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Remarks on Quantum Gravity*

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Abstract. Quantum gravity is analyzed from an axiomatic point of view. Under some general conditions imposed on the asymptotic structure of space–time a rigorous proof of the CPT theorem and a general discussion of the axiomatic approach to quantum gravity are presented.

I. Introduction

In this paper we analyze in detail a set of axioms for quantum gravity recently proposed by Hawking [1–3]. The interest of such an axiomatization is twofold: on the one hand it allows us to make general statements about quantum gravitational effects and to analyze what type of principles and results of ordinary quantum field theory apply to quantum gravity; and on the other hand it may give a more definitive answer to the question of whether the inclusion of nontrivial topological configurations of the gravitational field generate loss of quantum coherence, i.e. evolution of pure to mixed states. This possibility was suggested shortly after the discovery that black holes can emit particles in a thermal spectrum [4, 5]. The semiclassical calculation carried out so far in the presence of a black hole seems to suggest that such a process might take place. These arguments led Hawking to infer that such an evolution from a pure to a mixed state could also occur on a microscopic level due to quantum fluctuations of the metric. Since these gravitational bubbles can have nontrivial topologies, the virtual geometries they describe will not be globally hyperbolic, and thus, they could generate acausal poles in the Green's functions of fields propagating through them [6–8]. Hence, it seems plausible that an acausality intrinsic to quantum gravity, might generate the loss of quantum coherence. Equivalently, one may interpret the quantum gravitational bubbles as virtual black holes which form and evaporate.

Without a rigorous theory of quantum gravity, the most reasonable way to assess the validity of the above conclusions is to try to formulate a minimal set of

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axioms for the Green's functions of quantum gravity. Then one can examine whether the loss of quantum coherence could be a consequence of the axioms, or at least compatible with them. This approach has recently been explored by Hawking [2, 3].

In this paper we will further investigate the consequences of Hawking's axiomatic framework using the standard methods of axiomatic quantum field theory. In Sect. II we present a brief summary of Hawking's formalism and some interesting recent criticism [9–12] concerning the superscattering operator. Section III contains the consequences of the axioms, in particular, we present a fairly rigorous proof of the CPT theorem, and a more rigorous formulation of the axioms in [2, 3]. Finally, we introduce a new axiom which helps eliminate certain pathological cases where loss of unitarity would also follow in ordinary flat space. We call this the axiom of weak asymptotic completeness.

II. Axioms for Quantum Gravity

The analysis of scattering processes which might result in the evolution of pure states into mixed states can be most easily carried out by means of the superscattering operator \$ [1–3]. Let $\mathscr{H}^- \otimes \mathscr{H}^{-\prime}$ [13] be the Hilbert space of incoming density matrices defined in terms of states prepared in the asymptotic past region, and $\mathscr{H}^+ \otimes \mathscr{H}^{+\prime}$ be the corresponding space of outgoing density matrices. The $\mathscr{H}^+$ and $\mathscr{H}^-$ are assumed to have the structure of free field Fock spaces. All information about scattering processes is now described by the operator \$ which transforms ingoing into outgoing density matrices. Since density matrices are hermitian operators with unit trace, conservation of probability requires that \$ transform unit trace hermitian incoming matrices into unit trace hermitian outgoing density matrices:

$$\rho^{A}_{+B} = \$^{AD}_{BC}\rho^{C}_{-D}, \tag{1.a}$$

$$\$^{AD}_{AC} = \delta^{D}_{C}, \tag{1.b}$$

\$ being hermitian in each pair of indices (AB) and (CD). Quantum coherence is maintained only if $\mathrm{Tr}\rho^2$ is conserved, i.e.

$$\mathrm{Tr}(\$\rho_-)^2 = \mathrm{Tr}\rho^2. \tag{2}$$

In ordinary field theory in flat space–time, we usually have $\$ = S \otimes S^+, SS^+ = 1$, so (2) will always be satisfied, thus forbidding transitions from pure to mixed states.

The axiomatic formulation should provide us in principle with a method of computing \$ in terms of the Green's function of the theory. In order to define the latter, we'll invoke the Euclidean formulation of quantum gravity and the Euclidicity postulate [13, 14]. In ordinary quantum field theory it is well known that the Green's functions for any operator ordering can be obtained as boundary values of a single set of holomorphic functions. Thus, the ordinary Green's functions, Wightman functions and Schwinger functions can all be obtained from a single set of holomorphic functions (Wightman functions) [15]. Thus by appropriate analytic continuation one can get the functions of interest for the problem at hand. While these results have been rigorously established in Minkowski space [16, 17], quantum gravity hasn't as firm a foundation and one has to appeal to the Euclidicity postulate which basically gives the conditions under which the same results are expected to hold. It should be remarked that if we consider that the Green's functions are given by the Euclidean functional integral, we have to integrate over all topologies and geometries. Therefore the arguments of the Green's functions will not have a well defined meaning in general. Since we are here interested in the effects that gravitational vacuum fluctuations will have in scattering processes, we can restrict the functional integral so that it averages over all asymptotically Euclidean, positive definite metrics [2]. By asymptotically Euclidean it is meant that outside a compact region, the metric approaches the flat Euclidean metric in R^4 fast enough, (We will say more about regularity conditions later on.) In this way, the Euclidicity postulate gives us a way of computing in principle the Schwinger and Wightman functions of the theory, and thus, the Wightman functions $W_n(x_1,\ldots,x_n)$ with their arguments taking values in the asymptotic regions. In the remainder of this section we will present the axioms that Hawking [2] imposes on the "expectation values" $W_n(x_1,\ldots,x_n)$ obtained by the procedure outlined above. They are the following:

(a) Under a Poincaré transformation of the past and future asymptotic regions, the Wightman functions transform as they would in Minkowski space–time.

(b) The $W_n(x_1,\ldots,x_n)$'s satisfy the ordinary positivity and hermiticity requirements.

(c) If *all* the arguments of $W_n(x_1,\ldots,x_n)$ belong to to either of the asymptotic regions then the $W_n(x_1,\ldots,x_n)$'s satisfy the free field equations and commutation relations.[1] In other words, the theory is asymptotically noninteracting. This axiom allows the definition of in and out Hilbert spaces where the scattering data is given in terms of the creation and annihilation operators associated with the fields generating the given free field Wightman functions.

(d) Spectral condition: axioms (a), (b), and (c) imply that we can represent the Wightman functions in terms of field operators $A(x)$ (see below). Thus, one can represent the field $A(x)$ in the asymptotic past and future in terms of past and future creation and annihilation operators satisfying the canonical commutation relations, even though commutation relations between the creation and annihilation operators in the past with those of the future are not trivial. The creation and annihilation operators in the past and future are then obtained by the usual Yang–Feldman relations:

$$a_{\pm}(k) = -\frac{i}{(2\pi)^{3/2}} \int_{\Sigma_{\pm}} e^{-ikx} \overleftrightarrow{\nabla}_{\mu} A(x)\, d\Sigma^{\mu}_{\pm}. \tag{3}$$

Here $\Sigma_+(\Sigma_-)$ is a space–like or null Cauchy surface for the future (past) asymptotic region. The spectral condition is then formulated [2] by saying that for any string Q of creation and annihilation operators the corresponding expectation value $\langle Q \rangle = 0$ unless at each point in the string, the sum of the energies of the annihilation operators to the right of the point is less than or equal to the sum of the energies of the creation operators to its right. Similarly, one demands conservation of angular momentum and electric charge.

Another way of formulating the spectral condition is to say that given the unitary

1 The field equations have to be satisfied in each argument separately in the asymptotic region

representation of the Poincaré group which is induced by axiom (a) on the reconstructed Hilbert space, the support of the spectrum of the generator of translations is contained in the forward light cone. This axiom is very important in the sense that it allows us to prove that the Wightman functions are boundary values of holomorphic functions [18, 19]. Those are the usual axioms in ordinary quantum field theory, with only the axiom of asymptotic completeness omitted. It should be mentioned though that the asymptotic completeness is independent of the oridnary Wightman axioms plus the Haag–Ruelle scheme of defining the S-matrix (see Ref. [20]).

III. Reconstruction Theorems

Given the axiomatic framework, we can apply the reconstruction theorem of Wightman theory [18] in order to obtain the Hilbert space field operator, etc. We first present a simple outline of how this is done, and later we present a more complete construction. In order to construct the Hilbert space of states, we consider the space of test functions whose support is contained in the asymptotic regions. With each such test function f we can, in an abstract sense, associate a vector in a topological vector space $\phi(f)$. Using the positivity requirement, we can then define the scalar product between two vectors $\phi(f_1)\,\phi(f_2)$ in the following way:

$$\langle \phi(f_1)|\phi(f_2)\rangle = \int_{D(f_1)uD(f_2)} f_1^*(x_1,\ldots,x_n) f_2(y_1,\ldots,y_m) W_{n+m}(x_1,\ldots,x_n,y_1,\ldots,y_m), \tag{4}$$

where $D(f_1)$ is the support of $f_i i = 1, 2$. The integral is well defined because the functions involved are all well defined in the asymptotic region. The completion of the quotient of this space by its radical is our Hilbert space $\mathscr{H}$. Next, we can define the field operator $A(f)$ by linear extension of the relation:

$$A(f)\phi(g) = \phi(f \otimes g). \tag{5}$$

Poincaré invariance allows the definition of a continuous unitary representation $u(\Lambda,a)$ of the Poincaré group (Λ,a represents Lorentz rotations and a translations, respectively.) By construction, one can find a state $|0\rangle$ which is left invariant under the action of $u(\Lambda,a)$, and such that we can represent the distributions $W_n(x_1,\ldots,x_n)$ as:

$$W_n(x_1,\ldots,x_n) = \langle 0|A(x_1)\ldots A(x_n)|0\rangle, \tag{6}$$

proving CPT still requires some analyticity conditions which can be derived from the spectral conditions (more on this shortly). Notice that the asymptotic condition automatically implies that when the arguments of $W_n(x_1,\ldots,x_n)$ are all in the future or in the past region (M^+ and M^-, respectively), we obtain the Wightman functions of a free field theory; thus by considering test functions with support in either M^+ or M^- we apply the reconstruction theorem again, and obtain two Hilbert spaces $\mathscr{H}^+$ and $\mathscr{H}^-$, and two free fields A^+, A^- which represent the asymptotic states. This construction can be made in a more rigorous way, and we will present it here, because it is important for our arguments.

Recall that throughout, we are considering asymptotically Minkowskian manifolds, and therefore we can assume that the structure of $\mathscr{I}$[2] is that of Minkowski space. In order to ensure this, it is clear that one needs to impose certain regularity conditions on the space of gravitational bubbles we integrate over. In order to give a more concrete characterization of these manifolds, we can identify the asymptotic regions by saying that a point x belongs to $M^+(M^-)$ if its future (past) light cone intersects $\mathscr{I}^+(\mathscr{I}^-)$ in a shear free cross section. These regions (M^+, M^-) must be four dimensional and clearly form a neighborhood of $\mathscr{I}$. The meaning of the shear free condition can be most easily understood by recognizing that the passage of a beam of light through a region of gravitation can be described by idealizing gravity as a set of lenses [22] which produce among other effects, astigmatic focusing (i.e., shear). This can be avoided asymptotically by requiring the Weyl tensor to vanish sufficiently quickly. Since the gravitational bubbles are essentially concentrated in a compact region, the regions M^+, M^- will have a notrivial set in common, so that they are not totally disconnected.

For the manifolds considered it is possible to construct an abstract Minkowski space–time $\mathscr{M}$ by associating to each shear free cross section of $\mathscr{I}$ the tip of the light cone which would produce the cross section, had the original asymptotically Minkowskian manifold been true Minkowski space. (Recall that each point of Minkowski space is in one-to-one correspondence with a shear free cut of $\mathscr{I}$.) Due to our formulation of the geometrical asymptotic condition, M^+ and M^- can be identified with subsets of $\mathscr{M}$. Within this framework, the functional integral will provide the Wightman functions over the asymptotic regions and $\mathscr{I}^+$, $\mathscr{I}^-$. According to (c), the W's will then satisfy the free field equations in each argument, and thus we can use the values of W at $\mathscr{I}^+$ as initial data on the null hypersurface $\mathscr{I}^+$ for the free field equations. This allows one by standard formulae [23] to obtain a unique extension of W_n to the whole of $\mathscr{M}$. In this way, we obtain from W_n at $\mathscr{I}^+$ a set of Wightman distributions satisfying the free field equations throughout $\mathscr{M}$. The distributions so obtained will be denoted W_n^+. Since the W_n^+'s are defined in $\mathscr{M}$, we can now apply the standard reconstruction theorem and obtain a free field A^+ throughout $\mathscr{M}$ and a Hilbert space $\mathscr{H}^+$ whose vectors represent out states. Moreover, by standard arguments we can construct the free CPT operator θ^+ associated with A^+ [19]. Before we go on to the case when some of the arguments of W_n are in $\mathscr{I}^+$ and some in $\mathscr{I}^-$, there are several remarks worth pointing out:

(i) The axiom of Poincaré invariance can be formulated naturally for W_n^+ as a consequence of the asymptotic structure of the manifolds we sum over. In general relativity, there is an important asymptotic symmetry group defined for flat manifolds, the BMS group [24–26], which can be defined as the set of transformations preserving the strong conformal geometry of null infinity. In the generic

2 $\mathscr{I}$ is the three dimensional manifold that is obtained by conformal compactification of the original four-dimensional manifold M[28, 29]. Briefly, a conformal compactification cf (M,g) is given by a triad $(\tilde{M},\tilde{g},\Omega)$, where $\tilde{M}$ corresponds to the manifold M with all the "end points" of maximally extended geodesics in M which reach the asymptotic regions. Thus $\mathscr{I} = \partial\tilde{M}$ and M is diffeomorphic to $\tilde{M} - g$. Here Ω is a smooth positive field in $\tilde{M}$ so that $\tilde{g} = \Omega^2 g$ in M, and $\Omega = 0$ at $\mathscr{I}$ so that $\nabla_\mu\Omega(\mathscr{I}) \neq 0$. If we consider the parts of $\tilde{M}$ which correspond to null infinity, it is easy to see that it is formed by two pieces $\mathscr{I}^+$, $\mathscr{I}^-$, $\mathscr{I}^+(\mathscr{I}^-)$ corresponds to the end points of the maximally extended null geodesics towards the future (past) which are not captured inside space–time. The $\mathscr{I}^+$ and $\mathscr{I}^-$ have global topology $S^2 \times \mathbb{R}$

case, however, it is not known how to extract a canonical Poincaré subgroup of BMS; although it is well known [26] that if we consider those transformations which send shear free cross sections to shear free cross sections, we obtain a Poincaré group. For manifolds whose global asymptotic structure is Minkowskian, we can take this Poincaré sub-group as the Poincaré group of the theory; and in fact, the transformations induced in $\mathscr{I}$ by Poincaré transformations in $\mathscr{M}$ are exactly those preserving the shear free property of cross sections. Axiom (a) in this context implies that the Poincaré group has a well defined unitary action on $\mathscr{H}^+$, and the spectral condition can be formulated in the usual way in terms of the generator of translations in $\mathscr{H}^+$. Finally, we can find a Poincaré invariant state $|0^+\rangle$ so that W_n^+ may be represented as:

$$W_n^+(x_1,\ldots,x_n)=\langle 0^+|A^+(x_1)\ldots A^+(x_n)|0^+\rangle, \tag{7}$$

in $\mathscr{M}$, and coincides with the original W_n in M^+.

(ii) A totally similar construction can be applied *mutatis mutandis* to the Wightman functions $W_n^-(x,\ldots,x_n)$ whose arguments lie in $\mathscr{I}^-$. Using again the asymptotic structure of the manifolds consider the corresponding abstract Minkowski space is identical with $\mathscr{M}$ endowed with the usual Poincaré group. Thus there will exist $\mathscr{H}^-$, A^-, $|0^-\rangle$, θ^-, so that the states of $\mathscr{H}^-$ represent asymptotic in states, θ^- represents the in CPT operator, and throughout $\mathscr{M}$:

$$W_n^-(x_1,\ldots,x_n)=\langle 0^-|A^-(x_1)\ldots A^-(x_n)|0^-\rangle. \tag{8}$$

Both W_n^+, W_n^- are distributions defined on the usual space of test functions $S(\mathscr{M}^n)$.

Finally, we come to the far less trivial case of studying the case where $W_n(x_1,\ldots,x_n)$ has arguments at both M^+ and M^-. These distributions will somehow exhibit the effect of the gravitational bubbles. In this case, the W_n's will be distributions defined on $S(M^+UM^-)$. Since M^+ and M^- are subsets of $\mathscr{M}$, we have $S(M^+UM^-)\subset S(\mathscr{M})$, and the W_n's are linear functionals on $S(M^+UM^-)$. We can now apply the Hahn–Banach theorem (see, for instance [20] and references therein) and extend the set of distributions $\{W_n\}$ to a new set $\{\tilde{W}_n\}$ defined on $S(\mathscr{M})$ and such that on $S(M^+UM^-)$ $\tilde{W}_n$ and W_n coincide for all n. Poincaré invariance will imply that the extensions $\tilde{W}_n$ will also be Poincaré covariant; however, the Hahn–Banach theorem does not imply uniqueness. (This is somewhat similar to the situation encountered in the theorems concerning renormalized perturbation theory, where one first defines the Wick ordered distributions $G(x_1,\ldots,x_n)$ in $\hat{M}^n$, where $\hat{M}^n$ stands for Minkowski space with the points where the arguments coincide removed. Then one can define the renormalized distributions in $S(M^n)$ through the Hahn–Banach theorem [27].) With all our distributions now lifted from the original scenario to $\mathscr{M}$ we can analyze the consequences of the axioms in more detail. By using the Euclidicity postulate, or the weaker assumption of weak locality which states that (for simplicity we only consider a single Hermitian scalar field):

$$\langle 0|A(x_1)\ldots A(x_n)|0\rangle=\langle 0|A(-x_n)\ldots A(-x_1)|0\rangle \tag{9}$$

the analyticity condition and the spectral condition, one can now follow the ordinary proof of CPT [19]. Thus we have

$$\langle 0|A(x_1)\ldots A(x_n)|0\rangle=\langle 0|A^+(-x_1)\ldots A^+(-x_n)|0\rangle^*, \tag{10}$$

and an anti-unitary operator θ such that $\theta|0\rangle = |0\rangle$ and θ is defined by an anti-unitary extension of $\theta\phi_n(\phi) = \phi_n(\phi^-)$, where $\phi^-(x_1,\ldots,x_n) = \phi^*(-x_1,\ldots,-x_n)$ is a test function, and $\phi_n(\phi)$ is its associated state in the Hilbert space.

The generalization to higher spin fields is trivial and will not be presented here.

This concludes our proof of the CPT theorem for the axioms presented in [2, 3], following the methods of axiomatic field theory.

It is also interesting to point out that in [9, 11] a proof is given that CPT invariance in a strong sense is incompatible with loss of quantum coherence. Even though the original argument was basically concerned with the quantum effects related to black hole formation and evaporation, the argument goes through without change in the present context. For completeness we briefly reproduce the argument (more details can be found in [11, 12]. Following our conventions, $\rho_-(\rho_+)$ represents an incoming (outgoing) density matrix in $\mathscr{H}^-\otimes\mathscr{H}^{-\prime}(\mathscr{H}^+\otimes\mathscr{H}^{+\prime})$. If there exists a CPT symmetry, then there are two operators:

$$\theta_-:\mathscr{H}^-\otimes\mathscr{H}^{-\prime}\to\mathscr{H}^+\otimes\mathscr{H}^{+\prime}, \tag{11}$$

$$\theta_+:\mathscr{H}^+\otimes\mathscr{H}^{+\prime}\to\mathscr{H}^-\otimes\mathscr{H}^{-\prime}, \tag{12}$$

satisfying $\theta_-\theta_+ = \theta_+\theta_- = 1$. Writing $\theta = \theta_- = \theta_+^{-1}$ and applying the CPT symmetry one easily obtains:

$$\theta = \$\theta^{-1}\$, \tag{13}$$

$$\$^{-1} = \theta^{-1}\$\theta^{-1}. \tag{14}$$

It is now easy to show that \$ having an inverse is incompatible with loss of quantum coherence, for if \$ transforms a mixed state into a pure state (or vice-versa) using (13), (14) and hermiticity, it would follow that \$ is not one-to-one and therefore its inverse would not exist.

If one wants to find loss of quantum coherence in this general context, it is possible to propose weaker forms of CPT invariance, and in fact this is done in [9, 10].

It is at the present not known whether CPT will be satisfied in the strong sense or not (even though it does not seem it will) because a generalization of the LSZ [20] formalism for \$ in this context is lacking. However it is unlikely that this could be proven, because one can think of examples of field theories in flat space time where one would have loss of unitarity even though there is CPT invariance. The simplest example of this kind of field theories[3] consists of a set of two scalar fields A, B in interaction in ordinary flat space, with the condition that full A, B theory is asymptotically complete. If out of the A, B theory we just extract the Wightman functions of the A field, they satisfy all the axioms of Sect. II, although in the reduced theory there is loss of unitarity as can be seen by the following argument: In the truncated theory, we can only predict and calculate (at least naively) processes where the initial and final states are just A-mesons. It is a well known fact of elementary field theory that the optical theorem is a consequence of the unitarity of the S-matrix; therefore we could in principle calculate the forward scattering amplitude for two A-meson elastic scattering. Thus we obtain the total cross section for $A+$

3 We are grateful to S. Hawking for suggesting this example

$A \to$ anything. Since we have all the A's Wightman functions, we could in principle check whether processes involving only A-mesons in the final state saturate the total cross section for $A + A \to$ anything. Obviously in this example, this is not the case, because we know from the beginning that we could also have B particles in the final state, and thus the S-matrix that one would define using the ordinary Haag–Ruelle construction to the A Wightman functions would not be unitary.

It is also clear in this case which is the way out of the apparent loss of unitarity. By using simple analyticity arguments, we can study the cross section for $A + A \to$ anything as a function of the energy. As is well known we should find in all the Green's functions different cuts at the different thresholds, and therefore knowing all the Wightman functions for the A-field should allow us to infer the existence of the B-particle and most of its properties. In order to try and avoid the loss of quantum coherence due to this type of problem, one might try to postulate that all the fields appearing in the Wightman functions should be a complete set of fields at least in so far as to describe all the asymptotic states as seen by asymptotic observers. In other words, in the example explained, asymptotic observers with detectors only turning on to A-mesons would certainly observe energy momentum and angular momentum nonconservation, and one would certainly like to exclude this possibility by strengthening postulate I and imposing the condition of completeness of the asymptotic fields. It seems to us that one should add the condition that the asymptotic fields A^+, A^- should be a complete set of fields so that the observers in the asymptotic regions could describe the scattering states in terms of them in a way consistent with conservation of charge, energy, momentum and angular momentum. This postulate of weak asymptotic completeness would eliminate the pathologies mentioned before, and may help to pin down whether the topological degrees of freedom of the gravitational field do in fact generate loss of quantum coherence. Given the scheme presented before, it is not hard to extend the Haag–Ruelle construction of scattering states to our situation [19]. Using the $\tilde{W}_n$'s, we can obtain by using the standard reconstruction theorem, a Hilbert space $\mathscr{H}$ in which we can give sense to the Haag–Ruelle definition of asymptotic states. The spectral condition and the asymptotic behavior of our Wightman distributions implies the existence in $\mathscr{H}$ of the asymptotic Hilbert spaces $\mathscr{H}_{\text{in}}$, $\mathscr{H}_{\text{out}}$ related through CPT $\mathscr{H}_{\text{in}} = \theta\mathscr{H}_{\text{out}}$. The $\mathscr{H}_{\text{in}}$ and $\mathscr{H}_{\text{out}}$ are clearly unitarily related to $\mathscr{H}^-$ and $\mathscr{H}^+$ because they are defined by the same free Wightman distributions, and therefore we can also define a unitary isometry between $\mathscr{H}^-$ and $\mathscr{H}^+$: $\theta\theta_{\text{in}}$, which is not necessarily related to the S-matrix of the theory. It seems plausible that a construction of \$ *à la* Haag–Ruelle rather than using the LSZ formalism should immediately verify whether loss of quantum coherence is possible in the more interesting case of theories satisfying the axiom of weak asymptotic completeness. Work in this direction is in progress.

IV. Conclusions

We have shown that the axiomatics presented in [2, 3] can be expressed in a more rigorous manner if certain asymptotic conditions are imposed on the space–times one sums over in the functional integral. Under these conditions one can prove the

CPT theorem for quantum gravity. The result depends crucially on two facts:

(i) The Wightman functions obtained approach the free Wightman functions in the asymptotic past and future.

(ii) We have been summing only over a set of geometries with very strong asymptotic constraints; in particular, we were able to identify an abstract Minkowski space–time and consider M^+ and M^- as subsets of $\mathscr{M}$ so that the whole problem could be formulated on $\mathscr{M}$, and therefore the application of axiomatic field theory techniques is greatly simplified. We believe that this formulation could be made more accurate and perhaps some of its conditions relaxed by using the techniques of $\mathscr{H}$-space [30], although we have not pursued this approach.

If one wants to maintain CPT one can relax conditions (b) and (c) of Sect. II without affecting very much the proof, which basically depends on Poincaré invariance, weak locality, and the spectral condition. However, relaxing (b) and (c) might lead to acute problems in the identification of scattering states, the asymptotic description of the theory in terms of Fock spaces, and therefore the asymptotic description of the theory in terms of a particle interpretation.

Finally we formulated the axiom of weak asymptotic completeness in order to simplify the analysis of loss of quantum coherence in quantum gravity due to the topological degrees of freedom of the gravitational field. Wherever this line of thought may lead to, we believe that the axiomatic approach started in [2] will teach us many things about the structure of quantum gravity, and is certainly worth pursuing.

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Note added. The argument presented to prove CPT does not include the presence of gravitational radiation in the asymptotic states. It should be quite interesting to extend it to that case where the asymptotic covariance group would be the full BMS group. We thank the referee for pointing this out to us.