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Glueball Spectroscopy in Strongly Coupled Lattice Gauge Theories

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Abstract. We study the mass spectrum up to $-7\ (1-\varepsilon)\ \log\beta$ of pure three-dimensional lattice gauge theories with action $\beta\sum_P \chi(g_P)$ for real irreducible χ and small β. Besides the lowest excitation $m_0 \sim -4\log\beta$, we find two nearly degenerate excited states m_1, m_2 with $m_i \sim -6\log\beta$ $(i=1,2)$ and (m_1-m_2) at least $O(\beta)$.

1. Introduction

The existence of glueballs within QCD has been predicted already some time ago by Fritzsch and Gell-Mann [1]. They are receiving increasing attention, in the context of lattice gauge theories, since the pioneering work of Kogut et al. [2]. The information concerning the mass spectrum in the lattice case has come mainly from Monte Carlo calculations and strong coupling perturbation expansions. See, e.g. [3, 4] and references given there. Using appropriate selection rules, excited states have been obtained by locating the lowest excitation within each selection sector, but the methods were not suitable to find states with the same quantum numbers.

In two previous publications [5, 6], we started a nonperturbative study of the glueball spectrum in pure gauge lattice models with the Wilson action

$$S_\Lambda = \beta \sum_{P\subset\Lambda} \mathrm{Re}\,\chi(g_P) \tag{1.1}$$

making the simplifying assumption that the character χ is real irreducible and the space-time dimensionality is three (see Sect. 2 for notation). We found isolated one particle states in the full energy-momentum spectrum of the theory, if β is small enough. The particle mass $m_0(\beta)$ has the asymptotic behaviour $m_0(\beta) \sim -4\log\beta$ as $\beta\to 0$ and is the only spectrum (besides the vacuum) up to the threshold $-6(1-\varepsilon)\log\beta$.

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In the present paper, we investigate the mass spectrum up to the threshold $-7(1-\varepsilon)\log\beta$ and find precisely two excited states $m_1(\beta)$, $m_2(\beta)$ with almost degenerate masses: both m_1, m_2 are asymptotic to $-6\log\beta$ as $\beta\to 0$, and $|m_1-m_2|$ is at least $O(\beta)$. The actual asymptotic behaviour of m_1-m_2 is not determined in this work although the methods developed here are suitable to calculate in principle the power series expansion for the mass difference. These results are derived under the same simplifying assumptions alluded to before but we believe the methods can be adapted to handle more general cases.

We now explain briefly how the excited states are obtained. Precise statements can be found in Sect. 2. The two-dimensional lattice quantum field theory associated to the action (1.1) in three space-time dimensions has an obvious $Z(4)$ symmetry (corresponding to successive rotations R of $\pi/2$ around an axis parallel to the time direction and through the center of a plaquette). This symmetry induces a selection rule on the zero momentum states which can then be split into a direct sum of four subspaces, each transforming according to the irreducible representations of $Z(4)$. We denote these representations by E, A_1, A_2, A_3. They associate to the abstract group $\{R^0=I, R, R^2, R^3\}$, respectively, $\{1,1,1,1\}$, $\{1,-1,1,-1\}$, $\{1,i,-1,-i\}$, and $\{1,-i,-1,i\}$. The methods developed in [5, 6] are suitable to analyze only the lowest non-trivial excitation within each selection sector. In this way, we show that there is no mass spectrum below $-7(1-\varepsilon)\log\beta$ for vectors transforming according to A_2 or A_3. For vectors transforming as A_1, there is exactly one excitation with mass $m_2(\beta)$, asymptotic to $-6\log\beta$ as $\beta\to 0$. On the subspace corresponding to the identity representation, the method of [5, 6] gives only the already known particle $m_0(\beta)$. To analyse excited states within this subspace we implemented some ideas developed by Koch in the context of continuum quantum field theories [7]. As it turns out, there is exactly one such state [up to $-7(1-\varepsilon)\log\beta$] with mass $m_1(\beta)$, which is asymptotic to $-6\log\beta$ as $\beta\to 0$; $m_1(\beta)$ is obtained as the solution of a "perturbed" equation, whose "unperturbed" solution is $m_2(\beta)$. The estimate on the mass difference given above comes from this fact.

The organization of the paper is as follows. In Sect. 2, we give some definitions and present the statements leading to the results above, without proofs. Almost all of them require estimates on decay rates of appropriate Green's functions, which were obtained by using the decoupling procedure of [5, 6] extended suitably to handle the region of mass up to $-7(1-\varepsilon)\log\beta$. The required theorems for this part are given in Sect. 3. Finally, in Sect. 4, we give the missing proofs of Sect. 2.

2. Some Definitions and Main Results

We consider a d dimensional pure lattice theory with compact group G. A gauge field configuration associates to each oriented bound $\ell=(x_1,x_2)$ in $\mathbb{Z}^d$ a group element $g_\ell\in G$, with the convention $g_{\ell^{-1}}=g_\ell^{-1}$, where $\ell^{-1}=(x_2,x_1)$. If φ is a continuous function depending only on a finite number of bond variables (φ has "finite support)" we define the expectations

$$\langle\varphi\rangle(\Lambda;\{\beta_P\})=\frac{1}{Z_\Lambda}\int\varphi(g)\exp\left[\sum_{P\subset\Lambda}\beta_P\,\mathrm{Re}\,\chi(g_P)\right]dg_\Lambda, \tag{2.1}$$

where $\Lambda \subset \mathbb{Z}^d$ is a finite set containing the support of φ and χ is a character from an irreducible, unitary representation of G. The sum in the exponent is over all plaquettes in Λ, and to each plaquette P we associate a complex number β_P; g_P is the oriented product of group elements along the boundary of P and dg_Λ is a product of Haar measures, one for each bond in Λ. Z_Λ is a normalization factor, such that $\langle 1\rangle(\Lambda;\{\beta_P\})=1$. From the cluster expansion of Osterwalder and Seiler [8], there are constants β_0 (independent of Λ, φ) and C_φ (independent of Λ) such that (2.1) is analytic and uniformly bounded by C_φ on $|\beta_P|<\beta_0$. Also, truncated correlations have exponential decay rates: if the supports of φ, ψ are separated by a distance d, then on $|\beta_P|<\beta_0$,

$$|\langle\varphi\psi\rangle(\Lambda;\{\beta_P\})-\langle\varphi\rangle(\Lambda;\{\beta_P\})\langle\psi\rangle(\Lambda;\{\beta_P\})|\leqq C'_{\varphi\psi}e^{-Md} \tag{2.2}$$

for suitable constants M (independent of Λ, φ, ψ) and $C'_{\varphi\psi}$ (independent of Λ). In addition, setting all $\beta_P=\beta$, $0<\beta<\beta_0$ the expectations $\langle\varphi\rangle(\Lambda;\beta)$ converge uniformly to $\langle\varphi\rangle(\beta)$ as $\Lambda\to\mathbb{Z}^d$ and define a probability measure $d\mu$ on the Baire subsets Σ of $\mathscr{X}=\prod_{\ell\subset\mathbb{Z}^d} G_\ell$ (infinite product of G with itself, one factor for each bond in $\mathbb{Z}^d$). The "interacting measure" $d\mu$ is translation and time reflection invariant, so that these operations are implemented on the "path space" $\mathscr{E}=L^2(\mathscr{X},\Sigma,d\mu)$ by unitary operators, denoted $U(x)$, $x\in\mathbb{Z}^d$ and θ, respectively. The physical Hilbert space is the time zero "slice" of $\mathscr{E}$, i.e. $\mathscr{H}=L^2(\mathscr{X},\Sigma_0,d\mu)$, where Σ_0 is generated by continuous φ of finite support in the time zero hyperplane of $\mathbb{Z}^d$ and a Feynman-Kac formula holds for gauge invariant functions. Thus, for any $\varphi,\psi\in\mathscr{H}$ gauge invariant writing $x=(x_0,\mathbf{x})$ with $x_0\in\mathbb{Z}$, $\mathbf{x}\in\mathbb{Z}^{d-1}$,

$$(\varphi, U(x_0,\mathbf{x})\psi)_{\mathscr{E}}=(\varphi, U(-x_0,\mathbf{x})\psi)_{\mathscr{E}}=(\varphi, e^{-H|x_0|}e^{i\mathbf{P}\cdot\mathbf{x}}\psi)_{\mathscr{H}}, \tag{2.3}$$

where H and $\mathbf{P}$ are the energy and momentum operators, respectively. See [6] for more details on this formalism.

In the sequel, we only deal with gauge invariant functions of finite support in the time zero hyperplane. We define

$$G_{\varphi\psi}(x)=(\varphi, U(x)\psi)_{\mathscr{E}}-(\varphi,1)_{\mathscr{E}}(1,\psi)_{\mathscr{E}}, \tag{2.4}$$

$$\hat{G}_{\varphi\psi}(x_0)=\sum_{\mathbf{x}} G_{\varphi\psi}(x_0,\mathbf{x}), \tag{2.5}$$

and

$$\tilde{G}_{\varphi\psi}(p_0)=\sum_{x_0}\hat{G}_{\varphi\psi}(x_0)e^{ip_0x_0}, \tag{2.6}$$

so that $\tilde{G}_{\varphi\psi}(p_0)$ is the usual Fourier transform of $G_{\varphi\psi}(x)$ at zero momentum. From (2.2), there exists a constant $C_{\varphi\psi}$ such that for $0<\beta<\beta_0$,

$$\sum_x |G_{\varphi\psi}(x)|\leqq C_{\varphi\psi}. \tag{2.7}$$

Also, from (2.3), $\tilde{G}_{\varphi\psi}(p_0)$ has the integral representation

$$\tilde{G}_{\varphi\psi}(p_0)=(2\pi)^3\int_{(0,\infty)}\int_{(-\pi,\pi]^{d-1}}\frac{\sinh\lambda_0}{\cosh\lambda_0-\cos p_0}\,\delta(\boldsymbol{\lambda})d(\varphi, E(\lambda_0,\boldsymbol{\lambda})\varphi)_{\mathscr{H}}, \tag{2.8}$$

where $dE(\lambda_0, \boldsymbol{\lambda})$ is the joint energy-momentum spectral measure. Thus, the singularities of (2.8) are located at the spectrum of the energy operator at zero momentum (plotted in the imaginary axis), the poles corresponding to particles. It is the goal of this work to find the possible singularities of $\tilde{G}_{\varphi\psi}(p_0)$ up to the threshold $|\mathrm{Im}\, p_0| \leqq -7(1-\varepsilon)\log\beta$, for arbitrary φ. The analysis can be carried out due to the existence of selection rules operating on the zero momentum states. We assume the space-time dimensionality $d=3$ from now on.

From the invariance of the interacting measure under rotations of $\pi/2$ around an axis parallel to the time direction and through the center of a plaquette, it follows that

$$G_{\varphi\psi}(x_0, \mathbf{x}) = G_{R_{\mathbf{a}}\varphi, R_{\mathbf{a}}\psi}(x_0, R\mathbf{x}), \tag{2.9}$$

where $R_{\mathbf{a}}$ denotes rotation along an axis through $\mathbf{a}$ and $R\mathbf{x}$ is $\mathbf{x}$ rotated by $\pi/2$ around the origin. Thus, $\hat{G}_{\varphi\psi}(x_0) = \hat{G}_{R_{\mathbf{a}}\psi, R_{\mathbf{a}}\psi}(x_0)$. Actually,

$$\hat{G}_{\varphi\psi}(x_0) = \hat{G}_{R_{\mathbf{a}}\varphi, R_{\mathbf{b}}\psi}(x_0) \tag{2.10}$$

for arbitrary $\mathbf{a}$, $\mathbf{b}$. This is because $R_{\mathbf{b}} = R_{\mathbf{a}} U(\mathbf{z})$ provided $\mathbf{z} = (a_1 - a_2 + b_2 - b_1, a_1 + a_2 - b_1 - b_2)$ and so,

$$\hat{G}_{R_{\mathbf{a}}\varphi, R_{\mathbf{b}}\psi}(x_0) = \hat{G}_{R_{\mathbf{a}}\varphi, R_{\mathbf{a}}U(\mathbf{z})\psi}(x_0) = \hat{G}_{\varphi, U(\mathbf{z})\psi}(x_0) = \hat{G}_{\varphi\psi}(x_0).$$

Now, define

$$P_{\mathbf{a}}^{(1)} = (1/4)(1 + R_{\mathbf{a}} + R_{\mathbf{a}}^2 + R_{\mathbf{a}}^3), \qquad P_{\mathbf{a}}^{(2)} = (1/4)(1 - R_{\mathbf{a}} + R_{\mathbf{a}}^2 - R_{\mathbf{a}}^3),$$
$$P_{\mathbf{a}}^{(3)} = (1/4)(1 + iR_{\mathbf{a}} - R_{\mathbf{a}}^2 - iR_{\mathbf{a}}^3), \qquad P_{\mathbf{a}}^{(4)} = (1/4)(1 - iR_{\mathbf{a}} - R_{\mathbf{a}}^2 + iR_{\mathbf{a}}^3).$$

Clearly, $P_{\mathbf{a}}^{(i)} P_{\mathbf{a}}^{(j)} = \delta_{ij} P_{\mathbf{a}}^{(i)}$ and $\sum_i P_{\mathbf{a}}^{(i)} = 1$. Moreover, $\hat{G}_{P_{\mathbf{a}}^{(i)}\varphi, \psi}(x_0) = \hat{G}_{\varphi, P_{\mathbf{b}}^{(i)}}(x_0)$, so that $\hat{G}_{\varphi\psi}(x_0) = 0$ if $\varphi = P_{\mathbf{a}}^{(i)}\varphi$, $\psi = P_{\mathbf{b}}^{(j)}\psi$ with $i \neq j$. This is the selection rule referred to above. It reduces our problem to locating singularities of functions of the form $\tilde{G}_{\varphi_i\varphi_i}(p_0)$, $1 \leqq i \leqq 4$ with $\varphi_i = P_{\mathbf{a}_i}^{(i)}\varphi_i$ for some $\mathbf{a}_i$. As it turns out, the cases $i = 3, 4$ are the easiest to analyse [up to $-7(1-\varepsilon)\log\beta$]. Notice that since $\tilde{G}_{\varphi\varphi}(p_0 + 2\pi) = \hat{G}_{\varphi\varphi}(p_0) = \tilde{G}_{\varphi\varphi}(-p_0)$, it is sufficient to restrict p_0 to $|\mathrm{Re}\, p_0| \leqq \pi$, $\mathrm{Im}\, p_0 \geqq 0$, and in this region the singularities can lie only in the imaginary axis, see (2.8). In the following, we assume $\varepsilon \leqq 1/10$, $\varrho_0 \leqq 1/2$ and the character χ in (2.1) real irreducible. The reality assumption simplifies the analysis of the problem under investigation. See the beginning of Sect. 3, where we point out these simplifications.

Theorem 2.1. *There exists $\beta_1 \leqq \beta_0$ such that for $0 < \beta < \beta_1$, $\tilde{G}_{\varphi\varphi}(p_0)$ is analytic on $|\mathrm{Re}\, p_0| \leqq \pi$, $0 \leqq \mathrm{Im}\, p_0 \leqq -7(1-\varepsilon)\log\beta$ if $\varphi = P_{\mathbf{a}}^{(i)}\varphi$ for any $\mathbf{a}$ and $i = 3, 4$.* □

To analyse $\tilde{G}_{\varphi_i\varphi_i}(p_0)$ with $i = 1, 2$, we introduce the function $\chi_h(g) = \chi(g_w)$, where g_W is the oriented product of the six group elements along the boundary of the elementary horizontal "window" located at the origin of the time zero plane.

Let $\chi_1 = P_{\mathbf{0}}^{(1)}\chi_h$, $\chi_2 = P_{\mathbf{0}}^{(2)}\chi_h$. As will be apparent from the next few theorems, the study of the analytic structure up to $|\mathrm{Im}\, p_0| \leqq -7(1-\varepsilon)\log\beta$ of $\tilde{G}_{\varphi_2\varphi_2}(p_0)$ is completely analogous to the one developed in [5, 6] for $\tilde{G}_{\varphi_1\varphi_1}(p_0)$ up to $|\mathrm{Im}\, p_0| \leqq -6(1-\varepsilon)\log\beta$, the function χ_2 replacing the elementary plaquette χ of the latter case. Thus, we have

Theorem 2.2. *There exist positive constants k_1, k_2, d_1, d_2 and $\beta_2 \leqq \beta_1$ such that if* $0<\beta<\beta_2$,

$$k_1(d_1\beta)^{6|x_0|} \leqq \hat{G}_{\chi_2\chi_2}(x_0) \leqq k_2(d_2\beta)^{6|x_0|}. \quad \square$$

We define $m_2(\beta) = -\lim_{|x_0|\to\infty}(1/|x_0|)\log\hat{G}_{\chi_2\chi_2}(x_0)$, so that $\lim_{\beta\to 0} m_2(\beta)/-6\log\beta = 1$ and β_2 can be chosen such that

$$-\tfrac{11}{2}\log\beta < m_2(\beta) < -\tfrac{13}{2}\log\beta. \tag{2.11}$$

Let $\hat{\Gamma}_{\chi_2\chi_2}(x_0)$ be the convolution inverse of $-\hat{G}_{\chi_2\chi_2}(x_0)$. The existence of $\hat{\Gamma}_{\chi_2\chi_2}$ is part of the following

Theorem 2.3. *There exists $\beta_3 \leqq \beta_2$ such that for $0<\beta<\beta_3$, $\hat{\Gamma}_{\chi_2\chi_2}(p_0)$ is analytic on* $|\mathrm{Re}\,p_0| \leqq \pi$, $0 \leqq \mathrm{Im}\,p_0 \leqq -7\left(1-\frac{\varepsilon}{2}\right)\log\beta$. $\square$

From this last result, $\tilde{G}_{\chi_2\chi_2}(p_0)$ is meromorphic on $|\mathrm{Re}\,p_0| \leqq \pi$, $0 \leqq \mathrm{Im}\,p_0 \leqq -7(1-\varepsilon)\log\beta$. Using the integral representation (2.8) and (2.11) it is easy to see that $\tilde{G}_{\chi_2\chi_2}(p_0)$ has precisely one simple pole at $p_0 = im_2$ (see [5, 9] for more details). Thus, $\tilde{G}_{\chi_2\chi_2}(p_0)$ has the form

$$\tilde{G}_{\chi_2\chi_2}(p_0) = v_2(m_2)\frac{\sinh m_2}{\cosh m_2 - \cos p_0} + \int_{-7\left(1-\frac{\varepsilon}{2}\right)\log\beta}^{\infty} \frac{\sinh\lambda_0}{\cosh\lambda_0 - \cos p_0}\,dv_2(\lambda_0) \tag{2.12}$$

with $v_2(m_2)>0$ and $dv_2(\lambda_0) = (2\pi)^3 \int \delta(\boldsymbol{\lambda})d(\chi_2, E(\lambda_0,\boldsymbol{\lambda})\chi_2)_{\mathscr{H}}$. We write

$$\tilde{H}_1(p_0) = v_2(m_2)\frac{\sinh m_2}{\cosh m_2 - \cos p_0}, \tag{2.13}$$

$$\tilde{H}_2(p_0) = \int_{-7\left(1-\frac{\varepsilon}{2}\right)\log\beta}^{\infty} \frac{\sinh\lambda_0}{\cosh\lambda_0 - \cos p_0}\,dv_2(\lambda_0). \tag{2.14}$$

Theorem 2.4. *There are constants k_3, k_4 and $\beta_4 \leqq \beta_3$ such that for $0<\beta<\beta_4$*

(a) $0 < v_2(m_2) < k_3$,

(b) *$\tilde{H}_2(p_0)$ is analytic on $|\mathrm{Re}\,p_0| \leqq \pi$, $0 \leqq \mathrm{Im}\,p_0 \leqq -7(1-\varepsilon)\log\beta$ and $|\tilde{H}_2(p_0)| \leqq k_4$ there.* $\square$

To show that $p_0 = im_2$ is the only possible singularity on $|\mathrm{Re}\,p_0| \leqq \pi$, $0 \leqq \mathrm{Im}\,p_0 \leqq -7(1-\varepsilon)\log\beta$ of $\tilde{G}_{\varphi\varphi}(p_0)$ for arbitrary $\varphi = P_{\mathbf{a}}^{(2)}\varphi$, we proceed as in [6] defining

$$\tilde{F}_{\varphi\varphi}^{(2)}(p_0) = \tilde{G}_{\varphi\varphi}(p_0) + \tilde{G}_{\varphi\chi_2}(p_0)\tilde{\Gamma}_{\chi_2\chi_2}(p_0)\tilde{G}_{\chi_2\varphi}(p_0).$$

By using a spectral representation analogous to (2.8) one shows that $\tilde{G}_{\varphi\chi_2}(p_0)$ and $\tilde{G}_{\chi_2\varphi}(p_0)$ are analytic in the region above, with the possible exception of a simple pole at $p_0 = im_2$. Thus, the desired result for $\tilde{G}_{\varphi\varphi}(p_0)$ follows from the next theorem.

Theorem 2.5. *Assume $0<\beta<\beta_3$. Then, $\tilde{F}_{\varphi\varphi}^{(2)}(p_0)$ is analytic on $|\mathrm{Re}\,p_0| \leqq \pi$, $0 \leqq \mathrm{Im}\,p_0 \leqq -7(1-\varepsilon)\log\beta$.* $\square$

We next study functions of the form $\tilde{G}_{\varphi\varphi}(p_0)$, $\varphi = P_{\mathbf{a}}^{(1)}\varphi$. An analysis up to $|\mathrm{Im}\,p_0| \leqq -6(1-\varepsilon)\log\beta$ has already been done in [5, 6]. Thus, in the theorem below we just summarize the results, without giving the corresponding proofs in Sect. 4. These could also be obtained by adapting the proofs of Theorems 2.2, 2.3, and 2.5.

We denote simply by χ the function corresponding to the elementary plaquette located at the origin of the time zero plane, and by $\tilde{\Gamma}_{\chi\chi}(x_0)$ the convolution inverse of $-\tilde{G}_{\chi\chi}(x_0)$.

Theorem 2.6. *There exists $\beta_5 \leqq \beta_4$ such that for $0<\beta<\beta_5$, $\tilde{G}_{\varphi\varphi}(p_0)$ is analytic on $|\mathrm{Re}\,p_0| \leqq \pi$, $|\mathrm{Im}\,p_0| \leqq -6(1-\varepsilon)\log\beta$ except possibly for a simple pole at $p_0 = im_0(\beta)$. Here $\varphi = P_{\mathbf{a}}^{(1)}\varphi$, $m_0(\beta) = -\lim_{|x_0|\to\infty}(1/|x_0|)\log \tilde{G}_{\chi\chi}(x_0) < -5\log\beta$, and in fact $\lim_{\beta\to 0} m_0(\beta)/(-4\log\beta) = 1$. In addition, $\tilde{\Gamma}_{\chi\chi}(p_0)$ is analytic in the region above.* □

To investigate the analytic structure on $-6(1-\varepsilon)\log\beta \leqq \mathrm{Im}\,p_0 \leqq -7(1-\varepsilon)\log\beta$ for general $\tilde{G}_{\varphi_1\varphi_1}(p_0)$ we consider first $\tilde{G}_{\chi\chi}(p_0)$. We will show, to begin with, that $\tilde{\Gamma}_{\chi\chi}(p_0)$ is analytic on $|\mathrm{Re}\,p_0| \leqq \pi$, $0 \leqq \mathrm{Im}\,p_0 \leqq -7(1-\varepsilon)\log\beta$ except for a simple pole at $p_0 = i\varrho$, $|m_2 - \varrho| = O(\beta)$. With additional estimates, we then show that $\tilde{\Gamma}_{\chi\chi}(p_0)$ has also a simple zero nearby, at $p_0 = im_1$ with $|m_1 - m_2| = O(\beta)$. Thus, $\tilde{G}_{\chi\chi}(p_0)$ has two simple poles, at $p_0 = im_0$ and $p_0 = im_1$. We next show that this is also the case for $\tilde{G}_{\chi_1\chi_1}(p_0)$ (χ_1 as defined before) and finally extend the result for general $\tilde{G}_{\varphi_1\varphi_1}(p_0)$.

Let

$$\tilde{F}_{\chi_1\chi_1}(p_0) = \tilde{G}_{\chi_1\chi_1}(p_0) + \tilde{G}_{\chi_1\chi}(p_0)\tilde{\Gamma}_{\chi\chi}(p_0)\tilde{G}_{\chi\chi_1}(p_0). \tag{2.15}$$

Since χ is expected to couple to all excited states with the same quantum numbers, a naive analysis using the spectral theorem suggests that the subtraction from $\tilde{G}_{\chi_1\chi_1}$ performed in (2.15) should get rid of all physical poles (i.e. poles corresponding to particles) in the function $\tilde{F}_{\chi_1\chi_1}$. The possible singularities of $\tilde{F}_{\chi_1\chi_1}$ therefore should be related to those of $\tilde{\Gamma}_{\chi\chi}$. The next few theorems shows that this is indeed the case.

Let $\hat{\Phi}_{\chi_1\chi_1}(x_0)$ be the convolution inverse of $-\hat{F}_{\chi_1\chi_1}(x_0)$. Its existence is proven as part of the

Theorem 2.7. *There exists $\beta_6 \leqq \beta_5$ such that for $0<\beta<\beta_6$, $\tilde{\Phi}_{\chi_1\chi_1}(p_0)$ is analytic on $|\mathrm{Re}\,p_0| \leqq \pi$, $0 \leqq \mathrm{Im}\,p_0 \leqq -7(1-\varepsilon)\log\beta$.* □

Thus, $\tilde{F}_{\chi_1\chi_1}(p_0)$ is meromorphic on the region above. From the work in [6], the poles can only occur for $\mathrm{Im}\,p_0 > -6(1-\varepsilon)\log\beta$. To find them, we introduce the function

$$\tilde{K}(p_0) = \tilde{\Gamma}_{\chi_2\chi_2}(p_0) - \tilde{\Phi}_{\chi_1\chi_1}(p_0). \tag{2.16}$$

As will be seen, $\tilde{K}$ play a role analogous to the Bethe–Salpeter kernel in [7]. Define also

$$\tilde{T}_i(p_0) = \tilde{K}(p_0)\tilde{H}_i(p_0), \qquad i = 1, 2 \tag{2.17}$$

[see (2.13) and (2.14)].

Theorem 2.8. *There are constants k_5 and $\beta_7 \leqq \beta_6$ such that for $0<\beta<\beta_7$,*

(a) *$\tilde{K}(p_0)$ is analytic on $|\mathrm{Re}\,p_0| \leqq \pi$, $0 \leqq \mathrm{Im}\,p_0 \leqq -7(1-\varepsilon)\log\beta$ and $|\tilde{K}(p_0)| \leqq k_5\beta^{\left(\frac{7}{2}\varepsilon\right)}$ there.*

(b) *$|\tilde{T}_2(p_0)| \leqq \frac{1}{2}$ on the region of* (a). □

From the definition (2.16),

$$\tilde{F}_{\chi_1\chi_1}(p_0)=\frac{\tilde{G}_{\chi_2\chi_2}(p_0)}{1+\tilde{K}(p_0)\tilde{G}_{\chi_2\chi_2}(p_0)}=\frac{\tilde{H}_1(p_0)+\tilde{H}_2(p_0)}{1+\tilde{T}_1(p_0)+\tilde{T}_2(p_0)},$$

which can be written as

$$\tilde{F}_{\chi_1\chi_1}(p_0)=\frac{\tilde{H}_1(1+\tilde{T}_2)^{-2}}{[1+(1+\tilde{T}_2)^{-1}\tilde{T}_1]}+\tilde{H}_2(1+\tilde{T}_2)^{-1},$$

or more explicitly

$$\tilde{F}_{\chi_1\chi_1}(p_0)=\frac{v_2(m_2)(1+\tilde{T}_2)^{-2}}{\dfrac{\cosh m_2-\cos p_0}{\sinh m_2}+(1+\tilde{T}_2)^{-1}v_2(m_2)\tilde{K}}+\tilde{H}_2(1+\tilde{T}_2)^{-1}. \quad (2.18)$$

Thus, the singularities of $\tilde{F}_{\chi_1\chi_1}(p_0)$ on $|\mathrm{Re}\,p_0|\leqq\pi$, $0\leqq\mathrm{Im}\,p_0\leqq-7(1-\varepsilon)\log\beta$ are the same as the zeros of

$$f(p_0)=\frac{\cosh m_2-\cos p_0}{\sinh m_2}+v_2(m_2)(1+\tilde{T}_2)^{-1}\tilde{K}. \quad (2.19)$$

Notice that the entire function $g(p_0)=\dfrac{\cosh m_2-\cos p_0}{\sinh m_2}$ has only two zeros in the strip $|\mathrm{Re}\,p_0|\leqq\pi$, namely $p_0=\pm im_2$.

Theorem 2.9. *There exists $\beta_8\leqq\beta_7$ such that if $0<\beta<\beta_8$, $|v_2(m_2)(1+\tilde{T}_2)^{-1}\tilde{K}|\leqq\frac{1}{2}$ on $|\mathrm{Re}\,p_0|\leqq\pi$, $0\leqq\mathrm{Im}\,p_0\leqq-7(1-\varepsilon)\log\beta$, and $|g(p_0)|>1/2$ on the boundary of this region.* □

From Rouché's theorem, it then follows that

Corollary 2.10. *$\tilde{F}_{\chi_1\chi_1}(p_0)$ has precisely one simple pole on $|\mathrm{Re}\,p_0|<\pi$, $0<\mathrm{Im}\,p_0<-7(1-\varepsilon)\log\beta$, if $0<\beta<\beta_8$.* □

From (2.15), the pole of $\tilde{F}_{\chi_1\chi_1}(p_0)$ can lie only on the imaginary axis because, as follows from the arguments in [9], $\tilde{\Gamma}_{\chi\chi}(p_0)$ for $|\mathrm{Re}\,p_0|\leqq\pi$ is regular outside that axis. We denote the pole by $p_0=i\varrho$.

Theorem 2.11. *There are constants ζ and $\beta_9\leqq\beta_8$ such that if $0<\beta<\beta_9$, $|v_2(m_2)(1+\tilde{T}_2)^{-1}\tilde{K}|<|g|$ on the circle $|p_0-im_2|=\zeta\beta$.* □

Corollary 2.12. *If $0<\beta<\beta_9$, $|m_2-\varrho|<\zeta\beta$.* □

As remarked earlier, $p_0=i\varrho$ is expected to be a pole of $\tilde{\Gamma}_{\chi\chi}(p_0)$. To show that this is indeed the case, define

$$\tilde{L}_{\chi\chi_1}(p_0)=\tilde{\Gamma}_{\chi\chi}(p_0)\tilde{G}_{\chi\chi_1}(p_0)\tilde{\Phi}_{\chi_1\chi_1}(p_0), \quad (2.20)$$

and

$$\tilde{L}_{\chi_1\chi}(p_0)=\tilde{\Phi}_{\chi_1\chi_1}(p_0)\tilde{G}_{\chi_1\chi}(p_0)\tilde{\Gamma}_{\chi\chi}(p_0). \quad (2.21)$$

These two functions are actually the same because in general $\hat{G}_{\varphi\psi}(x_0)=\overline{\hat{G}_{\psi\varphi}(x_0)}$ and χ, χ_1 are real functions.

Theorem 2.13. *There exists a constant k_6 such that if $0<\beta<\beta_7$, $\tilde{L}_{\chi\chi_1}(p_0)$ is analytic on $|\mathrm{Re}\,p_0|\leqq\pi$, $0\leqq\mathrm{Im}\,p_0\leqq-7(1-\varepsilon)\log\beta$ and $|\tilde{L}_{\chi\chi_1}(p_0)|\leqq k_6\beta^{\frac{7}{2}\varepsilon}$ there.* □

Define $\tilde{M}(p_0)$ by the equation

$$\tilde{\Gamma}_{\chi\chi}(p_0)=\tilde{L}_{\chi\chi_1}(p_0)\tilde{F}_{\chi_1\chi_1}(p_0)\tilde{L}_{\chi_1\chi}(p_0)+\tilde{M}(p_0). \tag{2.22}$$

Theorem 2.14. *$\tilde{M}(p_0)$ is analytic on $|\mathrm{Re}\,p_0|\leqq\pi$, $0\leqq\mathrm{Im}\,p_0\leqq-7(1-\varepsilon)\log\beta$ if $0<\beta<\beta_7$.*

From this last theorem and (2.22), we see that the only possible singularity on $|\mathrm{Re}\,p_0|\leqq\pi$, $0\leqq\mathrm{Im}\,p_0\leqq-7(1-\varepsilon)\log\beta$ *of* $\tilde{\Gamma}_{\chi\chi}(p_0)$ is $p_0=i\varrho$. To show that this is indeed a singularity, we must check that $\tilde{L}_{\chi\chi_1}(i\varrho)\neq0$.

Theorem 2.15. *There exists $\beta_{10}\leqq\beta_9$ such that for $0<\beta<\beta_{10}$, $\tilde{L}_{\chi\chi_1}(p_0)\neq0$ if $|p_0-im_2|\leqq1$.* □

We are now in position to find the zeros of $\tilde{\Gamma}_{\chi\chi}(p_0)$. Inserting (2.18), (2.19) into (2.22), we get

$$\tilde{\Gamma}_{\chi\chi}=\frac{1}{f}[v_2(m_2)\tilde{L}_{\chi\chi_1}(1+\tilde{T}_2)^{-2}\tilde{L}_{\chi_1\chi}+(\tilde{L}_{\chi\chi_1}\tilde{H}_2(1+\tilde{T}_2)^{-1}\tilde{L}_{\chi_1\chi}+\tilde{M})f]. \tag{2.23}$$

From Theorem 2.6 we know that $\tilde{\Gamma}_{\chi\chi}(im_0)=0$ and this is the only zero on $0\leqq\mathrm{Im}\,p_0\leqq-6(1-\varepsilon)\log\beta$. Thus, to find new zeros there is no loss of generality in restricting p_0 to $|\mathrm{Re}\,p_0|\leqq\pi$, $-5\log\beta\leqq\mathrm{Im}\,p_0\leqq-7(1-\varepsilon)\log\beta$.

Theorem 2.16. *There are constants k_7, $\beta_{11}\leqq\beta_{10}$ such that if $0<\beta<\beta_{11}$ and $|\mathrm{Re}\,p_0|\leqq\pi$, $-5\log\beta\leqq\mathrm{Im}\,p_0\leqq-7(1-\varepsilon)\log\beta$,*

$$|\tilde{M}+\tilde{L}_{\chi\chi_1}\tilde{H}_2(1+\tilde{T}_2)^{-1}\tilde{L}_{\chi_1\chi}|\geqq(k_7/\beta)\,. \quad \square$$

Thus, the zeros of $\tilde{\Gamma}_{\chi\chi}$ in the region of Theorem 2.16 are the same as those for the function

$$f_1=f+\frac{v_2(m_2)\tilde{L}_{\chi\chi_1}(1+\tilde{T}_2)^{-2}\tilde{L}_{\chi_1\chi}}{(\tilde{L}_{\chi\chi_1}\tilde{H}_2(1+\tilde{T}_2)^{-1}\tilde{L}_{\chi_1\chi}+\tilde{M})}=g+h$$

with f and $g(p_0)$ as before [see definition after Eq. (2.19)] and

$$h(p_0)=v_2(m_2)\,(1+\tilde{T}_2)^{-1}\tilde{K}+\frac{v_2(m_2)\tilde{L}_{\chi\chi_1}(1+\tilde{T}_2)^{-2}\tilde{L}_{\chi_1\chi}}{\tilde{L}_{\chi\chi_1}\tilde{H}_2(1+\tilde{T}_2)^{-1}\tilde{L}_{\chi_1\chi}+\tilde{M}}.$$

Theorem 2.17. *There exists $\beta_{12}\leqq\beta_{11}$ such that for $0<\beta<\beta_{12}$, $|h(p_0)|\leqq1/2$ on $|\mathrm{Re}\,p_0|\leqq\pi$, $-5\log\beta\leqq\mathrm{Im}\,p_0\leqq-7(1-\varepsilon)\log\beta$ and $|g(p_0)|>1/2$ on the boundary of the same region.* □

Because of (2.11), we know that $g(p_0)$ has precisely one zero in the interior of the region above. Therefore, from Rouché's theorem, the same is true for $\tilde{\Gamma}_{\chi\chi}(p_0)$. Call this zero $p_0=im_1$. We have a result analogous to Theorem 2.11.

Theorem 2.18. *There are constants η and $\beta_{13}\leqq\beta_{12}$ such that if $0<\beta<\beta_{13}$, $|h(p_0)|<|g(p_0)|$ on $|p_0-im_2|=\eta\beta$.* □

Corollary 2.19. *If* $0<\beta<\beta_{13}$, $|m_1-m_2|<\eta\beta$. □

The theorems above prove that on $|\mathrm{Re}\,p_0|\leqq\pi$, $0\leqq\mathrm{Im}\,p_0<-7(1-\varepsilon)\log\beta$, $\tilde{G}_{\chi\chi}(p_0)$ has precisely two simple poles, at $p_0=im_0$ and $p_0=im_1$. The next result shows that this is also the case for $\tilde{G}_{\chi_1\chi_1}$:

Theorem 2.20. *There exists* $\beta_{14}\leqq\beta_{13}$ *such that if* $0<\beta<\beta_{14}$, $\tilde{G}_{\chi_1\chi_1}(p_0)$ *has exactly two simple poles on* $|\mathrm{Re}\,p_0|\leqq\pi$, $0\leqq\mathrm{Im}\,p_0<-7(1-\varepsilon)\log\beta$, *namely* $p_0=im_0$ *and* $p_0=im_1$. □

Finally, we extend this result to arbitrary $\tilde{G}_{\varphi_1\varphi_1}$:

Theorem 2.21. *If* $0<\beta<\beta_{14}$ *and* $\varphi=P_{\mathbf{a}}^{(1)}\varphi$ *for some* **a**, $\tilde{G}_{\varphi\varphi}(p_0)$ *is analytic on* $|\mathrm{Re}\,p_0|\leqq\pi$, $0\leqq\mathrm{Im}\,p_0<-7(1-\varepsilon)\log\beta$ *except possibly for simple poles at* $p_0=im_0$, $p_0=im_2$. □

3. Properties of Finite Volume Generalized Correlation Functions

The aim of this section is to calculate derivatives of

$$G_{\varphi\psi}(x,y;\Lambda;\{\beta_P\})=\langle\bar{\varphi}(x)\psi(y)\rangle(\Lambda;\{\beta_P\})-\langle\bar{\varphi}(x)\rangle\langle\psi(y)\rangle \tag{3.1}$$

with respect to β_P for a suitable choice of these variables. See the beginning of Sect. 2 for notation. This will be the main technical input to determine the domains of analyticity of the functions introduced in the last section. As usual, we assume φ, ψ of finite support in the time zero plane and e.g. $\varphi(x)$ denotes the translate of φ by the lattice vector $x=(x_0,\mathbf{x})$. We also assume the space-time dimensionality to be three and that the character χ in (2.1) is real irreducible. This assumption is responsible for the simple forms of Theorems 3.2 and 3.3 below. We use periodic boundary conditions in the space directions only, and without loss of generality, assume the region $\Lambda\subset\mathbb{Z}^3$ to be a rectangle along the coordinate axis. Setting all β_P equal to small β, the thermodynamic limit of (3.1) does not depend on this particular choice of boundary conditions, as follows easily from the cluster expansion [8].

The specific choice of $\{\beta_P\}$ variables we will adopt is the following. We set $\beta_P=w_q$ for all plaquettes P parallel to the time axis and located between $t=q$ and $t=q+1$. For plaquettes perpendicular to the time axis, we set $\beta_P=z$. In terms of duplicate variables, (3.1) can be written as

$$G_{\varphi\psi}(x,y;\Lambda)=\frac{1}{2Z_\Lambda^2}\iint(\bar{\varphi}(x)-\bar{\varphi}'(x))(\psi(y)-\psi'(y))e^{S_\Lambda(g,g')}dg_\Lambda dg'_\Lambda\equiv\frac{\mathcal{S}}{2\mathcal{D}}, \tag{3.2}$$

where e.g. φ' is φ at g' and

$$S_\Lambda(g,g')=\sum_q w_q\sum_{P\in\mathcal{P}''_q}(\chi(g_P)+\chi(g'_P))+z\sum_{P\in\mathcal{P}^\perp}(\chi(g_P)+\chi(g'_P)). \tag{3.3}$$

The sum over q is only for times defined within Λ, $\mathcal{P}''_q$ is the set of plaquettes in Λ parallel to the time direction and located between the time layers $t=q$, $t=q+1$, and $\mathcal{P}^\perp$ is the set of plaquettes in Λ perpendicular to the time direction. Also,

$$Z_\Lambda^2=\mathcal{D}=\iint e^{S_\Lambda(g,g')}dg_\Lambda dg'_\Lambda. \tag{3.4}$$

We want to find the structure of the coefficients of the power series expansion of (3.2) in a particular variable w_q. This has been done in [5, 6] up to the coefficient of fifth order. For the present work, we need the coefficient of the sixth order as well.

Write (3.3) in the form

$$S_\Lambda(g,g')=w_q \sum_{P\in\mathscr{P}''_q}(\chi(g_P)+\chi(g'_P))+S^{(q)}_\Lambda(g,g'), \tag{3.5}$$

and let $G(\Lambda)$ be the matrix whose elements are $G(x,y;\Lambda)$.

We will always assume $|z|$, $|w_q|<\beta_0$ so that the results of the cluster expansion applies to $G_{\varphi\psi}(x,y;\Lambda)$.

Theorem 3.1.

(a) $(\partial^m/\partial w_q^m)|_{w_q=0}G(\Lambda)=0$ *if* $1\leqq m\leqq 3$,

(b) $(\partial^m/\partial w_q^m)|_{w_q=0}G(x,y;\Lambda)=0$ *if* $x_0\leqq q<y_0$ *and* $0\leqq m\leqq 3$.

Proof. The numerator $\mathscr{S}$ of (3.2) has the expansion

$$\mathscr{S}=\sum_{n=0}^{\infty}\frac{1}{n!}\mathscr{S}^{(q)}_n(x,y)w_q^n, \tag{3.6}$$

with

$$\mathscr{S}^{(q)}_n(x,y)=\iint(\bar\varphi(x)-\bar\varphi(x))(\psi(y)-\psi'(y))e^{S^{(q)}_\Lambda(g,g')}\Big(\sum_{P\in\mathscr{P}''_q}(\chi(g_P)+\chi(g'_P))\Big)^n dg_\Lambda dg'_\Lambda. \tag{3.7}$$

To calculate (3.7) we use the Peter-Weyl orthogonality theorem together with the fact that $(\bar\varphi(x)-\bar\varphi'(x))(\psi(y)-\psi'(y))\exp(S^{(q)}_\Lambda(g,g'))$ do not depend on the bond variables parallel to the time direction and located between the time layers $t=q$, $t=q+1$. Thus, $\mathscr{S}^{(q)}_1(x,y)=0$. When $n=2$, it is clear that

$$\begin{aligned}\mathscr{S}^{(q)}_2(x,y)&=\sum_{P\in\mathscr{P}''_q}\iint(\bar\varphi(x)-\bar\varphi'(x))(\psi(y)-\psi'(y))e^{S^{(q)}_\Lambda}(\chi(g_P)^2+\chi(g'_P)^2)dg_\Lambda dg'_\Lambda\\&=\sum_{P\in\mathscr{P}''_q}2\mathscr{S}^{(q)}_0(x,y)=2\mathscr{A}(\Lambda)\mathscr{S}^{(q)}_0(x,y),\end{aligned} \tag{3.8}$$

where $\mathscr{A}(\Lambda)$ is the number of plaquettes in $\mathscr{P}''_q$ (which is just four times the area of a time layer). Similarly, when $n=3$, the only terms giving a nonzero contribution in the expansion of the third power in (3.7) are $\sum_P \chi(g_P)^3$ and $\sum_P \chi(g'_P)^3$. Therefore,

$$\mathscr{S}^{(q)}_3(x,y)=2\alpha\mathscr{A}(\Lambda)\mathscr{S}^{(q)}_0(x,y), \tag{3.9}$$

where $\alpha=\int\chi(g)^3dg$. The denominator in (3.2) has of course a similar expansion:

$$\mathscr{D}=\sum_{n=0}^{\infty}\frac{1}{n!}\mathscr{D}^{(q)}_n w_q^n, \tag{3.10}$$

with

$$\mathscr{D}^{(q)}_1=0,\quad \mathscr{D}^{(q)}_2=2\mathscr{A}(\Lambda)\mathscr{D}^{(q)}_0\quad\text{and}\quad \mathscr{D}^{(q)}_3=2\alpha\mathscr{A}(\Lambda)\mathscr{D}^{(q)}_0. \tag{3.11}$$

From this, it follows immediately that

$$G_{\varphi\psi}(x,y;\Lambda)=G_{\varphi\psi}(x,y;\Lambda)|_{w_q=0}+O(w_q^4), \tag{3.12}$$

which proves part (a) of the theorem. Part (b) also follows from (3.12), since if $x_0\leqq q\leqq y_0$, then clearly $G_{\varphi\psi}(x,y;\Lambda)|_{w_q=0}=0$. □

The next theorem was established in [5, 6]. Its simple proof is reproduced here for completeness. Let $\hat{e}_0$ denote the unit vector in the time direction.

Theorem 3.2. *Assume* $x_0 \leqq q \leqq y_0$. *There are constants* $c_4 > 0$ *and* c_5 *such that* $(m=4,5)$

$$\frac{\partial^m}{\partial w_q^m} G_{\varphi\psi}(x,y;\Lambda)|_{w_q=0} = c_m \sum_{t_0=q} G_{\varphi\chi}(x,t;\Lambda) G_{\chi\psi}(t+\hat{e}_0,y;\Lambda)|_{w_q=0}\,.$$

Proof. Since $\mathscr{S}_n^{(q)}(x,y)=0$ from $0 \leqq n \leqq 3$, we have $(m=4,5)$

$$\frac{\partial^m}{\partial w_2^m} G_{\varphi\psi}(x,y;\Lambda)|_{w_q=0} = \frac{1}{2}\,\frac{\mathscr{S}_m^{(q)}(x,y)}{\mathscr{D}_0^{(q)}}\,.$$

Expanding the fourth power in (3.7) with $n=4$, one shows that the only nonvanishing contributions come from terms of the form $\prod_{i=1}^{4} \chi(g_{P_i})$ or $\prod_{i=1}^{4} \chi(g'_{P_i})$ with the plaquettes $\{P_i\}_{i=1}^4$ disposed along the faces of cube. In this case, the product of characters can be integrated explicitly, yielding

$$\int \prod_{i=1}^{4} \chi(g_{P_i}) \prod_{\ell=1}^{4} dg_\ell = \frac{1}{d^4} \chi(g_{P_5})\chi(g_{P_6})\,. \tag{3.13}$$

The integration above is performed only with respect to bond variables parallel to the time direction, and P_5, P_6 are the remaining faces (which are perpendicular to the time direction) of the cube under consideration; d is the dimension of the representation defining χ. Thus, $\mathscr{S}_4^{(q)}(x,y)$ has the form

$$\mathscr{S}_4^{(q)}(x,y) = c_4 \sum_{t_0=q} \iint (\bar{\varphi}(x) - \bar{\varphi}'(x))(\psi(y)-\psi'(y))$$
$$\cdot e^{S_\Lambda^{(q)}} (\chi(t)\chi(t+\hat{\ell}_0) + \chi'(t)\chi'(t+\hat{\ell}_0)) dg_\Lambda dg'_\Lambda\,,$$

where c_4 is just a combinatorial factor [divided by d^4 from (3.13)], which counts the number of terms arising from (3.7) associated to a fixed elementary cube in the time slice $t=q$, $t=q+1$. This proves the theorem when $m=4$.

When $m=5$ we proceed as above with $\mathscr{S}_5^{(q)}(x,y)$. Expanding the fifth power in (3.7), nonvanishing contributions come only from terms like $\prod_{i=1}^{5} \chi(g_{P_i})$ or $\prod_{i=1}^{5} \chi(g'_{P_i})$ with the five plaquettes disposed along the faces of a cube. The side carrying a χ^2 term can be expanded in a Fourier series and the only term in this expansion giving a non-zero contribution to the integral $\int \prod_{i=1}^{5} \chi(g_{P_i}) \prod_{\ell=1}^{4} dg_\ell$ comes from the component along χ. Thus, the result is proportional to (3.13) and the theorem is proved also for $m=5$. □

The sixth derivative of $G_{\varphi\psi}(x,y;\Lambda)$ has a richer structure:

Theorem 3.3. *There are constants* c_6, c_7 $(c_7>0)$ *such that if* $x_0 \leqq q < y_0$, *then*

$$\frac{\partial^6}{\partial w_q^6} G_{\varphi\psi}(x,y;\Lambda)|_{w_q=0} = c_6 \sum_{t_0=q} G_{\varphi\chi}(x,t;\Lambda) G_{\chi\psi}(t+\hat{e}_0,y;\Lambda)|_{w_q=0}$$
$$+ c_7 \sum_{i=1}^{4} \sum_{t_0=q} G_{\varphi\chi_i}(x,t;\Lambda) G_{\chi_i\psi}(t+\hat{e}_0,y;\Lambda)|_{w_q=0}\,,$$

where $\chi_i = P_{\mathbf{0}}^{(i)}\chi_h$, $1 \leqq i \leqq 4$ *were defined in Sect. 2.*

Proof. We have

$$\frac{\partial^6}{\partial w_q^6} G_{\varphi\psi}(x, y; \Lambda)|_{w_q=0} = \frac{\mathcal{S}_6^{(q)}(x, y)}{2\mathcal{D}_0^{(q)}} - \frac{6!}{4!} \mathcal{A}(\Lambda) \frac{\mathcal{S}_4^{(q)}(x, y)}{2\mathcal{D}_0^{(q)}},$$

with

$$\mathcal{S}_6^{(q)}(x, y) = \sum_{P_\iota \in \mathcal{P}_q''} \iint (\bar{\varphi}(x) - \bar{\varphi}'(x))(\psi(y) - \psi'(y)) \cdot e^{S_\Lambda^{(q)}} \prod_{i=1}^{6} (\chi(g_{P_i}) + \chi(g'_{P_i})) dg_\Lambda dg'_\Lambda. \tag{3.14}$$

Expanding the product we see that cross terms of the form $\chi^2\chi'^4$ or $\chi^4\chi'^2$ give nonvanishing contributions proportional to $\mathcal{A}(\Lambda)\mathcal{S}_4^{(q)}(x, y)/(2\mathcal{D}_0^{(q)})$. Also the "pure" terms χ^6, χ'^6 have contributions proportional to $\mathcal{A}(\Lambda)\mathcal{S}_4^{(q)}(x, y)/(2\mathcal{D}_0^{(q)})$. This happens when the term is of the form $\chi^2(g_P) \prod_{i=1}^{4} \chi(g_{P_i})$ with $\{P_i\}$ disposed along the faces of a cube and P different from the P_i's. It is not difficult to show that the terms proportional to $\mathcal{A}(\Lambda)\mathcal{S}_4^{(q)}(x, y)/(2\mathcal{D}_0^{(q)})$ add up to zero. This is to be expected, since from the cluster expansion, $G_{\varphi\psi}(x, y; \Lambda; \{w_q\}, z)$ has a limit as $\Lambda \to \mathbb{Z}^3$, which is analytic in each variable (the limit is attained uniformly on compact subsets of $|w_q|$, $|z| < \beta_0$). Then $\lim_{\Lambda \to \mathbb{Z}^3} (\partial^6/\partial w_q^6)|_{w_q=0} G_{\varphi\psi}(x, y; \Lambda)$ exists, showing that extensive terms must cancel in the expression for the sixth derivative. There are still contributions proportional to $\mathcal{S}_4^{(q)}/(2\mathcal{D}_0^{(q)})$. This happens when all six plaquettes in $\prod_{i=1}^{6} \chi(g_{P_i})$ are disposed along the sides of a cube. We then proceed as in the $\mathcal{S}_5^{(q)}$ case by Fourier transforming the plaquettes containing χ with a power higher than one. The sum of all these terms gives a contribution to $(\partial^6/\partial w_q^6)|_{w_q=0} G_{\varphi\psi}(x, y; \Lambda)$ of the form $c_6 \sum_{t_0=q} G_{\varphi\chi}(x, t; \Lambda) G_{\chi\psi}(t + \hat{e}_0, y; \Lambda)|_{w_q=0}$ for a suitable constant c_6.

The other contribution from $\mathcal{S}_6^{(q)}(x, y)$ come when the six plaquettes in $\prod_{i=1}^{6} \chi(g_{P_i})$ are disposed along the faces of a parallelepiped. The calculation of such terms is similar to the one corresponding to $\mathcal{S}_4^{(q)}(x, y)$. Let χ_h and χ_v be the horizontal and vertical windows at the origin in the time zero plane. χ_h was defined after Theorem 2.1, and $\chi_v = R_0 \chi_h$. The contribution of these terms to $\mathcal{S}_6^{(q)}$ is given by

$$c'_7 \sum_{t_0=q} \iint (\bar{\varphi}(x) - \bar{\varphi}'(x))(\psi(y) - \psi'(y)) e^{S_\Lambda^{(q)}} (\chi_h(t)\chi_h(t + \hat{e}_0) + \chi_v(t)\chi_v(t + \hat{e}_0) + \chi'_h(t)\chi'_h(t + \hat{e}_0) + \chi'_v(t)\chi'_v(t + \hat{e}_0)) dg_\Lambda dg'_\Lambda,$$

where c'_7 is a combinatorial factor (divided by d^6) analogous to c_4 and therefore positive. Thus, the final expression for the sixth derivative is

$$\frac{\partial^6}{\partial w_q^6} G_{\varphi\psi}(x, y; \Lambda)|_{w_q=0} = c_6 \sum_{t_0=q} G_{\varphi\chi}(x, t; \Lambda) G_{\chi\psi}(t + \hat{e}_0, y; \Lambda)|_{w_q=0} + c'_7 \sum_{t_0=q} [G_{\varphi\chi_h}(x, t; \Lambda) G_{\chi_h\psi}(t + \hat{e}_0, y; \Lambda) + G_{\varphi\chi_v}(x, t; \Lambda) G_{\chi_v\psi}(t + \hat{e}_0, y; \Lambda)]|_{w_q=0}.$$

This can be reexpressed in terms of χ_i, $1 \leqq i \leqq 4$ by using the identity

$$\sum_{i=1}^{4} G_{\varphi\chi_i}(x,t;\Lambda)G_{\chi_i\psi}(t+\hat{e}_0,y;\Lambda)=\tfrac{1}{4}[G_{\varphi\chi_h}(x,t;\Lambda)G_{\chi_h\psi}(t+\hat{e}_0,y;\Lambda)$$
$$+G_{\varphi\chi_v}(x,t;\Lambda)G_{\chi_v}(t+\hat{e}_0,y;\Lambda)+G_{\varphi\chi_h}(x,t-\hat{e}_1;\Lambda)G_{\chi_h\psi}(t-\hat{e}_1+\hat{e}_0,y;\Lambda)$$
$$+G_{\varphi\chi_v}(x,t-\hat{e}_2;\Lambda)G_{\chi_v\psi}(t-\hat{e}_2+\hat{e}_0,y;\Lambda)],$$

where $\hat{e}_1$, $\hat{e}_2$ are the unit vectors along the horizontal and vertical directions, respectively. Using the periodicity in the space directions and setting $c_7=2c'_7$ completes the proof of the theorem. □

The partial Fourier transform (with respect to the space variables) at zero momentum is defined by

$$\hat{G}_{\varphi\psi}(x_0,y_0;\Lambda)=\sum_{\mathbf{y}} G_{\varphi\psi}(x,y;\Lambda), \tag{3.15}$$

the sum being clearly independent of $\mathbf{x}$ due to spatial translation invariance of the finite volume interacting measure (with complex parameters $\{w_q\}, z$). Let $\hat{G}_{\varphi\psi}(\Lambda)$ be the matrix whose elements are $\hat{G}_{\varphi\psi}(x_0,y_0;\Lambda)$. We summarize the results obtained in this section in the following

Theorem 3.4.
(a) $(\partial^m/\partial w_q^m)|_{w_q=0}\hat{G}_{\varphi\psi}(\Lambda)=0$ *if* $1\leqq m\leqq 3$.
(b) *If* $x_0\leqq q<y_0$, *then*

$$(\partial^m/\partial w_q^m)|_{w_q=0}\hat{G}_{\varphi\psi}(x_0,y_0;\Lambda)=0, \qquad 0\leqq m\leqq 3.$$

(c) *There are constants* $c_4>0$ *and* c_5 *such that if* $x_0\leqq q<y_0$, *then*

$$(\partial^m/\partial w_q^m)|_{w_q=0}\hat{G}_{\varphi\psi}(x_0,y_0;\Lambda)=c_m\hat{G}_{\varphi\chi}(x_0,q;\Lambda)\hat{G}_{\chi\psi}(q+1,y_0;\Lambda)|_{w_q=0}, \qquad m=4,5.$$

(d) *There are constants* c_6, $c_7>0$ *such that for* $x_0\leqq q<y_0$,

$$\frac{\partial^6}{\partial w_q^6}\hat{G}_{\varphi\psi}(x_0,y_0;\Lambda)|_{w_q=0}=c_6\hat{G}_{\varphi\chi}(x_0,q;\Lambda)\hat{G}_{\chi\psi}(q+1,y_0;\Lambda)|_{w_q=0}$$
$$+c_7\sum_{i=1}^{2}\hat{G}_{\varphi\chi_i}(x_0,q;\Lambda)\hat{G}_{\chi_i\psi}(q+1,y_0;\Lambda)|_{w_q=0}.$$

Proof. We have only to explain why there is no contribution due to χ_i, $i=3,4$ in the last sum in (d). This is because, e.g.

$$\hat{G}_{\varphi\chi_3}(x_0,y_0;\Lambda)=\sum_{\mathbf{y}}G_{\varphi\chi_3}(x,y;\Lambda)=\tfrac{1}{4}\sum_{\mathbf{y}}[G_{\varphi\chi_h}(x,y;\Lambda)$$
$$+iG_{\varphi\chi_v}(x,y;\Lambda)-G_{\varphi\chi_h}(x,y-\hat{e}_1;\Lambda)-iG_{\varphi\chi_v}(x,y-\hat{e}_2;\Lambda)]=0. \quad □$$

4. Proof of Theorems in Sect. 2

In this section, the missing proofs of the results stated in Sect. 2 will be presented. We start with the observation that an equation analogous to (2.9) holds even for finite volume correlation functions depending on the complex parameters $\{w_q\}, z$.

Namely,

$$G_{\varphi\psi}((x_0,\mathbf{x}),(y_0,\mathbf{y});\Lambda)=G_{R_\mathbf{a}\varphi,R_\mathbf{a}\psi}((x_0,R_\mathbf{a}\mathbf{x}),(y_0,R_\mathbf{a}\mathbf{y});\Lambda).$$

The reason is that the finite volume interacting measure (with complex parameters $\{w_q\},z)$ is invariant under a rotation of $\pi/2$ around an axis parallel to the time direction and through the center of a plaquette. Thus, the finite volume partial Fourier transforms (3.15) satisfy $\hat{G}_{\varphi\psi}(x_0,y_0;\Lambda)=\hat{G}_{R_\mathbf{a}\varphi,R_\mathbf{a}\psi}(x_0,y_0;\Lambda)$ and by the same argument as in Sect. 2,

$$\hat{G}_{\varphi\psi}(x_0,y_0;\Lambda)=\hat{G}_{R_\mathbf{a}\varphi,R_\mathbf{b}\psi}(x_0,y_0;\Lambda)$$

for arbitrary **a**, **b**. Also, from the cluster expansion, if $|w_q|,|z|<\beta_0$, $\sum_y|G_{\varphi\psi}(x,y;\Lambda)|$ is bounded independent of x and Λ. We call this bound $C_{\varphi\psi}$ [(2.7) thus follows from this fact]. Setting all w_q,z equal to the same β, $0<\beta<\beta_0$ we have $\lim_{\Lambda\to\mathbb{Z}^3}\hat{G}_{\varphi\psi}(x_0,y_0;\Lambda)=\hat{G}_{\varphi\psi}(y_0-x_0)$ defined by (2.4).

Proof of Theorem 2.1. If $\varphi=P_\mathbf{a}^{(i)}\varphi$ with $i=3$ or 4, then $\hat{G}_{\chi\varphi}(x_0,y_0;\Lambda)=G_{\varphi\chi_i}(x_0,y_0;\Lambda)=0$ for $i=1,2$. Hence, from Theorem 3.4, if $x_0\leqq q<y_0$, $(\partial^m/\partial w_q^m)|_{w_q=0}\hat{G}_{\varphi\varphi}(x_0,y_0;\Lambda)=0$ for $0\leqq m\leqq 6$. From the maximum modulus theorem, it then follows that $|\hat{G}_{\varphi\varphi}(x_0,y_0;\Lambda)|\leqq C_{\varphi\varphi}\prod_{x_0\leqq q<y_0}|w_q/\beta_0|^7$. Setting all w_q, z equal to β, $0<\beta<\beta_0$ and letting $\Lambda\to\mathbb{Z}^3$, we get $|\hat{G}_{\varphi\varphi}(x_0)|\leqq C_{\varphi\varphi}(\beta/\beta_0)^{7|x_0|}$. This implies $\hat{G}_{\varphi\varphi}(p_0)$ to be analytic on $|\mathrm{Im}\,p_0|<-7\log(\beta/\beta_0)$. Choosing $\beta<\beta_1\equiv\beta_0^{2/\varepsilon}$ guarantees that this region contains $|\mathrm{Im}\,p_0|<-7\left(1-\frac{\varepsilon}{2}\right)\log\beta$. □

Proof of Theorem 2.2. From Theorem 3.4, we have if $x_0\leqq q<y_0$:

$$\frac{\partial^m}{\partial w_q^m}\hat{G}_{\chi_2\chi_2}(x_0,y_0;\Lambda)|_{w_q=0}=0\quad\text{for}\quad 0\leqq m\leqq 5,\tag{4.1}$$

and

$$\frac{\partial^6}{\partial w_q^6}\hat{G}_{\chi_2\chi_2}(x_0,y_0;\Lambda)|_{w_q=0}=c_7\hat{G}_{\chi_2\chi_2}(x_0,q;\Lambda)\hat{G}_{\chi_2\chi_2}(q+1,y_0;\Lambda)|_{w_q=0}.\tag{4.2}$$

As in the proof of Theorem 2.1, (4.1) implies that for $0<\beta<\beta_0$,

$$|\hat{G}_{\chi_2\chi_2}(x_0)|\leqq C_{\chi_2\chi_2}(\beta/\beta_0)^{6|x_0|}.\tag{4.3}$$

On the other hand, setting $x_0=0$, $y_0=1$ in (4.2) yields

$$\hat{G}_{\chi_2\chi_2}(0,1;\Lambda;\{w_q\},z)=\frac{c_7}{6!}\hat{G}_{\chi_2\chi_2}(0,0;\Lambda;\{w_q\}_{q\neq 0},z)\cdot\hat{G}_{\chi_2\chi_2}(1,1;\Lambda;\{w_q\}_{q\neq 0},z)w_0^6+O(w_0^7),$$

from which we get

$$\lim_{\beta\downarrow 0}\frac{1}{\beta^6}\hat{G}_{\chi_2\chi_2}(0,1;\Lambda;\beta)=\frac{c_7}{6!}\hat{G}_{\chi_2\chi_2}(0,0;\Lambda;\beta=0)\hat{G}_{\chi_2\chi_2}(1,1;\Lambda;\beta=0).$$

The right hand side is easily calculable, since

$$\hat{G}_{\chi_2\chi_2}(0,0;\Lambda;\beta=0)=\hat{G}_{\chi_2\chi_2}(1,1;\Lambda;\beta=0)=1/2.$$

Letting $\Lambda\to\mathbb{Z}^3$, we conclude that

$$\lim_{\beta\to 0}\beta^{-6}\hat{G}_{\chi_2\chi_2}(x_0=1)=c_7/(4\cdot 6!).$$

Thus, there exists $\beta_2\leqq\beta_1$ such that

$$\hat{G}_{\chi_2\chi_2}(x_0=1)\geqq(c_7\beta^6)/(8\cdot 6!)$$

if $0<\beta<\beta_2$. From the integral representation [arising from (2.8)],

$$\hat{G}_{\chi_2\chi_2}(x_0)=\int_{(0,\infty)} e^{-\lambda_0|x_0|}dv_2(\lambda_0)$$

(with dv_2 a positive, finite measure) and Hölder's inequality, we have ($x_0\neq 0$)

$$\hat{G}_{\chi_2\chi_2}(x_0=1)^{|x_0|}\leqq\hat{G}_{\chi_2\chi_2}(x_0)\hat{G}_{\chi_2\chi_2}(x_0=0)^{|x_0|-1}\leqq\hat{G}_{\chi_2\chi_2}(x_0)C_{\chi_2\chi_2}^{|x_0|-1},$$

so that

$$\hat{G}_{\chi_2\chi_2}(x_0)\geqq C_{\chi_2\chi_2}\left(\frac{c_7}{C_{\chi_2\chi_2}\cdot 8\cdot 6!}\beta^6\right)^{|x_0|}. \tag{4.4}$$

The proof of Theorem 2.2 follows from (4.3) and (4.4). □

Proof of Theorem 2.3. We first show that the matrix $\hat{G}_{\chi_2\chi_2}(\Lambda)$ is invertible if $|w_q|$, $|z|$ are small enough. Let $P_{\chi_2\chi_2}(\Lambda)$ be the diagonal part of $\hat{G}_{\chi_2\chi_2}(\Lambda)$, i.e. $P_{\chi_2\chi_2}(x_0,y_0;\Lambda)=\hat{G}_{\chi_2\chi_2}(x_0,y_0;\Lambda)\delta_{x_0,y_0}$, and let $Q_{\chi_2\chi_2}(\Lambda)$ be defined by $\hat{G}_{\chi_2\chi_2}(\Lambda)=P_{\chi_2\chi_2}(\Lambda)+Q_{\chi_2\chi_2}(\Lambda)$. From the fundamental theorem of calculus,

$$\hat{G}_{\chi_2\chi_2}(x_0,x_0;\Lambda;\{w_q\},z)=\tfrac{1}{2}+\int_0^1 d\lambda\frac{d}{d\lambda}\hat{G}_{\chi_2\chi_2}(x_0,x_0;\Lambda;\{\lambda w_q\},\lambda z).$$

Restricting $\{w_q\}$, z to $|w_q|$, $|z|<\beta_0'\equiv\beta_0/N$ (N to be determined shorly), the derivative above can be estimated by Cauchy's formula

$$\frac{d}{d\lambda}\hat{G}_{\chi_2\chi_2}(x_0,x_0;\Lambda;\{\lambda w_q\},\lambda z)=\frac{1}{2\pi i}\oint_{|\eta|=N}\frac{\hat{G}_{\chi_2\chi_2}(x_0,x_0;\Lambda;\{\lambda w_q\},\eta z)}{(\eta-\lambda)^2}d\eta.$$

Since $0\leqq\lambda\leqq 1$, we have

$$\left|\frac{d}{d\lambda}\hat{G}_{\chi_2\chi_2}(x_0,x_0;\Lambda;\{\lambda w_q\},\lambda z)\right|\leqq\frac{NC_{\chi_2\chi_2}}{(N-1)^2}.$$

We choose $N>1$ to be the smallest number for which the right-hand side above is less than or equal to 1/4. This implies $\|P_{\chi_2\chi_2}(\Lambda)\|\geqq(1/4)$ so that $P_{\chi_2\chi_2}(\Lambda)$ is invertible, and

$$\hat{G}_{\chi_2\chi_2}(\Lambda)=P_{\chi_2\chi_2}(\Lambda)[1+P_{\chi_2\chi_2}(\Lambda)^{-1}Q_{\chi_2\chi_2}(\Lambda)].$$

Using $|\hat{G}_{\chi_2\chi_2}(x_0,y_0;\Lambda)|\leqq C_{\chi_2\chi_2}\prod_{x_0\leqq q<y_0}|w_q/\beta_0|^6$, we estimate $\|Q_{\chi_2\chi_2}(\Lambda)\|$ as follows:

$$\|Q_{\chi_2\chi_2}(\Lambda)\|\leqq\max_{x_0}\sum_{y_0}|Q_{\chi_2\chi_2}(x_0,y_0;\Lambda)|\leqq C_{\chi_2\chi_2}\max_{x_0}\sum_{y_0\neq x_0}(\beta_1'/\beta_0)^{6|x_0-y_0|},$$

assuming $|w_q|$, $|z|<\beta_1'$. Thus

$$\|Q_{\chi_2\chi_2}(\Lambda)\| \leqq 2C_{\chi_2\chi_2}\frac{(\beta_1'/\beta_0)^6}{1-(\beta_1'/\beta_0)^6}.$$

Choose $\beta_1' \leqq \beta_0'$ such that $\|Q_{\chi_2\chi_2}(\Lambda)\| \leqq 1/8$. Then $\|P_{\chi_2\chi_2}(\Lambda)^{-1}Q_{\chi_2\chi_2}(\Lambda)\| \leqq 1/2$, and $\hat{G}_{\chi_2\chi_2}(\Lambda)$ is invertible. Calling the inverse $-\hat{\Gamma}_{\chi_2\chi_2}(\Lambda)$, we have $\|\hat{\Gamma}_{\chi_2\chi_2}(\Lambda)\| \leqq 8$ if $|w_q|$, $|z|<\beta_1'$. Since $\hat{G}_{\chi_2\chi_2}(x_0, y_0; \Lambda)|_{w_q=0}=0$ if $x_0 \leqq q < y_0$, the same is true for $\hat{\Gamma}_{\chi_2\chi_2}(x_0, y_0; \Lambda)|_{w_q=0}$. From Leibniz's formula,

$$\frac{\partial^m}{\partial w_q^m}\hat{\Gamma}_{\chi_2\chi_2}(\Lambda) = \sum_{n=0}^{m-1}\binom{m}{n}\frac{\partial^n}{\partial w_q^n}\hat{\Gamma}_{\chi_2\chi_2}(\Lambda)\frac{\partial^{m-n}}{\partial w_q^{m-n}}\hat{G}_{\chi_2\chi_2}(\Lambda)\hat{\Gamma}_{\chi_2\chi_2}(\Lambda)$$

and Theorem 3.4, it follows immediately that

$$\frac{\partial^m}{\partial w_q^m}\hat{\Gamma}_{\chi_2\chi_2}(\Lambda)|_{w_q=0}=0 \quad \text{if} \quad 1 \leqq m \leqq 3,$$

and

$$\frac{\partial^m}{\partial w_q^m}\hat{\Gamma}_{\chi_2\chi_2}(\Lambda)|_{w_q=0} = \hat{\Gamma}_{\chi_2\chi_2}(\Lambda)\frac{\partial^m}{\partial w_q^m}\hat{G}_{\chi_2\chi_2}(\Lambda)\hat{\Gamma}_{\chi_2\chi_2}(\Lambda)|_{w_q=0} \qquad (4 \leqq m \leqq 6).$$

Since $(\partial^m/\partial w_q^m)|_{w_q=0}\hat{G}_{\chi_2\chi_2}(x_0, y_0; \Lambda)=0$ if $x_0 \leqq q < y_0$ and $m=4,5$, the same is true for $(\partial^m/\partial w_q^m)|_{w_q=0}\hat{\Gamma}_{\chi_2\chi_2}(x_0, y_0; \Lambda)$. In addition, if $x_0 \leqq q < y_0$,

$$\begin{aligned}
&\frac{\partial^6}{\partial w_q^6}\hat{\Gamma}_{\chi_2\chi_2}(x_0, y_0; \Lambda)|_{w_q=0} \\
&= \sum_{\substack{u_0 \leqq q \\ v_0 > q}} \hat{\Gamma}_{\chi_2\chi_2}(x_0, u_0; \Lambda)\frac{\partial^6}{\partial w_q^6}\hat{G}_{\chi_2\chi_2}(u_0, v_0; \Lambda)\hat{\Gamma}_{\chi_2\chi_2}(v_0, y_0; \Lambda)|_{w_q=0} \\
&= c_7 \sum_{u_0, v_0} \hat{\Gamma}_{\chi_2\chi_2}(x_0, u_0; \Lambda)\hat{G}_{\chi_2\chi_2}(u_0, q; \Lambda)\hat{G}_{\chi_2\chi_2}(q+1, v_0; \Lambda) \\
&\quad \cdot \hat{\Gamma}_{\chi_2\chi_2}(v_0, y_0; \Lambda)|_{w_q=0} = c_7\delta(x_0, q)\delta(q+1, y_0).
\end{aligned} \tag{4.5}$$

From the maximum modulus theorem,

$$|\hat{\Gamma}_{\chi_2\chi_2}(x_0, y_0; \Lambda)| = 8 \prod_{x_0 \leqq q < y_0} |w_q/\beta_1'|^7 \quad \text{if} \quad |x_0 - y_0| > 1,$$

and letting all w_q, z be equal to β, $0<\beta<\beta_1'$:

$$|\hat{\Gamma}_{\chi_2\chi_2}(x_0, y_0; \Lambda)| \leqq 8(\beta/\beta_1')^{7|x_0-y_0|} \quad \text{if} \quad |x_0 - y_0| > 1.$$

It is not hard to show that $\lim_{\Lambda \to \mathbb{Z}^3} \hat{\Gamma}_{\chi_2\chi_2}(x_0, y_0; \Lambda) \equiv \hat{\Gamma}_{\chi_2\chi_2}(y_0 - x_0)$ exists and is the convolution inverse of $-\hat{G}_{\chi_2\chi_2}(x_0)$. We have $|\hat{\Gamma}_{\chi_2\chi_2}(x_0)| \leqq 8(\beta/\beta_1')^{7|x_0|}$ if $|x_0|>1$, so that $\hat{\Gamma}_{\chi_2\chi_2}(p_0)$ is analytic on $|\mathrm{Im}\, p_0| \leqq -7\left(1-\frac{\varepsilon}{2}\right)\log\beta$ if $\beta < \beta_1'^{2/\varepsilon} \equiv \beta_3$. The proof is complete. □

Proof of Theorem 2.4. (a) We have $v_2(m_2)\sinh m_2/(\cosh m_2 - 1) \leqq \tilde{G}_{\chi_2\chi_2}(p_0=0) \leqq C_{\chi_2\chi_2}$, so that $v_2(m_2) \leqq C_{\chi_2\chi_2}\coth m_2$. Choose $\beta_2' \leqq \beta_0$ so small that $\coth m_2 < \sqrt{5}/2$ (for later convenience) for $0<\beta<\beta_2'$.

(b) From (2.14),

$$|\tilde{H}_2(p_0)| \leqq \int_{-7\left(1-\frac{\varepsilon}{2}\right)\log\beta}^{\infty} \frac{\sinh\lambda_0}{|\cosh\lambda_0 - \cos p_0|} dv_2(\lambda_0),$$

and $|\mathrm{Im}\,p_0| \leqq -7(1-\varepsilon)\log\beta$. Choose $\beta_3' \leqq \beta_0$ so small (depending on ε) that

$$\cosh[-7(1-\varepsilon)\log\beta] \leqq \tfrac{1}{2}\cosh\left[-7\left(1-\frac{\varepsilon}{2}\right)\log\beta\right]$$

for $0<\beta<\beta_3'$. Then,

$$|\cosh\lambda_0 - \cos p_0| \geqq \cosh\lambda_0 - \cosh\,\mathrm{Im}\,p_0 \geqq \tfrac{1}{2}\cosh\lambda_0 > \tfrac{1}{2}(\cosh\lambda_0 - 1)$$

if $\lambda_0 \geqq -7\left(1-\frac{\varepsilon}{2}\right)\log\beta$. Therefore $|\tilde{H}_2(p_0)| \leqq 2\tilde{G}_{\chi_2\chi_2}(p_0=0) \leqq 2C_{\chi_2\chi_2}$. Setting $\beta_4 = \min\{\beta_2', \beta_3', \beta_3\}$ completes the proof of the theorem. □

Proof of Theorem 2.5. Consider the finite volume approximation of $\hat{F}^{(2)}_{\varphi\varphi}(x_0)$. In matrix notation

$$\hat{F}^{(2)}_{\varphi\varphi}(\Lambda) = \hat{G}_{\varphi\varphi}(\Lambda) + \hat{G}_{\varphi\chi_2}(\Lambda)\hat{\Gamma}_{\chi_2\chi_2}(\Lambda)\hat{G}_{\chi_2\varphi}(\Lambda).$$

$\hat{F}^{(2)}_{\varphi\varphi}(x_0, y_0; \Lambda)$ is analytic on $|w_q|, |z| < \beta_1'$ (introduced in the proof of Theorem 2.3) and bounded there by $C_{\varphi\varphi} + 8C_{\varphi\chi_2}C_{\chi_2\varphi}$. Also, if all w_q, z are equal to β, $0<\beta<\beta_1'$, $\lim\limits_{\Lambda\to\mathbb{Z}^3} \hat{F}^{(2)}_{\varphi\varphi}(x_0, y_0; \Lambda) = \hat{F}^{(2)}_{\varphi\varphi}(y_0 - x_0)$. Clearly, if $x_0 \leqq q < y_0$, $\hat{F}^{(2)}_{\varphi\varphi}(x_0, y_0; \Lambda)|_{w_q=0} = 0$. From

$$\frac{\partial^m}{\partial w_q^m}\hat{F}^{(2)}_{\varphi\varphi}(\Lambda) = \frac{\partial^m}{\partial w_q^m}\hat{G}_{\varphi\varphi}(\Lambda) + \sum_{n_1+n_2+n_3=m} \frac{m!}{n_1!n_2!n_3!} \frac{\partial^{n_1}}{\partial w_q^{n_1}}\hat{G}_{\varphi\chi_2}(\Lambda)\frac{\partial^{n_2}}{\partial w_q^{n_2}}\hat{\Gamma}_{\chi_2\chi_2}(\Lambda)\frac{\partial^{n_3}}{\partial w_q^{n_3}}\hat{G}_{\chi_2\varphi}(\Lambda)$$

and Theorem 3.4, we see that $(\partial^m/\partial w_q^m)\hat{F}^{(2)}_{\varphi\varphi}(\Lambda)|_{w_q=0} = 0$ if $1 \leqq m \leqq 3$, and that

$$\begin{aligned}\frac{\partial^m}{\partial w_q^m}\hat{F}^{(2)}_{\varphi\varphi}(\Lambda)|_{w_q=0} &= \frac{\partial^m}{\partial w_q^m}\hat{G}_{\varphi\varphi}(\Lambda)|_{w_q=0} + \hat{G}_{\varphi\chi_2}(\Lambda)\hat{\Gamma}_{\chi_2\chi_2}(\Lambda)\frac{\partial^m}{\partial w_q^m}\hat{G}_{\chi_2\varphi}(\Lambda)|_{w_q=0}\\ &+ \hat{G}_{\varphi\chi_2}(\Lambda)\frac{\partial^m}{\partial w_q^m}\hat{\Gamma}_{\chi_2\chi_2}(\Lambda)\hat{G}_{\chi_2\varphi}(\Lambda)|_{w_q=0}\\ &+ \frac{\partial^m}{\partial w_q^m}\hat{G}_{\varphi\chi_2}(\Lambda)\hat{\Gamma}_{\chi_2\chi_2}(\Lambda)\hat{G}_{\chi_2\varphi}(\Lambda)|_{w_q=0},\end{aligned}$$

if $4 \leqq m \leqq 6$. Since $\varphi = P^{(2)}_{\mathbf{a}}\varphi$, Theorem 3.4 implies $(\partial^m/\partial w_q^m)|_{w_q=0}\hat{G}_{\varphi\varphi}(x_0, y_0; \Lambda) = 0$, if $0 \leqq m \leqq 5$ and $x_0 \leqq q < y_0$. Also $(\partial^2/\partial w_q^6)|_{w_q=0}\hat{G}_{\varphi\varphi}(x_0, y_0; \Lambda) = C_7\hat{G}_{\varphi\chi_2}(x_0, q; \Lambda) \cdot \hat{G}_{\chi_2\varphi}(q+1, y_0; \Lambda)|_{w_q=0}$, and a similar result holds for $\hat{G}_{\varphi\chi_2}$, $\hat{G}_{\chi_2\varphi}$. Hence, using the proof of Theorem 2.3 to calculate $(\partial^m/\partial w_q^m)|_{w_q=0}\hat{\Gamma}_{\chi_2\chi_2}(x_0, y_0; \Lambda)$, we find

$(\partial^m/\partial w_q^m)|_{w_q=0}\hat{F}^{(2)}_{\varphi\varphi}(x_0, y_0; \Lambda)=0$ for $0 \leqq m \leqq 5$ and $x_0 \leqq q < y_0$. In addition,

$$\frac{\partial^6}{\partial w_q^6}\hat{F}^{(2)}_{\varphi\varphi}(x_0, y_0; \Lambda)|_{w_q=0} = C_7 \hat{G}_{\varphi\chi_2}(x_0, q; \Lambda)\hat{G}_{\chi_2\varphi}(q+1, y_0; \Lambda)|_{w_q=0}$$
$$+ \sum_{\substack{u_0 \leqq q \\ v_0 \leqq q}} \hat{G}_{\varphi\chi_2}(x_0, u_0; \Lambda)\hat{\Gamma}_{\chi_2\chi_2}(u_0, v_0; \Lambda)C_7\hat{G}_{\chi_2\chi_2}(v_0, q; \Lambda)\hat{G}_{\chi_2\varphi}(q+1, y_0; \Lambda)|_{w_q=0}$$
$$+ \sum_{\substack{u_0 \leqq q \\ v_0 \leqq q}} \hat{G}_{\varphi\chi_2}(x_0, u_0; \Lambda)C_7\delta(u_0, q)\delta(q+1, v_0)\hat{G}_{\chi_2\varphi}(v_0, y_0; \Lambda)|_{w_q=0}$$
$$+ \sum_{\substack{u_0 > q \\ v_0 > q}} C_7\hat{G}_{\varphi\chi_2}(x_0, q; \Lambda)\hat{G}_{\chi_2\chi_2}(q+1, u_0; \Lambda)\hat{\Gamma}_{\chi_2\chi_2}(u_0, v_0; \Lambda)\hat{G}_{\chi_2\varphi}(v_0, y_0; \Lambda)|_{w_q=0}$$
$$=0.$$

Proceeding as in the proof of Theorem 2.1, we conclude that $\tilde{F}^{(2)}_{\varphi\varphi}(p_0)$ is analytic up to $|\mathrm{Im}\, p_0| < -7\log(\beta/\beta'_1)$, which includes the region $|\mathrm{Im}\, p_0| \leqq -7\left(1-\frac{\varepsilon}{2}\right)\log\beta$ if $\beta < \beta_1'^{2/\varepsilon} \equiv \beta_3$. The proof of the theorem is complete. □

Proof of Theorem 2.7. Consider the finite volume approximation to $\hat{F}_{\chi_1\chi_1}(x_0)$:

$$\hat{F}_{\chi_1\chi_1}(\Lambda) = \hat{G}_{\chi_1\chi_1}(\Lambda) + \hat{G}_{\chi_1\chi}(\Lambda)\hat{\Gamma}_{\chi\chi}(\Lambda)\hat{G}_{\chi\chi_1}(\Lambda). \tag{4.6}$$

We assume $|w_q|$, $|z| < \beta_5$ (given in Theorem 2.6) which is so chosen that $\hat{\Gamma}_{\chi\chi}(\Lambda)$ is analytic there and $\|\hat{\Gamma}_{\chi\chi}(\Lambda)\| \leqq 4$ (see [5] or proceed as in the proof of Theorem 2.3). Thus, $\hat{F}_{\chi_1\chi_1}(\Lambda)$ is analytic on the region above and setting all w_q, z equal to β, $0 < \beta < \beta_5$, $\lim_{\Lambda \to \mathbb{Z}^3} \hat{F}_{\chi_1\chi_1}(x_0, y_0, \Lambda) = \hat{F}_{\chi_1\chi_1}(y_0 - x_0)$. As in the proof of Theorem 2.3, there exists $\beta'_3 \leqq \beta_5$ such that $\hat{G}_{\chi_1\chi_1}(\Lambda)$ is invertible for $|w_q|$, $|z| < \beta'_3$ and if $-\hat{\Gamma}_{\chi_1\chi_1}(\Lambda)$ denotes the inverse, $\|\hat{\Gamma}_{\chi_1\chi_1}(\Lambda)\| \leqq 8$. Hence, we can write

$$\hat{F}_{\chi_1\chi_1}(\Lambda) = \hat{G}_{\chi_1\chi_1}(\Lambda)(1 - \hat{\Gamma}_{\chi_1\chi_1}(\Lambda)\hat{G}_{\chi_1\chi}(\Lambda)\hat{\Gamma}_{\chi\chi}(\Lambda)\hat{G}_{\chi\chi_1}(\Lambda)).$$

Now $\hat{G}_{\chi_1\chi}(\Lambda; \{w_q = 0\}, z=0) = 0$, so that

$$\hat{G}_{\chi_1\chi}(\Lambda) = \int_0^1 d\lambda \frac{d}{d\lambda}\hat{G}_{\chi_1\chi}(\Lambda; \{\lambda w_q\}, \lambda z).$$

The derivative can be estimated through a Cauchy integral as before. Thus, there exists $\beta'_4 \leqq \beta'_3$ such that $\|\hat{G}_{\chi_1\chi}(\Lambda)\|$, $\|\hat{G}_{\chi\chi_1}(\Lambda)\| \leqq 1/8$ for $|w_q|$, $|z| < \beta'_4$. This implies $\hat{F}_{\chi_1\chi_1}(\Lambda)$ invertible. Calling the inverse $-\hat{\Phi}_{\chi_1\chi_1}(\Lambda)$, we have $\|\hat{\Phi}_{\chi_1\chi_1}(\Lambda)\| \leqq 16$. Setting all w_q, z equal to β, $0 < \beta < \beta'_4$, one can verify that $\lim_{\Lambda \to \mathbb{Z}^3} \hat{\Phi}_{\chi_1\chi_1}(x_0, y_0; \Lambda) \equiv \hat{\Phi}_{\chi_1\chi_1}(y_0 - x_0)$ exists and is the convolution inverse of $-\hat{F}_{\chi_1\chi_1}(x_0)$. We now verify the analyticity properties of $\tilde{\Phi}_{\chi_1\chi_1}(p_0)$. From (4.6) and Theorem 3.4, it is clear that $\hat{F}_{\chi_1\chi_1}(x_0, y_0; \Lambda)|_{w_q=0} = 0$, if $x_0 \leqq q \leqq y_0$. Also $(\partial^m/\partial w_q^m)|_{w_q=0}\hat{F}_{\chi_1\chi_1}(\Lambda) = 0$, $1 \leqq m \leqq 3$ and

$$\frac{\partial^m}{\partial w_q^m}\hat{F}_{\chi_1\chi_1}(\Lambda)|_{w_q=0}$$
$$= \frac{\partial^m}{\partial w_q^m}\hat{G}_{\chi_1\chi_1}(\Lambda)|_{w_q=0} + \hat{G}_{\chi_1\chi}(\Lambda)\hat{\Gamma}_{\chi\chi}(\Lambda)\frac{\partial^m}{\partial w_q^m}\hat{G}_{\chi\chi_1}(\Lambda)|_{w_q=0}$$
$$+ \hat{G}_{\chi_1\chi}(\Lambda)\frac{\partial^m}{\partial w_q^m}\hat{\Gamma}_{\chi\chi}(\Lambda)\hat{G}_{\chi\chi_1}(\Lambda)|_{w_q=0} + \frac{\partial^m}{\partial w_q^m}\hat{G}_{\chi_1\chi}(\Lambda)\hat{\Gamma}_{\chi\chi}(\Lambda)\hat{G}_{\chi\chi_1}(\Lambda)|_{w_q=0} \tag{4.7}$$

for $4 \leqq m \leqq 6$. To calculate these derivatives, we need $(\partial^m/\partial w_q^m)\hat{\Gamma}_{\chi\chi}$, which can be obtained as in the proof of Theorem 2.3:

$$\frac{\partial^m}{\partial w_q^m}\hat{\Gamma}_{\chi\chi}(\Lambda)|_{w_q=0}=\hat{\Gamma}_{\chi\chi}(\Lambda)\frac{\partial^m}{\partial w_q^m}\hat{G}_{\chi\chi}(\Lambda)\hat{\Gamma}_{\chi\chi}(\Lambda)|_{w_q=0} \qquad (4 \leqq m \leqq 6),$$

hence, from Theorem 3.4, if $x_0 \leqq q < y_0$,

$$\frac{\partial^m}{\partial w_q^m}\hat{\Gamma}_{\chi\chi}(x_0, y_0; \Lambda)|_{w_q=0}=c_m\delta(x_0, q)\delta(q+1, y_0); \qquad m=4, 5, \tag{4.8}$$

and

$$\frac{\partial^6}{\partial w_q^6}\hat{\Gamma}_{\chi\chi}(x_0, y_0; \Lambda)|_{w_q=0}=c_6\delta(x_0, q)\delta(q+1, y_0)$$
$$+c_7\sum_{u_0, v_0}\hat{\Gamma}_{\chi\chi}(x_0, u_0; \Lambda)\hat{G}_{\chi\chi_1}(u_0, q; \Lambda)$$
$$\cdot\hat{G}_{\chi_1\chi}(q+1, v_0; \Lambda)\hat{\Gamma}_{\chi\chi}(v_0, y_0; \Lambda)|_{w_q=0}. \tag{4.9}$$

Inserting (4.8), (4.9) into (4.7), we find if $x_0 \leqq q < y_0$

$$\frac{\partial^m}{\partial w_q^m}\hat{F}_{\chi_1\chi_1}(x_0, y_0; \Lambda)|_{w_q=0}=0 \qquad (m=4, 5),$$

and after a lengthy computation

$$\frac{\partial^6}{\partial w_q^6}\hat{F}_{\chi_1\chi_1}(x_0, y_0; \Lambda)\Big|_{w_q=0}=c_7\hat{F}_{\chi_1\chi_1}(x_0, q; \Lambda)\hat{F}_{\chi_1\chi_1}(q+1, y_0; \Lambda)|_{w_q=0}.$$

As in the proof of Theorem 2.3, these results imply $(x_0 \leqq q < y_0)$

$$\frac{\partial^m}{\partial w_q^m}\hat{\Phi}_{\chi_1\chi_1}(x_0, y_0; \Lambda)|_{w_q=0}=0 \quad \text{if} \quad 0 \leqq m \leqq 5,$$

and

$$\frac{\partial^6}{\partial w_q^6}\hat{\Phi}_{\chi_1\chi_1}(x_0, y_0; \Lambda)|_{w_q=0}=c_7\delta(x_0, q)\delta(q+1, y_0). \tag{4.10}$$

By the same arguments as before, $\tilde{\Phi}_{\chi_1\chi_1}(p_0)$ is thus analytic on $|\mathrm{Im}\, p_0| < -7\log(\beta/\beta'_4)$, which includes $|\mathrm{Im}\, p_0| < -7\left(1-\frac{\varepsilon}{2}\right)\log\beta$ if $\beta < \beta_4'^{2/\varepsilon}$. Setting $\beta_6 = \beta_4'^{2/\varepsilon}$ completes the proof of the theorem. □

Proof of Theorem 2.8. Consider the finite volume approximation to $\hat{K}(x_0)$:

$$\hat{K}(\Lambda)=\hat{\Gamma}_{\chi_2\chi_2}(\Lambda)-\hat{\Phi}_{\chi_2\chi_2}(\Lambda). \tag{4.11}$$

We assume $|w_q|, |z| < \beta_6$ so that the bound $\|\hat{K}(\Lambda)\| \leqq 24$ holds. Also, from the proof of Theorems 2.3 and 2.7, if $x_0 \leqq q < y_0$,

$$(\partial^m/\partial w_q^m)|_{w_q=0}\hat{K}(x_0, y_0; \Lambda)=0, \qquad 0 \leqq m \leqq 5.$$

But also, due to (4.5) and (4.10)

$$\frac{\partial^6}{\partial w_q^6}\hat{K}(x_0, y_0; \Lambda)|_{w_q=0}=0 \quad \text{if} \quad x_0 \leqq q < y_0 .$$

This implies in the usual way $|\hat{K}(x_0)| \leqq 24(\beta/\beta_6)^{7|x_0|}$, and $\tilde{K}(p_0)$ is analytic on $|\mathrm{Im}\, p_0| < -7\left(1-\frac{\varepsilon}{2}\right)\log\beta$ provided $\beta < \beta_6^{2/\varepsilon}$. In addition, $\hat{K}(\Lambda; \{w_q=0\}, z=0)=0$ and by doing an estimate using the Cauchy formula as before, we find $\|\hat{K}(\Lambda)\| \leqq 96 \max\{|w_q|/\beta_6, |z|/\beta_6\}$, if $|w_q|, |z| < \beta_6/2$. This is automatically satisfied if $|w_q|, |z| < \beta_6^{2/\varepsilon}$, since we assumed from the beginning that $\beta_0 \leqq 1/2$ and $\varepsilon \leqq 1/10$. In particular, $|\hat{K}(x_0=0)| \leqq 96(\beta/\beta_6)$ for $0<\beta<\beta_6^{2/\varepsilon}$. Hence, if $|\mathrm{Im}\, p_0| \leqq -7(1-\varepsilon)\log\beta$,

$$\begin{aligned}|\tilde{K}(p_0)| &\leqq 96\left[(\beta/\beta_6)+2\sum_{n=1}^{\infty}\exp\left(7\log\frac{\beta}{\beta_6}+|\mathrm{Im}\, p_0|\right)n\right]\\ &\leqq 96\left[(\beta/\beta_6)+2\sum_{n=1}^{\infty}\exp\left(7\left(1-\frac{\varepsilon}{2}\right)\log\beta-7(1-\varepsilon)\log\beta\right)n\right]\\ &=96\left[\frac{\beta}{\beta_6}+\frac{2\exp(\frac{7}{2}\varepsilon\log\beta)}{1-\exp(\frac{7}{2}\varepsilon\log\beta)}\right] \leqq 96\left[\frac{\beta}{\beta_6}+4\exp(\tfrac{7}{2}\varepsilon\log\beta)\right],\end{aligned}$$

because $\beta^{\frac{7}{2}\varepsilon} < \beta_6^{\frac{7}{2}} < 1/2$. Also, since $\varepsilon \leqq 1/10$, $\beta^{\left(1-\frac{7}{2}\varepsilon\right)} < \beta_6^{\frac{2}{\varepsilon}-7} < \beta_6$, i.e. $\beta/\beta_6 < \beta^{\frac{7}{2}\varepsilon}$. We thus have $|\tilde{K}(p_0)| \leqq 480\beta^{\frac{7}{2}\varepsilon}$ and therefore, from Theorem 2.4 $|\tilde{T}_2(p_0)| \leqq 480 k_4 \beta^{(7\varepsilon/2)} \leqq 1/2$ if $\beta \leqq \beta_5'$ (this defines β_5'). Letting $\beta_7 = \min\{\beta_6^{2/\varepsilon}, \beta_5'\}$ completes the proof of the theorem. □

Proof of Theorem 2.9. From Theorems 2.4 and 2.8, $|v_2(m_2)(1+\tilde{T}_2)^{-1}\tilde{K}| \leqq 2k_3k_5\beta^{\frac{7}{2}\varepsilon}$ on $|\mathrm{Im}\, p_0| \leqq -7(1-\varepsilon)\log\beta$, if $\beta < \beta_7$. Thus, $|v_2(m_2)(1+\tilde{T}_2)^{-1}\tilde{K}| \leqq (1/2)$ if $\beta < \beta_6' \leqq \beta_7$ for an appropriate β_6'. Let

$$AB = \{p_0 : |\mathrm{Re}\, p_0| \leqq \pi, \mathrm{Im}\, p_0 = 0\};$$

$$BC = \{p_0 : \mathrm{Re}\, p_0 = \pi, 0 \leqq \mathrm{Im}\, p_0 \leqq -7(1-\varepsilon)\log\beta\};$$

$$CD = \{p_0 : |\mathrm{Re}\, p_0| \leqq \pi, \mathrm{Im}\, p_0 = -7(1-\varepsilon)\log\beta\}$$

and

$$AD = \{p_0 : \mathrm{Re}\, p_0 = -\pi, 0 \leqq \mathrm{Im}\, p_0 \leqq -7(1-\varepsilon)\log\beta\}.$$

On AB,

$$|g(p_0)| = (\cosh m_2 - \cos p_0)/\sinh m_2 \geqq 1-(1/\sinh m_2) > 1/2,$$

since from the proof of Theorem 2.4 $\coth m_2 < \sqrt{5}/2$ which implies $1/\sinh m_2 < 1/2$.

On BC, $|g(p_0)| = (\cosh m_2 + \cosh\, \mathrm{Im}\, p_0)/\sinh m_2 \geqq 1$, and similarly on AD. Finally, on CD,

$$\begin{aligned}|g(p_0)|^2 &= [\cosh^2 m_2 + \cos^2\, \mathrm{Re}\, p_0 + \sinh^2 7(1-\varepsilon)\log\beta\\ &\quad -2\cosh m_2 \cos\, \mathrm{Re}\, p_0 \cosh 7(1-\varepsilon)\log\beta]/\sinh^2 m_2\\ &\geqq [(\cosh m_2 - \cosh 7(1-\varepsilon)\log\beta)^2 - 1]/\sinh^2 m_2\\ &\geqq \left(\frac{\cosh 7(1-\varepsilon)\log\beta}{\sinh m_2} - \coth m_2\right)^2 - \frac{1}{4}.\end{aligned}$$

The right-hand side tends to infinity as $\beta \to 0$. Thus, it stays bigger than 1/2 if $\beta < \beta'_7 \leqq \beta'_6$ for an appropriate β'_7. We set $\beta_8 \equiv \beta'_7$ to finish the proof. □

Proof of Theorem 2.11. Let $p_0 = im_2 + r$, with $|r| = \zeta\beta$. ζ will be given explicitly below. Then,

$$|g(p_0)| = |r| \coth m_2 (1-\cos r)/r + i \sin r/r|$$

and $\lim_{\beta \to 0} |g(p_0)|/|r| = 1$. We choose $\beta'_8 \leqq \beta_8$ such that $|r| \leqq 1$ and $|g(p_0)| > (1/2)r$. On the other hand, from the proof of Theorem 2.8,

$$|\tilde{K}(p_0)| \leqq 96 \left[(\beta/\beta_6) + 2 \sum_{n=1}^{\infty} \exp[(7 \log(\beta/\beta_6) + m_2 + 1)n] \right].$$

From Theorem 2.2, $m_2 \leqq -6 \log(d_1 \beta)$, hence

$$\begin{aligned} |\tilde{K}(p_0)| &\leqq 96 \left[(\beta/\beta_6) + 2 \sum_{n=1}^{\infty} \exp[(\log\beta - \log(\beta_6^7 d_1^6) + 1)n] \right] \\ &= 96 \left[(\beta/\beta_6) + 2 \frac{(e\beta/\beta_6^7 d_1^6)}{1-(e\beta/\beta_6^7 d_1^6)} \right] \\ &\leqq 96[(\beta/\beta_6) + 4e\beta/(\beta_6^7 d_1^6)] \end{aligned}$$

if $\beta < \beta_9 \leqq \beta'_8$ is appropriately chosen. We set $\zeta = 192[1/\beta_6 + 4e/(\beta_6^7 d_1^6)]$, so that $|\tilde{K}(p_0)| \leqq (1/2)\zeta\beta$. In conclusion, we have $|\tilde{K}(p_0)| \leqq (1/2)\zeta\beta = (1/2)|r| < |g(p_0)|$ and the proof is complete. □

Proof of Theorem 2.13. Consider the finite volume approximations to $\hat{L}_{\chi_1\chi}(x_0)$:

$$\hat{L}_{\chi_1\chi}(\Lambda) = \hat{\Phi}_{\chi_1\chi_1}(\Lambda) \hat{G}_{\chi_1\chi}(\Lambda) \hat{\Gamma}_{\chi\chi}(\Lambda), \tag{4.12}$$

where $|w_q|$, $|z| < \beta_6$, so that $\|\hat{\Phi}_{\chi_1\chi_1}(\Lambda)\| \leqq 16$, $\|\hat{G}_{\chi_1\chi}(\Lambda)\| \leqq C_{\chi_1\chi}$, and $\|\hat{\Gamma}_{\chi\chi}(\Lambda)\| \leqq 4$, hence $\|\hat{L}_{\chi_1\chi}(\Lambda)\| \leqq 64 C_{\chi_1\chi}$. From (4.12) it follows that $\hat{L}_{\chi_1\chi}(x_0, y_0; \Lambda)|_{w_q=0} = 0$, if $x_0 \leqq q < y_0$ and $(\partial^m/\partial w_q^m)|_{w_q=0} \hat{L}_{\chi_1\chi}(\Lambda) = 0$, if $1 \leqq m \leqq 3$. Also,

$$\begin{aligned} \frac{\partial^m}{\partial w_q^m} \hat{L}_{\chi_1\chi}(\Lambda)|_{w_q=0} &= \hat{\Phi}_{\chi_1\chi_1}(\Lambda) \hat{G}_{\chi_1\chi}(\Lambda) \frac{\partial^m}{\partial w_q^m} \hat{\Gamma}_{\chi\chi}(\Lambda)|_{w_q=0} \\ &+ \hat{\Phi}_{\chi_1\chi_1}(\Lambda) \frac{\partial^m}{\partial w_q^m} \hat{G}_{\chi_1\chi}(\Lambda) \hat{\Gamma}_{\chi\chi}(\Lambda)|_{w_q=0} \\ &+ \frac{\partial^m}{\partial w_q^m} \hat{\Phi}_{\chi_1\chi_1}(\Lambda) \hat{G}_{\chi_1\chi}(\Lambda) \hat{\Gamma}_{\chi\chi}(\Lambda)|_{w_q=0}. \quad (4 \leqq m \leqq 6). \end{aligned}$$

A straightforward but tedius computation shows that

$$\frac{\partial^m}{\partial w_q^m} \hat{L}_{\chi_1\chi}(x_0, y_0; \Lambda)|_{w_q=0} = 0, \quad \text{if} \quad x_0 \leqq q < y_0 \quad \text{and} \quad 4 \leqq m \leqq 6.$$

Thus, in the usual way this implies $|\hat{L}_{\chi_1\chi}(x_0)| \leqq 64 C_{\chi_1\chi} (\beta/\beta_6)^{7|x_0|}$ if $0 < \beta < \beta_6$ and $\tilde{L}_{\chi_1\chi}(p_0)$ analytic up to $|\mathrm{Im}\, p_0| < -7(1-\varepsilon/2)\log\beta$ if $0 < \beta < \beta_7$ (remember $\beta_7 \leqq \beta_6^{2/\varepsilon}$). Also, $\hat{L}_{\chi_1\chi}(\Lambda;\ \{w_q = 0\},\ z = 0) = 0$. Proceeding in the same way as for $\hat{K}(\Lambda)$, we deduce, after an estimate using the Cauchy formula, that $|\hat{L}_{\chi_1\chi}(x_0 = 0)|$

$\leqq 256C_{\chi_1\chi}(\beta/\beta_6)$ if $0<\beta<\beta_7$, and this leads again as in the proof of Theorem 2.8 to $|\tilde{L}_{\chi_1\chi}(p_0)|\leqq 1280C_{\chi_1\chi}\beta^{\frac{7}{2}\varepsilon}$ for $|\mathrm{Im}\,p_0|\leqq -7(1-\varepsilon)\log\beta$. □

Proof of Theorem 2.14. Let

$$\hat{M}(\Lambda)=\hat{\Gamma}_{\chi\chi}(\Lambda)-\hat{L}_{\chi\chi_1}(\Lambda)\hat{F}_{\chi_1\chi_1}(\Lambda)\hat{L}_{\chi_1\chi}(\Lambda) \tag{4.13}$$

with $|w_q|$, $|z|<\beta_6$. Then $\|\hat{\Gamma}_{\chi\chi}(\Lambda)\|\leqq 4$, $\|\hat{L}_{\chi\chi_1}(\Lambda)\|\leqq 64C_{\chi\chi_1}$ and $\|\hat{F}_{\chi_1\chi_1}(\Lambda)\|\leqq C_{\chi_1\chi_1}+4C_{\chi_1\chi}C_{\chi\chi_1}$ [see (4.6)], so that $\|\hat{M}(\Lambda)\|\leqq k_M$ for and appropriate constant. Now, if $x_0\leqq q<y_0$, one shows by direct calculation that

$$\frac{\partial^m}{\partial w_q^m}\hat{M}(x_0,y_0;\Lambda)|_{w_q=0}=0 \quad \text{for} \quad 0\leqq m\leqq 3\,,$$

and

$$\frac{\partial^m}{\partial w_q^m}\hat{M}(x_0,y_0;\Lambda)|_{w_q=0}=c_m\delta(x_0,q)\delta(q+1,y_0) \quad \text{for} \quad 4\leqq m\leqq 6\,.$$

This implies, as before, that $|\hat{M}(x_0)|\leqq k_M(\beta/\beta_6)^{7|x_0|}$ if $|x_0|>1$ if $0<\beta<\beta_6$, and therefore $\tilde{M}(p_0)$ is analytic on

$$|\mathrm{Im}\,p_0|<-7(1-\varepsilon/2)\log\beta \quad \text{if} \quad 0<\beta<\beta_7\,. \quad \square$$

Proof of Theorem 2.15. Consider

$$\hat{G}_{\chi\chi_1}(x_0,x_0;\Lambda:\{w_q=0\},z)=\hat{G}_{\chi\chi_h}(x_0,x_0;\Lambda;\{w_q=0\},z)\,.$$

It is clear that

$$\begin{aligned}&\lim_{z\to 0}\hat{G}_{\chi\chi_1}(x_0,x_0;\Lambda;\{w_q=0\},z)/z\\&\quad=2\int\chi(g_{\mathscr{P}_\ell})\chi_h(g)\chi(g_{P_r})dg_\Lambda=(2/d)\end{aligned}$$

(d = dimension of the representation χ). Here, P_ℓ and P_r are the left and right plaquettes of the elementary horizontal window. From (4.12),

$$\begin{aligned}&\hat{L}_{\chi\chi_1}(x_0,x_0;\Lambda;\{w_q=0\},z)\\&\quad=\hat{\Gamma}_{\chi\chi}(x_0,x_0;\Lambda;\{w_q=0\},z)\\&\quad\quad\cdot\hat{G}_{\chi\chi_1}(x_0,x_0;\Lambda;\{w_q=0\},z)\hat{\Phi}_{\chi_1\chi_1}(x_0,x_0;\Lambda;\{w_q=0\},z)\,,\end{aligned}$$

and hence

$$\begin{aligned}&\lim_{z\to 0}z^{-1}\hat{L}_{\chi\chi_1}(x_0,x_0;\Lambda;\{w_q=0\},z)\\&\quad=2\lim_{z\to 0}z^{-1}\hat{G}_{\chi\chi_1}(x_0,x_0;\Lambda;\{w_q=0\},z)=(4/d)\,.\end{aligned}$$

Using the fact that $(\partial/\partial w_q)|_{w_q=0}\hat{L}_{\chi\chi_1}(\Lambda)=0$ for all q, the above result implies

$$\frac{d}{d\beta}\hat{L}_{\chi\chi_1}(x_0,x_0;\Lambda;\{w_q=\beta\},z=\beta)|_{\beta=0}=\frac{4}{d},$$

from which follows (taking the thermodynamic limit)

$$\frac{d}{d\beta}\hat{L}_{\chi\chi_1}(x_0=0)|_{\beta=0}=\frac{4}{d}.$$

We choose $\beta<\beta'_9(\leqq\beta_9)$ so that $|\hat{L}_{\chi\chi_1}(x_0=0)|\geqq(2/d)\beta$.

Now, consider $\hat{L}_{\chi\chi_1}(x_0=0,\ y_0=1;\ \Lambda;\ \{w_q=0\}_{q\neq 0},\ z=0)$. We claim that this is identically zero as a function of w_0. The reason is that $\hat{G}_{\chi\chi_1}(x_0,\ y_0;\Lambda;\{w_q=0\}_{q\neq 0},\ z=0)=0$ for all x_0, y_0. This is clear if $x_0<0$ or $y_0>1$ (or vice-versa), but is also true when $x_0=0$ and $y_0=1$ because integrals of the form

$$\int \chi(0,\mathbf{0})\chi_h(1,\mathbf{x})\exp\left[w_0\sum_{P\in\mathscr{P}''_0}\chi(g_P)\right]dg_\Lambda$$

vanish identically as is easy to see.

Write

$$\hat{L}_{\chi\chi_1}(x_0=0,y_0=1;\Lambda;\{w_q\},z)=\sum_{n=7}^{\infty}h_n(\Lambda;\{w_q\}_{q\neq 0},z)w_0^n, \tag{4.14}$$

with

$$h_n(\Lambda;\{w_q\}_{q\neq 0},z)=\frac{1}{h!}\frac{\partial^n}{\partial w_0^n}\hat{L}_{\chi\chi_1}(x_0=0,y_0=1;\Lambda;\{w_q\},z)|_{w_q=0}.$$

From the remark above, $h_n(\Lambda;\ \{w_q=0\}_{q\neq 0},\ z=0)=0\ \ \forall n$. Setting all w_q, z equal to the same β in (4.14), we get

$$\hat{L}_{\chi\chi_1}(x_0=0,y_0=1;\Lambda;\beta)=\sum_{n=7}^{\infty}h_n(\Lambda;\beta)\beta^n,$$

and $h_n(\Lambda;\beta=0)=0$. From this, we get the important result

$$\frac{\partial^7}{\partial\beta^7}\hat{L}_{\chi\chi_1}(x_0=0,y_0=1;\Lambda;\beta)|_{\beta=0}=0,$$

which implies, if $|\beta|<\beta_6:|\hat{L}_{\chi\chi_1}(x_0=0,\ \ y_0=1;\ \ \Lambda)|\leqq 64C_{\chi\chi_1}|\beta/\beta_6|^8$. This bound carries to the infinite volume limit:

$$|\hat{L}_{\chi\chi_1}(x_0=1)|\leqq 64C_{\chi\chi_1}(\beta/\beta_6)^8\quad\text{if}\quad 0<\beta<\beta_6.$$

Now, suppose $p_0=im_2+r$, with $|r|\leqq 1$. Then,

$$\begin{aligned}
&|\tilde{L}_{\chi\chi_1}(p_0)-\hat{L}_{\chi\chi_1}(x_0=0)|\\
&\leqq\left|\sum_{|x_0|=1}\hat{L}_{\chi\chi_1}(x_0)e^{ip_0x_0}+\sum_{|x_0|\geqq 2}\hat{L}_{\chi\chi_1}(x_0)e^{ip_0x_0}\right|\\
&\leqq 128C_{\chi\chi_1}\left(\frac{\beta}{\beta_6}\right)^8\exp(m_2+1)+128C_{\chi\chi_1}\sum_{n=2}^{\infty}\exp\left[\left(7\log\left(\frac{\beta}{\beta_6}\right)+m_2+1\right)n\right],
\end{aligned}$$

where we used results from the proof of Theorem 2.13. Proceeding now as in the proof of Theorem 2.11, using $m_2 \leqq -6\log(d_1\beta)$, we get

$$\begin{aligned}
&|\tilde{L}_{\chi\chi_1}(p_0)-\hat{L}_{\chi\chi_1}(x_0=0)| \\
&\leqq 128 C_{\chi\chi_1}\left[\beta^2\left(\frac{e}{\beta_6^8 d_1^6}\right)+\frac{(e\beta/\beta_6^7 d_1^6)^2}{1-(e\beta/\beta_6^7 d_1^6)}\right] \\
&\leqq 128 C_{\chi\chi_1}\left[\left(\frac{e}{\beta_6^8 d_1^6}\right)+2\left(\frac{e}{\beta_6^7 d_1^6}\right)^2\right]\beta^2 \equiv k_L\beta^2 \quad \text{if} \quad \beta<\beta_9 .
\end{aligned}$$

Thus, if $\beta<\beta_9'$, we have

$$|\tilde{L}_{\chi\chi_1}(p_0)| \geqq (2/d)\beta - k_L\beta^2 ,$$

and we choose $\beta_{10} \leqq \beta_9'$ so that $|\tilde{L}_{\chi\chi_1}(p_0)|>0$ for $0<\beta<\beta_{10}$. □

Proof of Theorem 2.16. First, notice that if $0<\beta<\beta_7$, $|\tilde{L}_{\chi\chi_1}\tilde{H}_2(1+\tilde{T}_2)^{-1}\tilde{L}_{\chi_1\chi}| \leqq a\beta^{7\varepsilon}$ for some constant a and $|\mathrm{Im}\, p_0| \leqq -7(1-\varepsilon)\log\beta$. This follows from Theorems 2.4, 2.8, and 2.13. From the proof of Theorem 2.14, $|\hat{M}(x_0)| \leqq k_M(\beta/\beta_6)^{7|x_0|}$ if $|x_0|>1$. Thus,

$$\tilde{M}(p_0)=\hat{M}(x_0=0)+\sum_{|x_0|=1}\hat{M}(x_0)e^{ip_0x_0}+\tilde{M}_1(p_0), \tag{4.15}$$

with $\tilde{M}_1(p_0)=\sum_{|x_0|\geqq 2}\hat{M}(x_0)e^{ip_0x_0}$ being bounded on $|\mathrm{Im}\, p_0| \leqq -7(1-\varepsilon)\log\beta$ by

$$|\hat{M}_1(p_0)| \leqq \sum_{|x_0|\geqq 2} k_M \exp[(7\log(\beta/\beta_6)-7(1-\varepsilon)\log\beta)\,|x_0|] \leqq 4k_M\beta^{7\varepsilon}$$

(as in the proof of Theorem 2.8). Now, from the definition (4.13),

$$\hat{M}(\Lambda;\{w_q=0\},z=0)=\hat{\Gamma}_{\chi\chi}(\Lambda;\{w_q=0\},z=0)=-I ,$$

so that $\lim_{\beta\to 0}\hat{M}(x_0=0)=-1$. Also, from the results in the proof of Theorem 2.15,

$$\begin{aligned}
&\hat{M}(x_0=0,y_0=1;\Lambda;\{w_q=0\}_{q\neq 0},z=0) \\
&=\hat{\Gamma}_{\chi\chi}(x_0=0,y_0=1;\Lambda;\{w_q=0\}_{q\neq 0},z=0) \\
&\quad -\hat{L}_{\chi\chi_1}(x_0=0,y_0=0;\Lambda;\{w_q=0\}_{q\neq 0},z=0) \\
&\quad \cdot\hat{F}_{\chi_1\chi_1}(x_0=0,y_0=1;\Lambda;\{w_q=0\}_{q\neq 0},z=0) \\
&\quad \cdot\hat{L}_{\chi_1\chi}(x_0=1,y_0=1;\Lambda;\{w_q=0\}_{q\neq 0},z=0) .
\end{aligned}$$

From the proof of Theorem 2.7, we know that

$$\hat{F}_{\chi_1\chi_1}(x_0=0,y_0=1;\Lambda;\{w_q=\}_{q\neq 0},z=0) \quad \text{is} \quad O(w_0^6) .$$

Also, from

$$\frac{\partial^4}{\partial w_0^4}\hat{\Gamma}_{\chi\chi}(\Lambda)|_{w_q=0}=\hat{\Gamma}_{\chi\chi}(\Lambda)\frac{\partial^4}{\partial w_0^4}\hat{G}_{\chi\chi}(\Lambda)\hat{\Gamma}_{\chi\chi}(\Lambda)|_{w_q=0} ,$$

we have

$$(\partial^4/\partial w_0^4)|_{w_0=0}\hat{\Gamma}_{\chi\chi}(x_0=0,y_0=1;\Lambda;\{w_q=0\}_{q\neq 0},z=0)=c_4 ,$$

i.e.

$$\hat{\Gamma}_{\chi\chi}(x_0=0, y_0=1; \Lambda; \{w_q=0\}_{q\neq 0}, z=0)=O(w_0^4).$$

Therefore,

$$\lim_{w_0\to 0} w_0^{-4}\tilde{M}(x_0=0, y_0=1; \Lambda; \{w_q=0\}_{q\neq 0}, z=0)=c_4/4!,$$

which implies

$$\lim_{\beta\to 0} \beta^{-4}\tilde{M}(x_0=0, y_0=1; \Lambda; \beta)=c_4/4!.$$

In this last expression, $\hat{M}$ is calculated setting all w_q, z equal to the same β. Taking the thermodynamic limit, we conclude that

$$\lim_{\beta\to 0} \frac{1}{\beta^4}\hat{M}(x_0=1)=\frac{1}{4!}c_4.$$

Now, from (4.15)

$$|\hat{M}(p_0)-2\hat{M}(x_0=1)\cos p_0| \leqq |\hat{M}(x_0=0)|+4k_M\beta^{7\varepsilon}.$$

Since $\operatorname{Im} p_0 \geqq -5\log\beta$, $|\cos p_0| \geqq \sinh(-5\log\beta)$ and hence,

$$\begin{aligned}|\hat{M}(p_0)+\tilde{L}_{\chi\chi_1}\tilde{H}_2(1+\tilde{T}_2)^{-1}\tilde{L}_{\chi_1\chi}| \\ \geqq 2|\hat{M}(x_0=1)|\sinh(-5\log\beta)-(|\hat{M}(x_0=0)|+4k_M\beta^{7\varepsilon}+a\beta^{7\varepsilon})\end{aligned}$$

and we can choose $\beta<\beta_{11}\leqq\beta_{10}$ such that the right hand side is bigger than $c_4/(5!\beta)$. □

The proof of Theorems 2.17 and 2.18 are very similar to the ones for Theorems 2.9 and 2.11, after using the bound in Theroem 2.16. We therefore omit them.

Proof of Theorem 2.20. From (2.15), $\tilde{G}_{\chi_1\chi_1}=\tilde{F}_{\chi_1\chi_1}-\tilde{G}_{\chi_1\chi}\tilde{\Gamma}_{\chi\chi}\tilde{G}_{\chi\chi_1}$ and the possible singularities of $\tilde{G}_{\chi_1\chi_1}$ on $0\leqq \operatorname{Im} p_0 \leqq -7(1-\varepsilon)\log\beta$, $|\operatorname{Re} p_0|\leqq\pi$ are $p_0=im_0$, im_1 and $i\varrho$. We show that $G_{\chi_1\chi_1}$ is regular at $p_0=i\varrho$:

$$\begin{aligned}\lim_{p_0\to i\varrho}(p_0-i\varrho)\tilde{G}_{\chi_1\chi_1} &= \lim_{p_0\to i\varrho}(p_0-i\varrho)\tilde{F}_{\chi_1\chi_1} \\ &\quad -\tilde{G}_{\chi_1\chi}(i\varrho)\tilde{G}_{\chi\chi_1}(i\varrho)\lim_{p_0\to i\varrho}(p_0-i\varrho)\tilde{\Gamma}_{\chi\chi}.\end{aligned} \tag{4.16}$$

On the other hand, from (2.22)

$$\lim_{p_0\to i\varrho}(p_0-i\varrho)\tilde{\Gamma}_{\chi\chi}=\tilde{L}_{\chi\chi_1}(i\varrho)\tilde{L}_{\chi_1\chi}(i\varrho)\lim_{p_0\to i\varrho}(p_0-i\varrho)\tilde{F}_{\chi_1\chi_1}. \tag{4.17}$$

Since

$$\begin{aligned}\tilde{L}_{\chi\chi_1} &= \tilde{\Gamma}_{\chi\chi}\tilde{G}_{\chi\chi_1}\tilde{\Phi}_{\chi_1\chi_1}=(\tilde{L}_{\chi\chi_1}\tilde{F}_{\chi_1\chi_1}\tilde{L}_{\chi_1\chi}+\tilde{M})\tilde{G}_{\chi\chi_1}\tilde{\Phi}_{\chi_1\chi_1} \\ &= -\tilde{L}_{\chi\chi_1}\tilde{L}_{\chi_1\chi}\tilde{G}_{\chi\chi_1}+\tilde{M}\tilde{G}_{\chi\chi_1}\tilde{\Phi}_{\chi_1\chi_1},\end{aligned}$$

we have (remembering that

$$\tilde{\Phi}_{\chi_1\chi_1}(i\varrho)=0):\tilde{L}_{\chi_1\chi}(i\varrho)=-1/\tilde{G}_{\chi\chi_1}(i\varrho). \tag{4.18}$$

Taking (4.17) and (4.18) into (4.16) shows that $\lim_{p_0\to i\varrho}(p_0-i\varrho)\tilde{G}_{\chi_1\chi_1}=0$. Thus, the only possible singularities are $p_0=im_0, im_1$. We show that these are indeed singularities of $\tilde{G}_{\chi_1\chi_1}$. For suppose $\tilde{G}_{\chi_1\chi}$ is regular at $p_0=im_1$. Then, from $\tilde{L}_{\chi_1\chi}=\tilde{\Phi}_{\chi_1\chi_1}\tilde{G}_{\chi_1\chi}\tilde{\Gamma}_{\chi\chi}$ we would get $\tilde{L}_{\chi_1\chi}(im_1)=0$, which is impossible due to Theorem 2.15 since $|m_1-m_2|=O(\beta)$. Similarly, if $\tilde{G}_{\chi_1\chi}$ is regular at $p_0=im_0$ then $\tilde{L}_{\chi_1\chi}(im_0)=0$. But recall that $|\tilde{L}_{\chi_1\chi}(x_0)|\leqq 64C_{\chi_1\chi}\left(\frac{\beta}{\beta_6}\right)^{7|x_0|}$ (from the proof of Theorem 2.13) and $\lim_{\beta\to 0}(1/\beta)\hat{L}_{\chi_1\chi}(x_0=0)=4/d$ (proof of Theorem 2.15). Thus, from

$$|\tilde{L}_{\chi_1\chi}(im_0)-\tilde{L}_{\chi_1\chi}(x_0=0)|\leqq 64C_{\chi_1\chi}\sum_{x_0\neq 0}\left(\frac{\beta}{\beta_6}\right)^{7|x_0|}e^{m_0|x_0|},$$

and the fact that $m_0\sim -4\log\beta$ as $\beta\to 0$, it is clear that $\tilde{L}_{\chi_1\chi}(im_0)\neq 0$ if $\beta>0$ is small enough. □

Proof of Theorem 2.21. Introduce the function

$$\tilde{F}^{(1)}_{\varphi\varphi}=\tilde{G}_{\varphi\varphi}+\tilde{G}_{\varphi\chi}\tilde{\Gamma}_{\chi\chi}\tilde{G}_{\chi\varphi}+\tilde{F}_{\varphi\chi_1}\tilde{\Phi}_{\chi_1\chi_1}\tilde{F}_{\chi_1\varphi},$$

where

$$\tilde{F}_{\varphi\chi_1}=\tilde{G}_{\varphi\chi_1}+\tilde{G}_{\varphi\chi}\tilde{\Gamma}_{\chi\chi}\tilde{G}_{\chi\chi_1}.$$

By going to the finite volume approximation, one can show by direct but tedious calculations that if $x_0\leqq q<y_0$,

$$\frac{\partial^m}{\partial w_q^m}\hat{F}^{(1)}_{\varphi\varphi}(x_0,y_0;\Lambda)|_{w_q=0}=0\quad\text{for}\quad 0\leqq m\leqq 6.$$

Thus, $\tilde{F}^{(1)}_{\varphi\varphi}$ is analytic on $|\mathrm{Im}\,p_0|<-7(1-\varepsilon)\log\beta$ and the only possible singularities of $\tilde{G}_{\varphi\varphi}$ on this region are therefore $p_0=im_0, im_1, i\varrho$. To show that $\tilde{G}_{\varphi\varphi}$ is regular at $i\varrho$, we calculate

$$\begin{aligned}0=&\lim_{p_0\to i\varrho}(p_0-i\varrho)\tilde{G}_{\varphi\varphi}+\tilde{G}_{\varphi\chi}(i\varrho)\tilde{G}_{\chi\varphi}(i\varrho)\lim_{p_0\to i\varrho}\tilde{\Gamma}_{\chi\chi}\\&+\left[\lim_{p_0\to i\varrho}(p_0-i\varrho)\tilde{F}_{\varphi\chi_1}\right]\left[\lim_{p\to i\varrho}\frac{\tilde{\Phi}_{\chi_1\chi_1}}{(p_0-i\varrho)}\right]\left[\lim_{p_0\to i\varrho}(p_0=i\varrho)\tilde{F}_{\chi_1\varphi}\right].\end{aligned}\tag{4.19}$$

Notice that e.g.

$$\lim_{p_0\to i\varrho}(p_0-i\varrho)\tilde{F}_{\varphi\chi_1}=\tilde{G}_{\varphi\chi}(i\varrho)\tilde{G}_{\chi\chi_1}(i\varrho)\lim_{p_0\to i\varrho}(p_0-i\varrho)\tilde{\Gamma}_{\chi\chi},\tag{4.20}$$

hence

$$\lim_{p_0\to i\varrho}(p_0-i\varrho)^{-1}\tilde{\Phi}_{\chi_1\chi_1}=-\left[\tilde{G}_{\chi_1\chi}(i\varrho)\tilde{G}_{\chi\chi_1}(i\varrho)\lim_{p_0\to i\varrho}(p_0-i\varrho)\tilde{\Gamma}_{\chi\chi}\right]^{-1}.\tag{4.21}$$

Taking (4.20), (4.21) into (4.19) completes the proof of Theorem 2.21. □

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