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Computing the Topological Entropy of Maps

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Abstract. We give an algorithm for determining the topological entropy of a unimodal map of the interval given its kneading sequence. We also show that this algorithm converges exponentially in the number of letters of the kneading sequence.

It is by now well known that iterated maps of an interval, when viewed as dynamical systems, account for some of the irregular behaviour observed in physics. There are three commonly used indicators for the complexity of such systems: The metric entropy, the Liapunov exponent, and the topological entropy. Here we shall discuss an efficient method for calculating the weakest of these notions, namely the topological entropy.

A possible way of defining the topological entropy $h(f)$ of a function f is given by

$$h(f)=\lim_{n\to\infty}\frac{1}{n}\log_2 N(f^n), \tag{1}$$

where f^n denotes the n^{th} iterate of f, and $N(g)$ is the number of monotone pieces of the graph of the function g. Thus the topological entropy, if positive, measures the exponential growth rate of the number of laps of f^n as n increases. If f is continuous and has a single extremum, then $h(f)$ takes values in $[0, 1]$. If $h(f)$ is positive, then the map f has complex behaviour in the following sense:

1) f has infinitely many different types of aperiodic and periodic orbits. In particular, even if f has a stable periodic orbit, complicated transient behaviour will be observed.

2) Although the topological entropy gives essentially no information about attractors, it indicates, when positive, a sensitivity of the dynamical system to external noise [2, 4].

In this note, we prove that $h(f)$ can be computed efficiently from the orbit of the critical point of f, using the so-called kneading determinant of Milnor and Thurston [5].[1] Our theorem below shows that $h(f)$ can be computed with an error

1 See also [1] for background material

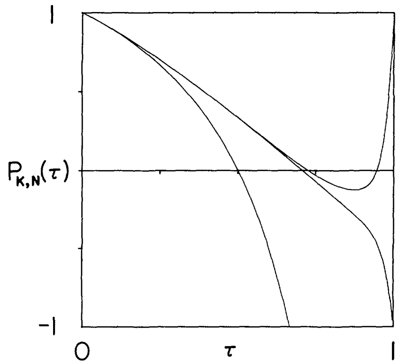

Fig. 1. Kneading determinant $P_{\mathbf{K},N}(\tau)$ vs. τ for the mapping $x_{n+1}=1-\mu x_n^2$: (left to right in the picture) $\mu=2.0$, $1.5436889\ldots$, 1.46610. The first value corresponds to the maximum allowed height of the map, here $P_{\mathbf{K},N}(\tau)$ has a zero at $\tau=1/2$ and the topological entropy $h(f)=\log_2(1/\tau)=1.0$. The second is the parameter where two bands merge [2], $P_{\mathbf{K},N}(\tau)$ has a zero at $\tau=1/\sqrt{2}$ and the topological entropy $h(f)=1/2$. The last value is in a regime where there are two distinct bands comprising the attractor and so $h(f)<1/2$. Each $P_{\mathbf{K},N}(\tau)$ calculated with $N=40$

which exponentially decreases with the number of iterates of the extremum. Previous numerical calculations of $h(f)$ have suggested that it can be computed with several different, rapidly convergent algorithms [6]. Our theorem will be based on the estimate of the smallest positive root $\tau(f)$ of a certain polynomial (called the characteristic polynomial of f), for which it is known that $h(f)=-\log_2\tau(f)$, (see e.g. [5]).

We present next the definition of characteristic polynomials for functions with one critical point, as given in [5]. In the sequel, assume f is a continuous map of $[-1,1]$ to itself, satisfying $f(0)=1$, with f strictly increasing on $x<0$ and strictly decreasing on $x>0$. The *kneading sequence* $\mathbf{K}(f)=K_1K_2K_3\ldots$ of f is the sequence of symbols K_i defined by

$$K_i=\begin{cases} R & \text{if} \quad f^i(0)>0, \\ C & \text{if} \quad f^i(0)=0, \\ L & \text{if} \quad f^i(0)<0. \end{cases}$$

We also define, for $i=1,2,\ldots$

$$\varepsilon_i=\begin{cases} +1 & \text{if} \quad K_i=L, \\ -1 & \text{if} \quad K_i=R, \end{cases}$$

and, recursively

$$\varepsilon_k=\prod_{i=1}^{k-1}\varepsilon_i \quad \text{if} \quad K_k=C.$$

Finally, given $\mathbf{K}$, the polynomials $P_{\mathbf{K},N}$ are defined by

$$P_{\mathbf{K},N}(\tau)=1+\sum_{n=1}^{N}\left(\prod_{j=1}^{n}\varepsilon_j\right)\tau^n.$$

Given f, the analytic function

$$P_{\mathbf{K}(f)}(\tau)=\lim_{N\to\infty} P_{\mathbf{K}(f),N}(\tau)$$

is called the *kneading determinant* [or (formal) *characteristic polynomial* of f].

Theorem 1 [5]. *The topological entropy of f equals $-\log\tau$, where τ is the smallest positive root of $P_{\mathbf{K}(f)}$.*

We shall show below that the smallest positive root of $P_{\mathbf{K}(f),N}$ rapidly approaches τ, providing thus an efficient means of computing τ. We shall see that the speed of convergence will depend (in a controllable manner) on $\mathbf{K}(f)$. To state our result, we need the following definition. If $\mathbf{A}$, $\mathbf{B}$ are two kneading sequences, we shall say that $\mathbf{A}<\mathbf{B}$ if there is an s such that $A_i=B_i$ for $i=1,2,\ldots,s$ and if

either $A_{s+1}<B_{s+1}$ and an even number of A_i's, $i\leqq s$ are equal to R,

or $A_{s+1}>B_{s+1}$ and an odd number of A_i's, $i\leqq s$ are equal to R.

This defines an ordering of kneading sequences, and one has

Lemma 2 [5]. *If $\mathbf{K}(f)<\mathbf{K}(g)$, then $h(f)\leqq h(g)$.*

Our main result is the

Theorem 3. *If $\mathbf{K}$ is a kneading sequence and $\mathbf{K}>RLR^{\infty}$, then $P_{\mathbf{K},N}$ has, for every $n\geqq 18$ a smallest positive root $\tau_{\mathbf{K},n}$. This root is less than one, and $|\tau_{\mathbf{K}}-\tau_{\mathbf{K},n}|<18.6\cdot 2^{-n/2}$.*

(Better bounds are sketched in the proof.)

Before indicating the algorithmic application of the theorem, we extend it to $\mathbf{K}\leqq RLR^{\infty}$. One defines [1], for any $\mathbf{K}$

$$R*\mathbf{K}=R\check{K}_1R\check{K}_2R\check{K}_3\ldots,$$

where

$$\check{K}_i=\begin{cases} L & \text{if} \quad K_i=R\,,\\ C & \text{if} \quad K_i=C\,,\\ R & \text{if} \quad K_i=L\,.\end{cases}$$

(Under technical conditions, if $\mathbf{K}(f)=R*\mathbf{A}$, then $\mathbf{K}(f(1)^{-1}f\circ f(f(1)\cdot))=\mathbf{A}$ and f exchanges two subintervals of $[-1,1]$.). Denote $R^{*m}=R*R*\ldots*R$ (m times). We have now the

Corollary 4. *Let $\mathbf{K}$ be a kneading sequence and suppose $\mathbf{K}>R^{*\infty}$.*

(i) *One can decide from the first 2^{m+1} symbols of $\mathbf{K}$ whether $\mathbf{K}\geqq R^{*m}*RL^{\infty}$, $m\geqq 0$.*

(ii) *If $\mathbf{K}$ is in the interval $R^{*(m-1)}*RL^{\infty}\geqq\mathbf{K}>R^{*m}*RL^{\infty}$, then $|\tau_{\mathbf{K}}-\tau_{\mathbf{K},n2^{m-1}}|<18.6\cdot 2^{-n/2}$, provided $n\geqq 18$.*

We now outline the algorithm for evaluating $\tau_{\mathbf{K}}$ given $\mathbf{K}=\mathbf{K}(f)$. Set $\mathbf{K}'=\mathbf{K}$, $m=0$, and perform the following steps.

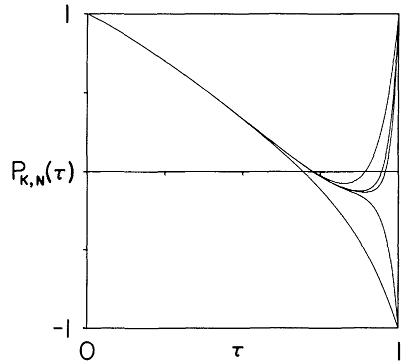

Fig. 2. Kneading determinant $P_{\mathbf{K},N}(\tau)$ vs. τ calculated with various numbers of terms at $\mu=1.46610$. From left to right, for those $P(\tau)$ with $P(1)=1$ in the figure, the number of terms are $N=20$, 40, and 50; for those with $P(1)=-1$, the number of terms are $N=10$ and 30, respectively

Step 1. Check if $K'_1K'_2K'_3K'_4 \geqq RLRR$. If not proceed to Step 3.

Step 2. We must have $\mathbf{K}' \geqq RLR^\infty$, and hence $\tau_{\mathbf{K}'} \leqq 2^{-1/2}$. Apply Theorem 3 to $\mathbf{K}'$, noting that

$$h(f) = -\frac{1}{2^m}\log_2 \tau_{\mathbf{K}'}$$

(cf. proof of Corollary 4).

Step 3. Increase m by 1. We now know

$$h(f) \leqq \frac{1}{2^m}\log 2$$

(cf. proof of Corollary 4). Replace $\mathbf{K}'$ by $\mathbf{K}''$ which is defined by the equation $R*\mathbf{K}'' = \mathbf{K}'$. Proceed to Step 1.

The proof of Theorem 3 will be based on the implicit function theorem. The main ingredient for the proof is Lemma 5 below. Denote by $\mathbf{J}_1, \ldots, \mathbf{J}_8$ the finite sequences given in Table 1. One can check [1] that every kneading sequence $\mathbf{K} \geqq RLR^\infty$ *either* starts out as $\mathbf{J}_i \ldots$ for some i *or* is equal to one of the "limiting sequences" of Table 1. In this latter case, the root $\tau_\mathbf{K}$ of $P_\mathbf{K}$ is equal to a root of a polynomial of finite degree [5], and an approximate value for this root σ_j is given in the last column of Table 1. Denote by n_j the number of letters in $\mathbf{J}_j$.

Lemma 5. *If* $\mathbf{K}$ *is a kneading sequence, and* $\mathbf{K} = \mathbf{J}_j\mathbf{B}$ *for some* $j \in \{1, 2, \ldots, 8\}$, *then we have for all* $n \geqq n_j$:

$$\frac{d}{d\tau} P_{\mathbf{K},n}|_{[-1/2,\sigma_j+0.001]} < -0.13\,.$$

The proof of Lemma 5 is a numerical verification of a finite number of cases, as we shall sketch now. Consider $\mathbf{K} = \mathbf{J}_j\mathbf{B}$. Then, for $n \geqq n_j$,

$$P_{\mathbf{K},n}(\tau) = P_{\mathbf{J}_j,n_j}(\tau) + \sum_{i=n_j+1}^{n} \zeta_i \tau^i,$$

Table 1

	Limiting sequences	Values of $\tau_{\mathbf{K}}$ for limiting sequence
	RLR^{∞}	$\sigma_0 = 1/\sqrt{2}$
$\mathbf{J}_1 = RLR^8$		
	$(RLR^8L)^{\infty}$ $(RLR^6C)'$	$\sigma_1 \sim 0.6938106281$
$\mathbf{J}_2 = RLR^6L$		
	$(RLR^4C)^{\infty}$	$\sigma_2 \sim 0.6823278038$
$\mathbf{J}_3 = RLR^4L$		
	$(RLR^2C)^{\infty}$	$\sigma_3 \sim 0.6609925319$
$\mathbf{J}_4 = RLR^2L$		
	$(RLC)^{\infty}$	$\sigma_4 = 2/(\sqrt{5}+1) \sim 0.6180339887$
$\mathbf{J}_5 = RLLRLLR$		
	$(RLLRLC)^{\infty}$	$\sigma_5 = \sigma_4$
$\mathbf{J}_6 = RLLRLR$		
	$(RLLRC)^{\infty}$	$\sigma_6 \sim 0.580691832$
$\mathbf{J}_7 = RLLRR$		
	$(RLLC)^{\infty}$	$\sigma_7 \sim 0.5436890127$
$\mathbf{J}_8 = RLLL$		
	RL^{∞}	$\sigma_8 = 1/2.$

with $\zeta_i = \pm 1$. We bound the derivative of $P_{\mathbf{K},n}(\tau)$ by

$$-\partial_\tau P_{\mathbf{K},n}(\tau) > -\partial_\tau P_{\mathbf{J}_j,n_j}(\tau) - \sum_{i=n_j+1}^{n} i|\tau|^{i-1}. \tag{2}$$

The first term on the right hand side of (2) is bounded explicitly, and the second analytically. Putting in the corresponding choices of $\mathbf{J}_j$, and σ_j yields the result. (It seems that any subdivision which is coarser than the one in Table 1 does not lead to a strictly negative bound for the derivative.)

We need one more general result about kneading sequences for the proof of Theorem 3.

Lemma 6. [5]. (i) *Let* $\mathbf{K}$ *be a kneading sequence. Then the smallest root* $\tau_{\mathbf{K}}$ *in modulus of* $P_{\mathbf{K}}$ *is real, positive and less than or equal to* 1.

(ii) *If* $\mathbf{K} = \mathbf{J}_j\mathbf{B}$ *with* $\mathbf{J}_j$ *from Table 1, then* $\tau_{\mathbf{K}} \in [\sigma_{j-1}, \sigma_j]$.

(This latter statement follows also from Lemma 2.)

Remark. At first sight one could think of applying the lemma directly to the $P_{\mathbf{K},n}$. The following example shows that this is not possible. Consider a kneading sequence $\mathbf{K} = RLR^2L\ldots$, and try to show that the root with smallest modulus of $P_{\mathbf{K},5}$, say $\tau_{\mathbf{K},5}$ is positive. One could be tempted to consider $\mathbf{K}' = (RLR^2LC)^{\infty}$, and then $P_{\mathbf{K}'}$ has a smallest root $\tau_{\mathbf{K}'} = \tau_{\mathbf{K},5}$, as is easily seen. Although $P_{\mathbf{K}'}$ has the same smallest positive root as $P_{\mathbf{K},5}$, we cannot apply Lemma 6 because $\mathbf{K}'$ is *not* a kneading sequence: No map f can have $\mathbf{K}(f) = \mathbf{K}'$.

Proof of Theorem 3. Suppose $\mathbf{K} = \mathbf{J}_j\mathbf{B}$.

We use the notation $G_n(\tau) = P_{\mathbf{K},n}(\tau)$. By definition,

$$G_n(\tau) = P_{\mathbf{K}}(\tau) + (P_{\mathbf{K},n}(\tau) - P_{\mathbf{K}}(\tau)) = P_{\mathbf{K}}(\tau) + \sum_{j=n+1}^{\infty} \zeta_j \tau^j, \tag{3}$$

with $\zeta_j = \pm 1$. Since $P_{\mathbf{K}}(\tau_{\mathbf{K}})=0$ and $\tau_{\mathbf{K}} < 2^{-1/2}$ by Lemma 6, we find

$$|G_n(\tau_{\mathbf{K}})| \leqq \sum_{k=n+1}^{\infty} \tau_{\mathbf{K}}^k \leqq 2^{-(n+1)/2}/(1-2^{-1/2}). \tag{4}$$

On the other hand, by Lemma 5

$$|G_n'(\tau)| > 0.13 \tag{5}$$

for all $\tau \in [-\frac{1}{2}, \sigma_j + 0.001]$. These two estimates imply that $G_n(\tau)$ has a unique zero in $[-\frac{1}{2}, \sigma_j + 0.001]$, provided we choose n so large that

$$|G_n(\tau_{\mathbf{K}})| < 0.0076 < 0.001/0.13,$$

($n \geqq 18$ is sufficient). This proves Theorem 3 (i). We also find, combining (4) and (5)

$$|\tau_{\mathbf{K}} - \tau_{\mathbf{K},n}| < 2^{-n/2} 2^{-1/2}/((1-2^{-1/2}) \cdot 0.13) < 18.6 \cdot 2^{-n/2}.$$

Proof of Corollary 4. The point (i) follows by inspection of the definition of $R*$. Now point (ii) is an obvious consequence of the fact that if $\mathbf{K} = R^{*m} * \mathbf{K}'$ with $\mathbf{K}' \geqq RLR^{\infty}$, then only every 2^m-th digit of $\mathbf{K}$ effectively contributes to $\mathbf{K}'$. But $h_{\text{top}}(\mathbf{K}) = \frac{1}{2^m} h_{\text{top}}(\mathbf{K}')$. The result follows, by reduction to the case $\mathbf{K}' \geqq RLR^{\infty}$ using $P_{\mathbf{K}}(\tau) = P_{\mathbf{K}'}(\tau) Q(\tau)$, where all zeros of Q lie on the unit circle. Q.E.D.

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References

1. Collet, P., Eckmann, J.-P.: Iterated maps on the interval as dynamical systems. Boston: Birkhäuser 1980
2. Crutchfield, J.P., Farmer, J.D., Huberman, B.A.: Fluctuations and simple chaotic dynamics. Physics Reports (to appear)
   Crutchfield, J.P., Huberman, B.A.: Fluctuations and the onset of chaos. Phys. Lett. **77** A, 407 (1980)
3. Dieudonné, J.: Foundations of modern analysis. New York, London: Academic Press 1969
4. Mayer-Kress, G., Haken, M.: The influence of noise on the logistic equation. J. Stat. Phys. **26**, 149 (1981)
5. Milnor, J., Thurston, W.: On iterated maps of the interval. I, II. Preprint Institute for Advanced Studies, Princeton University (1977)
6. Crutchfield, J.P., Shaw. R.: Unpublished

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