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# A New Proof of the Existence and Nontriviality of the Continuum $\varphi_2^4$ and $\varphi_3^4$ Quantum Field Theories

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**Abstract.** We use Schwinger-Dyson equations combined with rigorous "perturbation-theoretic" correlation inequalities to give a new and extremely simple proof of the existence and nontriviality of the weakly-coupled continuum $\varphi_2^4$ and $\varphi_3^4$ quantum field theories, constructed as subsequence limits of lattice theories. We prove an asymptotic expansion to order $\lambda$ or $\lambda^2$ for the correlation functions and for the mass gap. All Osterwalder-Schrader axioms are satisfied except perhaps Euclidean (rotation) invariance.

## 1. Introduction

The proof of existence of the superrenormalizable $\varphi_3^4$ quantum field theory along with the analysis of some of its physical properties (mass gap, particle structure, symmetry breaking...) is one of the grand achievements of the Constructive Quantum Field Theory program. We direct the reader to [1, 2] and to the references cited in [2, 3] for background. Even a casual inspection of that literature will reveal how difficult and clever are the methods invented and used by previous workers on $\varphi_3^4$.

In this paper we present a novel – and, we believe, extremely simple – approach to the $\varphi_3^4$ quantum field model. We have tried hard to make our presentation comprehensible to experts and non-experts alike. We therefore beg the expert's indulgence as we review some well-known facts. We reassure the non-expert that any technical terms used in this Introduction will be defined in an accessible manner in the main body of the paper.

We begin with a brief summary of our methods, since these are possibly more interesting than our results. Indeed, *all* of our results are known ones; what was previously unknown was that they could be obtained so easily.

Theoretical physicists have long made use of a system of coupled non-linear integral equations, called the Schwinger-Dyson equations (or "field equations"), in

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their non-rigorous studies of quantum field theory. Of course, the theories in question were always assumed implicitly to exist. In the present paper we use the Schwinger-Dyson equations as a tool in the *construction* and *rigorous* study of a quantum field model. (See also [3] for related ideas.) We do this by truncating the Schwinger-Dyson system rigorously with the help of "perturbation-theoretic" correlation inequalities developed in two previous papers [4, 5]. (To read the present paper it is *not* necessary to have read these earlier papers.) The result is a closed system of non-linear integral inequalities which can be analyzed explicitly, yielding upper and lower bounds which imply the existence and non-triviality of the continuum limit.

Previous constructions of quantum field models have always proceeded through a study of unnormalized quantities which behave badly in the appropriate (ultraviolet and infinite-volume) limits. However, it is a quotient of two such quantities which is of primary interest. In our construction, by contrast, we work directly with the normalized quantities (as proposed earlier in [6–9, 3]), thereby avoiding many technical difficulties.

Our methods permit us to show that, *at weak (bare) coupling*, the correlation ($\equiv$Euclidean Green's) functions of an infinite-volume $\varphi_3^4$ lattice field theory, mass-renormalized according to second-order pertrubation theory, are bounded uniformly in the lattice spacing as the lattice spacing tends to zero. From this it follows [6, 3] that a continuum limit exists (by subsequences) and satisfies all Osterwalder-Schrader axioms [10–13, 2] except perhaps Euclidean (rotation) invariance. Furthermore, we show that any such continuum limit is *non-Gaussian* (for non-zero bare coupling $\lambda$) and has a strictly positive mass gap. We achieve all these results by proving that perturbation theory is asymptotic (to order $\lambda^2$ for the two-point function, to order $\lambda$ for higher-point functions) uniformly in the lattice spacing. By using simple Griffiths inequalities and infrared bounds, we obtain as a corollary the existence (but not the non-Gaussianness) of the continuum limit for *arbitrary (not necessarily weak) coupling*, provided the theory lies in the single-phase region. (With some additional effort, one can also construct the theory in an "external magnetic field.") However, we fall short of a complete construction of $\varphi_3^4$, even at weak coupling, because we are unable (as yet) to prove that the theory in the continuum limit is rotation-invariant. Moreover, we do not know, at present, how to extend our construction to the two-phase region of the $\varphi_3^4$ theory.

The philosophy underlying our construction is a modified version of that proposed in [3]. The $\varphi_3^4$ theory in the continuum limit is constructed by proving uniform bounds on correlation functions of $\varphi_3^4$ lattice theories and appealing to weak compactness to obtain a convergent subsequence [6]. (A constructive version of passing to the continuum limit does, however, appear possible – see Sect. 9.) The non-triviality (i.e. non-Gaussianness) of the continuum theory is established by proving that its connected four-point function is close to the value predicted by lowest-order perturbation theory (that is, the tree diagram), up to radiative corrections which are ultraviolet convergent and are small for weak bare coupling. However – and this is where we differ from [3] – we here perform the renormalizations of ultraviolet divergences (in particular, the mass renormalization) *explicitly*, using the conventional counterterms suggested by perturbation theory (see Sect. 4). Thus, our results are stronger than those of [3] (and also make no use of unproven correlation inequalities). On the other hand, we have to work harder to get them (but not very much harder). That is, we have to *prove* that our choice of counterterms yields a theory which lies in the single-phase region and whose correlation functions neither diverge to infinity nor converge to identically zero as the lattice spacing tends to zero. To do this, it suffices, by the Gaussian and Griffiths inequalities (see Sect. 8), to control the two-point function. As indicated above, we shall accomplish this – for weak bare coupling – by combining the Schwinger-Dyson equation for the two-point function [14, 7, 3], which expresses the two-point function in terms of the four-point function, with the skeleton inequalities [5], which bound the four-point function both above and below in terms of the two-point function. This yields non-linear integral inequalities which bound the two-point function both above and below in terms of itself. These integral inequalities tell us (Proposition 6.2) that the two-point function is either very close to the free propagator or else very far from it (that is, there is a "forbidden region" for the two-point function); how close or far depends on the bare coupling constant $\lambda$ but *not* on the lattice spacing. We shall prove, in addition, that for each *fixed* value of the lattice spacing, the two-point function is a continuous function of $\lambda$; and, of course, it is equal to the free propagator when $\lambda = 0$. Combining these facts, we derive an estimate on the two-point function that is valid *uniformly in the lattice spacing*, provided that $\lambda \geqq 0$ is sufficiently small (how small does not depend on the lattice spacing). This estimate (Theorem 6.1) is the key technical result; all else follows quite easily from it.

The major contribution of this paper is perhaps pedagogical: it shows that, in spite of non-trivial ultraviolet renormalizations, the construction of the super-renormalizable $\varphi_3^4$ model can be made so simple that it could be taught in a first-year graduate course. Apart from its pedagogical ambitions, this paper makes a contribution to the understanding of the use of Schwinger-Dyson equations and skeleton expansions in the analysis of quantum field models. It has grown out of our attempts to synthesize Symanzik's program for the construction of Euclidean $\varphi^4$ theory [15, 16], as developed and improved in [17, 18, 4, 19], with the ideas proposed in [3].

Before stating the main results of this paper, let us introduce some preliminary notations: We let $\mathbb{Z}_\varepsilon^d$ denote the $d$-dimensional simple-cubic lattice with lattice spacing $\varepsilon$; $C^{(\varepsilon)}$ denotes the Euclidean propagator (two-point function) of the free (Gaussian) lattice theory with mass $m_0$; $S_{2,\lambda}^{(\varepsilon)}$ denotes the Euclidean propagator (two-point function) of conventionally renormalized $\varphi_d^4$ lattice theory with bare coupling constant $\lambda$; $u_{4,\lambda}^{(\varepsilon)}$ denotes the connected four-point function (Ursell function) of this same theory; $u_{4,\lambda,\mathrm{tree}}^{(\varepsilon)}$ denotes the perturbative tree-graph contribution to $u_{4,\lambda}^{(\varepsilon)}$ (it is a certain "generalized convolution" of free propagators $C^{(\varepsilon)}$, with coefficient $-6\lambda$); and finally, $m^{(\varepsilon)} = m^{(\varepsilon)}(\lambda)$ denotes the physical mass gap of this theory. Constants independent of $\varepsilon$ and $\lambda$ will be denoted by $c_1, c_2, \ldots$. We define a norm $|||\cdot|||$ as follows: Let $f$ be a function in $l^1(\mathbb{Z}^d)$. We set

$$|||f||| = \sup_{x \in \mathbb{Z}_\varepsilon^d} |f(x)| + \varepsilon^d \sum_{x \in \mathbb{Z}_\varepsilon^d} |f(x)| . \tag{1.1}$$

The dependence of various quantities on the bare mass $m_0$ will be suppressed in the following. (By scaling, one can set $m_0 = 1$ in two and three dimensions; see Sect. 8. The dimensionless bare coupling constant is $\lambda m_0^{d-4}$.)

We may now state the main result of this paper:

**Theorem 1.1.** *Let $\lambda \geqq 0$ be sufficiently small. Then the correlation functions of the $\varphi^4_{2,3}$ lattice field theories (renormalized as described in Sect. 4, below) are bounded uniformly in the lattice spacing $\varepsilon$ and satisfy the estimates*

$$|||S^{(\varepsilon)}_{2,\lambda} - C^{(\varepsilon)}||| \leqq c_1 \lambda^2 \tag{1.2}$$

*and*

$$\varepsilon^{3d} \sum_{x,y,z} |u^{(\varepsilon)}_{4,\lambda}(0,x,y,z) - u^{(\varepsilon)}_{4,\lambda,\mathrm{tree}}(0,x,y,z)| \leqq c_2 \lambda^2 . \tag{1.3}$$

*Moreover, the physical mass gap of the lattice theory satisfies*

$$|m^{(\varepsilon)} - m_0| \leqq c_3 \lambda^2 \tag{1.4}$$

*uniformly in $\varepsilon$. A continuum limit exists (for a suitable sequence $\varepsilon_i \to 0$) and satisfies all Osterwalder-Schrader axioms except perhaps rotation invariance. Moreover, the obvious analogues of the estimates* (1.2)–(1.4) *hold for any such continuum limit; in particular, if $\lambda \neq 0$ then the limiting theory is non-Gaussian.*

*Remarks.* 1. We actually show considerably stronger estimates than (1.2)–(1.4); see Sect. 6 through 8.

2. It has been shown in [20–24] (by other, much more difficult methods) that, for $\lambda \geqq 0$ small, *all* Osterwalder-Schrader axioms are valid. For results which hold also at strong coupling, see [25–27].

We define the "susceptibility" $\chi^{(\varepsilon)}_\lambda$ by

$$\chi^{(\varepsilon)}_\lambda \equiv \varepsilon^d \sum_{x \in \mathbb{Z}^d} S^{(\varepsilon)}_{2,\lambda}(x) . \tag{1.5}$$

As a corollary of Theorem 1.1 we obtain a result for arbitrary (not necessarily weak) couplings which lie in the single-phase region:

**Theorem 1.2.** *Let $\lambda \geqq 0$ be such that*

$$\limsup_{\varepsilon \to 0} \chi^{(\varepsilon)}_\lambda < \infty . \tag{1.6}$$

*Then a continuum limit exists (for a suitable sequence $\varepsilon_i \to 0$) and satisfies all Osterwalder-Schrader axioms except perhaps rotation invariance. Moreover, the correlation functions for the limiting theory are not identically zero.*

*Remarks.* 1. As before, we actually prove a somewhat stronger result than that stated above; see Sect. 8. In three dimensions there exists a "critical" theory, i.e. one for which the mass gap $m$ vanishes; see Sect. 9.

2. Under the same hypothesis, a continuum limit of the $\varphi^4_{2,3}$ lattice theory in an "external magnetic field" exists and satisfies all Osterwalder-Schrader axioms except perhaps rotation invariance. This can be proven by using a variant of the methods developed in this paper, but we shall not describe the construction here.

It is of some interest to compare our construction of $\varphi^4_3$ with the previous literature on this model. There are, at present, five main approaches to $\varphi^4_3$:

1) the "traditional" and earliest method, due to Glimm and Jaffe [1], with further work by Feldman and Osterwalder [28, 20, 25, 24], Magnen and Sénéor [21, 22] and others [23, 26, 27, 29–33, 12];

2) the "Italian" method [34–38];

3) Bałaban's method [39–41];

4) the Battle-Federbush method [42, 43]; and

5) our method, based on Schwinger-Dyson equations and correlation inequalities.

The first four of these approaches all involve a considerable amount of "hard analysis," and all are based on one form or another of *phase-space cell localization.* (The renormalization-group philosophy is particularly apparent in methods 2, 3, and 4.) All work initially with unnormalized quantities, and a key step in each method is the proof of ultraviolet and infrared stability for the partition function (or a variant thereof in method 4). These methods (or at least method 1, and potentially the others) yield very strong information about the model, including Euclidean invariance [20, 21] and Borel summability [22]. Moreover, the techniques are applicable to other superrenormalizable models. Our approach, by contrast, is considerably simpler: it is inspired by, and for this model makes rigorous, elementary mass-renormalized perturbation theory. We make no use of renormalization-group insights: $\varphi_3^4$ is sufficiently simple (we now know) that no such heavy machinery is needed. We work only with normalized quantities (i.e. correlation functions), thus avoiding much technical complication. Of course, our results are much weaker than those of method 1. There is no hope whatsoever of proving Borel summability by our methods (at least in their present form), since correlation inequalities are inherently restricted to real values of the coupling constant $\lambda$. Euclidean invariance is by no means hopeless (see our comments in Sect. 9), but we are at present quite far from proving it. Our method is embarrassingly limited to $\varphi^4$ models (with $N=0$, 1 or 2 components): at present we cannot even see how to treat $P(\varphi)_2$ models other than $\varphi_2^4$. (This is because, in the other cases, the required correlation inequalities are either unknown, or else are known to be false!) Thus, our work in no way renders the previous work (or possible future extensions of it) obsolete; it merely gives a simpler approach to obtaining *some* of the major results on the $\varphi_3^4$ model.

Our methods apply to $\varphi_d^4$ theories in any space-time dimension $d<4$ (in fact, they get simpler as $d$ is reduced). But the traditional construction of $\varphi_2^4$ [2, 44] (see also [4] for mass gap) is already fairly simple, so our method has fewer advantages in that case.

We should also remark that somewhat related ideas have been used, in a less delicate context, by Bricmont et al. [45].

The plan of this paper is the following: In Sects. 2 through 4 we review the definition of the $\varphi_d^4$ lattice model, introduce the Feynman-diagram notation, and discuss the continuum limit and renormalization. We have tried to make the exposition accessible to non-experts; more information can be found in [2, 3] and the references cited there. In Sect. 5 we introduce the three main technical tools employed in our analysis: the field equation for the 2-point function, the "skeleton inequalities" for the 4-point function, and the $l^1$ continuity of the lattice 2-point function in the bare parameters. In Sect. 6 we show that mass-renormalized perturbation theory for the 2-point function is asymptotic to order $\lambda^2$ as $\lambda \to 0$, uniformly in the lattice spacing $\varepsilon$; this section is the heart of the paper. In Sect. 7 we refine the analysis of Sect. 6 so as to derive an asymptotic expansion to order $\lambda^2$ for the mass gap; this section may be omitted on a first reading. In Sect. 8 we complete the proofs of Theorems 1.1 and 1.2, using standard methods. In Sect. 9 we remark on various corollaries and possible extensions of our analysis. In the appendix we collect some classical inequalities of real analysis and prove some generalizations which are needed in Sect. 7.

## 2. Lattice $\varphi_d^4$ Field Theory

The $\varphi_d^4$ field theory, $d=2$ or 3, will be constructed as a limit of finite-volume lattice field theories as the region $\Lambda \subset \mathbb{R}^d$ increases to $\mathbb{R}^d$ and the lattice spacing $\varepsilon$ tends to 0. We shall first take the infinite-volume limit ($\Lambda \nearrow \mathbb{R}^d$) and subsequently pass to the continuum limit ($\varepsilon \searrow 0$), but our methods would probably permit us, with extra work, to reverse the order of these limits. In the literature, the continuum limit has usually been taken before the infinite-volume limit.

First, we construct a theory on the finite lattice

$$L \equiv (\varepsilon \mathbb{Z})^d \cap \Lambda \subset \mathbb{R}^d .$$

Points in the lattice are denoted by $x, y$, etc., and are labelled by their Cartesian coordinates in $\mathbb{R}^d$. To each point $x \in L$ is associated a field $\varphi(x)$ which is a real random variable. The collection of all these random variables, $\varphi = \{\varphi(x) : x \in L\}$, is distributed according to the probability measure

$$\prod_{x \in L} d\varphi(x) e^{-S_L(\varphi)} / Z_L ,$$

where $Z_L$ is a normalization factor (it is called the "partition function") and the "action" $S_L$ is given by

$$S_L(\varphi) \equiv \frac{1}{2} \sum_{\langle xy \rangle} \varepsilon^{d-2} (\varphi(x) - \varphi(y))^2 + \frac{1}{2} a \sum_{x \in L} \varepsilon^d \varphi(x)^2 + \frac{\lambda}{4} \sum_{x \in L} \varepsilon^d \varphi(x)^4 , \tag{2.1}$$

where $\lambda \geqq 0$, $a \in \mathbb{R}$. The sum over $\langle xy \rangle$ is a sum over nearest neighbors *in the infinite lattice* $(\varepsilon \mathbb{Z})^d$ (each pair is counted *twice* in the sum, once in each order). We extend the definition of $\varphi$ to $x \notin L$ by setting $\varphi(x) = 0$ outside $L$. This procedure imposes *Dirichlet* boundary conditions. The action $S_L(\varphi)$ is a finite-difference approximation to

$$S(\varphi) \equiv \int \left\{ \frac{1}{2} [\nabla \varphi(x)]^2 + \frac{1}{2} a \varphi(x)^2 + \frac{\lambda}{4} \varphi(x)^4 \right\} d^d x , \tag{2.2}$$

and the above probability measure can be thought of as a finite-difference approximation to a measure on continuum fields *heuristically* given by

$$\text{“} Z^{-1} e^{-S(\varphi)} \prod_{x \in \mathbb{R}^d} d\varphi(x) \text{”} .$$

We define the *expectation* for the finite-volume lattice theory by

$$\langle F\rangle_L \equiv \frac{1}{Z_L}\int \prod_{x\in L} d\varphi(x) e^{-S_L(\varphi)} F(\varphi), \tag{2.3}$$

where $F$ is an arbitrary function of $\varphi$. If $F(\varphi)$ is a polynomial in $\{\varphi(x): x\in L\}$ with positive coefficients, and $L'$ is a lattice containing $L$ (but of the same lattice spacing $\varepsilon$), then

$$\langle F\rangle_L \leqq \langle F\rangle_{L'}, \tag{2.4}$$

a consequence of the second Griffiths inequality [44, 46]. Moreover, it is known [47–49] that such expectations are bounded uniformly in $L$. Therefore the limit

$$\langle F\rangle^{(\varepsilon)} \equiv \lim_{L\nearrow(\varepsilon\mathbb{Z})^d} \langle F\rangle_L \tag{2.5}$$

exists and is, by construction, invariant under translations which preserve $(\varepsilon\mathbb{Z})^d$. This extends to arbitrary polynomials $F$. For a summary of the properties of the lattice $\varphi_d^4$ field theory, see [3].

*Remarks 1.* Our notation in the present paper is slightly different from that in [3, 5]. In particular, we here denote by $\lambda/4$ what was previously denoted by $\lambda_0/4! = \lambda_0/24$. The reader should bear this in mind when reading Sect. 5.

2. In [3, 9], the coefficient of the $(\nabla\varphi)^2$ term in $S(\varphi)$ is considered to be a third free parameter of the model (after $\lambda$ and $a$); this allows for "field-strength renormalization." However, for superrenormalizable $\varphi_d^4$ models in dimension $d<4$, the desired field-strength renormalization is the trivial one, i.e. the coefficient of the $(\nabla\varphi)^2$ term is just a fixed positive number. We have here introduced this simplification from the beginning, in (2.1) and (2.2).

## 3. Feynman Diagrams for the Lattice $\varphi_d^4$ Theory

If in $\langle - \rangle^{(\varepsilon)}$ we set $a=m_0^2$, $\lambda=0$, the resulting theory is the *free lattice field theory of mass $m_0$*. It is given by a Gaussian measure, and all moments of this measure are determined by the covariance ($\equiv$"free two-point function"$\equiv$"free propagator")

$$C^{(\varepsilon)}(x-y) \equiv \langle \varphi(x)\varphi(y)\rangle^{(\varepsilon)}. \tag{3.1}$$

The covariance $C^{(\varepsilon)}$ can be calculated explicitly and turns out to be

$$C^{(\varepsilon)}(x-y) = (2\pi)^{-d} \int_{\left[-\frac{\pi}{\varepsilon},\frac{\pi}{\varepsilon}\right]^d} \left[m_0^2 + 2\varepsilon^{-2}\sum_{i=1}^{d}(1-\cos\varepsilon k_i)\right]^{-1} e^{ik\cdot(x-y)} d^dk, \tag{3.2}$$

which is best thought of as the kernel of $(m_0^2-\Delta)^{-1}$, where $\Delta$ is the finite-difference Laplacian for the infinite lattice. If $d\leqq 2$, $m_0$ must be strictly positive; if $d>2$, $m_0=0$ is allowed as well.

When the coupling constant $\lambda$ does not vanish, $a$ can be an arbitrary real number, and the covariance

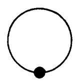
$$S^{(\varepsilon)}(x-y) \equiv \langle \varphi(x)\varphi(y)\rangle^{(\varepsilon)} \tag{3.3}$$

is known as the "interacting two-point function" or "interacting propagator."

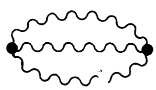
We now introduce the Feynman-graph notation which is extremely helpful in organizing the terms in a perturbative expansion of $\langle - \rangle^{(\varepsilon)}$ in $\lambda$. We set up the correspondence:

$$C^{(\varepsilon)}(x-y) \leftrightarrow x \text{———} y\,,$$
$$S^{(\varepsilon)}(x-y) \leftrightarrow x \sim\!\sim\!\sim\!\sim y\,.$$

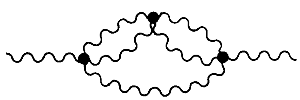
Examples of the use of Feynman graphs to represent algebraic expressions are:


$x$

$$\leftrightarrow C^{(\varepsilon)}(x-x) = C^{(\varepsilon)}(0)\,,$$

$x$  $y \leftrightarrow S^{(\varepsilon)}(x-y)^3\,.$

We adopt the *convention* that unlabelled vertices of graphs are summed over the lattice (with weight $\varepsilon^d$ at each site). Thus

$x$  $y \leftrightarrow$

$$\iiint S^{(\varepsilon)}(x-z_1) S^{(\varepsilon)}(z_1-z_2) \cdot S^{(\varepsilon)}(z_1-z_3)^2 S^{(\varepsilon)}(z_3-z_2)^2 S^{(\varepsilon)}(z_2-y)\, dz_1\, dz_2\, dz_3\,,$$

where we have also used the

*Convention.* Riemann sums $\sum_z \varepsilon^d f(z)$ over the lattice $\varepsilon\mathbb{Z}^d$ are denoted by integrals $\int f(z) d^d z$.

## 4. Preliminaries for the Continuum Limit

We consider a sequence $(L_i)$ of infinite lattices

$$\begin{aligned} &L_i \equiv (\varepsilon_i \mathbb{Z})^d\,,\\ &L_1 \subset L_2 \subset \ldots\,, \end{aligned} \tag{4.1}$$

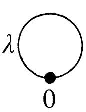
i.e., each lattice refines its predecessor. (This condition is convenient but not really necessary; see [3].) The goal is to construct a limit

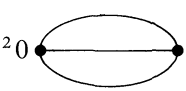
$$S_n(x_1, \ldots, x_n) \equiv \lim_{i\to\infty} \left\langle \prod_{j=1}^n \varphi(x_j) \right\rangle^{(\varepsilon_i)}\,, \tag{4.2}$$

where $\{x_1, \ldots, x_n\}$ is contained in $L_{i_0}$, for some arbitrary, finite $i_0$. (Equivalently, the right-hand and left-hand sides of (4.2) can be considered as distributions in $\mathscr{S}'(\mathbb{R}^{nd})$, and the limit is taken in this space; see [3].) By soft analysis, it can be shown that the distributions $S_n$, called the Euclidean Green's functions (or *Schwinger functions*) of the continuum theory, are the moments of a probability measure (see [3]). In this paper we shall construct a continuum limit by proving bounds on $\left\langle \prod_{j=1}^n \varphi(x_j) \right\rangle^{(\varepsilon_i)}$ which are uniform in $i$ and then appealing to a compactness argument. In proving those bounds one meets a complication: It is known that the expectations $\left\langle \prod_{j=1}^{n} \varphi(x_j) \right\rangle^{(\varepsilon_i)}$ converge to 0, as $i \to \infty$, unless the bare mass parameter $a$ in the definition of $\langle - \rangle^{(\varepsilon_i)}$ is chosen to depend on $\varepsilon_i$ in a certain fashion. Indeed, it must be chosen to diverge to $-\infty$, as $i \to \infty$. The precise way in which it is taken to $-\infty$ is a rather delicate matter and is correctly predicted by perturbation theory in $\lambda$. The fact that perturbation theory correctly predicts the $\varepsilon$-dependence of the bare mass $a(\varepsilon)$ is a simplifying feature of $\varphi^4$ theory in two or three as opposed to four dimensions, or more generally of "super-renormalizable" as opposed to "renormalizable" field theories. For $d=2$ or 3, perturbation theory yields

$$a(\varepsilon) = \begin{cases} m_0^2 + \delta m_1^2(\varepsilon), & d=2 \\ m_0^2 + \delta m_1^2(\varepsilon) + \delta m_2^2(\varepsilon), & d=3, \end{cases} \tag{4.3}$$

where $m_0^2 > 0$ is the mass parameter appearing in the free propagator $C^{(\varepsilon)}(x-y)$, and

$\delta m_1^2 \equiv -3\lambda$  , (4.4)

$\delta m_2^2 \equiv 6\lambda^2$  . (4.5)

By (3.2),

$$\delta m_1^2(\varepsilon) \sim \begin{cases} -O(|\ln \varepsilon|), & d=2 \\ -O(\varepsilon^{-1}), & d=3, \end{cases}$$

and

$$\delta m_2^2(\varepsilon) \sim \begin{cases} O(1), & d=2 \\ O(|\ln \varepsilon|), & d=3. \end{cases}$$

Note that $\delta m_2^2(\varepsilon)$ remains finite, as $\varepsilon \searrow 0$, in two dimensions; for this reason it is omitted (for simplicity) from (4.3), although it could equally well be included. The process of choosing $a(\varepsilon)$, and other parameters on which $\langle - \rangle^{(\varepsilon)}$ depends, as a function of the lattice spacing $\varepsilon$ (in a way which may well diverge as $\varepsilon \searrow 0$) is called *renormalization*. See [2, 3] for an account of renormalization theory.

## 5. The Main Tools

Our proof of the existence and nontriviality of the continuum limit for $\varphi_2^4$ and $\varphi_3^4$ employs three main ingredients: the field equation (or "Schwinger-Dyson equation") for the 2-point function [14, 7, 3]; the "skeleton inequalities" for the 4-point function [5]; and the $l^1$-continuity of the lattice 2-point function in the bare parameters (Proposition 5.1 below). In this section we discuss briefly each of these items.

The field equations can be derived by integration by parts in the defining integral (2.3). *Formally*,

$$\langle \cdot \rangle = \frac{\langle \cdot e^{-V} \rangle_G}{\langle e^{-V} \rangle_G}, \tag{5.1}$$

where $\langle \cdot \rangle_G$ denotes expectation in the Gaussian (lattice) measure with covariance $C \equiv C^{(\varepsilon)}$, and

$$V = \frac{\lambda}{4} \int \varphi(x)^4 d^d x + \frac{a - m_0^2}{2} \int \varphi(x)^2 d^d x . \tag{5.2}$$

(Here we have used our convention that lattice sums with factors $\varepsilon^d$ are denoted by integrals.) Inside a Gaussian expectation, $\varphi(x)$ is equivalent to $(C^* \delta/\delta\varphi)(x)$, i.e.

$$\langle \varphi(x) F(\varphi) \rangle_G = \int d^d y C(x-y) \left\langle \frac{\delta F}{\delta \varphi(y)} \right\rangle_G ; \tag{5.3}$$

this is a consequence of integration by parts. We apply it to the 2-point function by writing

$$S(x-y) \equiv S^{(\varepsilon)}(x-y) = \frac{\langle \varphi(x)\varphi(y) e^{-V} \rangle_G}{\langle e^{-V} \rangle_G}, \tag{5.4}$$

and using (5.3) with

$$F(\varphi) = \varphi(y) e^{-V} . \tag{5.5}$$

We obtain

$$\begin{aligned} S(x-y) &= C(x-y) - \int d^d z C(x-z) \left\langle \frac{\delta V}{\delta \varphi(z)} \varphi(y) \right\rangle \\ &= C(x-y) - (a - m_0^2) \int d^d z C(x-z) S(z-y) \\ &\quad - \lambda \int d^d z C(x-z) \langle \varphi(z)^3 \varphi(y) \rangle . \end{aligned} \tag{5.6}$$

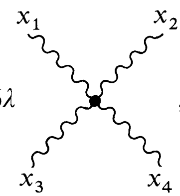
Of course, this discussion has been purely formal, because of its cavalier manipulation of infinite-volume sums and integrals. A more careful treatment, which works first in finite volume and then uses the DLR equations to handle the infinite-volume measure, is given in [3]; the upshot is that (5.6) is rigorously valid.

The field equation (5.6) is an *identity* which expresses one unknown quantity (the 2-point function $S$) in terms of another unknown quantity (the 4-point function $\langle \varphi(z)^3 \varphi(y) \rangle$). At first glance this may not seem particularly useful. It becomes more useful, however, when we combine it with the "skeleton inequalities" [5], which bound the 4-point function both above and below in terms of the 2-point function. These inequalities are most easily expressed in terms of the *connected* 4-point function

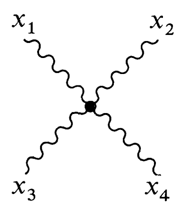
$$\begin{aligned} u_4(x_1, x_2, x_3, x_4) &\equiv \langle \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) \rangle \\ &\quad - \begin{matrix} x_1 \sim\sim x_2 \\ x_3 \sim\sim x_4 \end{matrix} - \begin{matrix} x_1 \sim\sim x_3 \\ x_2 \sim\sim x_4 \end{matrix} - \begin{matrix} x_1 \sim\sim x_4 \\ x_2 \sim\sim x_3 \end{matrix} . \end{aligned} \tag{5.7}$$

Then the first three skeleton inequalities are:

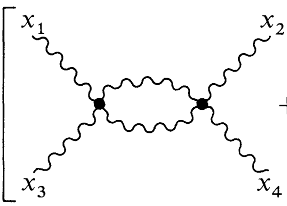
$$u_4(x_1, x_2, x_3, x_4) \leqq 0, \tag{5.8}$$

$$u_4(x_1, x_2, x_3, x_4) \geqq -6\lambda$$  , (5.9)



$$u_4(x_1, x_2, x_3, x_4) \leqq -6\lambda$$ 

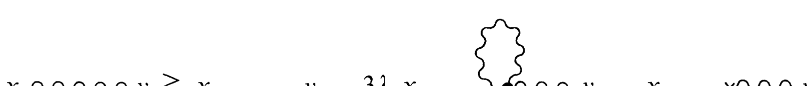


$+18\lambda^2$ [  + two permutations ] . (5.10)

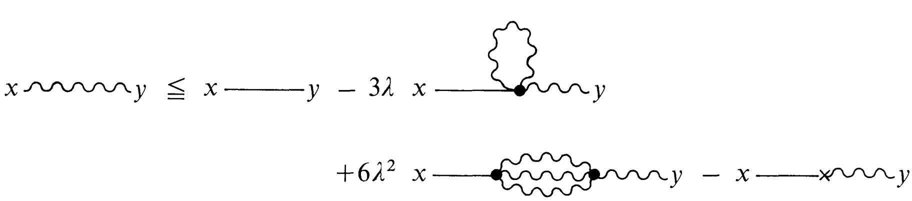


A detailed proof of (5.8)–(5.10) is given in [5], and is based on the random-walk expansion of [4]. Let us simply note here that the right-hand sides of (5.8)–(5.10) are precisely the low-order perturbation-theory expressions for $u_4$, except that interacting propagators (i.e. 2-point functions) appear in place of free propagators.

By combining the field equation (5.6) with the skeleton inequalities (5.8)–(5.10), we can obtain inequalities which bound the 2-point function above and below in terms of itself. Indeed, inserting (5.7) and (5.8)–(5.10) into (5.6), we find

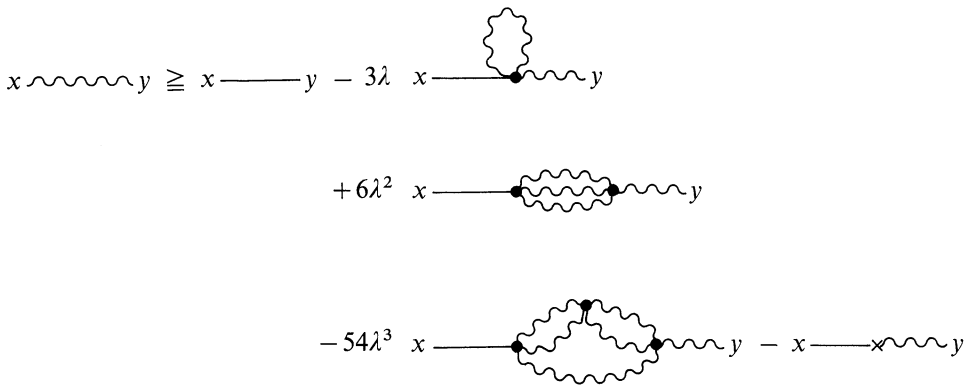
 (5.11)



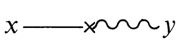
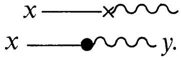


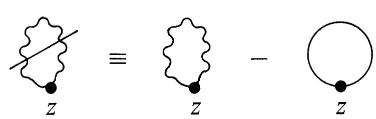


(5.12)

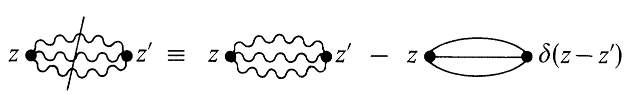


(5.13)

where  is shorthand for the mass counterterm $(a-m_0^2)$ . We can now see the rationale for the choices (4.4)/(4.5) of the mass counterterm: they are designed precisely to cancel the ultraviolet divergences on the right-hand side of (5.11)–(5.13). That they succeed in doing so is not all obvious; but it is true, as we shall show in Sect. 6! In any case, we are free to introduce the notation

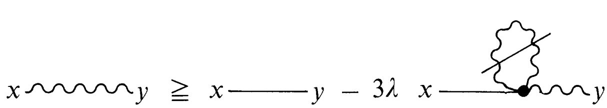


(5.14)

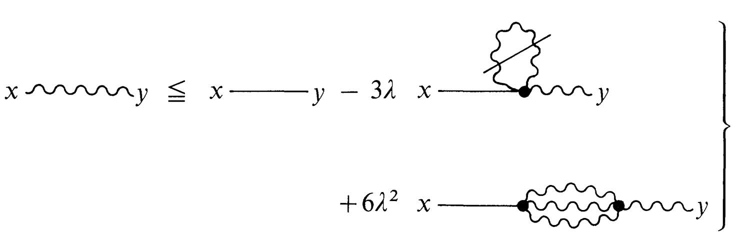


(5.15)

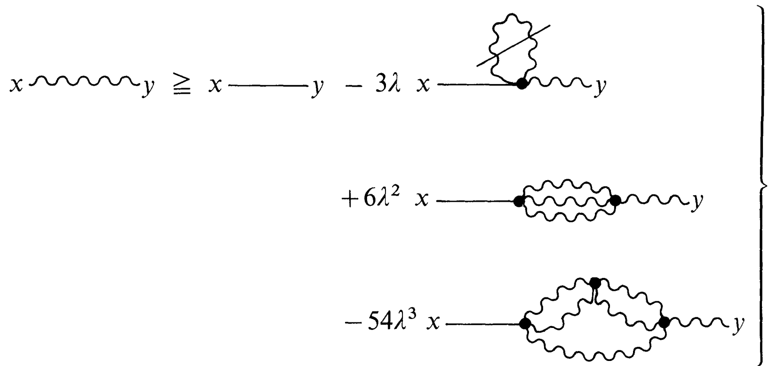
Then, for $\varphi_2^4$, (5.11)–(5.13) become



(5.16)

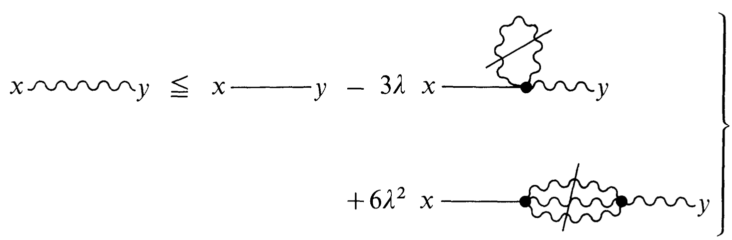


(5.17)

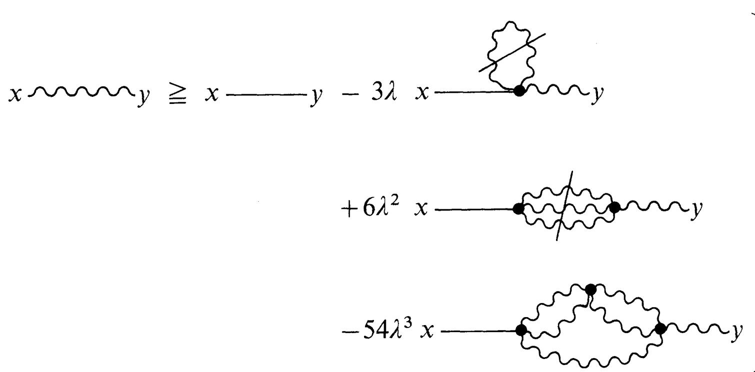



(5.18)

For $\varphi_3^4$, (5.11) is useless, but (5.12)–(5.13) become




(5.19)




(5.20)

We call (5.16)–(5.18) and (5.19) and (5.20) the *propagator inequalities* for $\varphi_2^4$ and $\varphi_3^4$, respectively.

In the following section we shall use the propagator inequalities to control the difference between the interacting propagator ∿∿∿∿ and the free propagator ———, for sufficiently small $\lambda$. Surprisingly, we do not need any a priori information on the interacting propagator other than its continuity in the bare parameters $\lambda, a$ (for *fixed* lattice spacing $\varepsilon$). Let $\mathscr{B}$ be the space of allowable bare parameters[1]:

1 For lattice dimension $d > 2$, the point $\lambda = 0$, $a = 0$ (massless free field) can also be included in $\mathscr{B}$, if desired

$$\mathscr{B}=\{(\lambda,a):\lambda>0,a\in\mathbb{R}\}\cup\{(\lambda,a):\lambda=0,a>0\}. \tag{5.21}$$

We give $\mathscr{B}$ its usual topology as a subset of $\mathbb{R}^2$.

**Proposition 5.1.** *Fix the lattice spacing $\varepsilon>0$. Then:*

(a) *The set*

$$\mathscr{B}_0=\{(\lambda,a)\in\mathscr{B}:\|S(\lambda,a)\|_1<\infty\} \tag{5.22}$$

*is a nonempty, connected, open subset of $\mathscr{B}$.*

(b) *The map $(\lambda,a)\mapsto\|S(\lambda,a)\|_1$ is a continuous map from $\mathscr{B}$ into $[0,+\infty]$.*

(c) *The map $(\lambda,a)\mapsto S(\lambda,a)$ is a continuous map from $\mathscr{B}_0$ into $l^1$.*

Here $\mathscr{B}_0$ is the single-phase region minus the critical surface; the subtlest part of this proposition is the assertion that $\|S\|_1$ (which in the single-phase region is just the susceptibility) diverges continuously as the critical surface is approached.

*Proof.* Since the lattice spacing plays no role, we might as well set $\varepsilon=1$. Now by the Simon-Lieb inequality [50, 51] in the version of [4, 19], the 2-point function satisfies

$$\langle\varphi(0)\varphi(x)\rangle\leqq\langle\varphi(0)\varphi(x)\rangle_A+\sum_{\substack{z\in A\\ z'\notin A}}\langle\varphi(0)\varphi(z)\rangle_A J_{zz'}\langle\varphi(z')\varphi(x)\rangle, \tag{5.23}$$

where

$$J_{zz'}=\begin{cases}1 & \text{if } |z-z'|=1\\ 0 & \text{otherwise},\end{cases} \tag{5.24}$$

and $0\in A\subset\mathbb{Z}^d$. [Here $\langle\cdot\rangle_A$ is the expectation in the lattice model in the region $A$, with zero ($\equiv$ Dirichlet) boundary conditions.] As explained in [50], if

$$\sum_{\substack{z\in A\\ z'\notin A}}\langle\varphi(0)\varphi(z)\rangle_A J_{zz'}<1, \tag{5.25}$$

then (5.23) can be iterated to obtain an upper bound for $\langle\varphi(0)\varphi(x)\rangle$ that explicitly exhibits exponential decay (and hence $\|S\|_1<\infty$); moreover, this upper bound depends smoothly on the finite-volume expectations $\langle\varphi(y)\varphi(z)\rangle_A$. Now if $(\lambda,a)$ are such that $\|S(\lambda,a)\|_1<\infty$, then (5.25) can clearly be made to hold by choosing the volume $A$ sufficiently large; and then (5.25) also holds for $(\lambda',a')$ in a small neighborhood of $(\lambda,a)$, since the finite-volume expectations are continuous functions of the bare parameters. Thus $\mathscr{B}_0$ is an open subset of $\mathscr{B}$, and $\|S(\lambda,a)\|_1$ is *locally bounded* on $\mathscr{B}_0$. (This argument is essentially already contained in [51]; we repeat it here for the convenience of the reader.)

Clearly $\mathscr{B}_0$ is nonempty, since it contains the points $\lambda=0$, $a>0$ (massive free field). Moreover, $\|S(\lambda,a)\|_1$ is a decreasing function of $\lambda$ and $a$ (by the Griffiths inequality), from which it follows that any two points $(\lambda,a)$, $(\lambda',a')\in\mathscr{B}_0$ can be connected in $\mathscr{B}_0$ by a pair of line segments, one vertical and one horizontal. This proves (a).

We now note that $S(\lambda,a)$ is the monotone limit, pointwise in $x$-space, of the finite-volume 2-point function

$$S_V(\lambda,y)(x)\equiv\langle\varphi(0)\varphi(x)\rangle_{V,\lambda,a}, \tag{5.26}$$

as $V \uparrow \mathbb{Z}^d$. By the monotone convergence theorem, $\|S_V(\lambda, a)\|_1 \to \|S(\lambda, a)\|_1$ as $V \uparrow \mathbb{Z}^d$ (irrespective of whether $\|S(\lambda, a)\|_1$ is finite or $+\infty$); and if $\|S(\lambda, a)\|_1 < \infty$, then $S_V(\lambda, a) \to S(\lambda, a)$ in $l^1$ norm. Since $S_V(x) \geqq 0$,

$$\|S_V(\lambda, a)\|_1 = \sum_{x \in V} \langle \varphi(0)\varphi(x) \rangle_{V, \lambda, a}, \tag{5.27}$$

so

$$\frac{\partial}{\partial \lambda} \|S_V(\lambda, a)\|_1 = \sum_{x \in V} \frac{\partial}{\partial \lambda} \langle \varphi(0)\varphi(x) \rangle_{V, \lambda, a} = -\frac{1}{4} \sum_{x, y \in V} \langle \varphi(0)\varphi(x); \varphi(y)^4 \rangle_{V, \lambda, a}, \tag{5.28}$$

where we have used the convenient shorthand $\langle A; B \rangle \equiv \langle AB \rangle - \langle A \rangle \langle B \rangle$. By the Griffiths inequality and the strong Gaussian inequality [52, 53, 4, 5], we get

$$\begin{aligned} 0 \geqq \frac{\partial}{\partial \lambda} \|S_V(\lambda, a)\|_1 &\geqq -3 \sum_{x, y \in V} \langle \varphi(0)\varphi(y) \rangle_{V, \lambda, a} \langle \varphi(y)\varphi(x) \rangle_{V, \lambda, a} \langle \varphi(y)^2 \rangle_{V, \lambda, a} \\ &\geqq -3 \sum_{x, y} \langle \varphi(0)\varphi(y) \rangle_{\lambda, a} \langle \varphi(y)\varphi(x) \rangle_{\lambda, a} \langle \varphi(y)^2 \rangle_{\lambda, a} \\ &= -3 \|S(\lambda, a)\|_1^2 S(\lambda, a)(0) \\ &\geqq -3 \|S(\lambda, a)\|_1^3 . \end{aligned} \tag{5.29}$$

Similarly one can show that

$$0 \geqq \frac{\partial}{\partial a} \|S_V(\lambda, a)\|_1 \geqq -\|S(\lambda, a)\|_1^2 . \tag{5.30}$$

Now

$$\|S_V(\lambda_2, a)\|_1 - \|S_V(\lambda_1, a)\|_1 = \int_{\lambda_1}^{\lambda_2} d\lambda \frac{\partial}{\partial \lambda} \|S_V(\lambda, a)\|_1 , \tag{5.31}$$

which is bounded by const $\times |\lambda_2 - \lambda_1|$ for $(\lambda_1, a)$, $(\lambda_2, a)$ lying in a small neighborhood in $\mathscr{B}_0$, by (5.29) and the local boundedness of $\|S(\lambda, a)\|_1$ on $\mathscr{B}_0$. Similarly (5.30) handles variation of $a$ with $\lambda$ fixed. Letting $V \uparrow \mathbb{Z}^d$, one concludes that $\|S(\lambda, a)\|_1$ is a continuous (in fact, locally Lipschitz) function *on the set* $\mathscr{B}_0$.

We still have to prove that $\|S(\lambda_n, a_n)\|_1 \to \|S(\lambda, a)\|_1 = +\infty$ whenever

$$(\lambda_n, a_n) \to (\lambda, a) \in \mathscr{B} \backslash \mathscr{B}_0 .$$

But it is easy: for assume otherwise; then passing to a subsequence we can assume that $\|S(\lambda_n, a_n)\|_1$ is bounded by $K$. But then

$$\|S_V(\lambda_n, a_n)\|_1 \leqq \|S(\lambda_n, a_n)\|_1 \leqq K ,$$

so $\|S_V(\lambda, a)\|_1 \leqq K$ by the continuity of the finite-volume expectations. Taking $V \uparrow \mathbb{Z}^d$, we find $\|S(\lambda, a)\|_1 \leqq K$, contradicting the definition of $(\lambda, a)$. This completes the proof of (b).

The proof of (c) is virtually identical to that of the first part of (b):

$$\left\|\frac{\partial}{\partial\lambda}S_V(\lambda,a)\right\|_1 = \sum_{x\in V}\left|\frac{\partial}{\partial\lambda}\langle\varphi(0)\varphi(x)\rangle_{V,\lambda,a}\right|$$
$$= \frac{1}{4}\sum_{x,y\in V}\langle\varphi(0)\varphi(x);\varphi(y)^4\rangle_{V,\lambda,a}$$
$$\leqq 3\|S(\lambda,a)\|_1^3, \tag{5.32}$$

and

$$\|S_V(\lambda_2,a)-S_V(\lambda_1,a)\|_1 \leqq \int_{\lambda_1}^{\lambda_2} d\lambda \left\|\frac{\partial}{\partial\lambda}S_V(\lambda,a)\right\|_1 \tag{5.33}$$

take the place of (5.29) and (5.31). Similarly one handles variation of $a$ with $\lambda$ fixed. Using these equations and the local boundedness of $\|S(\lambda,a)\|_1$ on $\mathscr{B}_0$, and then letting $V\uparrow\mathbb{Z}^d$, one concludes that $S(\lambda,a)$ is a continuous (in fact, locally Lipschitz) $l^1$-valued function on $\mathscr{B}_0$. □

*Remarks.* 1. The proof of (b) can be summarized in "high-falutin'" language as follows: The function $\|S(\lambda,a)\|_1$, being a continuous function on the open set $\mathscr{B}_0$ and equal to $+\infty$ on $\mathscr{B}\backslash\mathscr{B}_0$, is therefore *upper semicontinuous*. But $\|S(\lambda,a)\|_1$, being the supremum of the continuous functions $\|S_V(\lambda,a)\|_1$, is necessarily *lower semicontinuous*. Hence it is continuous.

2. The proof of [3, Proposition 2.1] is incomplete; it requires an argument similar to the one given here.

3. The proof given here shows continuity not only for nearest-neighbor interactions, but in fact in the cone of ferromagnetic pair interactions of any fixed finite range. (Probably it works also for suitably-decaying infinite-range interactions, but we haven't bothered to work out the details.)

4. A stronger form of part (b) is valid: $\|S(\lambda,a)\|_1^{-1}$ is a locally Lipschitz function of $\lambda$ and $a$ on the entire set $\mathscr{B}$. [This bounds from below the rate of divergence of $\|S(\lambda,a)\|_1$ as $(\lambda,a)$ approaches the boundary of $\mathscr{B}_0$.] The proof is simple: using (5.29)–(5.31) and letting $V\uparrow\mathbb{Z}^d$, one finds (on $\mathscr{B}_0$)

$$0 \leqq \|S(\lambda,a_2)\|_1^{-1} - \|S(\lambda,a_1)\|_1^{-1} \leqq \int_{a_1}^{a_2} da \frac{\|S(\lambda,a)\|_1^2}{\|S(\lambda,a_1)\|_1\|S(\lambda,a_2)\|_1}$$
$$\leqq |a_2-a_1|\frac{\|S(\lambda,a_1)\|_1}{\|S(\lambda,a_2)\|_1}, \tag{5.34}$$

and analogously for variation of $\lambda$ with a fixed. Subdividing the interval $[a_1,a_2]$ into $N$ smaller intervals, applying (5.34) for each smaller interval, and then letting $N\to\infty$ (using the continuity of $\|S(\lambda,a)\|_1$ on $\mathscr{B}_0$), we obtain

$$0 \leqq \|S(\lambda,a_2)\|_1^{-1} - \|S(\lambda,a_1)\|_1^{-1} \leqq |a_2-a_1| \tag{5.35}$$

for $(\lambda,a_1)$, $(\lambda,a_2)\in\mathscr{B}_0$. But since $\|S(\lambda,a)\|_1^{-1}$ is continuous on all of $\mathscr{B}$, we can let $(\lambda,a_1)$ approach the boundary of $\mathscr{B}_0$, and conclude that (5.35) holds for *all* $a_1,a_2$. (This implies, by the way, the critical-exponent inequality $\gamma\geqq 1$ [54–56], whose rigorous proof seems therefore to be a bit trickier than heretofore believed. One could avoid some of the subtlety of the foregoing proofs by working with periodic instead of Dirichlet boundary conditions [55], since for periodic b.c. the "fluctuation-dissipation relation" for $(\partial/\partial a)\|S_V(\lambda, a)\|_1$ is rigorously valid in finite volume, and hence

$$0 \geqq \frac{\partial}{\partial a}\|S_V(\lambda, a)\|_1 \geqq -\|S_V(\lambda, a)\|_1^2; \tag{5.36}$$

but then one has to work quite hard to prove that $\|S_V(\lambda, a)\|_1$ converges to $\|S(\lambda, a)\|_1$ as $V \uparrow \mathbb{Z}^d$.)

## 6. Uniform Bounds on the 2-Point Function

This section contains the main technical ideas of the paper. We shall show that mass-renormalized perturbation theory for the 2-point function is asymptotic to order $\lambda^2$ as $\lambda \to 0$, uniformly in the lattice spacing $\varepsilon$. Once this is achieved, similar statements for the $n$-point functions will follow from the "skeleton inequalities" of [5].

In this section we shall set $m_0 = 1$; this is no loss of generality, since any other nonzero value of $m_0$ can be obtained by scaling lengths. Our estimates of correlation functions on the lattice will use the $L^1 \cap L^\infty$ norm,

$$|||f||| \equiv \|f\|_1 + \|f\|_\infty \equiv \varepsilon^d \sum_{x \in \mathbb{Z}^d} |f(x)| + \sup_{x \in \varepsilon\mathbb{Z}^d} |f(x)|. \tag{6.1}$$

(For each fixed $\varepsilon$, this is equivalent to the $l^1$ norm; but since we seek estimates uniform in $\varepsilon$, the $L^1 \cap L^\infty$ norm is strictly stronger.)

The main result of this section is the following:

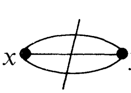
**Theorem 6.1.** *There exist universal constants $\lambda_0 > 0$, $c_1$, $c_2$ such that if $0 \leqq \lambda \leqq \lambda_0$, then*

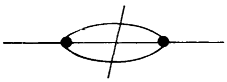
$$|||S^{(\varepsilon)} - C^{(\varepsilon)}||| \leqq c_1 \lambda^2, \tag{6.2}$$

$$|||S^{(\varepsilon)} - (C^{(\varepsilon)} + 6\lambda^2 C^{(\varepsilon)} * \psi^{(\varepsilon)} * C^{(\varepsilon)})||| \leqq c_2 \lambda^3 \tag{6.3}$$

*for all $\varepsilon \geqq 0$.*

Here

$\psi^{(\varepsilon)}(x, y) \equiv x$  $y \equiv x$  $y - 0$  $\delta^{(\varepsilon)}(x-y)$. (6.4)

Notice that $6\lambda^2 C^{(\varepsilon)} * \psi^{(\varepsilon)} * C^{(\varepsilon)} \equiv 6\lambda^2$  is precisely the second-order perturbation-theory contribution to $S^{(\varepsilon)}$.

The main technical estimate is contained in the following proposition. It is a straightforward consequence of taking the $|||\cdot|||$ norms of both sides of the propagator inequalities (5.19) and (5.20). We define

$$E^{(\varepsilon)} \equiv S^{(\varepsilon)} - C^{(\varepsilon)}. \tag{6.5}$$

**Proposition 6.2.** *There exist polynomials $P_1$, $P_2$, $P_3$, $Q_1$, $Q_2$, $Q_3$ with positive universal coefficients such that*

$$|||E^{(\varepsilon)}||| \leqq \sum_{n=1}^{3} \lambda^n P_n(|||E^{(\varepsilon)}|||), \tag{6.6}$$

$$|||E^{(\varepsilon)} - 6\lambda^2 C^{(\varepsilon)} * \psi^{(\varepsilon)} * C^{(\varepsilon)}||| \leqq \sum_{n=1}^{3} \lambda^n Q_n(|||E^{(\varepsilon)}|||) \tag{6.7}$$

*for all $\varepsilon > 0$. Moreover, the polynomials $P_1$, $Q_1$, and $Q_2$ have zero constant term.*

We emphasize that the estimates (6.6) and (6.7) are uniform in $\varepsilon$.

*Proof of Theorem 6.1 assuming Proposition 6.2.* Choose $\lambda_0 > 0$ so that

$$\sum_{n=1}^{3} \lambda_0^n P_n(2) \leqq 1 .$$

If $\lambda \in [0, \lambda_0]$, then by Proposition 6.2, $|||E^{(\varepsilon)}||| \leqq 2$ implies $|||E^{(\varepsilon)}||| \leqq 1$; in other words, $|||E^{(\varepsilon)}|||$ cannot lie in the interval $(1, 2]$. This is true for all $\varepsilon$. Now Proposition 5.1 implies that, for each $\varepsilon$, $|||E^{(\varepsilon)}|||$ is a continuous function of $\lambda$. Since at $\lambda = 0$ we have $|||E^{(\varepsilon)}||| = 0$, it follows that $|||E^{(\varepsilon)}||| \leqq 1$ for all $\lambda \in [0, \lambda_0]$. This is true for all $\varepsilon$. The estimates (6.2) and (6.3) then follow from (6.6) and (6.7). □

Let us once again emphasize the two ingredients in the proof of Theorem 6.1: an estimate uniform in $\varepsilon$ (Proposition 6.2); and the continuity of $|||E^{(\varepsilon)}|||$ in $\lambda$ *for each fixed* $\varepsilon$, with no uniformity required (Proposition 5.1).

We now turn to the proof of Proposition 6.2. The idea is to substitute $S^{(\varepsilon)} = C^{(\varepsilon)} + E^{(\varepsilon)}$ on the right-hand side of the propagator inequalities (5.19) and (5.20) and then to estimate $|||\cdot|||$ norms. Those terms involving only $C^{(\varepsilon)}$ are either ultraviolet convergent (i.e. bounded uniformly in $\varepsilon$) or are made ultraviolet finite by virtue of the mass counterterms. Those terms involving at least one $E^{(\varepsilon)}$ are all ultraviolet convergent, the intuitive reason being that functions in $L^1 \cap L^\infty$ are much less singular than the free propagator $C$. The actual estimates are very simple applications of the Hölder and Young inequalities, which imply (among other things) that $L^1 \cap L^\infty$ is a normed algebra with respect to both multiplication and convolution. From now on we suppress the $\varepsilon$-dependence in our notation; all estimates are uniform in $\varepsilon$.

**Lemma 6.3.** *The function $\psi = \psi^{(\varepsilon)}$, defined in* (6.4), *satisfies the following bounds:*

(a) $|\tilde{\psi}(k)| \leqq c_0 \log(|k| + 1)$.

(b) $|\partial^{\mathbf{m}} \tilde{\psi}(k)| \leqq c_{\mathbf{m}} (|k|^2 + 1)^{-|\mathbf{m}|/2}$ *for $k \in [-\pi/\varepsilon, \pi/\varepsilon]^3$ and any multi-index $\mathbf{m}$ with* $|\mathbf{m}| \geqq 1$. $\left[\textit{Here } |\mathbf{m}| \equiv \sum_{i=1}^{3} m_i.\right]$

(c) $\|C * \psi\|_{L^1 \cap L^2} \leqq c'$.

(d) $\|C * \psi * C\|_{L^1 \cap L^\infty} \leqq c''$.

*The constants $c_0$, $c_{\mathbf{m}}$, $c'$, $c''$ are all universal, i.e. independent of $\varepsilon$.*

*Proof.* By (6.4),

$$\tilde{\psi}(k) = (2\pi)^{-6} \iint_{q_1, q_2 \in [-\pi/\varepsilon, \pi/\varepsilon]^3} d^3 q_1 d^3 q_2 \tilde{C}(q_1) \tilde{C}(q_2) [\tilde{C}(k - q_1 - q_2) - \tilde{C}(-q_1 - q_2)],$$

where

$$\tilde{C}(q)=\left[1+2\varepsilon^{-2}\sum_{i=1}^{3}(1-\cos\varepsilon q_i)\right]^{-1}.$$

By the fundamental theorem of calculus, we can write

$$\tilde{\psi}(k)=(2\pi)^{-6}\int_0^1 d\alpha \iint_{q_1,q_2\in[-\pi/\varepsilon,\pi/\varepsilon]^3} d^3q_1 d^3q_2 \tilde{C}(q_1)\tilde{C}(q_2)\frac{d}{d\alpha}\tilde{C}(\alpha k-q_1-q_2),$$

and bound it in absolute value using

$$|\tilde{C}(q)|\leqq \text{const}\times(1+q^2)^{-1},$$

$$\left|\frac{d}{d\alpha}\tilde{C}(\alpha k-q_1-q_2)\right|\leqq \text{const}\times\frac{|k|}{[1+(\alpha k-q_1-q_2)^2]^2},$$

provided $k$, $q_1$, $q_2\in[-\pi/\varepsilon,\pi/\varepsilon]^3$. Thus

$$\begin{aligned}|\tilde{\psi}(k)|&\leqq \text{const}\times\int_0^1 d\alpha|k|\iint_{q_1,q_2\in[-\pi/\varepsilon,\pi/\varepsilon]^3} d^3q_1 d^3q_2\\&\quad\cdot(1+q_1^2)^{-1}(1+q_2^2)^{-1}[1+(\alpha k-q_1-q_2)^2]^{-2}\\&\leqq \text{const}\times\int_0^1 d\alpha\frac{|k|}{1+\alpha|k|}\\&\leqq \text{const}\times\log(|k|+1).\end{aligned}$$

This holds for $k\in[-\pi/\varepsilon,\pi/\varepsilon]^3$; but since $\tilde{\psi}(k)$ is periodic in $k$, the bound is manifestly true for all $k$. This proves (a); similar computations prove (b).

Now by the Plancherel theorem,

$$\begin{aligned}\|C*\psi\|_2=\|\tilde{C}\psi\|_2&\leqq \text{const}\times\left[\int d^3p\frac{1}{(1+p^2)^2}\log^2(|p|+1)\right]^{1/2}\\&\leqq \text{const}<\infty\end{aligned}$$

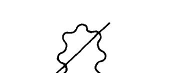
by part (a). Moreover, a similar argument using (b) shows that $\||x|^N C*\psi\|_2\leqq c_N<\infty$ (with a universal constant) for any $N\geqq 0$. Then

$$\|C*\psi\|_1\leqq\|(1+|x|^2)^{-1}\|_2\|(1+|x|^2)C*\psi\|_2\leqq \text{const}<\infty$$

by Hölder's inequality. This proves (c). (d) is now an immediate consequence of (c), Young's inequality, and the universal bound on $\|C\|_{L^1\cap L^2}$. □

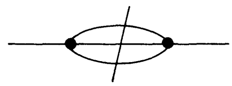
*Proof of Proposition 6.2.* The propagator inequalities (5.19) and (5.20) give upper and lower bounds for $E$ which are sums of terms of orders $\lambda$, $\lambda^2$, and $\lambda^3$. We treat each of these three classes of terms in turn, and estimate the $|||\cdot|||$ norms.

The term of order $\lambda$ can be bounded as



$$\begin{aligned}|||&\leqq\|E\|_\infty|||C*S|||\\&\leqq|||E|||\,(|||C*C|||+|||C*E|||)\\&\leqq|||E|||\,(c_1+c_2|||E|||),\end{aligned}$$

where we have used Young's inequality and the universal bound $\|C\|_{L^1 \cap L^2} \leqq \text{const.}$

One of the terms of order $\lambda^2$ is  $\equiv C*\psi*C$, which is bounded in $|||\cdot|||$ norm by Lemma 6.3(d). Another of the terms is $C*\psi*E$, which is controlled by Lemma 6.3(c) and Young's inequality:

$$|||C*\psi*E||| \leqq \|C*\psi\|_1 |||E||| \leqq c_3 |||E||| .$$

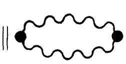
The remaining terms are $C*(S^3 - C^3)*S$:

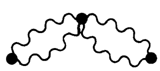
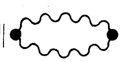
$$\begin{aligned} |||C*(S^3 - C^3)*S||| &\leqq |||C*S||| \, \|S^3 - C^3\|_1 \\ &= |||C*C + C*E||| \, \|3C^2E + 3CE^2 + E^3\|_1 \\ &\leqq (c_1 + c_2|||E|||)(c_4|||E||| + c_5|||E|||^2 + c_6|||E|||^3) \end{aligned}$$

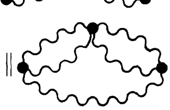
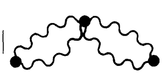
by Young's and Hölder's inequalities and the universal bound $\|C\|_{L^1 \cap L^2} \leqq \text{const.}$

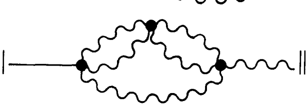
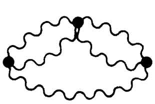
We bound the term of order $\lambda^3$ by the following sequence of inequalities:

$\|$  $\|_{5/2} = \|C + E\|_{5/2} \leqq c_7 + |||E|||$,

$\|$  $\|_{5/4} \leqq \|$  $\|_{5/2}^2 \leqq (c_7 + |||E|||)^2$,

$\|$  $\|_{5/3} \leqq \|$  $\|_{5/4}^2 \leqq (c_7 + |||E|||)^4$,

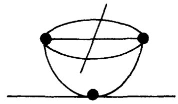
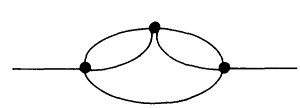
$\|$  $\|_1 \leqq \|$  $\|_{5/3} \|$  $\|_{5/2} \leqq (c_7 + |||E|||)^5$,

$|||$  $||| \leqq |||C*S||| \, \|$  $\|_1$

$\leqq (c_1 + c_2|||E|||)(c_7 + |||E|||)^5 .$

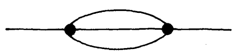
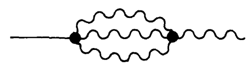
Here we have repeatedly used Hölder's and Young's inequalities along with the universal bound $\|C\|_{L^1 \cap L^{5/2}} \leqq \text{const.}$ □

*Remarks.* 1. If we had taken a general lattice dimension $d$ (instead of $d = 3$), then we would find that all of the above estimates are valid for $d < 10/3$. However, for $d \geqq 10/3$ the graphs

 and 

are ultraviolet divergent, and additional mass counterterms are required to cancel them and thereby to restore the validity of Lemma 6.3(d) and the estimate of the order-$\lambda^3$ term, respectively. Further graphs begin to diverge at the dimensions $d = 4 - 2/n$, $n$ integer [57] and require corresponding counterterms. Indeed, one of the beauties of our method is its close correspondence with perturbation-theory power-counting. (*Note:* The reader who dislikes flights of fancy should ignore our comments regarding non-integral dimensions.)

2. Analogous but even easier estimates establish the analogues of Proposition 6.2 and Theorem 6.1 for $\varphi_2^4$. Nothing like Lemma 6.3 is needed, since the graphs

 and 

are ultraviolet convergent in dimension $d<3$. Moreover, to prove the fundamental bound $|||E||| \leqq \text{const} \times \lambda^2$ one needs the propagator inequalities only to order $\lambda^2$ [i.e. (5.16) and (5.17)]; the order-$\lambda^3$ inequality (5.18) is needed only for the order-$\lambda^3$ bound. The results are the following:

**Proposition 6.4.** *For the $\varphi_2^4$ theory, there exist polynomials $P_1$, $P_2$, $Q_1$, $Q_2$, $Q_3$ with positive universal coefficients such that*

$$|||E^{(\varepsilon)}||| \leqq \sum_{n=1}^{2} \lambda^n P_n(|||E^{(\varepsilon)}|||), \tag{6.8}$$

$$|||E^{(\varepsilon)} - 6\lambda^2 C^{(\varepsilon)} * C^{(\varepsilon)3} * C^{(\varepsilon)}||| \leqq \sum_{n=1}^{3} \lambda^n Q_n(|||E^{(\varepsilon)}|||) \tag{6.9}$$

*for all $\varepsilon>0$. Moreover, the polynomials $P_1$, $Q_1$, and $Q_2$ have zero constant term.*

**Theorem 6.5.** *For the $\varphi_2^4$ theory, there exist universal constants $\lambda_0>0$, $c_1$, $c_2$ such that if $0 \leqq \lambda \leqq \lambda_0$, then*

$$|||S^{(\varepsilon)} - C^{(\varepsilon)}||| \leqq c_1 \lambda^2, \tag{6.10}$$

$$|||S^{(\varepsilon)} - (C^{(\varepsilon)} + 6\lambda^2 C^{(\varepsilon)} * C^{(\varepsilon)3} * C^{(\varepsilon)})||| \leqq c_2 \lambda^3 \tag{6.11}$$

*for all $\varepsilon>0$.*

*Remarks (continued).* 3. The invariant meaning of the weak-coupling hypothesis in Theorem 6.1 and 6.5 is that our methods break down for theories too close to the critical surface. (With our mass-renormalization convention (4.3), this means that $\lambda$ cannot be too large, at least in the case of $\varphi_2^4$: in this model it is known that the two-phase region is reached for large $\lambda$ [58, 59], and that the critical surface is crossed for some intermediate value of $\lambda$ [60]. For $\varphi_3^4$, analogous estimates on the location of the critical surface have apparently not yet been carried out.) We can, in any case, prove some *weaker* results for theories in the entire single-phase region (including near the critical surface); see Sect. 8.

4. The proofs in this section show essentially that the propagator inequalities (5.19) and (5.20) [or (5.16)–(5.18)] can be iterated to yield convergent upper and lower bounds; indeed, this is to be expected, since the number of diagrams at $n^{\text{th}}$ order clearly grows no worse than $K^n$ (and no ultraviolet troubles can occur). The continuity argument in the proof of Theorems 6.1 and 6.5 then ensures that $0 \leqq \lambda \leqq \lambda_0$ is within the region of convergence of these series. However, the proofs are much simplified by not making this iteration explicit.

## 7. The Mass Gap

In this section we strengthen Theorems 6.1 and 6.5 so as to exhibit explicitly the exponential decay of the 2-point function; in particular, we obtain strong twosided bounds on the mass gap. As in the preceding section, we work always with the lattice theory but seek estimates which are uniform in the lattice spacing $\varepsilon$. We continue to fix $m_0=1$.

It is useful to introduce the exponentially-weighted $L^p$ norms

$$\|f\|_{p,\alpha}\equiv\|\cosh(\alpha x_1)f(x)\|_p = \begin{cases} \left(\varepsilon^d \sum\limits_{x\in\varepsilon\mathbb{Z}^d} |\cosh(\alpha x_1)f(x)|^p\right)^{1/p} & \text{for} \quad 1<p<\infty \\ \sup\limits_{x\in\varepsilon\mathbb{Z}^d} \cosh(\alpha x_1)|f(x)| & \text{for} \quad p=\infty\,, \end{cases} \tag{7.1}$$

where $\alpha>0$. We also introduce an exponentially-weighted generalization of the $L^1\cap L^\infty$ norm used in the preceding section, namely

$$|||f|||_\alpha=\|f\|_{1,\alpha}+\|f\|_{\infty,\alpha}\,. \tag{7.2}$$

For each fixed $\varepsilon$, this is equivalent to the $l^1_\alpha$ norm; but since we seek estimates uniform in $\varepsilon$, it is strictly stronger. Some useful facts about these norms are summarized in the appendix; the key fact is that there exist versions of the Young and Hölder inequalities for convolution and pointwise multiplication, respectively.

We need an analogue of Proposition 5.1:

**Proposition 7.1.** *Fix the lattice spacing* $\varepsilon>0$. *Then, for each* $\alpha\geqq 0$:

(a) *The set*

$$\mathscr{B}_\alpha=\{(\lambda,a)\in\mathscr{B}:\|S(\lambda,a)\|_{1,\alpha}<\infty\}$$

*is a nonempty, connected, open subset of* $\mathscr{B}$.

(b) *The map* $(\lambda,a)\mapsto\|S(\lambda,a)\|_{1,\alpha}$ *is a continuous map from* $\mathscr{B}$ *into* $[0,+\infty]$.

(c) *The map* $(\lambda,a)\mapsto S(\lambda,a)$ *is a continuous map from* $\mathscr{B}_\alpha$ *into* $l^1_\alpha$.

*Proof.* Since the proof is virtually identical to that of Proposition 5.1, we merely make a few remarks. The Simon-Lieb inequality ensures that if $\|S(\lambda,a)\|_{1,\alpha}<\infty$, then in fact $\|S(\lambda,a)\|_{1,\alpha'}<\infty$ for some $\alpha'>\alpha$; moreover, this bound depends only on some finite-volume expectation $\langle\cdot\rangle_\Lambda$, and so is stable under small variations of $(\lambda,a)$. Thus $\mathscr{B}_\alpha$ is open, and $\|S(\lambda,a)\|_{1,\alpha}$ is locally bounded on $\mathscr{B}_\alpha$. The $\mathscr{B}_\alpha$ is nonempty because it contains the points $\lambda=0$, $a$ sufficiently large. The rest of the argument goes through virtually unchanged, by virtue of the convolution inequality (A.9). □

*Remark.* We shall define the mass gap as

$$m=\sup\{\alpha:\|S\|_{1,\alpha}<\infty\}\,. \tag{7.3}$$

The Simon-Lieb argument alluded to above shows that $\|S\|_{1,m}=+\infty$. Using the Schrader–Messager–Miracle-Sole inequalities, this definition of the mass gap can be shown to be equivalent to the more usual definition

$$m=\liminf_{|x_1|\to\infty}\frac{-\log S(x_1,0,\ldots,0)}{|x_1|}\,; \tag{7.4}$$

however, this argument relies on the nearest-neighbor nature of the interaction, which we have otherwise avoided using.

We can now obtain bounds on the mass gap by proceeding as in Sect. 6, but using everywhere the norms $|||\cdot|||_\alpha$ and $\|\cdot\|_{1,\alpha}$. To illustrate the method, we consider first the $\varphi_2^4$ theory, which is a bit simpler than $\varphi_3^4$. We define

$$m_0^{(\varepsilon)} = \frac{1}{\varepsilon}\cosh^{-1}\left(1+\frac{\varepsilon^2}{2}\right); \tag{7.5}$$

this is the mass gap for the $\lambda$, $a=0$ theory with bare mass $m_0=1$. Note that $m_0^{(\varepsilon)} \to m_0 = 1$ as $\varepsilon \to 0$, but that $m_0^{(\varepsilon)} < 1$ for $\varepsilon > 0$; thus, we shall have to be a bit careful in stating bounds which are valid for all $\varepsilon$. From now on, we suppress the $\varepsilon$-dependence in our notation, except in regard to $m_0^{(\varepsilon)}$; all estimates are uniform in $\varepsilon$.

The following result can be obtained by virtually copying the arguments of Sect. 6, using everywhere $|||\cdot|||_\alpha$ in place of $|||\cdot|||$:

**Theorem 7.2.** *Fix $\delta > 0$. Then, for the $\varphi_2^4$ theory, there exist universal constants $\lambda_0 > 0$, $c_1, c_2$ (depending on $\delta$) such that if $0 \leqq \lambda \leqq \lambda_0$ and $\alpha \leqq (1-\delta)m_0^{(\varepsilon)}$, then*

$$|||S - C|||_\alpha \leqq c_1 \lambda^2, \tag{7.6}$$

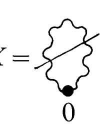
$$|||S - (C + 6\lambda^2 C * C^3 * C)|||_\alpha \leqq c_2 \lambda^3 \tag{7.7}$$

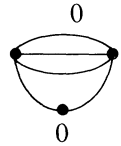
*for all $\varepsilon > 0$.*

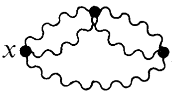
*Proof.* Everything goes through as in the proof of Proposition 6.4 (or 6.2), except that inequalities (A.8)–(A.10) play the role of the Young and Hölder inequalities. The key fact is that $\|C\|_{1,\alpha}$, $\|C\|_{2,\alpha}$, and $\|C\|_{5/2,\alpha}$ are all bounded uniformly in $\varepsilon$ and in $\alpha$, for $\alpha \leqq (1-\delta)m_0^{(\varepsilon)}$. $\square$

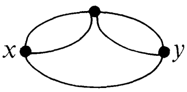
Theorem 7.2 implies that there is a mass gap $m(\lambda) > 0$ (uniformly in $\varepsilon$) and that in fact $\liminf_{\lambda\to 0} m(\lambda) \geqq m_0^{(\varepsilon)}$ (also uniformly in $\varepsilon$). This is nice, but with a little extra work we can do much better, and show that the mass gap is *exactly* that predicted by second-order perturbation theory plus an error which is uniformly of order $\lambda^3$. It is worth recalling how the perturbative calculation of the mass gap goes: one computes perturbatively the self-energy part ($\equiv$ the radiative corrections to the *inverse* propagator), and computes order-by-order the location of the first zero of the inverse propagator (i.e. the first pole of the propagator) in pure-imaginary ($\equiv$ Minkowski) momentum space. Our proof will follow the same pattern, but will employ rigorous inequalities in place of formal-power-series equalities.

We first define a few quantities which will arise in our computations:

$X=$  $= S(0) - C(0),$ (7.8)

$Y=$  $= (C * C^3 * C)(0),$ (7.9)

$\theta(x-y) = x$  $y,$ (7.11)

$$\theta_0(x-y) = x$$  $$y\,, \tag{7.11}$$

$$\Phi_2(\alpha) = \|C\|_{1,\alpha}^{-1} - 6\lambda^2 \|C^3\|_{1,\alpha}, \tag{7.12}$$

$$\Phi_3(\alpha) = \Phi_2(\alpha) + 18\lambda^3 Y + 54\lambda^3 \|\theta_0\|_{1,\alpha}. \tag{7.13}$$

Note that the quantities involving $C$ are explicitly computable, and are in fact nothing other than Feynman diagrams (for a lattice theory). For example, $\Phi_2(\alpha)$ and $\Phi_3(\alpha)$ are the second-order and third-order perturbative approximations to the inverse propagator $\tilde{S}(i\alpha, 0, \ldots, 0)^{-1}$.

We can now state a strengthened version of Theorem 7.2:

**Theorem 7.3.** *For the $\varphi_2^4$ theory, there exist universal constants $\lambda_0 > 0$, $c_1, \ldots, c_7$ such that if $0 \leqq \lambda \leqq \lambda_0$, then:*

$$\|S\|_{1,\alpha} \leqq [\Phi_2(\alpha) - c_1\lambda^4]^{-1}, \tag{7.14}$$

$$\|S\|_{1,\alpha} \geqq [\Phi_3(\alpha) + c_2\lambda^4]^{-1}, \tag{7.15}$$

$$|||S - C|||_\alpha \leqq c_3\lambda^2 \|C\|_{1,\alpha} [\Phi_2(\alpha) - c_4\lambda^3]^{-1}, \tag{7.16}$$

$$|||S - (C + 6\lambda^2 C^* C^{3*} C)|||_\alpha \leqq c_5\lambda^3 \|C\|_{1,\alpha}^2 + c_6\lambda^4 \|C\|_{1,\alpha}^2 [\Phi_2(\alpha) - c_4\lambda^3]^{-1} \tag{7.17}$$

*for any $\alpha$ $(0 \leqq \alpha < m_0^{(\varepsilon)})$ for which the bracket on the right-hand side is well-defined and positive. In particular, the mass gap satisfies*

$$|m - (m_0^{(\varepsilon)} - \lambda^2 m_2^{(\varepsilon)})| \leqq c_7\lambda^3, \tag{7.18}$$

*where*

$$m_2^{(\varepsilon)} = 6\|C^3\|_{1,m_0^{(\varepsilon)}} \Big/ \left( -\frac{d}{d\alpha} \|C\|_{1,\alpha}^{-1} \right)_{\alpha = m_0^{(\varepsilon)}} = 3\|C^3\|_{1,m_0^{(\varepsilon)}} \left(1 + \frac{\varepsilon^2}{4}\right)^{-1/2}. \tag{7.19}$$

*Remark.* The advantage of Theorem 7.3 over Theorem 7.2 is that it controls explicitly the rate of blow-up as $\alpha \uparrow m_0^{(\varepsilon)}$. To interpret (7.14)–(7.17), it is useful to know that

$$\|C\|_{1,\alpha} \sim (m_0^{(\varepsilon)} - \alpha)^{-1} \tag{7.20}$$

as $\alpha \uparrow m_0^{(\varepsilon)}$, and that $\Phi_2(\alpha)$ and $\Phi_3(\alpha)$ have simple zeros at locations slightly below $m_0^{(\varepsilon)}$ [more precisely, at $m_0^{(\varepsilon)} - \lambda^2 m_2^{(\varepsilon)} + O(\lambda^3)$].

*Proof of Theorem 7.3.* As before, we write $E = S - C$. The propagator inequalities (5.16)–(5.17) together yield

$$|||E|||_\alpha \leqq |||C^* S|||_\alpha (3\lambda |X| + 6\lambda^2 \|S^3\|_{1,\alpha}), \tag{7.21}$$

while (5.17)–(5.18) together yield

$$|||E - 6\lambda^2 C^* C^{3*} C|||_\alpha \leqq |||C^* S|||_\alpha (3\lambda|X| + 6\lambda^2 \|S^3 - C^3\|_{1,\alpha} + 54\lambda^3 \|\theta\|_{1,\alpha}) + 6\lambda^2 |||C^* E|||_\alpha \|C^3\|_{1,\alpha}. \tag{7.22}$$

Now

$$|||C^* E|||_\alpha \leqq \|C\|_{1,\alpha} |||E|||_\alpha \tag{7.23}$$

by (A.10), while

$$|||C^*S|||_\alpha \leqq |||C^*C|||_\alpha + |||C^*E|||_\alpha$$
$$\leqq (\|C\|_{1,\alpha}^2 + \|C\|_{2,\alpha}^2) + \|C\|_{1,\alpha}|||E|||_\alpha$$
$$\leqq (\text{const} \times \|C\|_{1,\alpha}^2) + \|C\|_{1,\alpha}|||E|||_\alpha \tag{7.24}$$

by (A.9), (A.10), and (A.20). Likewise, by (A.20),

$$\|C^3\|_{1,\alpha} = \|C\|_{3,\alpha/3}^3 \leqq \text{const} \tag{7.25}$$

and

$$\|S^3 - C^3\|_{1,\alpha} = \|3C^2E + 3CE^2 + E^3\|_{1,\alpha}$$
$$\leqq \text{const} \times (|||E|||_{\alpha/3} + |||E|||_{\alpha/3}^3) \tag{7.26}$$

by (A.8) and the uniform bound (A.20) on $\|C\|_{1,\alpha/3}$ and $\|C\|_{2,\alpha/3}$ for $\alpha \leqq m_0^{(\varepsilon)}$ [indeed, for $\alpha \leqq (3-\delta)m_0^{(\varepsilon)}$]. Combining (7.25) and (7.26) bounds $\|S^3\|_{1,\alpha}$. Finally, by imitating the last step in the proof of Proposition 6.2, we can show that

$$\|\theta_0\|_{1,\alpha} \leqq \text{const} \tag{7.27}$$

and

$$\|\theta - \theta_0\|_{1,\alpha} \leqq \text{const} \times (|||E|||_{\alpha/3} + |||E|||_{\alpha/3}^5) \tag{7.28}$$

uniformly for $\alpha \leqq m_0^{(\varepsilon)}$ [indeed, for $\alpha \leqq (3-\delta)m_0^{(\varepsilon)}$]. Combining (7.27) and (7.28) bounds $\|\theta\|_{1,\alpha}$.

By Theorem 6.5,

$$X = 6\lambda^2 Y + O(\lambda^3) \tag{7.29}$$

for $0 \leqq \lambda \leqq \lambda_0$, where the $O(\lambda^3)$ error is uniform in the lattice spacing $\varepsilon$; and

$$0 \leqq Y \leqq \text{const} \tag{7.30}$$

by an easy application of the Young and Hölder inequalities. Likewise, by Theorem 7.2,

$$|||E|||_{\alpha/3} \leqq \text{const} \times \lambda^2 \tag{7.31}$$

uniformly for $\alpha \leqq m_0^{(\varepsilon)}$ [indeed, for $\alpha \leqq (3-\delta)m_0^{(\varepsilon)}$]. Inserting (7.24)–(7.26) and (7.29)–(7.31) into (7.21), we obtain

$$|||E|||_\alpha \leqq [(\text{const} \times \|C\|_{1,\alpha}^2) + \|C\|_{1,\alpha}|||E|||_\alpha]\,[6\lambda^2\|C^3\|_{1,\alpha} + O(\lambda^3)]$$
$$\leqq \text{const} \times \|C\|_{1,\alpha}^2\lambda^2 + \|C\|_{1,\alpha}[6\lambda^2\|C^3\|_{1,\alpha} + O(\lambda^3)]\,|||E|||_\alpha\,. \tag{7.32}$$

Assume now that

$$\|C\|_{1,\alpha}[6\lambda^2\|C^3\|_{1,\alpha} + O(\lambda^3)] < 1\,. \tag{7.33}$$

Then it follows from (7.32) that

$$|||E|||_\alpha \leqq \text{const} \times \|C\|_{1,\alpha}^2\lambda^2(1 - \|C\|_{1,\alpha}[6\lambda^2\|C^3\|_{1,\alpha} + O(\lambda^3)])^{-1} \tag{7.34}$$

provided only that $|||E|||_\alpha < \infty$; but since $|||E|||_\alpha = 0$ for $\lambda = 0$ and is continuous in $\lambda$ (by Proposition 7.1), (7.34) must hold whenever (7.33) does. (Equivalently, one can argue from Theorem 7.2 and continuity in $\alpha$.) This proves (7.16). Similarly, inserting (7.23)–(7.31) and (7.34) in (7.22), we obtain

$$\begin{aligned}|||E - 6\lambda^2 C^* C^3 {}^* C|||_\alpha &\leqq [(\text{const} \times \|C\|_{1,\alpha}^2) + \|C\|_{1,\alpha} |||E|||_\alpha] \times O(\lambda^3) \\ &\quad + 6\lambda^2 \|C\|_{1,\alpha} \|C^3\|_{1,\alpha} |||E|||_\alpha \\ &\leqq (\text{const} \times \|C\|_{1,\alpha}^2 \lambda^3) + (\text{const} \times \lambda^2 \|C\|_{1,\alpha} |||E|||_\alpha) \\ &\leqq (\text{const} \times \|C\|_{1,\alpha}^2 \lambda^3) \\ &\quad + \text{const} \times \|C\|_{1,\alpha}^3 \lambda^4 (1 - \|C\|_{1,\alpha} [6\lambda^2 \|C^3\|_{1,\alpha} + O(\lambda^3)])^{-1} \end{aligned} \tag{7.35}$$

for $0 \leqq \lambda \leqq \lambda_0$ and $\alpha \leqq m_0^{(\varepsilon)}$, provided (7.33) is satisfied. This proves (7.17).

To prove (7.14) and (7.15), we return to the propagator inequalities (5.16)–(5.18). Since these inequalities hold pointwise in $x$-space, and since $e^{ip\cdot x} > 0$ for pure imaginary $p$, they hold also in *pure-imaginary p-space:*

$$\tilde{S}(ip) \geqq \tilde{C}(ip) - 3\lambda \tilde{C}(ip) \tilde{S}(ip) X, \tag{7.36}$$

$$\begin{aligned}\tilde{S}(ip) \leqq \tilde{C}(ip) &- 3\lambda \tilde{C}(ip) \tilde{S}(ip) X \\ &+ 6\lambda^2 \tilde{C}(ip) \tilde{S}(ip) \widetilde{S^3}(ip),\end{aligned} \tag{7.37}$$

$$\begin{aligned}\tilde{S}(ip) \geqq \tilde{C}(ip) &- 3\lambda \tilde{C}(ip) \tilde{S}(ip) X \\ &+ 6\lambda^2 \tilde{C}(ip) \tilde{S}(ip) \widetilde{S^3}(ip) - 54\lambda^3 \tilde{C}(ip) \tilde{S}(ip) \tilde{\theta}(ip)\end{aligned} \tag{7.38}$$

for all real $p$ for which they are well-defined (i.e. finite without resorting to analytic continuation). In particular, if we take $p = (\alpha, 0, \ldots, 0)$, then $\tilde{S}(ip) = \|S\|_{1,\alpha}$ and likewise for $C$, $S^3$, and $\theta$, since all these functions are even and nonnegative. Thus (7.36)–(7.38) can be interpreted as inequalities for $\|\cdot\|_{1,\alpha}$ norms. Inequality (7.37) becomes

$$\begin{aligned}\|S\|_{1,\alpha} &\leqq \|C\|_{1,\alpha} + \|C\|_{1,\alpha} \|S\|_{1,\alpha} [6\lambda^2 \|S^3\|_{1,\alpha} - 3\lambda X] \\ &\leqq \|C\|_{1,\alpha} + \|C\|_{1,\alpha} \|S\|_{1,\alpha} [6\lambda^2 \|C^3\|_{1,\alpha} - 18\lambda^3 Y + O(\lambda^4)].\end{aligned} \tag{7.39}$$

Assume now that

$$\|C\|_{1,\alpha} [6\lambda^2 \|C^3\|_{1,\alpha} + O(\lambda^4)] < 1. \tag{7.40}$$

Then, by (7.39) and (7.30),

$$\|S\|_{1,\alpha} \leqq \|C\|_{1,\alpha} (1 - \|C\|_{1,\alpha} [6\lambda^2 \|C^3\|_{1,\alpha} + O(\lambda^4)])^{-1}, \tag{7.41}$$

provided only that $\|S\|_{1,\alpha} < \infty$; but the finiteness of $\|S\|_{1,\alpha}$ follows as before, by continuity in $\lambda$ (or in $\alpha$) together with (7.41) itself. This proves (7.14). Likewise, (7.38) becomes

$$\begin{aligned}\|S\|_{1,\alpha} &\geqq \|C\|_{1,\alpha} + \|C\|_{1,\alpha} \|S\|_{1,\alpha} [6\lambda^2 \|S^3\|_{1,\alpha} - 3\lambda X - 54\lambda^3 \|\theta\|_{1,\alpha}] \\ &\geqq \|C\|_{1,\alpha} + \|C\|_{1,\alpha} \|S\|_{1,\alpha} [6\lambda^2 \|C^3\|_{1,\alpha} - 18\lambda^3 Y - 54\lambda^3 \|\theta_0\|_{1,\alpha} - O(\lambda^4)]\end{aligned} \tag{7.42}$$

by virtue of (7.26), (7.28), (7.29), and (7.31); this immediately implies (7.15) provided that $\|S\|_{1,\alpha} < \infty$. On the other hand, if $\|S\|_{1,\alpha} = +\infty$, then (7.15) holds trivially. The bounds (7.18) and (7.19) on the mass gap are an immediate consequence of (7.14) and (7.15) together with

$$\|C\|_{1,\alpha}^{-1}=1+2\varepsilon^{-2}(1-\cosh\varepsilon\alpha), \tag{7.43}$$

and a uniform bound on $\frac{d}{d\alpha}\|C^3\|_{1,\alpha}$ for $\alpha\leqq m_0^{(\varepsilon)}$. □

*Remarks.* 1. Inequality (7.14) actually holds with $c_1=0$, provided the lattice spacing $\varepsilon$ is not too *large* (e.g. let's say we take $\varepsilon\leqq 1$). For then $Y$ is uniformly strictly positive, and we can take $\lambda_0$ small enough so that $18\lambda^3 Y$ outweighs the $O(\lambda^4)$ term in (7.39).

2. Evaluating the propagator inequality (5.16) at $x=y$, we obtain $X\geqq 0$, valid for *all* $\lambda$. This might be of some use for strong coupling, where (7.29) no longer applies.

This completes the discussion of the mass gap for $\varphi_2^4$. Now $\varphi_3^4$ is not much more difficult; we need only an exponentially-weighted generalization of Lemma 6.3.

**Lemma 7.4.** *The function $\psi=\psi^{(\varepsilon)}$, defined in* (6.4), *satisfies the following bounds uniformly for $0\leqq\alpha\leqq m_0^{(\varepsilon)}$:*

(a) $|\tilde{\psi}(k\pm i\alpha)|\leqq c_0\log(|k|+1)$,

(b) $|\partial^{\mathbf{m}}\tilde{\psi}(k\pm i\alpha)|\leqq c_{\mathbf{m}}(|k|^2+1)^{-|\mathbf{m}|/2}$ *for $k\in[-\pi/\varepsilon,\pi/\varepsilon]^3$ and any multi-index* $\mathbf{m}$ *with $|\mathbf{m}|\geqq 1$. $\left[\text{Here } |\mathbf{m}|\equiv\sum_{i=1}^{3} m_i.\right]$*

(c) $\|C*\psi\|_{1,\alpha}+\|C*\psi\|_{2,\alpha}\leqq c'\|C\|_{1,\alpha}$.

(d) $|||C*\psi*C|||_\alpha\leqq c''\|C\|_{1,\alpha}^2$.

*(Here we have used $\alpha$ also to denote the vector $(\alpha,0,0)$.) The constants $c_0$, $c_{\mathbf{m}}$, $c'$, $c''$ are all universal, i.e. independent of $\varepsilon$.*

*Proof.* The proof is virtually identical to that of Lemma 6.3: the point is that the imaginary part of the momentum flowing in each of the three free propagators is $\pm\alpha/3$, which is bounded away from $m_0^{(\varepsilon)}$; and so the propagators satisfy the same bounds as before. This proves (a) and (b). If we write

$$F_\alpha(x)=e^{\alpha x_1}F(x) \tag{7.44}$$

for any function $F$, then

$$\|C*\psi\|_{p,\alpha}=\|(C*\psi)_\alpha\|_p=\|C_\alpha^*\psi_\alpha\|_p, \tag{7.45}$$

so (c) and (d) also follow as before, using (A.20). □

Again, we have first an easy result:

**Theorem 7.5.** *Fix $\delta>0$. Then, for the $\varphi_3^4$ theory, there exist universal constants $\lambda_0>0$, $c_1$, $c_2$ (depending on $\delta$) such that if $0\leqq\lambda\leqq\lambda_0$ and $\alpha\leqq(1-\delta)m_0^{(\varepsilon)}$, then*

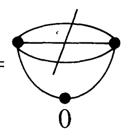
$$|||S-C|||_\alpha\leqq c_1\lambda^2, \tag{7.46}$$

$$|||S-(C+6\lambda^2C*\psi*C)|||_\alpha\leqq c_2\lambda^3 \tag{7.47}$$

*for all $\varepsilon>0$.*

The proof is a simple modification of that of Theorem 6.1.

In order to state the more precise result, we use the definitions (7.8), (7.10), (7.11) and

$Y'=$  $=(C^*\psi^*C)(0)\,,$ (7.48)

$$\Phi_2'(\alpha)=\|C\|_{1,\alpha}^{-1}-6\lambda^2\tilde{\psi}(i\alpha)\,, \tag{7.49}$$

$$\Phi_3'(\alpha)=\Phi_2'(\alpha)+18\lambda^3 Y'+54\lambda^3\|\theta_0\|_{1,\alpha}\,. \tag{7.50}$$

[Here we have again used $\alpha$ to denote also the vector $(\alpha,0,0)$.] Then:

**Theorem 7.6.** *For the $\varphi_3^4$ theory, there exist universal constants $\lambda_0>0$, $c_1,\ldots,c_7$ such that if $0\leqq\lambda\leqq\lambda_0$, then:*

$$\|S\|_{1,\alpha}\leqq[\Phi_2'(\alpha)-c_1\lambda^4]^{-1}\,, \tag{7.51}$$

$$\|S\|_{1,\alpha}\geqq[\Phi_3'(\alpha)+c_2\lambda^4]^{-1}\,, \tag{7.52}$$

$$|||S-C|||_\alpha\leqq c_3\lambda^2\|C\|_{1,\alpha}[\|C\|_{1,\alpha}^{-1}-c_4\lambda^2]^{-1}\,, \tag{7.53}$$

$$|||S-(C+6\lambda^2C^*\psi^*C)|||_\alpha\leqq c_5\lambda^3\|C\|_{1,\alpha}^2+c_6\lambda^4\|C\|_{1,\alpha}^2[\|C\|_{1,\alpha}^{-1}-c_4\lambda^2]^{-1}\,, \tag{7.54}$$

for any $\alpha$ $(0\leqq\alpha<m_0^{(\varepsilon)})$ for which the bracket on the right-hand side is well-defined and positive. In particular, the mass gap satisfies

$$|m-(m_0^{(\varepsilon)}-\lambda^2 m_2^{(\varepsilon)\prime})|\leqq c_7\lambda^3\,, \tag{7.55}$$

where

$$\begin{aligned} m_2^{(\varepsilon)\prime}&=6\tilde{\psi}(im_0^{(\varepsilon)})\bigg/\left(-\frac{d}{d\alpha}\|C\|_{1,\alpha}^{-1}\right)_{\alpha=m_0^{(\varepsilon)}}\\ &=3\tilde{\psi}(im_0^{(\varepsilon)})\left(1+\frac{\varepsilon^2}{4}\right)^{-1/2}\,. \end{aligned} \tag{7.56}$$

*Remark.* The error bounds (7.53) and (7.54) are *weaker* than the analogous bounds (7.16) and (7.17) for $\varphi_2^4$ in that they do not exhibit the correct order-$\lambda^2$ coefficient for the mass-gap shift. The reason for this, as will be seen in the proof, is that $\psi$, unlike $C^3$, does not have a definite sign; indeed, by definition $\psi$ has total integral equal to zero, but it certainly does not vanish identically! The same holds for $C^*\psi$. It is $(\widetilde{C^*\psi})(i\alpha)$ that appears in the order-$\lambda^2$ formula for the mass gap; but it is unfortunately $\|C^*\psi\|_{1,\alpha}$, which is *strictly larger*, that appears in the estimates leading to (7.53) and (7.54). Nevertheless, (7.51) and (7.52) do exhibit the correct order-$\lambda^2$ shift, so we are able to obtain the correct order-$\lambda^2$ asymptotic formula for the mass gap, (7.55) and (7.56).

*Proof.* The propagator inequalities (5.19) and (5.20) together yield

$$\begin{aligned} |||E-6\lambda^2C^*\psi^*C|||_\alpha\leqq{}&|||C^*S|||_\alpha(3\lambda|X|+6\lambda^2\|S^3-C^3\|_{1,\alpha}+54\lambda^3\|\theta\|_{1,\alpha})\\ &+6\lambda^2\|C^*\psi\|_{1,\alpha}|||E|||_\alpha\,, \end{aligned} \tag{7.57}$$

and hence, by Lemma 7.4(d),

$$|||E|||_\alpha \leqq \text{const} \times \lambda^2 \|C\|_{1,\alpha}^2 + |||C^*S|||_\alpha (3\lambda|X| + 6\lambda^2 \|S^3 - C^3\|_{1,\alpha} + 54\lambda^3 \|\theta\|_{1,\alpha}) + 6\lambda^2 \|C^*\psi\|_{1,\alpha} |||E|||_\alpha . \tag{7.58}$$

The rest of the argument for (7.53) and (7.54) is exactly analogous to that in Theorem 7.3, except that $\|C^*\psi\|_{1,\alpha}$ occurs in place of $\|C\|_{1,\alpha}\|C^3\|_{1,\alpha}$; we use Lemma 7.4(c) to bound this factor. This explains the remark made above. The proof of (7.51) and (7.52) likewise follows almost word-for-word the pattern from Theorem 7.3. Here, however, it is really $\tilde{\psi}(i\alpha)$ which arises, not the $\|\cdot\|_{1,\alpha}$ norm, so the correct order-$\lambda^2$ term is obtained. □

*A Final Remark.* Although we prove strong bounds on the *location* of the first singularity of the propagator in pure-imaginary momentum space, we are unable to prove in general that this singularity is a pole, i.e. that there exist one-particle states. Glimm and Jaffe [55] have proven that for almost every physical mass, there is a one-particle pole in the two-point function (although their argument does not imply that it is isolated). All we can say in general (i.e. without the "almost every" qualifier) is that the spectral weight beginning at $m$ cannot be *too* soft, because $\tilde{S}(ip)$ diverges as this singularity is approached from below. The traditional proofs of the existence of a one-particle pole (and of an upper mass gap between that pole and the continuum) proceed by analyzing one-particle-irreducible (1PI) correlation functions and showing that these have decay rate strictly greater than $m$ (see [2] and the references cited there). The counterpart in our approach is the analysis of [3, Sect. 3.3] (based on earlier work of [14, 7]); it uses, however, an unproved correlation inequality for the partially-1PI six-point function $G_6^{1\text{PI}}$ [3, Conjecture 3.2]. The ambitious reader is invited to try to prove (or disprove) the $G_6^{1\text{PI}}$ conjecture; see also [3, Chap. 5] for some warm-up problems.

## 8. Completion of the Proof

In Sects. 5 through 7 we have proven that, for $\alpha < m_0 = 1$, there is a positive constant $\lambda_\alpha$ *independent of the lattice spacing* $\varepsilon$, such that for $0 \leqq \lambda \leqq \lambda_\alpha$,

$$|||S^{(\varepsilon)} - C^{(\varepsilon)}|||_\alpha \leqq c(\alpha)\lambda^2 \tag{8.1}$$

for some finite constant $c(\alpha)$. The norm $|||\cdot|||_\alpha$ is defined in (7.2). This bound says that the interacting propagator (2-point Schwinger function) $S^{(\varepsilon)}(x-y)$ behaves like the free propagator $C^{(\varepsilon)}(x-y)$, up to an error term which decays at least like $\exp(-\alpha|x-y|_\infty)$ and is bounded uniformly by $\text{const} \times \lambda^2$ in the sup norm. In particular, the leading short-distance ($x \approx y$) behavior of $S^{(\varepsilon)}$ is identical to that of $C^{(\varepsilon)}$.

In the following we shall show that this result suffices to establish the existence of a continuum $\lambda\varphi_d^4$ theory ($d=2, 3$) satisfying all the Osterwalder-Schrader axioms except perhaps rotation invariance, and which moreover is nontrivial (i.e. non-Gaussian) and has a mass gap at least $\alpha$. All these results are for the *weakly coupled* theory, $0 \leqq \lambda \leqq \lambda_\alpha$. However, at the end of this section we shall show how the existence (though not the non-Gaussianness or the mass gap) can be extended to the entire single-phase region.

We begin by reminding the reader that the bound (8.1) for the 2-point functions $S^{(\varepsilon)} = S_2^{(\varepsilon)}$ implies a corresponding bound for the $2n$-point functions $S_{2n}^{(\varepsilon)}$. Indeed, by the first Griffiths inequality [44, 46] and the Gaussian inequality [52, 53, 61, 4, 5],

$$0 \leqq S_{2n}^{(\varepsilon)}(x_1, \ldots, x_{2n}) \leqq \sum_{\text{pairings}} \prod S_2^{(\varepsilon)}(x_\alpha, x_\beta). \tag{8.2}$$

This bound is already sufficient to guarantee the existence of the continuum limit for $S_{2n}$, but that limit might conceivably be identically zero! To rule out such a pathology, we use the slightly less crude lower bound

$$S_{2n}^{(\varepsilon)}(x_1, \ldots, x_{2n}) \geqq \frac{1}{(2n-1)!!} \sum_{\text{pairings}} \prod S_2^{(\varepsilon)}(x_\alpha, x_\beta), \tag{8.3}$$

which is obtained by repeated application of the second Griffiths inequality followed by symmetrization. [Here $(2n-1)!!$ is just the number of ways of pairing $2n$ objects.] Actually, we can give a much more accurate bound for $S_{2n}^{(\varepsilon)}$; see Eq. (8.5) below.

Finally, we note that the $S_{2n}^{(\varepsilon)}$ satisfy a cluster property with exponential rate at least $\alpha$. This follows from the truncated Gaussian inequality [52, 61, 5]

$$\begin{aligned} 0 &\leqq S_{2n}^{(\varepsilon)}(x_1, \ldots, x_{2n}) - S_j^{(\varepsilon)}(x_1, \ldots, x_j) S_{2n-j}^{(\varepsilon)}(x_{j+1}, \ldots, x_{2n}) \\ &\leqq \sum_{\text{pairings}}{}' \prod S_2^{(\varepsilon)}(x_\alpha, x_\beta), \end{aligned} \tag{8.4}$$

where $\sum'$ ranges over all pairings of $\{1, \ldots, 2n\}$ which connect at least one element of $\{1, \ldots, j\}$ with at least one element of $\{j+1, \ldots, 2n\}$.

Thus, if the arguments $x_{j+1}, \ldots, x_{2n}$ are replaced by $x_{j+1} + a, \ldots, x_{2n} + a$, where $a$ is some vector, then the left side of (8.4) decays at least as rapidly as $\exp(-\alpha|a|_\infty)$ as $|a| \to \infty$, since at least one factor of $S_2^{(\varepsilon)}$ must link the variables $x_1, \ldots, x_j$ with the variables $x_{j+1}, \ldots, x_{2n}$. In fact, if $j$ is even, the decay rate is at least $2\alpha$, since at least two factors of $S_2^{(\varepsilon)}$ must link these sets of variables [62].

It is now a standard fact [6, 3] that the bounds (8.1) and (8.2) imply the compactness of the set of $S_{2n}^{(\varepsilon)}$, considered as lying in the Schwartz distribution space $\mathscr{S}'(\mathbb{R}^{2nd})$. We can thus extract a sequence $\varepsilon_i \to 0$ such that all of the $S_{2n}^{(\varepsilon_i)}$ converge to limits $S_{2n}$. These $S_{2n}$ satisfy all the Osterwalder-Schrader axioms [10–13] except perhaps Euclidean (rotation) invariance. (The translation invariance is slightly subtle, but it does hold [3].) Moreover, the $S_{2n}$ are moments of a translation-invariant probability measure $\mu$ on $\mathscr{S}'(\mathbb{R}^d)$. The bounds (8.1)–(8.4) manifestly carry over to the continuum limit, so the continuum theory has a cluster property with exponential rate at least $\alpha$ (in particular, the mass gap is at least $\alpha$). In fact, all the bounds of Sects. 6 and 7 carry over to the continuum limit, so we have actually shown that $S_2$ is given by the usual second-order perturbation expansion plus an error which is of order $\lambda^3$.

We remark that this construction of the continuum limit by compactness and subsequences is somewhat distasteful: aside from its inherent nonconstructiveness, certain natural and desirable properties (such as the uniqueness of the limit) go unestablished. We discuss further in Sect. 9 the possibility of establishing the existence of the full limit $\varepsilon \to 0$.

It remains only to establish the non-Gaussianness of the continuum-limit model (always, of course, for small $\lambda > 0$). But this is an immediate consequence of the skeleton inequality (5.10) combined with the bound (8.1): for (8.1) guarantees that the internal integrations in (5.10) are convergent for $d < 4$, *uniformly in* $\varepsilon$, hence for sufficiently small $\lambda$ the order-$\lambda$ term dominates the order-$\lambda^2$ term, and $u_4$ is explicitly nonzero. This is just the strategy of [3]. In fact, (5.9) and (5.10) together with (8.1) show that $u_4$ is given exactly by its first-order perturbation expansion (which is the tree graph with *free* propagators) plus an error which is of order $\lambda^2$ (in a suitable norm). This also establishes that $u_4$ is nonzero for suitable *noncoinciding* arguments, which in turn guarantees that the reconstructed Minkowski-space quantum field theory [10] is not a generalized free field.

Similar bounds can also be established for the $S_{2n}$. Indeed, the analogue of (5.9) is the first-order skeleton inequality [5]

$$S_{2n}^{(\varepsilon)}(x_1, \ldots, x_{2n}) \geqq \sum_{\text{pairings}} \prod S_2^{(\varepsilon)}(x_\alpha, x_\beta) - 6\lambda \sum_H I_H, \tag{8.5}$$

where the sum ranges over all Feynman diagrams $H$ with a single internal vertex of order 4 and with external vertices at $x_1, \ldots, x_{2n}$, and $I_H$ is the corresponding Feynman amplitude with propagator $S_2^{(\varepsilon)}$. This greatly improves the crude bound (8.3); in fact, it is in a certain sense optimal, because the right side of (8.5) is precisely the first-order perturbation expansion for $S_{2n}^{(\varepsilon)}$. A second-order skeleton inequality for $S_{2n}^{(\varepsilon)}$, analogous to (5.10) for $S_4^{(\varepsilon)}$, is also valid (although in [5] we did not bother to write out the proof in detail); this, combined with (8.5) and (8.1), implies that $S_{2n}$ takes a non-Gaussian value which is in fact that predicted by first-order perturbation theory plus an error of order $\lambda^2$. (Note, however, that this does not establish that the fully connected correlation function $u_{2n}$ is nonzero, for $n \geqq 3$; to do this would require carrying the asymptotic expansion to higher order in $\lambda$, in fact to an order which increases with $n$.)

We now show how to prove the existence (but not the nontriviality) of the continuum limit throughout the single-phase region. First, we generalize our previous definition of the model by adding an extra mass term $-\frac{1}{2}\sigma\varphi^2$ to the lattice action; that is, (4.3) now becomes

$$a(\varepsilon) = \begin{cases} m_0^2 - \sigma + \delta m_1^2(\varepsilon), & d = 2 \\ m_0^2 - \sigma + \delta m_1^2(\varepsilon) + \delta m_2^2(\varepsilon), & d = 3, \end{cases} \tag{8.6}$$

where $\delta m_1^2(\varepsilon)$ and $\delta m_2^2(\varepsilon)$ continue to be given by (4.4) and (4.5) and so are independent of the new parameter $\sigma$. Let $S_\sigma^{(\varepsilon)}(x-y)$ denote the 2-point function for the model just described, and let $\tilde{S}_\sigma^{(\varepsilon)}(k)$ denote its (lattice) Fourier transform with respect to $x-y$. Then for arbitrary $\varepsilon > 0$, $\lambda \geqq 0$, $m_0 > 0$, and $\sigma$, the infrared bound [27] states that we have

$$0 \leqq \tilde{S}_\sigma^{(\varepsilon)}(k) \leqq \Delta(k)^{-1} + c\delta(k), \tag{8.7}$$

where

$$\Delta(k) \equiv 2\varepsilon^{-2} \sum_{i=1}^{d} [1 - \cos(\varepsilon k_i)], \tag{8.8}$$

and

$$c = c(\lambda, m_0, \sigma, \varepsilon) \equiv (2\pi)^d \lim_{|x-y| \to \infty} S_\sigma^{(\varepsilon)}(x-y) \geqq 0. \tag{8.9}$$

Here $c(\lambda, m_0, \sigma, \varepsilon)$ is called the *long-range order*. If $\lambda$, $m_0$, and $\sigma$ are such that

$$\lim_{\varepsilon \to 0} c(\lambda, m_0, \sigma, \varepsilon) = 0, \tag{8.10}$$

we say that the model is in the *single-phase region*. If, moreover,

$$\limsup_{\varepsilon \to 0} \int S_\sigma^{(\varepsilon)}(x) d^d x < \infty, \tag{8.11}$$

we say that the model is in the *strict single-phase region*. Clearly (8.11) implies that the long-range order vanishes for all sufficiently small $\varepsilon$. We set

$$\chi = \chi(\lambda, m_0, \sigma, \varepsilon) \equiv \int S_\sigma^{(\varepsilon)}(x) d^d x = \tilde{S}_\sigma^{(\varepsilon)}(0). \tag{8.12}$$

Thus, if $\chi < \infty$, we may combine (8.7) with the trivial inequality

$$\tilde{S}_\sigma^{(\varepsilon)}(k) \leqq \tilde{S}_\sigma^{(\varepsilon)}(0) = \chi \tag{8.13}$$

[which is a consequence of the first Griffiths inequality $S_\sigma^{(\varepsilon)}(x) \geqq 0$], and obtain

$$0 \leqq S_\sigma^{(\varepsilon)}(k) \leqq \min(\chi, \Delta(k)^{-1}). \tag{8.14}$$

Finally, we remark that in the single-phase region and for dimension $d > 2$, the $p$-space bound (8.7) together with correlation inequalities implies a corresponding uniform $x$-space bound [3, Appendix A].

We can now proceed to show the existence of the continuum limit $\varepsilon \to 0$ (by compactness and subsequences, as always). Indeed, for dimension $d > 2$, the massless propagator $\Delta(k)^{-1}$ is locally integrable with a uniform bound as $\varepsilon \to 0$, so (8.7) is a uniform distributional bound on the 2-point function as $\varepsilon \to 0$, provided that the model is in the single-phase region. For dimension $d \leqq 2$, the bound (8.14) yields the same conclusion, provided that the model is in the strict single-phase region. Then (8.2) and (8.3) imply uniform distributional bounds on all of the $2n$-point functions. By choice of subsequences we may therefore define a continuum limit $\varepsilon_i \to 0$ for all correlations; this limiting theory satisfies all the Osterwalder-Schrader axioms except perhaps Euclidean (rotation) invariance and clustering. (For $d > 2$, the $x$-space infrared bound [3, Lemma A.3] together with (8.4) imply that the theory *does* cluster at least as rapidly as the massless free field.)

This much of the argument is valid, in fact, for $\varphi^4$ theories in *any* dimension $d$, with *any* choice of mass and coupling-constant renormalization, provided only that the field-strength renormalization is bounded [so that (8.7) holds up to a bounded multiplicative factor]. The problem, of course, is that the continuum $S_{2n}$ constructed in this way could well be identically zero! (Or they could be delta functions concentrated at coinciding arguments, which likewise leads to an identically-zero Minkowski-space quantum field theory [10].) Indeed, the purpose of renormalization is to avoid precisely such trivial limits, and this is why the mass renormalization $a(\varepsilon)$ must be chosen to have the specific form (8.6) [up to possible finite additions]. In this case, we have constructed – by hard work – nonvanishing continuum limits for $0 \leqq \lambda \leqq \lambda_0$, $m_0 = 1$, and $\sigma = 0$. Now the 2-point function $S_\sigma^{(\varepsilon)}(x)$

is monotone increasing in $\sigma$, for all $x$, by the second Griffiths inequality, so

$$S_{\sigma}^{(\varepsilon)}(x) \geqq S_{\sigma=0}^{(\varepsilon)}(x) \tag{8.15}$$

for $\sigma \geqq 0$. If, furthermore, $0 \leqq \lambda \leqq \lambda_0$ and $m_0 = 1$, the right-hand side of (8.15) is bounded below in $x$-space, uniformly in $\varepsilon$, by our previous construction, e.g. (8.1). By (8.3), this implies nonvanishing distributional lower bounds on all $2n$-point functions, and thus nonzero continuum limits.

*Remark.* In order to show the existence of the continuum limit (by subsequences) for $d > 2$, it suffices that the long-range order $c(\lambda, m_0, \sigma, \varepsilon)$ be *bounded* as $\varepsilon \to 0$; it need not go to zero. So this may cover also certain parameter values in the two-phase region. But we do not know how to guarantee that the *connected* 2-point function $S(x) - c$ is not identically zero.

The last step is to show that given any set of values $(\lambda, m_0, \sigma)$, we can find another set of values $(\lambda', m_0', \sigma')$ which is equivalent to the first set by scaling and for which $\lambda' \leqq \lambda_0$, $m_0' = 1$, $\sigma' \geqq 0$. The argument is based on two simple observations:

1) A $\varphi_d^4$ lattice theory (2.1) is specified by the parameters $\varepsilon$, $a$, and $\lambda$. The mass $m_0$ does not appear in the definition of the model, but can be set arbitrarily ($>0$); then (8.6) determines $\sigma$.

2) Fix $\mu > 0$. Then the $\varphi_d^4$ lattice model $(\varepsilon, a, \lambda)$ is equivalent to the model $(\varepsilon', a', \lambda')$ with

$$\varepsilon' = \mu\varepsilon, \qquad a' = \mu^{-2} a, \qquad \lambda' = \mu^{d-4}\lambda, \tag{8.16}$$

provided that we make the identification

$$\varphi'(x) = \mu^{1-\frac{d}{2}} \varphi(\mu^{-1} x). \tag{8.17}$$

Lengths have been rescaled by a factor $\mu$ and field strengths by a factor $\mu^{1-\frac{d}{2}}$, but otherwise nothing has been changed. Thus, any estimates which are valid for the model $(\varepsilon', a', \lambda')$ will also be valid for the model $(\varepsilon, a, \lambda)$ (up to factors of $\mu$).

Thus, given $(\lambda, m_0, \sigma)$, the strategy is to choose $\mu$ large enough so that $\lambda' \equiv \mu^{d-4}\lambda \leqq \lambda_0$ (here $d < 4$); we then impose $m_0' = 1$ and see what value of $\sigma'$ results. We need the formulas

$$\delta m_1^2 = -\varepsilon^{-2}(\lambda\varepsilon^{4-d}) f_1(m_0\varepsilon), \tag{8.18}$$

$$\delta m_2^2 = \varepsilon^{-2}(\lambda\varepsilon^{4-d})^2 f_2(m_0\varepsilon), \tag{8.19}$$

which display explicitly the scaling behavior of the mass counterterms: they have dimension $(\text{length})^{-2} = (\text{mass})^2$, while the combinations $\lambda\varepsilon^{4-d}$ and $m_0\varepsilon$ are dimensionless. Here $f_1$ and $f_2$ are given by

$$f_1(x) = 3 \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \frac{1}{x^2 + 2\sum_{i=1}^{d} (1 - \cos k_i)}, \tag{8.20}$$

$$f_2(x)=6 \iint_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \left[\frac{1}{x^2+2\sum_{i=1}^{d}(1-\cos k_i)} \cdot \frac{1}{x^2+2\sum_{i=1}^{d}(1-\cos l_i)} \frac{1}{x^2+2\sum_{i=1}^{d}(1-\cos(k_i+l_i))}\right]. \tag{8.21}$$

We thus have

$$a=m_0^2-\sigma-\lambda\varepsilon^{2-d} f_1(m_0\varepsilon)+\lambda^2\varepsilon^{6-2d} f_2(m_0\varepsilon), \tag{8.22}$$

and

$$a'=m_0'^2-\sigma'-\lambda'\varepsilon'^{2-d} f_1(m_0'\varepsilon')+\lambda'^2\varepsilon'^{6-2d} f_2(m_0'\varepsilon'), \tag{8.23}$$

where for $d=2$ we omit the terms involving $f_2$. Combining these equations with (8.16) and imposing $m_0'=1$, we find

$$\sigma'=1+\frac{\sigma-m_0^2}{\mu^2}+\frac{\lambda\varepsilon^{2-d}}{\mu^2}[f_1(m_0\varepsilon)-f_1(\mu\varepsilon)] - \frac{\lambda^2\varepsilon^{6-2d}}{\mu^2}[f_2(m_0\varepsilon)-f_2(\mu\varepsilon)]. \tag{8.24}$$

For $d=2$, we have

$$f_1(x)=b_1 \log\frac{1}{x}+O(x^2) \tag{8.25}$$

for $0<x\ll 1$, where $b_1$ is a positive constant. Then (8.24) reads

$$\sigma'\approx 1+\frac{\sigma-m_0^2}{\mu^2}+\frac{b_1\lambda}{\mu^2}\log\frac{\mu}{m_0}. \tag{8.26}$$

Thus, for sufficiently large $\mu$ [depending on the initial parameter set $(\lambda, m_0, \sigma)$ but *not* depending on $\varepsilon$], $\sigma'$ is positive, and we are in the region of applicability of the nonvanishing lower bound (8.15). [Of course, we must have $\varepsilon\ll\min(m_0^{-1},\mu^{-1})$ so that the approximation (8.25)/(8.26) is valid; but it is precisely small $\varepsilon$ which interests us.] This completes the proof of Theorem 1.2 for $d=2$. Finally, for $d=3$ we have

$$f_1(x)=b_2-b_3x+O(x^2), \tag{8.27}$$

$$f_2(x)=b_4\log\frac{1}{x}+O(x) \tag{8.28}$$

for $0<x\ll 1$, where $b_2$, $b_3$, and $b_4$ are positive constants. Then (8.24) reads

$$\sigma'\approx 1+\frac{\sigma-m_0^2}{\mu^2}+\frac{b_3\lambda}{\mu^2}(\mu-m_0)-\frac{b_4\lambda^2}{\mu^2}\log\frac{\mu}{m_0}. \tag{8.29}$$

This, too, is positive for sufficiently large $\mu$ (independent of $\varepsilon$). The proof of Theorem 1.2 is thus complete.

## 9. Further Results and Open Problems

In this section we briefly sketch several extensions of our results. Some of the ideas described in the following are somewhat speculative.

(1) *Asymptoticity of Perturbation Theory for the $\varphi_2^4$ and $\varphi_3^4$ Theories to Arbitrary Order in $\lambda$*

As discussed in [5, Sect. 6], we expect that there are skeleton inequalities for the Schwinger functions $S_{2m}^{(\varepsilon)}(x_1, \ldots, x_{2m})$ valid to arbitrary order in $\lambda$. More precisely, let $G_{2m}(k)$ be an arbitrary Feynman diagram with $k$ internal vertices of order four and $2m$ external vertices located at the points $x_1, \ldots, x_{2m}$, which does not contain any self-energy subdiagram [i.e. $G_{2m}(k)$ is a skeleton diagram]. With each diagram $G_{2m}(k)$ we associate a Feynman amplitude by associating with each line of $G_{2m}(k)$ the exact propagator $S_2^{(\varepsilon)}(x, y)$, and multiplying the resulting integral with the usual combinatorial coefficient. Then it is expected that the following skeleton inequalities hold for arbitrary $n$:

$$\sum_{k=0}^{2n-1} (-\lambda)^k \sum_{G_{2m}(k)} I_{G_{2m}(k)} \leqq S_{2m}^{(\varepsilon)}(x_1, \ldots, x_{2m}) \leqq \sum_{k=0}^{2n} (-\lambda)^k \sum_{G_{2m}(k)} I_{G_{2m}(k)} \,. \tag{9.1}$$

In this subsection we sketch how the skeleton inequalities (9.1) can be employed to prove the asymptoticity of perturbation theory to all orders in $\lambda$, with error bounds that are uniform in the lattice spacing.

The Schwinger-Dyson equation for the propagator is [see (5.6)]

$$\begin{aligned} S_2^{(\varepsilon)}(x-y) = {} & C^{(\varepsilon)}(x-y) - (a - m_0^2) \int d^d z C^{(\varepsilon)}(x-z) S_2^{(\varepsilon)}(z-y) \\ & - \lambda \int d^d z C^{(\varepsilon)}(x-z) S_4^{(\varepsilon)}(z, z, z, y) \,. \end{aligned} \tag{9.2}$$

We obtain upper and lower bounds on $S_2^{(\varepsilon)}$ in terms of itself by inserting into (9.2) the skeleton inequalities for $S_4^{(\varepsilon)}(z, z, z, y)$ to order $l-1$ and $l$, with $l \geqq 1$ when $d=2$ and $l \geqq 2$ when $d=3$. We then iterate these bounds, choosing at each stage *one* of the propagators $S_2^{(\varepsilon)}$ which occurs on the right side and inserting for it the upper or lower bound (which one depends on the sign of the coefficient – here we make use of the fact that $C^{(\varepsilon)}$, $S_2^{(\varepsilon)} \geqq 0$). After a finite number of steps, we reach a stage at which all terms of order $\lambda^l$ or lower contain only *free* propagators $C^{(\varepsilon)}$; these terms are precisely the mass-renormalized perturbation expansion for $S_2^{(\varepsilon)}$ through order $\lambda^l$ (this is a consequence of the fact that the upper and lower bounds are identical through order $\lambda^l$). There are various (finitely many) remainder terms of order $\lambda^{l+1}$ and higher, each containing some combination of propagators $C^{(\varepsilon)}$ and $S_2^{(\varepsilon)}$; what is crucial to note is that these also contain only "renormalized" Feynman amplitudes, in the sense that each divergent (sub)graph, made out of *any* combination of free and interacting propagators, is accompanied by the corresponding mass counterterm. Thanks to Theorem 6.1 and 6.5, the mass counterterms which we have chosen cancel the divergences in *all* such diagrams, whether the propagators are free or interacting. [Otherwise put, if we write everywhere $S_2^{(\varepsilon)} = C^{(\varepsilon)} + E^{(\varepsilon)}$, we find that diagrams containing only propagators $C^{(\varepsilon)}$ are explicitly renormalized, while diagrams containing at least one $E^{(\varepsilon)}$ are finite due to the strong bound (6.2) on $E^{(\varepsilon)}$.] This proves the asymptotic expansion for $S_2^{(\varepsilon)}$

through order $\lambda^l$, with estimates that are uniform in the lattice spacing. It is worth noting that the continuity argument in the proof of Theorem 6.1 or 6.5 need only be made *once*, in order to get the *zeroth-order* bound on $S_2^{(\varepsilon)}$; all higher-order bounds can then be obtained by simple iteration of the propagator inequalities.

The corresponding asymptotic expansion for $S_{2m}^{(\varepsilon)}$ is now trivially obtained by inserting the asymptotic expansion for $S_2^{(\varepsilon)}$ through order $\lambda^l$ into the skeleton inequalities (9.1) with $2n-1 \geqq l$. Since the graphs $G_{2m}(k)$ contain no divergent subdiagrams (because $d<4$), and the terms in the asymptotic expansion for $S_2^{(\varepsilon)}$ are at least as well behaved as the free propagator $C^{(\varepsilon)}$, it follows that no ultraviolet divergences can occur.

This completes the sketch of the proof.

(2) *Construction of a Continuum Limit without Subsequences*

When one analyzes the continuum limit of $\varphi_d^4$ lattice theories one is obliged to compare Schwinger functions $S_{2n}^{(\varepsilon_1)}(f_1, \ldots, f_{2n})$ and $S_{2n}^{(\varepsilon_2)}(f_1, \ldots, f_{2n})$ of two different lattice theories on lattices $\mathbb{Z}_{\varepsilon_i}^d$, $i=1,2$ (with $\varepsilon_1 \geqq \varepsilon_2$). Here

$$S_{2n}^{(\varepsilon)}(f_1, \ldots, f_{2n}) = \sum_{x_1 \ldots x_{2n} \in \mathbb{Z}^d} S_{2n}^{(\varepsilon)}(x_1, \ldots, x_{2n}) \prod_{j=1}^{2n} \varepsilon^d f_j(x_j),$$

and $\{f_j(x)\}_{j=1}^{2n}$ are Schwartz-space functions on $\mathbb{R}^d$.

*Constructing* the continuum limit means proving that, for arbitrary $\delta>0$,

$$|S_{2n}^{(\varepsilon_1)}(f_1, \ldots, f_{2n}) - S_{2n}^{(\varepsilon_2)}(f_1, \ldots, f_{2n})| < \delta, \tag{9.3}$$

whenever $\varepsilon_1, \varepsilon_2 < \varepsilon(\delta)$, for some $\varepsilon(\delta)>0$. In the following we speculate about an idea of how one might go about proving (9.3): We choose rational numbers $\varepsilon_1$ and $\varepsilon_2$ and pick some $\varepsilon$ such that $\varepsilon_i = n_i \varepsilon$, where $n_1$ and $n_2$ are positive integers, for $i=1,2$. We now embed the lattices $\mathbb{Z}_{\varepsilon_i}^d$ in $\mathbb{Z}_\varepsilon^d$ in such a way that they all have a common origin. Let $\Lambda$ be some compact subset of $\mathbb{R}^d$. We define $\Lambda_\varepsilon = \mathbb{Z}_\varepsilon^d \cap \Lambda$ as the subset of points in $\mathbb{R}^d$ belonging to $\mathbb{Z}_\varepsilon^d$ and contained in $\Lambda$. Next, we formulate the $\varphi_d^4$ theories on $\mathbb{Z}_{\varepsilon_1}^d$ and $\mathbb{Z}_{\varepsilon_2}^d$ as theories on $\mathbb{Z}_\varepsilon^d$ and then try to interpolate between the two theories in two stages. We let $e_\mu$ be the unit lattice vectors of $\mathbb{Z}^d$ in the direction of the $\mu^{\text{th}}$ lattice axis, $\mu=1, \ldots, d$. We define the lattice $\varphi_d^4$ actions by setting

$$S^{(i)}(\varphi) = S_0^{(i)}(\varphi) + S_I^{(i)}(\varphi),$$

where

$$S_0^{(i)}(\varphi) = \tfrac{1}{2} \sum_{x \in \Lambda_\varepsilon, \mu} \varepsilon_i^{d-2} (\varphi(x) - \varphi(x+\varepsilon_i e_\mu))^2 + \tfrac{1}{2} \sum_{x \in \Lambda_\varepsilon} \varepsilon_i^d \varphi(x)^2,$$

and

$$S_I^{(i)}(\varphi) = \sum_{x \in \Lambda_{\varepsilon_i}} \varepsilon_i^d \{\tfrac{1}{2} \delta m_i^2 \varphi(x)^2 + \tfrac{1}{4} \lambda \varphi(x)^4\}.$$

An interpolating action is defined by

$$S_{\varrho,\mu}(\varphi) = \varrho S_0^{(2)}(\varphi) + (1-\varrho) S_0^{(1)}(\varphi) + \mu S_I^{(2)}(\varphi) + (1-\mu) S_I^{(1)}(\varphi),$$ [2]

2 Up to a $\varrho$- and $\mu$-dependent change in the mass counterterms

and an interpolating expectation by

$$\langle\!\langle(\cdot)\rangle\!\rangle(\varrho,\mu)=Z(\varrho,\mu)^{-1}\int\prod_{x\in\Lambda_\varepsilon}d\varphi(x)\,[e^{-S_{\varrho,\mu}(\varphi)}(\cdot)],$$

with $Z(\varrho,\mu)$ chosen such that $\langle\!\langle 1\rangle\!\rangle(\varrho,\mu)=1$.

Now note that $\langle\!\langle(\cdot)\rangle\!\rangle(\varrho,\mu)$ is a *ferromagnetic*, even lattice $\varphi_d^4$ expectation. Therefore *all* our correlation inequalities apply. The idea is now to try to estimate the differences

$$\left.\begin{aligned}&\langle\!\langle(\cdot)\rangle\!\rangle(1,0)-\langle\!\langle(\cdot)\rangle\!\rangle(0,0)=\int d\varrho\frac{\partial}{\partial\varrho}\langle\!\langle(\cdot)\rangle\!\rangle(\varrho,0),\\ &\text{and}\\ &\langle\!\langle(\cdot)\rangle\!\rangle(1,1)-\langle\!\langle(\cdot)\rangle\!\rangle(1,0)=\int d\mu\frac{\partial}{\partial\mu}\langle\!\langle(\cdot)\rangle\!\rangle(1,\mu).\end{aligned}\right\}\tag{9.4}$$

The integrands on the right side of (9.4) involve truncated correlations. One might try to use the Schwinger-Dyson equations and appropriate correlation inequalities – strong enough to yield convergent bounds on truncated correlations – to estimate the integrands on the right side of (9.4). Among the technical tools that one could use in this task are skeleton inequalities and an (expected) extension of Theorem 6.1 to the two-point function $\langle\varphi(x)\varphi(y)\rangle(\varrho,\mu)$. All these tools are available, in principle, because $\langle\!\langle(\cdot)\rangle\!\rangle(\varrho,\mu)$ is ferromagnetic, and the correlations $\langle\varphi(x_1)\dots\varphi(x_{2n})\rangle(\varrho,\mu)$ admit the standard random-walk representation.

Among the difficulties which we have not succeeded in bypassing, yet, is the circumstance that correlation inequalities for truncated expectations are not sufficiently sharp and, in repeated applications of the Schwinger-Dyson equations, uncancelled (divergent) subdiagrams proliferate.

One meets similar difficulties when one tries to control the infinite-volume limit constructively. Likewise, to prove Euclidean invariance of the continuum limit, one would have to control the difference between the theories on two lattices, one of which is rotated relative to the other; this seems even more difficult.

### (3) *Finite-Volume Version of our Construction*

Errico Presutti (private communication) has pointed out to us that our use of Proposition 5.1 can be avoided by carrying out the entire analysis of Sect. 6 in *finite* volume. This is most easily done using periodic boundary conditions, since they maintain translation invariance (otherwise Sect. 6 would have to be radically rewritten). The estimates of Sect. 6 would then manifestly be uniform in the volume as well as in the lattice spacing, so the infinite-volume and continuum limits could be taken simultaneously (or in either order) by compactness and subsequences. This method of proof has, however, the disadvantage of unnecessarily employing subsequences in performing the infinite-volume limit.

### (4) *Existence of a Critical $\varphi_3^4$ Theory*

Within the framework of the traditional approach to constructive quantum field theory, McBryan and Rosen [60] have demonstrated that in the $\varphi_2^4$ and $\varphi_3^4$ models there exists a critical point $\sigma_c$ such that the physical mass $m(\sigma)$ decreases continuously to zero as $\sigma \to \sigma_c$ from below, and also that, in the case of $\varphi_3^4$, there exists a theory *at* $\sigma = \sigma_c$ which has zero physical mass. In this subsection we sketch a proof of a version of this latter result within our approach; it is taken almost verbatim from a paper of Glimm and Jaffe [55].

In Sect. 8 we studied the $\varphi^4$ model with an extra mass term $-\frac{1}{2}\sigma\varphi^2$ added to the lattice action[3], and showed that for $(\lambda, m_0, \sigma)$ in the *single-phase region* we could construct a continuum-limit theory. The single-phase region was defined as the set of those parameters $(\lambda, m_0, \sigma)$ for which the long-range order $c(\lambda, m_0, \sigma, \varepsilon)$ [see (8.9)] vanishes as $\varepsilon \to 0$. By Griffiths' second inequality, this set is, for each fixed $(\lambda, m_0)$, an interval of the form $(-\infty, \sigma_e)$ or $(-\infty, \sigma_e]$, where

$$\sigma_e \equiv \sup \left\{ \sigma : \lim_{\varepsilon \to 0} c(\lambda, m_0, \sigma, \varepsilon) = 0 \right\}.$$

[From now on we fix $(\lambda, \mathrm{m}_0)$ and omit all reference to them.] We shall show that there exists $\sigma_c \leqq \sigma_e$ such that $m(\sigma)$ [the physical mass of the continuum theory] decreases to zero as $\sigma \nearrow \sigma_c$; and we shall further show that *if* $\sigma_c < \infty$, then there exists a continuum $\varphi_3^4$ theory (constructed as a limit $\sigma \nearrow \sigma_c$) with zero physical mass. (It is known that $\sigma_c < \infty$ [27], but we do not know how to prove this fact within our approach.) For technical reasons we do not use the theories constructed in Sect. 8, but rather the analogues obtained by replacing everywhere Dirichlet with *periodic* boundary conditions [see Subsect. (3) of Sect. 9]; conceivably the $\sigma_e$ and $\sigma_c$ obtained in this way could be smaller than those obtained using Dirichlet boundary conditions.

*Step 1.* We begin with the lattice $\varphi_3^4$ theory in a periodic box $\Lambda$ in spatial directions and the infinite-volume limit already taken in the time direction. In order to lighten the notation we unify the volume and ultraviolet cutoffs into a single symbol $\kappa = (\Lambda, \varepsilon)$; we shall write $\{\kappa_i\} \nearrow \infty$ as a shorthand for $\{\Lambda_i\} \nearrow \mathbb{R}^{d-1} = \mathbb{R}^2$ and $\{\varepsilon_i\} \searrow 0$.

Now the Fourier-transformed two-point function $\tilde{S}_{\kappa,\sigma}(k)$ has a spectral representation

$$\tilde{S}_{\kappa,\sigma}(k) = \int \frac{d\varrho_{\kappa,\sigma,\mathbf{k}}(a)}{2\varepsilon^{-2}(1-\cos \varepsilon k_0) + a}, \tag{9.5}$$

where $d\varrho_{\kappa,\sigma,\mathbf{k}}(a)$ is a positive measure supported on $[0, \infty)$. The physical mass of the finite-spatial-volume lattice theory, $m_\kappa(\sigma)$, is defined by

$$2\varepsilon^{-2}[\cosh \varepsilon m_\kappa(\sigma) - 1] = \inf \operatorname{supp} d\varrho_{\kappa,\sigma,0}.$$

The strength of the one-particle pole is

$$Z_\kappa(\sigma) \equiv \varrho_{\kappa,\sigma,\mathbf{0}}(\{2\varepsilon^{-2}[\cosh \varepsilon m_\kappa(\sigma) - 1]\}). \tag{9.6}$$

Taking $k_0 = \pi/\varepsilon$ in (9.5) and comparing with the infrared bound (8.7)/(8.8), we conclude that

3 Note that our definition of $\sigma$ is the negative of that used by McBryan and Rosen [60] and Glimm and Jaffe [55]

$$\int \frac{d\varrho_{\kappa,\sigma,\mathbf{k}}(a)}{1+\frac{a\varepsilon^2}{4}} \leqq 1\,, \tag{9.7}$$

and hence that

$$0 \leqq Z_\kappa(\sigma) \leqq 1 + \tfrac{1}{2}[\cosh \varepsilon m_\kappa(\sigma) - 1]\,. \tag{9.8}$$

Moreover, by the construction of Sects. 7 and 8 generalized to finite-spatial-volume periodic theories, we know that for some $\sigma_0$ (sufficiently large and negative),

$$0 < M_1 \leqq m_\kappa(\sigma_0) \leqq M_2 < \infty \tag{9.9}$$

uniformly in $\Lambda$ and in $\varepsilon \leqq 1$. Since $m_\kappa(\sigma)$ is a decreasing function of $\sigma$ (by Griffiths' second inequality), it follows from (9.8) that

$$0 \leqq Z_\kappa(\sigma) \leqq \text{const} \tag{9.10}$$

uniformly in $\Lambda$, $\varepsilon \leqq 1$ and $\sigma \geqq \sigma_0$. Henceforth we assume always that $\varepsilon \leqq 1$.

*Step 2.* By a simple but extremely clever argument due to Glimm and Jaffe [55], it is shown that

$$0 \leqq -\frac{dm_\kappa^2(\sigma)}{d\sigma} \leqq Z_\kappa(\sigma)\,. \tag{9.11}$$

By (9.10) this implies that the functions $m_\kappa^2(\sigma)$ are (uniformly) equicontinuous on the interval $[\sigma_0, \infty)$, and so by the Arzelà-Ascoli theorem there exists a sequence $\{\kappa_i\} \nearrow \infty$ such that $m_{\kappa_i}^2(\sigma)$ converges pointwise to a limit which we shall call $m_\infty^2(\sigma)$. The function $m_\infty^2(\sigma)$ is uniformly Lipschitz-continuous; note that it is defined for *all* $\sigma \geqq \sigma_0$. Clearly $m_\infty^2(\sigma)$ is a decreasing function of $\sigma$, and by (9.9) it is not identically zero. We now define

$$\sigma_c \equiv \sup\{\sigma : m_\infty^2(\sigma) > 0\} = \inf\{\sigma : m_\infty^2(\sigma) = 0\}\,. \tag{9.12}$$

If $\sigma_c < \infty$, then clearly $m_\infty^2(\sigma_c) = 0$ and $\sigma_c$ is the first zero of $m_\infty^2(\sigma)$.

*Step 3.* We claim that $\sigma_c \leqq \sigma_e$. Indeed, the spectral representation (9.5) with $\mathbf{k} = \mathbf{0}$ yields in $x$-space the bound

$$\int_\Lambda d^{d-1}\mathbf{x} S_{\kappa,0}(x_0, \mathbf{x}) = \int \frac{e^{-M(a)|x_0|}}{(4a + \varepsilon^2 a^2)^{1/2}} d\varrho_{\kappa,\sigma,\mathbf{0}}(a)\,, \tag{9.13}$$

where $M(a)$ is defined by

$$2\varepsilon^{-2}[\cosh \varepsilon M(a) - 1] = a\,. \tag{9.14}$$

Using (9.7), a little calculation shows that if $m_{\kappa_i}^2(\sigma) \to m_\infty^2(\sigma) > 0$, then (9.13) is uniformly bounded by a function which vanishes as $|x_0| \to \infty$; that is, the long-range order must vanish for the infinite-volume continuum theory defined via the sequence of cutoffs $\{\kappa_i\}$. Thus, if $\sigma < \sigma_c$, then $\sigma \leqq \sigma_e$, proving the claim. (Here $\sigma_e$ has to be defined as the sup over $\sigma$ for which the long-range order vanishes for *at least one* sequence of cut-offs $\{\kappa_i\}$.)

*Step 4. Assume* that $m_\infty^2(\sigma)$ is not constant in any lower neighborhood of $\sigma_c$ [if $\sigma_c < \infty$ this assumption is an immediate consequence of the definition of $\sigma_c$ and the continuity of $m_\infty^2(\sigma)$]. Then we claim that there exists a sequence $\{\sigma_n\} \nearrow \sigma_c$ and a subsequence of cutoffs $\{\kappa_{i_j}\}$ such that

$$\left.\frac{dm_{\kappa_{i_j}}^2(\sigma)}{d\sigma}\right|_{\sigma=\sigma_n} \geqq \alpha_n > 0 \tag{9.15}$$

for all $j$ (and some sequence of numbers $\alpha_n > 0$). Indeed, by the assumption we can choose an increasing sequence $\{\sigma_n^*\} \nearrow \sigma_c$ such that $m_\infty^2(\sigma_1^*) > m_\infty^2(\sigma_2^*) > \dots$.

Then by (9.10)/(9.11) and Fatou's lemma,[4]

$$\begin{aligned}\int_{\sigma_1^*}^{\sigma_2^*} \limsup_{i\to\infty}\left[-\frac{dm_{\kappa_i}^2(\sigma)}{d\sigma}\right] d\sigma &\geqq \limsup_{i\to\infty}\int_{\sigma_1^*}^{\sigma_2^*} -\left[\frac{dm_{\kappa_i}^2(\sigma)}{d\sigma}\right] d\sigma\\ &= \limsup_{i\to\infty}\,[m_{\kappa_i}^2(\sigma_1^*) - m_{\kappa_i}^2(\sigma_2^*)]\\ &= m_\infty^2(\sigma_1^*) - m_\infty^2(\sigma_2^*)\\ &> 0\,;\end{aligned} \tag{9.16}$$

thus $\limsup_{i\to\infty}(-dm_{\kappa_i}^2(\sigma)/d\sigma) > 0$ on a nonnull (hence nonempty) set of $\sigma \in [\sigma_1^*, \sigma_2^*]$; so there exists $\sigma_1 \in [\sigma_1^*, \sigma_2^*]$ and a subsequence $\{\kappa_{i_j^{(1)}}\}$ such that $(-dm_{\kappa_{i_j^{(1)}}}^2(\sigma)/d\sigma)|_{\sigma=\sigma_1} \geqq \alpha_1 > 0$ for all $j$. We now repeat this argument to choose $\sigma_2 \in [\sigma_2^*, \sigma_3^*]$ as above, making sure to choose $\{\kappa_{i_j^{(2)}}\}$ to be a subsequence of $\{\kappa_{i_j^{(1)}}\}$, and so on; and then we apply the diagonal argument to get a subsequence $\{\kappa_{i_j}\}$ which works simultaneously for all of the $\sigma_n$.

*Step 5.* We now take the infinite-volume and continuum limits simultaneously at $\sigma = \{\sigma_n\}$ along some common subsequence $\{\kappa_{i_j'}\}$ of the sequence $\{\kappa_{i_j}\}$ (under the same assumption as in Step 4). By the infrared bound, the Gaussian inequality and the diagonal argument, such a common convergent subsequence can always be extracted. We claim that, for each $n$, the physical mass of the resulting theory at $\sigma = \sigma_n$ is not greater than $m_\infty(\sigma_n)$. Indeed, its Fourier-transformed two-point function at zero spatial momentum, $\tilde{S}_{\sigma_n}(k_0, \mathbf{0})$, satisfies

$$\begin{aligned}\tilde{S}_{\sigma_n}(k_0,\mathbf{0}) &= \lim_{j\to\infty} \tilde{S}_{\kappa_{i_j'},\sigma_n}(k_0,\mathbf{0})\\ &\geqq \liminf_{j\to\infty} \frac{Z_{\kappa_{i_j'}}(\sigma_n)}{2\varepsilon_{i_j'}^{-2}[\cosh \varepsilon_{i_j} m_{\kappa_{i_j'}}(\sigma_n) - \cos \varepsilon_{i_j'} k_0]}\\ &\geqq \frac{\alpha_n}{k_0^2 + m_\infty^2(\sigma_n)}\end{aligned} \tag{9.17}$$

by the spectral representation (9.5) and (9.6), the Glimm-Jaffe inequality (9.10) and (9.11), and the lower bound (9.15). Again by the spectral representation (this time for the infinite-volume continuum theory), it follows that the exponential decay rate of $S_{\sigma_n}(x)$ cannot be greater than $m_\infty(\sigma_n)$.

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4 This lemma in its usual (lim inf) form can be applied to the functions $c\chi_{[\sigma_1^*, \sigma_2^*]} + dm_{\kappa_i}^2/d\sigma \geqq 0$, where $c$ is the constant in (9.10). This yields (9.16)

*Step 6.* We now let $n \to \infty$ and again extract a subsequence for which all correlation functions converge. The resulting theory has zero long-range order (by the infrared bound). *We claim that it also has zero physical mass, provided that $\sigma_c < \infty$.* Indeed, by the second Griffiths inequality we have $S_{\mathrm{limit}}(x) \geqq S_{\sigma_n}(x)$ for all $x$ and all $n$, so the exponential decay rate of the limiting theory cannot be greater than $m_\infty(\sigma_n)$. If $\sigma_c < \infty$, we have $\lim_{n\to\infty} m_\infty(\sigma_n) = 0$ by continuity of $m_\infty^2(\sigma)$. This completes the construction. □

We remark that the foregoing construction dependent crucially on the integrability at $p=0$ of the infrared bound (8.7)/(8.8); for this reason it is applicable only in dimension $d>2$. We again emphasize that this proof is entirely due to Glimm and Jaffe [55]; we have carried it over virtually without change into our approach to $\varphi_3^4$.

(5) *Zero-Component (Edwards Model) and Two-Component $|\varphi|_d^4$ Models, $d=2,3$*

As already noticed in [5, Sect. 6], the random-walk representation and the correlation inequalities extend to the Edwards model (zero-component $|\boldsymbol{\varphi}|_d^4$ model) and the isotropic two-component $|\boldsymbol{\varphi}|_d^4$ model, and it is not hard to prove the Schwinger-Dyson equations which relate the two-point functions to the four-point functions in these models. (See also [63].) All the results of this paper appear to extend therefore to these models and yield very simple existence proofs.

## Appendix. Some Real Analysis

In this appendix we review some classical inequalities and prove some generalizations which will be needed in Sect. 7. All estimates are valid for both integrals (continuum) and sums (lattice); we use the continuum notation.

The $L^p$ norm is defined for $1 \leqq p \leqq \infty$ by

$$\|f\|_p = \begin{cases} (\int |f(x)|^p d^d x)^{1/p} & \text{for} \quad 1 \leqq p < \infty \\ \sup_x |f(x)| & \text{for} \quad p = \infty\,. \end{cases} \tag{A.1}$$

Hölder's inequality states that

$$\|fg\|_r \leqq \|f_p\|g\|_q\,, \tag{A.2}$$

where $1 \leqq p, q, r \leqq \infty$ and $1/p + 1/q = 1/r$. Young's inequality states that

$$\|f*g\|_s \leqq c_{p,q,d}\|f\|_p\|g\|_q\,, \tag{A.3}$$

where $1 \leqq p, q, s \leqq \infty$, $1/p + 1/q - 1 = 1/s$, and $c_{p,q,d}$ is a (finite) universal constant. Here $*$ denotes convolution:

$$(f*g)(x) = \int f(x-y)g(y)d^d y\,. \tag{A.4}$$

Proofs and discussion of these inequalities can be found in [64, 65].

In Sect. 7 we employ the exponentially-weighted $L^p$ norms

$$\|f\|_{p,\alpha}=\|\cosh(\alpha x_1)f(x)\|_p=\begin{cases}(\int|\cosh(\alpha x_1)f(x)|^p d^d x)^{1/p} & \text{for } 1\leqq p<\infty \\ \sup_x \cosh(\alpha x_1)\,|f(x)| & \text{for } p=\infty\,,\end{cases} \tag{A.5}$$

where $\alpha\geqq 0$. If $|f|$ is an even function, $\cosh(\alpha x_1)$ can obviously be replaced by $\exp(\alpha x_1)$. In particular, if $f$ is even and nonnegative,

$$\|f\|_{1,\alpha}=\tilde{f}(i\alpha)\,, \tag{A.6}$$

where we have abused notation to let $\alpha$ denote also the vector $(\alpha, 0, \ldots, 0)$. We also employ the exponentially-weighted $L^1\cap L^\infty$ norm

$$|||f|||_\alpha=\|f\|_{1,\alpha}+\|f\|_{\infty,\alpha}\,. \tag{A.7}$$

These norms obey analogues of the Hölder and Young inequalities:

$$\|fg\|_{r,\alpha+\beta}\leqq 2\|f\|_{p,\alpha}\|g\|_{q,\beta}\,, \tag{A.8}$$

$$\|f*g\|_{s,\alpha}\leqq c'_{p,q,d}\|f\|_{p,\alpha}\|g\|_{q,\alpha}\,, \tag{A.9}$$

$$|||f*g|||_\alpha\leqq 2\|f\|_{1,\alpha}|||g|||_\alpha\,, \tag{A.10}$$

where $p,q,r$ [or $p,q,s$] obey the same relations as before. Inequality (A.8) is an immediate consequence of the definition (A.5) and the ordinary Hölder inequality (A.2); the factor 2 arises from $\cosh(a+b)\leqq 2(\cosh a)(\cosh b)$, and may be replaced by 1 if $f$ and $g$ are both even. Inequality (A.9) follows from the ordinary Young inequality (A.3) and the definition (A.5) together with the identity

$$e^{\alpha x_1}(f*g)(x)=\int[e^{\alpha(x_1-y_1)}f(x-y)]\,[e^{\alpha y_1}g(y)]d^d y\,. \tag{A.11}$$

Inequality (A.10) is an immediate consequence of (A.9) applied to $s=q=1, \infty$.

We conclude by presenting some estimates on the free lattice propagator $C=C^{(\varepsilon)}$ with mass $m_0=1$. Its Fourier transform $\tilde{C}(k)$ is

$$\tilde{C}(k)=\left[1+2\varepsilon^{-2}\sum_{i=1}^{d}(1-\cos\varepsilon k_i)\right]^{-1}\,. \tag{A.12}$$

This is analytic in $k_1$ in the strip $|\mathrm{Im}\,k_1|<m_0^{(\varepsilon)}$ with $k_2, \ldots, k_d$ real, where $m_0^{(\varepsilon)}>0$ is the solution of

$$1+2\varepsilon^{-2}(1-\cosh\varepsilon m_0^{(\varepsilon)})=0\,. \tag{A.13}$$

Thus, by a Paley-Wiener theorem [65], $C(x)$ decays roughly as $\exp(-m_0^{(\varepsilon)}|x_1|)$ in the $x_1$ direction (and likewise in the other directions). Note that $m_0^{(\varepsilon)}\uparrow m_0=1$ as $\varepsilon\to 0$ but that $m_0^{(\varepsilon)}\neq m_0$ for $\varepsilon>0$.

Now let $\alpha\in(-m_0^{(\varepsilon)}, m_0^{(\varepsilon)})$, and define

$$C_\alpha(x)=\cosh(\alpha x_1)C(x)\,, \tag{A.14}$$

so that

$$\tilde{C}_\alpha(k)=\tfrac{1}{2}[\tilde{C}(k+i\alpha)+\tilde{C}(k-i\alpha)]\,. \tag{A.15}$$

Since $C_\alpha(x) \geqq 0$ (this is a consequence, for example, of the Griffiths inequality), we have

$$\begin{aligned}\|C\|_{1,\alpha} &= \|C_\alpha\|_1 = \tilde{C}_\alpha(0)\\ &= [1+2\varepsilon^{-2}(1-\cosh\varepsilon\alpha)]^{-1}\\ &= [2\varepsilon^{-2}(\cosh\varepsilon m_0^{(\varepsilon)} - \cosh\varepsilon\alpha)]^{-1},\end{aligned} \tag{A.16}$$

hence

$$1 \leqq \|C\|_{1,\alpha} \leqq c_1(m_0^{(\varepsilon)} - |\alpha|)^{-1}, \tag{A.17}$$

where $c_1$ is independent of $\varepsilon$ and $\alpha$ (recall that $|\alpha| < m_0^{(\varepsilon)}$).

Next we note that for $\alpha \in (-m_0^{(\varepsilon)}, m_0^{(\varepsilon)})$ and $k$ real,

$$|\tilde{C}(k_1 + i\alpha, k_2, \ldots, k_d)| \leqq \|C\|_{1,\alpha}\tilde{C}(k); \tag{A.18}$$

this can be shown by a straightforward calculation using (A.12). Finally, we note that for $k \in [-\pi/\varepsilon, \pi/\varepsilon]^d$,

$$c_2 \frac{1}{k^2+1} \leqq \tilde{C}(k) \leqq c_3 \frac{1}{k^2+1}, \tag{A.19}$$

where $c_2$ and $c_3$ are independent of $\varepsilon$. [Strictly speaking, the upper bound in (A.19) is true only if $\varepsilon$ is not too *large*. But $\varepsilon$ large is of no interest to us, so we just assume that, say, $\varepsilon \leqq 1$.]

Now we can estimate the norms $\|C\|_{p,\alpha}$:

**Lemma A.1.** *Let $d < 4$, $1 \leqq p < d/(d-2)$ (or $1 \leqq p \leqq \infty$ if $d = 1$), and $0 \leqq \alpha < m_0^{(\varepsilon)}$. Then there exist strictly positive constants $c_4, c_5, c_6$ which depend on $d$ and $p$ but not on $\varepsilon$ ($\varepsilon \leqq 1$) or $\alpha$, such that*

$$c_4 \leqq \|C\|_{p,\alpha} \leqq c_5 \|C\|_{1,\alpha} \leqq c_6(m_0^{(\varepsilon)} - \alpha)^{-1}. \tag{A.20}$$

*Proof.* The upper bound on $\|C\|_{1,\alpha}$ is just (A.17). To get the upper bound on $\|C\|_{p,\alpha}$ for the case $1 < p \leqq 2$, we calculate $\|C\|_{2,\alpha}$ by the Plancherel formula and use interpolation. Thus,

$$\|C\|_{2,\alpha} = \|C_\alpha\|_2 = \text{const} \times \|\tilde{C}_\alpha\|_2; \tag{A.21}$$

but by (A.15), (A.18), and (A.19),

$$\|\tilde{C}_\alpha\|_2 \leqq \|C\|_{1,\alpha} \|\tilde{C}\|_2 \leqq \text{const} \times \|C\|_{1,\alpha} \tag{A.22}$$

for $d < 4$. This proves the upper bound in (A.20) for $p = 2$. Interpolation between $p = 1$ and $p = 2$ (e.g. by Hölder's inequality) proves the upper bound in (A.20) for $1 \leqq p \leqq 2$. The upper bound for the case $2 \leqq p < d/(d-2)$ (or $2 \leqq p \leqq \infty$ if $d = 1$) can be done by first using the Hausdorff-Young inequality [65]

$$\|C\|_{p,\alpha} = \|C_\alpha\|_p \leqq \text{const} \times \|\tilde{C}_\alpha\|_q, \tag{A.23}$$

where $p^{-1} + q^{-1} = 1$ (hence $1 \leqq q \leqq 2$); then $\|\tilde{C}_\alpha\|_q$ can be estimated as above, using (A.18) and (A.19).

For the lower bound, note first that

$$\|C\|_{p,\alpha} \geqq \|C\|_p. \tag{A.24}$$

Now $C(x) \equiv C^{(\varepsilon)}(x) \geqq 0$ and clearly $C^{(\varepsilon)}$ is not identically zero. Thus $\|C^{(\varepsilon)}\|_p$ is strictly positive for all $\varepsilon$, and is a continuous function of $\varepsilon$ which approaches a nonzero value (the $L^p$ norm of the free continuum propagator) as $\varepsilon \to 0$. It follows that $\|C^{(\varepsilon)}\|_p$ has a strictly positive lower bound on the interval $0 < \varepsilon \leqq 1$. By (A.24), this completes the proof of Lemma A.1. □

*Remark.* The bound (A.20), although sufficient for our purposes, is far from the best possible. We *conjecture* that the best possible bound is

$$\|C\|_{p,\alpha} \leqq \text{const} \times \begin{cases} (m_0^{(\varepsilon)} - \alpha)^{-\frac{d+1-(d-1)p}{2p}} & \text{if} \quad 1 \leqq p < \frac{d+1}{d-1} \\ |\log(m_0^{(\varepsilon)} - \alpha)|^{1/p} & \text{if} \quad p = \frac{d+1}{d-1} \\ 1 & \text{if} \quad \frac{d+1}{d-1} < p < \frac{d}{d-2}. \end{cases} \tag{A.25}$$

For $p \geqq 2$ this can be proven by the Hausdorff-Young argument used above; and for $d < 3$ the remaining cases can be obtained by interpolation between $p = 1$ and $p = 2$. But we have been unable to prove (A.25) for $d \geqq 3$ and $1 < p < 2$.

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