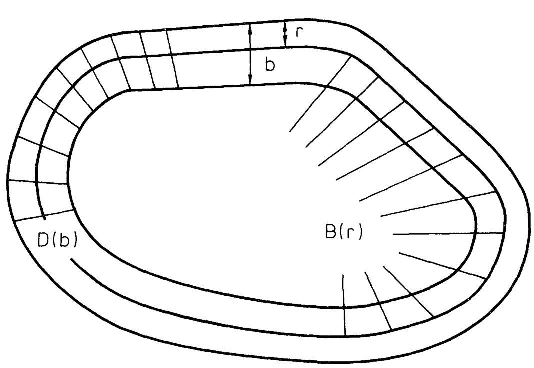

Fig. 2

When $k=1$, by Lemma 6, (38), and Lemma 4, (32), we have

$$\begin{aligned}\sup_{y\in S(r)} E^y[e_q(T_3),\ T_2<\tau_D] &= \sup_{y\in S(r)} E^y\{e_q(T_2)E^{x(T_2)}[e_q(T_1)],\ T_2<\tau_D\}\\ &\leqq (4/3)\sup_{y\in S(r)} E^y[e_q(T_2),\ T_2<\tau_D]\\ &\leqq (4/3)\Big\{\sup_{y\in S(r)} P^y(T_2<\tau_D)+\sup_{y\in S(r)} E^y[e_{|q|}(\tau_{D(b)})-1]\Big\}\\ &\leqq 8/9\,.\end{aligned}$$

Suppose (39) is true for $k\geqq 1$.

$$\begin{aligned}&\sup_{y\in S(r)} E^y[e_q(T_{2k+3}),\ T_{2k+2}<\tau_D]\\ &\quad= \sup_{y\in S(r)} E^y\{e_q(T_3)E^{x(T_3)}[e_q(T_{2k+1}),\ T_{2k}<\tau_D],\ T_2<\tau_D\}\\ &\quad\leqq (8/9)^k \sup_{y\in S(r)} E^y[e_q(T_3),\ T_2<\tau_D]\\ &\quad\leqq (8/9)^{k+1}\,.\end{aligned}$$

By Theorem 7 in [9], $M=\sup_{x\in D} E^x e_q(\tau_D)<+\infty$.

For all $x\in B(r)$, by Lemma 6, we have

$$M\geqq E^x e_q(\tau_D)=E^x\{e_q(T_1)E^{x(T_1)}[e_q(\tau_D)]\}\geqq (2/3)E^x e_q(T_1)\,.$$

Then

$$\sup_{x\in B(r)} E^x e_q(T_1)\leqq (3/2)M\,. \tag{40}$$

For all $x\in B(r)$, $z\in\partial D$, by Lemma 4, (31) and (6), we have

$$\begin{aligned}E_z^x e_q(\tau_D) &= \sum_{k=1}^{\infty} E_z^x[e_q(\tau_D),\ \tau_D=T_{2k}]\\ &= \sum_{k=1}^{\infty} E_z^x\{e_q(T_{2k-1})E_z^{x(T_{2k-1})}[e_q(T_2),\ \tau_D=T_2],\ T_{2k-2}<\tau_D\}\\ &\leqq (4/3)\sum_{k=1}^{\infty}\frac{1}{K(x,z)}E^x[e_q(T_{2k-1})K(x(T_{2k-1}),z),\ T_{2k-2}<\tau_D]\,.\end{aligned}$$

By the Harnack inequality, there exist two positive numbers b_1 and b_2 such that for all $x\in B(r)\cup S(r)$, $z\in\partial D$, $b_1\leqq K(x,z)\leqq b_2$. Continuing the above inequalities and using (39), (40),

$$\begin{aligned}E_z^x e_q(\tau_D) &\leqq \frac{4b_2}{3b_1}\sum_{k=1}^{\infty} E^x[e_q(T_{2k-1}),\ T_{2k-2}<\tau_D]\\ &\leqq \frac{4b_2}{3b_1}\sum_{k=0}^{\infty} E^x\{e_q(T_1)E^{x(T_1)}[e_q(T_{2k+1}),\ T_{2k}<\tau_D]\}\\ &\leqq \frac{4b_2}{3b_1}\sum_{k=0}^{\infty}(8/9)^k E^x e_q(T_1)\\ &\leqq \frac{4b_2}{3b_1}(3/2)M\frac{1}{1-(8/9)}=18M(b_2/b_1)\,.\end{aligned} \tag{41}$$

For all $x \in D \backslash B(r)$, $z \in \partial D$, by (41) and Lemma 4, (31), we have

$$\begin{aligned} E_z^x e_q(\tau_D) &= E_z^x[e_q(\tau_{D(b)}), \tau_{D(b)} = \tau_D] + E_z^x[e_q(\tau_D), \tau_{D(b)} < \tau_D] \\ &\leqq (4/3) + E_z^x\{e_q(\tau_{D(b)}) E_z^{x(\tau_{D(b)})}[e_q(\tau_D)], \tau_{D(b)} < \tau_D\} \\ &\leqq (4/3) + 18M(b_2/b_1) E_z^x[e_q(\tau_{D(b)}), \tau_{D(b)} < \tau_D] \\ &\leqq (4/3) + 24M(b_2/b_1). \end{aligned} \tag{42}$$

It follows from (41) and (42) that

$$\sup_{\substack{x \in D \\ z \in \partial D}} E_z^x e_q(\tau_D) < +\infty . \quad \square$$

Acknowledgements. I would like to thank my adviser K. L. Chung for his guidance and encouragement, N. Falkner and P. Li for helpful discussions.

References

1. Aizenman, N., Simon, B.: Brownian motion and Harnack inequality for Schrödinger operators. Commun. Pure Appl. Math. **35**, 209–273 (1982)
2. Chung, K.L., Rao, K.M.: Feynman-Kac functional and Schrödinger equation. In: Seminar on Stochastic Process. Boston: Birkhäuser 1981
3. Doob, J.L.: Conditional Brownian motion and the boundary limits of harmonic functions. Bull. Soc. Math. France **85**, 431–458 (1957)
4. Falkner, N.: Feynman-Kac functional and positive solutions of $\frac{1}{2}\Delta u + qu = 0$. Z. Wahrscheinlichkeitstheorie verw. Gebiete (to appear)
5. Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. Berlin, Heidelberg, New York: Springer 1977
6. Port, S.C., Stone, C.J.: Brownian motion and classical potential theory. New York Academic Press 1978
7. Simon, B.: Schrödinger semigroup. Bull. Am. Math. Soc. (N.S.) **7**, 447–526 (1982)
8. Widman, K.-O.: Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations. Math. Scand. **21**, 17–37 (1967)
9. Zhao, Z.: Conditional gauge with unbounded potential. Z. Wahrscheinlichkeitstheorie verw. Gebiete (to appear)

Communicated by B. Simon

Received March 24, 1983; in revised form August 31, 1983