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Removable Singularities of Coupled Yang-Mills Fields in R^3

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Abstract. We consider isolated point singularities of the coupled Yang-Mills equations in R^3. Under appropriate conditions on the curvature and the Higgs field, a removable singularity theorem is proved.

Introduction

The original removable singularity theorem of Uhlenbeck [19] in R^4, states that apparent point singularities in *finite action pure* Yang-Mills fields may be removed by a gauge transformation. Uhlenbeck's theorem was extended by Parker [13] to *coupled* Yang-Mills fields in R^4.

In R^3, finite action is too stringent a condition and may be relaxed to the assumption that the solution (i.e., the curvature) is in $L^{3/2}$. In recent work [17], it was shown that point singularities of solutions in $L^{3/2}$ of the *pure* Yang-Mills equations are removable.

In the following, we consider the *coupled* Yang-Mills equations in R^3. From the point of view of mathematical physics, our equations describe the *Higgs model* and have been studied extensively by Jaffe and Taubes [11]. We prove an isolated removable singularity theorem for solutions of these equations. The *sign* of the dominant lower order non-linear term plays a crucial role in this problem. In one case, no assumptions whatsoever are needed on the Higgs field to remove the singularity. In the other, a little more smoothness than is expected is required and an example of a singular solution is given which shows that the requirement is necessary. In both cases, we assume that the curvature is in $L^{3/2}$.

To prove the theorem, we first show that the Higgs field is bounded. This implies that its covariant derivative is square integeable and satisfies a strong growth condition on small balls about the puncture. This is then used to show that

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the curvature is in L^p for $p > 3/2$. Once this is known, a theorem of Uhlenbeck [20] may be applied to obtain a "Hodge gauge" and this can be done without *twisting* the underlying bundle. Then, we are able to apply the results of Hildebrandt and Widman [8–10] for systems in diagonal form to conclude that ϕ and F are C^∞ in a neighborhood of the puncture.

The first section consists of preliminary geometric and analytic results. Section 2 is devoted to showing that the Higgs field is bounded. This is proved for any $n \geqq 3$. In Sect. 3, scalar subelliptic theory is used to obtain a first growth condition. This section is independent of dimension and the results are true for coupled Yang-Mills fields in dimension n, provided the curvature belongs to $L^{n/2}$ and the Higgs field to $H_1^{n/2}$. In Sect. 4, we use the method of *broken Hodge gauges* [17, 19] to obtain the final growth condition. To illustrate the method, we do it first in four dimensions. Because this is a purely L^2 argument, and our solutions are in $L^{3/2}$ in dimension three, we are required to work with weighted L^2 norms and the proof is technically more complicated. In the last section, all results are combined to prove the theorem.

We note that the corresponding theorem in higher dimensions does *not* follow directly from the techniques used here. However, by keeping track explicitly of all constants involved, the theorem can be extended to dimensions $n = 5$, 6, and 7. See [17], where this is carried out for the pure Yang-Mills equations.

We also obtain as a corollary a result for *pure* Yang-Mills fields in R^4 having an apparent *line* singularity; namely, if the field is independent of x_4 and the curvature is in $L^{3/2}$ in dimension four, then a possible singularity on the x_4 axis is removable by a gauge transformation. This follows from *dimensional reduction* (see [11, II.6]) to a coupled field in R^3 with a point singularity.

1. Preliminary Results

Let M be a domain in R^n, and η a vector bundle over M, with compact structure group G, and Lie algebra g. Let d be *exterior differentiation*, δ its *adjoint* and denote by $[\,,\,]$ the Lie bracket in G.

A *connection* A is a Lie algebra valued one-form which locally defines a *covariant* derivative $D = d + A$ in η. On p-forms ω,

$$D\omega = d\omega + [A, \omega]. \tag{1.1}$$

The operator adjoint to D is the *Yang-Mills operator* D^*. On p-forms, ω,

$$D^*\omega = \delta\omega + *[A, *\omega]. \tag{1.2}$$

The *curvature* F of the connection is a Lie algebra valued two-form defined by

$$F = dA + \tfrac{1}{2}[A, A]. \tag{1.3}$$

Curvature forms of connections automatically satisfy the Bianchi identities:

$$DF = 0. \tag{1.4}$$

Gauge transformations are sections of Autη which act on connections and curvature forms according to the transformations:

(a) $A^g = g^{-1}Ag + g^{-1}dg$,
(b) $F^g = g^{-1}Fg$.

The pair (A, F) is *gauge equivalent* to $(\bar{A}, \bar{F})$ if there is a gauge transformation g such that $\bar{A} = A^g$ and $\bar{F} = F^g$.

The determinant of the volume bundle over M is a line bundle of *conformal weight n*. We denote by L, the determinant bundle raised to the $1/n$ power. Sections of this bundle are constant in a fixed coordinate system but have *weight 1* under scale transformations.

The *Higgs field* ϕ is a section of $\eta \otimes L$. Therefore, in a fixed coordinate system, ϕ may be regarded as a matrix valued function. Under scale changes, $y = rx$, $\phi(y) = \phi(x)/r$.

The mass m is defined to be a section of L, and hence, constant in a fixed coordinate system, but having weight 1 under scale changes.

(For a careful and rigorous discussion of conformal weights, see Parker [13, 14].)

With these definitions, the *Yang-Mills-Higgs equations* are

$$D^*F = [D\phi, \phi], \tag{YMH$_1$}$$

$$D^*D\phi = \frac{\lambda}{2}(|\phi|^2 - m^2)\phi, \tag{YMH$_2$}$$

where λ is a physical constant.

Since d increases weights by 1, the equations are invariant under scale transformations of the form $y = rx$.

We will make use of the fact that certain *norms* are invariant under scale transformations. For example, $\|\phi\|_{L^n}$ is invariant, and if ψ is any p-form, $\|\psi\|_{L^{n/p}}$ is invariant. This leads us to

Lemma 1.1. *Suppose $\psi \in L^{n/p}$ with $\|\psi\|_{L^{n/p}}$ invariant. Then, given any $\gamma > 0$, there is a metric g_0, conformally equivalent to the Euclidian metric, in which on bounded sets in R^n,*

$$\int |\psi|^{n/p} dx < \gamma. \tag{1.5}$$

The lemma follows from invariance and the continuity of the L^P norms (see [19]).

In the following, we will assume that γ has been chosen sufficiently small for our purposes, and we point out, as we go along, the bounds needed for γ in the proof.

Many of our estimates are obtained by using scalar subelliptic theory. We require several known inequalities [11, 19] valid for Lie-algebra valued sections and p-forms:

$$|\nabla(|\psi|)| \leqq |D\psi|. \tag{1.6}$$

Letting $\nabla^2 = D^*D + DD^* + \text{curvature}$, denote the covariant derivative Laplacian, we find that

$$\tfrac{1}{2}\Delta(|\psi|^2) = (\psi, \nabla^2\psi) + |D\psi|^2 \geqq (\psi, \nabla^2\psi), \tag{1.7}$$

$$|\psi|\Delta(|\psi|) = (\psi, \nabla^2\psi) + |D\psi|^2 - |\nabla|\psi||^2 \geqq (\psi, \nabla^2\psi), \tag{1.8}$$

where ∇ and Δ are the ordinary gradient and Laplacian on functions. The relation between solutions of equations whose principle part is the covariant derivative Laplacian and scalar subsolutions is given by:

Lemma 1.2. *Let ψ be a p-form with values in g which satisfies an equation of the form*

$$\nabla^2\psi + G_1(x, \psi, D\psi) = G_2(x, \psi)\psi, \tag{1.9}$$

where G_1 is a p-form with values in g, and G_2 is a scalar function. Then, the scalar function $|\psi|$ is a solution of the sub-elliptic inequality

$$\Delta(|\psi|) + |G_1(x, \psi, D\psi)| \geqq G_2(x, \psi)\,|\psi|. \tag{1.10}$$

Proof. Taking inner product with ψ in (1.9), we obtain

$$(\psi, \nabla^2\psi) + (G_1, \psi) = G_2|\psi|^2.$$

From inequality (1.8) and the Schwarz inequality,

$$|\psi|\Delta(|\psi|) + |G_1|\,|\psi| \geqq G_2|\psi|^2.$$

Dividing by $|\psi|$, proves (1.10) and the lemma.

We will require the Morrey-Moser iteration [12, Theorem 5.3.1] for subsolutions and next state the version of it that we use.

A function $f(x)$ satisfies a *Morrey growth condition* if on small balls in M,

$$\int_{B(x_0, \varrho)} |f|^{n/2} dx \leqq k\varrho^\alpha, \tag{1.11}$$

with $\alpha > 0$ and k independent of ϱ.

Remark. If $f \in L^p$ with $p > \frac{n}{2}$, then (1.11) is automatically satisfied.

Theorem 1.3. *Let $U \in H_1^2(M)$ with $U \geqq 0$, and suppose that for some λ, $1 \leqq \lambda < 2$, $W = U^\lambda$ is a subsolution of an elliptic equation, i.e.,*

$$\int_M (\nabla W \cdot \nabla\zeta + fW\zeta)dx \leqq 0, \tag{1.12}$$

for all non-negative $\zeta \in C_0^\infty(M)$, where f satisfies (1.11). *Then U is bounded on compact subdomains of M, and, for $x \in B(x_0, \varrho)$,*

$$|U(x)|^2 < \frac{C}{a^n} \int_{B(x_0, \varrho + a)} |U(y)|^2 dy. \tag{1.13}$$

(Note that the constant C depends on k and α.)

We frequently use two basic inequalities for functions $g \in L^{n/2}$ and $w \in H_0^1$. With C_n = Sobolev's constant,

$$\int |g|\,|w|^2 dx \leqq C_n \|g\|_{n/2} \int |\nabla w|^2 dx. \tag{1.14}$$

This follows from Hölder's inequality and the Sobolev inequality. Also, for any $\mu > 0$, there is a constant $C(\mu)$ such that

$$\int |g|\,|w|^2 dx \leqq \mu \int |\nabla w|^2 dx + C(\mu) \int |w|^2 dx. \tag{1.15}$$

2. A Regularity Theorem for the Higgs Field

In this section, we assume that the Higgs field is a C^∞ solution of the field equation

$$D^*D\phi = \frac{\lambda}{2}(|\phi|^2 - m^2)\phi \qquad (\mathrm{YMH}_2)$$

in the punctured unit ball $B-\{0\}$. Assumptions on ϕ at the origin depend upon the sign of λ. (Note that $\lambda \geqq 0$ is the case considered by Jaffe and Taubes [11].)

The main result of this section is

Theorem 2.1. *Let ϕ be a C^∞ solution of* (YMH_2) *in $B-\{0\}$. We assume*
(a) *no conditions on ϕ if $\lambda>0$,*
(b) *$\phi \in L^{3+\varepsilon}(B)$ for some $\varepsilon>0$, if $n=3$ and $\lambda<0$,*
(c) *$\phi \in L^n(B)$ if $n \geqq 4$ and $\lambda<0$,*
(d) *$\phi \in L^{n/n-2}(B)$ if $\lambda=0$.*
Then ϕ is bounded in B.

That condition (b) is the right one follows from the following

Example 2.2. Suppose the structure group G is commutative and $n=3$. Then, there are solutions of (YMH_2) which are C^∞ in $B-\{0\}$, belong to $L^3(B)$, having singularities at the origin which are not removable.

The example follows from results of Aviles [1], who has shown the existence of solutions of the equation

$$\Delta u + u^3 = 0\,,$$

satisfying the inequality

$$\frac{C_1}{|x|(-\log|x|)}^{1/2} \leqq u(x) \leqq \frac{C_2}{|x|(-\log|x|)}^{1/2}.$$

The inequality shows that $u \in L^3$, but $u \notin L^{3+\varepsilon}$, for any $\varepsilon>0$. It is also known that the Dirichlet problem has multiple solutions (see [18]).

The result (d) for $\lambda=0$ is due to Joel Spruck (private communication).

To prove Theorem 2.7, we make strong use of the fact that $|\phi|$ is a scalar subharmonic function.

From (YMH_2) and Lemma 1.2 with $G_1 \equiv 0$ and $G_2 = \frac{\lambda}{2}(|\phi|^2 - m^2)$, we find that

$$\Delta(|\phi|) \geqq \frac{\lambda}{2}(|\phi|^2 - m^2)\,|\phi|\,. \qquad (2.1)$$

First, we dispose of case (a). The function $V = k|\phi|$ is a solution of

$$-\Delta V + V^3 \leqq \text{const} \qquad (2.2)$$

for an appropriate k. Boundedness follows from the following

Theorem (Brezis and Veron [3]). *Let V be a C^∞ solution of* (2.2) *in $B-\{0\}$. Then, V is bounded in B.*

Next, assume $\lambda<0$ and letting $h=-\frac{\lambda}{2}(|\phi|^2-m^2)$, and $u=|\phi|$, we find from (2.1),

$$-\Delta u \leqq hu. \tag{2.3}$$

Integrating by parts, u is a non-negative subsolution of

$$\int \nabla u \cdot \nabla \zeta dx \leqq \int hu\zeta dx \tag{2.4}$$

for all non-negative $\zeta \in C_0^\infty(B-\{0\})$.

To prove (b) and (c), we will show that $u^q \in H_1^2(B)$ for sufficiently large q depending on dimension and is a weak solution of (2.4) in all of B. We then apply Theorem 1.3 of Morrey and Moser.

Throughout this section, we assume that the invariant norm $\int_B |\phi|^n dx \leqq \gamma < \gamma_1$, where γ_1 depends on dimension.

Proposition 2.3. (i) *If condition* (b) *is satisfied, then* $\nabla u \in L^2(B)$ *and for* $\eta \in C_0^\infty(B)$,

$$\int_B \eta^2 |\nabla u|^2 dx \leqq K \int_B |\nabla \eta|^2 u^2 dx. \tag{2.5}$$

ii) *If condition* (c) *is satisfied,* $\nabla u^q \in L^2(B)$ *for* $\frac{n-2}{*\ 2} < q \leqq \frac{n}{2}$ *with* $n \geqq 4$, *and for* $\eta \in C_0^\infty(B)$,

$$\int_B \eta^2 |\nabla u^q|^2 dx \leqq K \int_B |\nabla \eta|^2 u^{2q} dx. \tag{2.6}$$

First, we show

Proposition 2.3 implies Theorem 2.1. In case (i), the estimate (2.5) shows that (2.4) holds with $\zeta \in C_0^\infty(B)$, or u is an H_1^2 weak subsolution in all of B. Since $h \in L^{3/2+\varepsilon/2}$, a Morrey growth condition holds and u is bounded by Theorem 1.3. In case (ii), since $q>1$, u^q is a subsolution satisfying (2.4). The estimate (2.6) shows again that u^q is an H_1^2 weak subsolution in all of B. Since $q>\frac{n-2}{2}$, it follows from Sobolev's lemma that $|\phi| \in L^p$ for $p>n$, and therefore $h \in L^p$ for $p>n/2$. As before, u is bounded by Theorem 1.3.

The remainder of this section is devoted to the proof of Proposition 2.3. Following Gidas and Spruck [7], we will make use of the Serrin test function [15, 16].

For $u \geqq 0$, let

$$F(u)=\begin{cases} u^q & \text{for} \quad 0 \leqq u \leqq l \\ \dfrac{1}{q_0}(q l^{q-q_0} u^{q_0} + (q_0-q) l^q) & \text{for} \quad l \leqq u. \end{cases}$$

We assume $\frac{1}{2} < q_0 < q$ and let $G(u)=F(u)F'(u)$. We obtain the following properties of F and G:

$$F \leqq \frac{q}{q_0} l^{q-q_0} u^{q_0}, \tag{2.7a}$$

$$uF' \leqq qF \quad \text{and hence,} \quad uG \leqq qF^2, \tag{2.7b}$$

$$G' \geqq C'F'^2, \quad \text{with} \quad C' > 0. \tag{2.7c}$$

[Note that (2.7c) fails if $q_0 = 1/2$.]

We will also use a sequence $\bar{\eta}_k$ of test functions which vanish for $|x| \leqq \varepsilon_k$, tend to 1 as ε_k tends to zero, and such that $\int |\nabla \bar{\eta}_k|^n dx \to 0$, $k \to \infty$. (Such a sequence is constructed in [7].)

Proof of Proposition 2.3 (i). Let $\eta \in C_0^\infty(B)$ and $\bar{\eta}$ be a C^∞ function vanishing in a neighborhood of the origin. With $q_0 = \frac{1}{2} + \frac{\varepsilon}{6}$ and $q = 1$, we use the test function $\zeta = (\eta\bar{\eta})^2 G(u)$ in (2.4). Using the properties (2.7), we find that

$$\begin{aligned} k \int (\eta\bar{\eta})^2 |\nabla F|^2 dx &\leqq \int |2\eta\bar{\eta}\nabla F| \, |\nabla(\eta\bar{\eta})F| dx + \int (\eta\bar{\eta})^2 h F^2 dx \\ &= I_1 + I_2 . \end{aligned}$$

Now, $I_1 \leqq \mu \int (\eta\bar{\eta})^2 |\nabla F|^2 dx + C(\mu) \int |\nabla(\eta\bar{\eta})|^2 F^2 dx$, and the first term on the right may be absorbed on the left.

Also,

$$\begin{aligned} I_2 &\leqq \|\phi\|_{L^3}^{1/2} \|(\eta\bar{\eta})F\|_{L_6}^2 + K_1 \|(\eta\bar{\eta})F\|_{L^2}^2 \\ &\leqq \gamma_1^{1/2} \|\eta\bar{\eta})\nabla F\|_{L^2}^2 + K_2 \|\eta\bar{\eta}F\|_{L^2}^2 , \end{aligned}$$

and for γ_1 sufficiently small, the first term on the right may be absorbed on the left. With a new constant we obtain

$$k' \int (\eta\bar{\eta})^2 |\nabla F|^2 dx \leqq \int \bar{\eta}^2 |\nabla \eta|^2 F^2 dx + \int \eta^2 |\nabla \bar{\eta}|^2 F^2 dx .$$

Using (2.7a),

$$\begin{aligned} \int \eta^2 |\nabla \bar{\eta}|^2 F^2 dx &\leqq \frac{l^{1-q_0}}{q_0} \int |\nabla \bar{\eta}|^2 u^{2q_0} dx \\ &\leqq C(l, q^0) \left(\int |\nabla \bar{\eta}|^3 dx\right)^{2/3} \left(\int u^{6q_0} dx\right)^{1/3} . \end{aligned}$$

From our choice of q_0, $6q_0 = 3 + \varepsilon$, and choosing $\bar{\eta} = \bar{\eta}_k$ defined above, we see that the right hand side tends to zero. In the limit,

$$k' \int \eta^2 |\nabla F|^2 dx \leqq \int |\nabla \eta|^2 F^2 dx . \tag{2.8}$$

We now let $l \to \infty$. F converges strongly to u in L^2. By Lebesgue dominated convergence, ∇F converges strongly to ∇u in L^2, and Proposition 2.3(i) is proved.

Proof of Proposition 2.3 (ii). *Now let* $\zeta = (\eta\bar{\eta})^2 G(u)$ with $q_0 = \frac{n-2}{2}$ and $\frac{n-2}{2} < q \leqq \frac{n}{2}$. Repeating the argument, for $n \geqq 4$, we obtain the inequality (2.8). Since, $2q \leqq n$, F converges to u^q in L^2, and Proposition (2.3)(ii) is proved.

An important consequence of Theorem 2.1 which will be used later is

Corollary 2.4. *Under the hypothesis of Theorem 2.1,* $D\phi \in L^2(B)$.

Proof. Integrating by parts in (YMH_2),

$$\int (D\phi, D\zeta) = \int \frac{\lambda}{2}(|\phi|^2 - m^2)(\phi, \zeta). \tag{2.9}$$

Letting $\zeta = (\eta\bar{\eta})^2\phi$ with $\bar{\eta} = 0$ in a neighborhood of the origin, we find that

$$\int (\eta\bar{\eta})^2 (D\phi, D\phi) dx \leqq K \int (\eta\bar{\eta})^2 |\phi|^2 dx + |\int (D\phi, (2\eta\bar{\eta}) d(\eta\bar{\eta})\phi)|.$$

With new constants,

$$\int (\eta\bar{\eta})^2 |D\phi|^2 dx \leqq K \int ((\eta\bar{\eta})^2 + |\nabla(\eta\bar{\eta})|^2) |\phi|^2 dx.$$

Since ϕ is bounded, we can let $\bar{\eta} \to 1$, and

$$\int \eta^2 |D\phi|^2 dx \leqq K \int (\eta^2 + |\nabla\eta|^2) |\phi|^2 dx,$$

which proves the corollary.

3. A Sub-Elliptic Estimate for (F, ϕ)

In this section, we assume that (F, ϕ) is a smooth solution in $B - \{0\}$ in R^n, of (YMH_1) and (YMH_2), and that F and $D\phi$ belong to $L^{n/2}(B)$. We define the *total field* $h(x) = |F| + |D\phi| + |\phi|^2$. The main result of this section is a preliminary growth estimate which shows that $|x|^2 h(x) = o(1)$ at the origin.

Denote by $V_\varrho = \{x | \varrho/2 \leqq |x| \leqq 2\varrho\}$ the *reference ring* about the puncture. Let C_n be the Sobolev constant in dimension n. We require that $\|h\|_{n/2} \leqq \gamma < \gamma_2$, where γ_2 is an explicitly given constant depending on λ, C_n and dimension. The main theorem of this section is

Theorem 3.1. *There is a constant C such that for $|x| = r$*

$$|x|^2 h(x) \leqq C \|h\|_{L^{n/2}(V_r)}. \tag{3.1}$$

To prove Theorem 3.1, we consider solutions of the Higgs model in a bundle over the *unit* reference ring $V_1 \{y | 1/2 \leqq |y| \leqq 2\}$. We will obtain a bound on the L^∞ norm of the total field h, which we state in the following:

Proposition 3.2. *Let h be the total field of the smooth pair (F, ϕ) in a bundle over V_1. If $\|h\|_{n/2} < \gamma_2$, then there is a constant C such that*

$$h(y) \leqq C \|h\|_{L^{n/2}(V_1)} \tag{3.2}$$

for y belonging to the unit sphere in V_1, $|y| = 1$.

Before proving Proposition 3.2, we show

Proposition 3.2 implies Theorem 3.1. Map the reference ring V_r onto V_1 by the scale transformation $y = x/r$. The field equations are invariant under this transformation. By assumption, and norm invariance,

$$\|h\|_{L^{n/2}(V_1)} = \|h\|_{L^{n/2}(V_r)} \leqq \gamma < \gamma_2 .$$

Therefore, in y coordinates, F, ϕ, and h satisfy the hypothesis of Proposition 3.2. Pulling back to V_r, and using the fact that $h(y) = r^2 h(x)$, the inequality (3.2) becomes the inequality (3.1). This proves the theorem.

To prove the proposition, we want to apply scalar elliptic theory and the Morrey-Moser iteration to the scalar function $h(x)$. The first step is

Lemma 3.3. *The scalar function h is a solution of the subelliptic inequality*

$$\Delta h + (ah + b)h \geqq 0 . \tag{3.3}$$

Proof. We use the notation and basic identites of [11, Chap. 4, Sect. 9]. Let $f = *F$, $g = D\phi$, and $w = \frac{1}{2}(m^2 - |\phi|^2)$,

(a) $$\nabla f + [[f, \phi], \phi] - 2*(g \wedge g + f \wedge f) = 0 ,$$

(b) $$\nabla^2 g + [[g, \phi], \phi] - \lambda\phi(\phi, g) + \lambda w g - 2*(f \wedge g + g \wedge f) = 0 .$$

Applying (1.10) of Lemma 1.2, and the triangle inequality,

(a′) $$\Delta |f| + (|\phi|^2 + 2|f|)|f| + 2|g|^2 \geqq 0 ,$$

(b′) $$\Delta |g| + ((1 + |\lambda|)|\phi|^2 + |\lambda|\,|w| + 4|f|)|g| \geqq 0 .$$

Using the field equation (YMH_2) for ϕ and inequality (1.7),

(c′) $$\tfrac{1}{2}\Delta(|\phi|^2) + \frac{|\lambda|}{2}(|\phi|^2 + m^2)|\phi|^2 \geqq 0 .$$

Adding the three equations gives (3.3) with $a = 10 + 2|\lambda|$ and $b = |\lambda| m^2$.

In the following, $B(y_0, r) = \{y \mid |y - y_0| \leqq r\}$ always denote balls which are strictly contained in V_1.

Lemma 3.4. *If* $\gamma < \gamma_2$, *there is a constant k such that*

$$\int_{B(y_0, p)} |\nabla(h^p)|^2 dy \leqq \frac{k}{a^2} \int_{B(y_0, \varrho + a)} h^{2p} dy , \tag{3.4}$$

where $p = n/4$.

Proof. Integrating by parts in (3.3),

$$\int \nabla h \cdot \nabla \zeta dy \leqq \int ah + b)h\zeta dy \tag{3.3′}$$

for non-negative $\zeta \in C_0^\infty$.

By a limiting argument, we may choose $\zeta = \eta^2 h^{2p-1}$ with η arbitrary to obtain

$$\begin{aligned} \int \eta^2 |\nabla(h^p)|^2 dy &\leqq k_1 \int |a| \eta^2 h^{2p+1} dy \\ &\quad + k_2 \int |\eta \nabla(h^p)|\,|\nabla\eta h^p| dy \\ &\quad + k_3 \int b\eta^2 h^{2p} dy \\ &= k_1 I_1 + k_2 I_2 + k_3 I_3 . \end{aligned}$$

Estimating I_1 using (1.14)

$$I_1 \leqq C_n \|h\|_{n/2} \int |\nabla(\eta h^p)|^2 dy \leqq k\gamma \int |\nabla(\eta h^p)|^2 dy. \tag{3.6}$$

I_2 is estimated using Young's inequality. For $\gamma < \gamma_2$, we find

$$\int \eta^2 |\nabla(h^p)|^2 dy \leqq C \int (|\nabla\eta|^2 + \eta^2) h^{2p} dy. \tag{3.7}$$

Letting $\eta = 1$ on $B(y_0, \varrho)$ with support in $B(y_0, \varrho + a)$ with $|\nabla\eta| \leqq 2/a$ completes the proof of Lemma 3.4.

Proof of Proposition 3.2. By Lemma 3.4, $ah+b$ satisfies the Morrey growth condition (1.11). We apply Theorem 1.3 with $U = h^{3/4}$ and $W = U^{4/3}$ if $n=3$, and $U = W = h^{n/4}$ if $n \geqq 4$. Therefore, h is bounded and (1.13) implies that on compact subdomains of V_1,

$$h(x) \leqq \frac{C}{a^2} \left(\int_{B(x_0, \varrho + a)} |h(y)|^{n/2} dy \right)^{2/n} \tag{3.8}$$

for $x \in B(x_0, \varrho)$. Now, cover the unit sphere in V_1 by a finite number of balls, to obtain, for $|y| = 1$,

$$|h(y)| \leqq C \|h\|_{L^{n/2}(V_1)}.$$

This proves the proposition and therefore, Theorem 3.1.

Corollary 3.5. *$|x|^2 |F(x)|$ and $|x|^2 |D\phi(x)|$ are $o(1)$ near the origin.*

(We note that by working in the reference ring V_1 we are able to obtain estimates which are independent of the distance to the puncture. If one works directly in the ring V_r, one has to keep track of the dependence of constants on r.)

4. An Elliptic Estimate

In this section, we improve our results to obtain a final growth condition on the curvature F and on $D\phi$. Dimension is now restricted to $n=3$ or 4. We assume that $F \in L^{n/2}$, ϕ is bounded, and, hence, $D\phi \in L^2$ by Corollary 2.4. Since integration by parts is crucial here, we are forced to work in an L^2 setting. This is natural if $n=4$, but not if $n=3$, in which case, we use *weighted* L^2 norms. While the L^2 argument can be carried out if $n = 5$, 6, or 7 (see [17]) to prove the theorem, it is not strong enough to obtain the corresponding result if $n \geqq 8$.

Our first aim in this section is to obtain a growth condition on the Higgs field. This will then be used to estimate the curvature. Integrating by parts, we find

$$\int_{|x| \leqq 1} |D\phi|^2 dx = \int_{|x| \leqq 1} (\phi, D^* D\phi) + \int_{|x| = 1} \phi_S \wedge *(D\phi)_S. \tag{4.1}$$

Using the field equation (YMH_2) and Schwarz inequality

$$\int_{|x| \leqq 1} |D\phi|^2 dx \leqq \int_{|x| \leqq 1} \frac{|\lambda|}{2} |\phi|^2 (|\phi|^2 + m^2) dx + \tfrac{1}{2} \int_{|x| = 1} (|\phi|^2 + |D\phi|^2) dS. \tag{4.2}$$

Making the change of variables, $y = \varrho x$, with $\varrho < 1$, we find that

$$\int_{|y| \leqq \varrho} (|D\phi|^2 dy \leqq C \int_{|y| \leqq \varrho} |\phi|^4 dy + \int_{|y| \leqq \varrho} |y|^{-2} |\phi|^2 dy + \frac{\varrho}{2} \int_{|y| = \varrho} (|D\phi|^2 + |y|^{-2} |\phi|^2) dS_y. \tag{4.3}$$

Denoting the left hand side by $f(\varrho)$, and using the fact that φ is bounded, (4.3) becomes the differential inequality

$$f(\varrho) \leqq k_1 \varrho + \tfrac{1}{2}\varrho f'(\varrho), \quad \text{if} \quad n=3, \tag{4.4a}$$

$$f(\varrho) \leqq k_2 \varrho^2 + \tfrac{1}{2}\varrho f'(\varrho), \quad \text{if} \quad n=4, \tag{4.4b}$$

or,

$$0 \leqq k_1 \varrho^{-2} + \tfrac{1}{2}\frac{d}{d\varrho}\left(\frac{f(\varrho)}{\varrho^2}\right), \quad \text{if} \quad n=3, \tag{4.4a'}$$

$$0 \leqq k_2 \varrho^{-1} + \tfrac{1}{2}\frac{d}{d\varrho}\left(\frac{f(\varrho)}{\varrho^2}\right), \quad \text{if} \quad n=4. \tag{4.4b'}$$

Integrating from $\varrho = r$ to $\varrho = 1$ gives

Theorem 4.1. *The Higgs field satisfies the growth condition*

$$\int_{|x| \leqq r} |D\phi|^2 dx \leqq Cr, \qquad \textit{if} \quad n=3, \tag{4.5}$$

$$\int_{|x| \leqq r} |D\phi|^2 dx \leqq Cr^2 \log\left(\frac{1}{r}\right), \quad \textit{if} \quad n=4. \tag{4.6}$$

In three dimensions, we find from Hölder's inequality,

$$\left(\int_{|x| \leqq r} |D\phi|^{3/2} dx\right)^{2/3} \leqq C'r, \tag{4.7}$$

which we will require in Sect. 5.

The remainder of this section is devoted to proving

Theorem 4.2. *If* $n=3$, *for any* $\alpha > 1$, $\int_{|x| \leqq 1} |x|^\alpha |F(x)|^2 dx < \infty$ *and*

$$\int_{|x| \leqq 1} |x|^\alpha |F(x)|^2 dx \leqq C_1 \int_{|x| \leqq 1} |x|^\alpha (|D\phi|^2 + |\phi|^4) dx + C_2 \int_{|x|=1} |F|^2 dS. \tag{4.8}$$

If $n=4$,

$$\int_{|x| \leqq 1} |F(x)|^2 dx \leqq C_1 \int_{|x| \leqq 1} (|D\phi|^2 + |\phi|^4) dx + C_2 \int_{|x|=1} |F|^2 dS. \tag{4.9}$$

From Theorem 4.2, we obtain our final growth condition

Corollary 4.3. *If* $n=3$,

$$\int_{|x| \leqq r} |x|^\alpha |F(x)|^2 dx \leqq Kr^\beta, \tag{4.10}$$

with $\beta > 0$ *and* K *and* β *independent of* α. *If* $n=4$,

$$\int_{|x| \leqq r} |F(x)|^2 dx \leqq Kr^\beta \tag{4.11}$$

with $\beta > 0$.

We first show that

Theorem 4.2 implies Corollary 4.3. With $n=3$, make the change of variable $y=\varrho x$ in (4.8) to obtain

$$\int_{|y|\leqq\varrho} |y|^\alpha |F(y)|^2 dy \leqq C_1 \int_{|y|\leqq\varrho} |y|^\alpha (|D\phi|^2+|\phi|^4)dx + C_2\varrho \int_{|y|=\varrho} |y|^\alpha |F|^2 dS_y. \tag{4.8'}$$

Using (4.5), this gives the differential inequality

$$f(\varrho)\leqq a\varrho^2+b\varrho f'(\varrho) \tag{4.8''}$$

with a and b constants. Since $f'\geqq 0$, we may assume $b>1$ to obtain

$$0\leqq \frac{a\varrho^{1-1/b}}{b}+\frac{d}{d\varrho}\left(\frac{f(\varrho)}{\varrho^{1/b}}\right). \tag{4.8'''}$$

Integrating proves (4.10) with $\beta=1/b$. The same argument proves (4.11).

The remainder of this section is devoted to the proof of Theorem 4.2. The basic idea of "broken Hodge gauges" is due to Uhlenbeck [19] with modifications if $n=3$ which were proved in [17]. We recall the necessary results without proof in a sequence of lemmas.

We first consider an eigenvalue problem for a 1-form ω defined over a reference ring $U=\{x|1\leqq|x|\leqq\tau\}$. We denote by ω_s the tangential component of a form on the boundary.

Problem I. Find ω satisfying in U, the

(a) equations: $\delta\omega=0$ and $\delta d\omega+\mu\omega=0$,

(b) boundary conditions: $\delta_s\omega_s=0$ for $|x|=1$ and $|x|=\tau$,

(c) homology condition: $\int_{|x|=\varrho}(*\omega)_s=0,\ 1\leqq\varrho\leqq\tau$.

Lemma 4.4. *The eigenvalues of this problem are strictly positive if* $n\geqq 3$. *If* $n=3$, *the first eigenvalue is greater than or equal to* 2.

The lemma is proved in [17].

Now, let $U^i=\left\{x\left|\frac{1}{\tau^i}\leqq|x|\leqq\frac{1}{\tau^{i-1}}\right.\right\}$ and $S^i=\left\{x\left||x|=\frac{1}{\tau^i}\right.\right\}$. The next lemma expresses the *existence* of broken Hodge gauges over $B=\bigcup_{i=1}^{\infty}U^i$. Here γ_3 is an additional restriction on γ which comes from applying the Implicit Function Theorem, μ is the first eigenvalue of Problem I, and v is the first eigenvalue of the Laplacian on co-closed 1-forms on S^{n-1}.

Lemma 4.5 (Broken Hodge Gauges [19]). *There exist gauges for* η/U^i *such that*

(a) $\delta A^i=0$,

(b) $\delta_s A^i_s=0$ *on* S^i *and* S^{i-1},

(c) $\int(*A^i)_s=0$ *on absolute cycles,*

(d) $|A^i(x)|\leqq\gamma_3\tau^i$,

(e) $\int_{U^i}|A^i(x)|^2dx\leqq\frac{1}{\tau^{2i}(\mu-\gamma_3)}\int_{U^i}|F^i|^2dx$, (e′) $\int_{U^i} |x|^\alpha |A^i(x)|^2 dx \leqq \frac{\tau^\alpha}{\tau^{2i}(\mu-\gamma_3)} \int_{U^i} |x|^\alpha |F^i(x)|^2 dx$, *if* $n=3$, *for any* $\alpha>1$,

(f) *the gauges agree on boundary spheres* S^i,

(g) $\int_{S^0} |A_s^1|^2 ds \leqq \frac{1}{v-\gamma_3} \int_{S^0} |F^1|^2 dS.$

The proof of the lemma is in [19] except for (e′) which involves the weighted L^2 norm and is proved in [17].

A consequence of Lemma 4.5 is the inequality

$$\text{if } n=3, \quad \left(\int_{U^i} |x|^\alpha |A^i(x)|^4 dx\right)^{1/2} \leqq \gamma_3 \left(\frac{\tau^\alpha}{\mu-\gamma_3}\right)^{1/2} \left(\int_{U^i} |x|^\alpha |F^i(x)| dx\right)^{1/2}, \tag{4.12}$$

$$\text{if } n=4, \quad \left(\int_{U^i} |A^i(x)|^4 dx\right)^{1/2} \leqq \gamma_3 \left(\frac{1}{\mu-\gamma_3}\right)^{1/2} \left(\int_{U^i} |F^i(x)|^2 dx\right)^{1/2}. \tag{4.12′}$$

We now turn our attention to the proof of Theorem 4.1. First, let $n=4$. We integrate by parts over each U^i to obtain:

$$\begin{aligned}\int_{U^i} |F^i(x)|^2 dx &= \int_{U^i} (A^i, D^*F^i) - \int_{U^i} (\tfrac{1}{2}[A^i, A^i], F^i) \\ &\quad + \int_{S^{i-1}} - \int_{S^i} A_s^i \wedge (*F^i)_s. \\ &= I_1 + I_2 + \text{boundary terms}.\end{aligned} \tag{4.13}$$

Using the field equation (YMH$_1$), $D^*F=[D\phi,\phi]$, we find

$$\begin{aligned} I_1 &= \int_{U^i} (A^i, [D\phi,\phi]) \leqq \int_{U^i} |D\phi|^2 dx + \int_{U^i} |A^i|^2 |\phi|^2 dx \\ &\leqq \int |D\phi|^2 dx + \int_{U^i} |A^i|^4 dx + \int_{U^i} |\phi|^4 dx.\end{aligned}$$

From (4.12′),

$$I_1 \leqq \gamma_3^2 \left(\frac{1}{\mu-\gamma_3}\right) \int_{U^i} |F^i|^2 dx + \int_{U^i} (|D\phi|^2 + |\phi|^4) dx,$$

$$I_2 \leqq \frac{\gamma_3}{2} \left(\frac{1}{\mu-\gamma_3}\right)^{1/2} \int_{u^i} |F^i(x)|^2 dx,$$

using the Schwarz inequality and (4.12′).

Combining terms and replacing small constants by ε, we find,

$$(1-\varepsilon) \int_{U^i} |F^i(x)|^2 dx \leqq \int_{U^i} |D\phi|^2 dx + \int_{U^i} |\phi|^4 dx + \int_{S^{i-1}} - \int_{S^i} A_s^i \wedge (*F^i)_s. \tag{4.14}$$

Adding the integrals over each U^i, we see that intermediate boundary terms cancel, the boundary integrals tend to zero as i tends to infinity, and we are left with

$$(1-\varepsilon) \int_{|x|\leqq 1} |F(x)|^2 dx \leqq \int_{|x|\leqq 1} (|D\phi|^2 + |\phi|^4) dx + \int_{S^0} |A_s^1||F_s| dS. \tag{4.15}$$

Using Schwarz' inequality and (g) of Lemma 4.5 proves the inequality (4.9) of Theorem 4.2.

Next, let $n=3$. We now require that $\tau<2$ and we also make an additional restriction on γ; namely, we assume $\gamma<\gamma_4$, where

$$\left(\frac{\tau}{2-\gamma_4}\right)^{1/2}\left(1+\frac{\gamma_4}{2}\right)<1.$$

We again integrate by parts over each U^i to obtain

$$\begin{aligned}\int_{U^i}|x|^\alpha|F^i(x)|^2dx&=\int_{U^i}(A^i,D^*(|x|^\alpha F^i))-\int_{U^i}(\tfrac{1}{2}[A^i,A^i],|x|^\alpha F^i)\\&\quad+\int_{S^{i-1}}-\int_{S^i}A^i_s\wedge|x|^\alpha(*F^i)_s\\&=I_1+I_2+\text{boundary terms}.\end{aligned}\tag{4.13'}$$

Now,

$$\begin{aligned}I_1&\leqq\int_{U^i}(A^i,|x|^\alpha[D\phi,\phi])+\int_{U^i}\alpha|x|^{\alpha-1}|A^i||F^i|dx\\&\leqq\left(\frac{\tau^\alpha}{2-\gamma_4}\right)^{1/2}\left(\alpha+\frac{\gamma_4}{2}\right)\int_{U^i}|x|^\alpha|F^i|^2dx+\int_{U^i}|x|^\alpha(|D\phi|^2+|\phi|^4)dx.\end{aligned}$$

By the assumption on γ_4, and for α close to 1, the coefficient of the first integral is small, and combining terms,

$$(1-\varepsilon')\int_{U^i}|x|^\alpha|F^i(x)|^2dx\leqq\int_{U^i}|x|^\alpha(D\phi|^2+|\phi|^4)dx+\text{boundary terms}.\tag{4.14'}$$

The rest of the proof is exactly analogous to the 4 dimensional case, and we obtain (4.8), and hence, Theorem 4.2.

5. Statement and Proof of the Removable Singularity Theorem

Let $n=3$ or 4. In this section, we combine the preceding results to prove:

Theorem 5.1 (Removable Singularities). *Let η be a bundle over $B-\{0\}$ with compact structure group G. Suppose that (F,ϕ) is a smooth solution of the Yang-Mills-Higgs equations in $B-\{0\}$. We assume in all cases that $F\in L^{n/2}$, $n=3,4$. If $\lambda>0$, we make no assumptions on ϕ or $D\phi$ in a neighborhood of the origin. If $\lambda<0$, we assume that $\phi\in L^{3+\varepsilon}(B)$ for some $\varepsilon>0$ if $n=3$, and $\phi\in L^4(B)$ if $n=4$. If $\lambda=0$, we assume $\phi\in L^{n/n-2}(B)$. Then, there is a continuous gauge transformation such that (F,ϕ) is gauge equivalent to a C^∞ pair over B, and η extends continuously to a bundle over B.*

We now put all previous estimates together to obtain

Proposition 5.2. *For some $\delta>0$,*

$$|x|^{2-\delta}(|F(x)|+|D\phi(x)|)\leqq C.\tag{5.1}$$

Proof. From (3.1) with $|x|=r$, we obtain

$$\begin{aligned}|x|^2(|F(x)|+|D\phi(x)|)&\leqq C\|h\|_{L^{n/2}(V_r)}\leqq C_1\|F\|_{L^{n/2}(V_r)}\\&\quad+C_2\|D\phi\|_{L^{n/2}(V_r)}+C_3\|\phi^2\|_{L^{n/2}(V_r)}.\end{aligned}\tag{5.2}$$

We now use the fact that ϕ is bounded and that $D\phi$ satisfies (4.7) if $n=3$ and (4.6) if $n=4$.

If $n=4$, from (4.11),

$$|x|^2(|F(x)|+|D\phi(x)|)\leqq k_1 r^{\beta/2}+k_2 r\left(\log\frac{1}{r}\right)^{1/2}+k_3 r^2\,,$$

where $\beta>0$.

If $n=3$, from Hölder's inequality and (4.10),

$$\begin{aligned}|x|^2(|F(x)|+|D\phi(x)|)&\leqq k_4 r^{(1-\alpha)/2}\Big(\int_{|x|\leqq 2r}|x|^\alpha|F(x)|^2dx\Big)^{1/2}\\&\quad+k_5 r+k_6 r^2\\&\leqq k_7 r^{(1-\alpha+\beta)/2}+k_5 r+k_6 r^2\,,\end{aligned}$$

with $\beta>0$ independent of α. Choosing α sufficiently close to one proves the proposition.

Corollary 5.3. *The curvature F is in L^p for $n/2\leqq p<n/(2-\delta)$ and (F,ϕ) is a weak solution of the field equations in the full ball B.*

(The proof is elementary.)

Proposition 5.4. *If $F\in L^p(B)\cap C^\infty(B-\{0\})$ with $p>n/2$, then there is a connection $A\in H_1^p(B)$ with $p>n/2$.*

Proof. Using the broken Hodge gauge construction (Lemma 4.5), we obtain on each U^i, a connection $A^i\in L^{2p}$ for $p>n/2$ and norm uniformly bounded by the L^p norm of F. Since $dA^i=F^i-\frac{1}{2}[A^i,A^i]$, $dA^i\in L^p$ for $p>n/2$. This, together with the equation $\delta A^i=0$ implies that $\nabla A^i\in L^p$ for $p>n/2$. Letting $A=\{A^i(x),x\in U^i\}$ proves the proposition.

We next apply the following theorem of Uhlenbeck [20],

Proposition 5.5. *Suppose $\tilde{F}$ is the curvature form of a connection $\tilde{A}$, with $L^{n/2}$ norm sufficiently small. If $\tilde{F}\in L^p$ for $p>n/2$, then $(\tilde{F},\tilde{A})$ is gauge equivalent by a continuous gauge transformation to (F,A), where*

(i) $\delta A=0$,

(ii) $\|A\|_{H_1^p}\leqq C\|F\|_{L^p}$, $p>n/2$.

From Proposition 5.5, we find in the new gauge that (A,ϕ) satisfies the system of equations

$$(d\delta+\delta d)A+\tfrac{1}{2}\delta[A,A]+*[A,*F]=[D\phi,\phi]\,, \tag{5.3a}$$

$$\delta d\phi+\delta[A,\phi]+*[A,*[A,\phi]]=\frac{\lambda}{2}(|\phi|^2-m^2)\phi\,. \tag{5.3b}$$

Computations similar to those in Sect. 3 applied to (5.3a) show that $W=1+|A|$ is a subsolution of an inequality:

$$\int(\nabla W\cdot\nabla\zeta+fW\zeta)dx\leqq 0 \tag{5.4}$$

for all non-negative $\zeta \in C_0^\infty(B)$. From (ii) of Proposition 5.5, the boundedness of ϕ, and the growth conditions (4.5) and (4.6), it is not hard to see that a Morrey growth condition (1.11) is satisfied by f. Therefore, W, and hence, A, is bounded in B.

We now turn to Eq. (5.3b) which we write in component form:

$$\Delta\phi^i = F^i(x, \phi, \nabla\phi). \tag{5.5}$$

We want to apply the results of Hildebrandt and Widman [10] on regularity of solutions of systems in diagonal form. Since the connection A is bounded, F^i is bounded, measurable. In the notation of [8], $A^{\alpha\beta} = \delta^{\alpha\beta}$ in our case, and therefore, the ellipticity constant $\lambda \equiv 1$. More importantly, F^i depends *linearly* on $\nabla\phi$. Therefore, if ϕ is bounded by M, we find that

$$|F^i(x, \phi, \nabla\phi| \leqq \varepsilon|\nabla\phi|^2 + b \tag{5.6}$$

with $2M\varepsilon < 1$.

We conclude [8, Theorem 6.6(iii)] that ϕ and $D\phi$ are Hölder continuous in B.

Returning to (5.3a), we find that the components of A satisfy a system exactly of the form (5.5) with $A^{\alpha\beta} = \delta^{\alpha\beta}$, $\lambda \equiv 1$, F^i linear in ∇A^i, and also, F^i bounded since ϕ and $\nabla\phi$ are bounded. By the same theorem of Hildebrandt-Widman, A and DA are Hölder continuous. Standard elliptic theory now implies ϕ and A are C^∞ in B. This completes the proof of Theorem 5.1.

Note added. The corresponding theorem for these equations in two dimensions has been proved by P. D. Smith and will appear in a forthcoming paper.

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