Commun. Math. Phys. 88, 235-256 (1983)

Communications in Mathematical Physics

Irreducible Kernels and Nonperturbative Expansions in a Theory with Pure $m \to m$ Interaction

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Abstract. Recent results on the structure of the S matrix at the m-particle threshold ($m \geqq 2$) in a simplified $m \to m$ scattering theory with no subchannel interaction are extended to the Green function F on the basis of off-shell unitarity, through an adequate mathematical extension of some results of Fredholm theory: local two-sheeted or infinite-sheeted structure of F around $s=(m\mu)^2$ depending on the parity of $(m-1)(v-1)$ (where $\mu>0$ is the mass and v is the dimension of space-time), off-shell definition of the irreducible kernel U which is the analogue of the K matrix in the two different parity cases $(m-1)(v-1)$ odd or even, and related local expansion of F, for $(m-1)(v-1)$ even, in powers of $\sigma^\beta \ln \sigma (\sigma=(m\mu)^2-s)$. It is shown that each term in this expansion is the dominant contribution to a Feynman-type integral in which each vertex is a kernel U. The links between the kernel U and Bethe-Salpeter type kernels G of the theory are exhibited in both parity cases, as also the links between the above expansion of F and local expansions, in the Bethe-Salpeter type framework, of F_λ in terms of Feynman-type integrals in which each vertex is a kernel G and which include both dominant and subdominant contributions.

1. Introduction

The two-sheeted, square-root type structure of the S matrix at the two-particle threshold of a $2 \to 2$ process in space-time dimension $v=4$ is an old result of the sixties [1–4] derived from two-body unitarity. The result has also been extended to the Green function F of the process (whose mass shell restriction is the scattering function T) either [5] in the Bethe-Salpeter framework in which the Bethe-Salpeter kernel G is assumed to be irreducible (i.e. analytic at thereshold) or [4] on the basis of off-shell unitarity (= asymptotic completeness).

Further results on the structure of the multiparticle S matrix and Green functions near other Landau singularities have been obtained in axiomatic S matrix [6–8] and field theory [4, 9, 10]. They include in particular local decompositions in terms of "Feynman-type" contributions, or of analogous quantities defined in terms of pure on-shell S matrix elements in S matrix theory, which are singular along well specified Landau surfaces and whose singularities have a well defined nature. Such results all refer, however, to cases where the only relevant Landau singularities are associated to graphs with sets of at most one or two lines between any pair of vertices, and rely, whenever sets of two lines are involved, on the known square-root nature of two-particle thresholds in dimension 4. There is so far no similar information, even from a heuristic viewpoint, on m-particle thresholds for $m \geqq 3$. In parallel with investigations on the subject in the actual theory (see in particular [9, 10]), it has then appeared, as discussed in [11], that some interesting features could already be exhibited in the easier analysis of the m-particle threshold in a simplified $m \to m$ scattering theory with locally a pure $m \to m$ interaction (i.e. no $m' \to m''$ interaction with $m' \leqq m$, $m'' < m$ or $m' < m$, $m'' \leqq m$). A theory with only one type of particle, a boson of mass $\mu > 0$, is considered for simplicity. The simplified theory coincides with the exact one at $m=2$, and its study may thus also yield, as a byproduct, some supplementary results on 2-particle thresholds in the exact theory: e.g. in the present work, the off-shell extension of the K matrix and the analysis of its links with the Bethe-Salpeter kernel G.

Two possible approaches [12, 13] of the simplified theory, based respectively on unitarity-type and Bethe-Salpeter type equations, have been proposed by analogy with the case $m=2$. As in the latter case [5], they are closely linked, although not completely equivalent. The first approach has been developed in [12] (see also [13]) in the on-shell framework. The purpose of the present work is to extend some of its results to the off-shell Green function F of the theory (Sects. 4 and 5) and to clarify the links between these results and the Bethe-Salpeter type approach [13] (Sects. 6 and 7). In order to exploit, in Sects. 4 and 5, the off-shell unitarity-type equation, a mathematical extension of results of Fredholm theory, which is different from that presented in [13, 14] in the Bethe-Salpeter type approach and is inspired by a method of [4], will be needed. It has its own mathematical interest and is thus presented first, in Sect. 2.

The first approach of the simplified theory is then presented in more detail, in the off-shell framework, in Sect. 3 where some preliminary notations and results are also included.

In Sect. 4, the local maximal analyticity of F around $s=(m\mu)^2$ with either two sheets if $(m-1)(\nu-1)$ is odd, or an infinite number of sheets if $(m-1)(\nu-1)$ is even, is then derived from off-shell unitarity. (This result, which extends the previous result of [12] on the scattering function T to the Green function F, is analogous to that obtained in [13] in the Bethe-Salpeter type approach.)

In Sect. 5, a class of kernels U, which are adequate off-shell analogues of the K matrix in the two different parity cases $(m-1)(\nu-1)$ odd or even, is introduced. These kernels are shown to be meromorphic and uniform around $s=(m\mu)^2$ as a consequence of off-shell unitarity, with possibly a singularity at $s=(m\mu)^2$ and a discrete set of poles that may accumulate near $s=k^2=(m\mu)^2$. For $\beta = \dfrac{(m-1)\nu - m - 1}{2} > 0$ and under a boundedness condition on F near the threshold, these singularities are excluded locally, in a complex neighborhood of $s=(m\mu)^2$, for a subclass of kernels which are thus irreducible. This is the case in particular for the kernel U which is well defined by the integral equation:

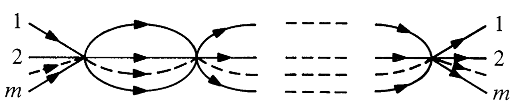
$$F = U + f(\sigma) F * U\,, \tag{1}$$

where $f(\sigma) = \frac{1}{2}$ if $(m-1)(v-1)$ is odd, (β half-integer), $f(\sigma) = \frac{i}{2\pi} \ln \sigma$ if $(m-1)(v-1)$ is even (β integer), $\sigma = (m\mu)^2 - s$ and $*$ denotes on-mass-shell convolution over m internal energy-momenta. The following convergent expansion of F in powers of $\sigma^\beta f(\sigma)$, which is the Neumann series of F in Eq. (1), is then obtained near $\sigma = 0$:

$$F = \sum_{n=0}^{\infty} U^{\hat{*}(n+1)} (\sigma^\beta f(\sigma))^n\,, \tag{2}$$

where $* = \sigma^\beta \hat{*}$ and all coefficients $U^{\hat{*}(n+1)}$ are, like U, locally analytic ($U^{\hat{*}(n+1)} = U \hat{*} U \ldots \hat{*} U$, $n+1$ factors).

These results are the off-shell extension of previous on-shell results of [12], and are presented here independently of the symmetry assumption used there for simplicity. The local expansion (2) of F is of interest mainly in the case $(m-1)$ $(v-1)$ even, and then exhibits F as an infinite convergent sum of well specified contributions that behave like $(\sigma^\beta \ln \sigma)^n, n = 0, 1, 2, \ldots$. These contributions can be associated with the graphs

$G_n^{(m)} =$ 

(3)

with n sets of m internal lines and $n+1$ vertices, which all give rise to the same Landau surface $s = (m\mu)^2$: each term is in fact an on-mass-shell convolution integral $U^{*(n+1)}$ of kernels U associated with each vertex, multiplied by $(f(\sigma))^n = \left(\frac{i}{2\pi} \ln \sigma\right)^n$. In the off-shell framework, it will also be shown that these terms are the "dominant" contributions to Feynman-type integrals $U^{\circ(n+1)}$ ($\equiv U \circ U \ldots \circ U$, $n+1$ factors, $\circ$ denoting Feynman-type integration) associated with the graphs $G_n^{(m)}$: $U^{\circ(n+1)}$ is in fact equal, in the case $(m-1)(v-1)$ even, to $U^{*(n+1)}$ $\left(\frac{i}{2\pi} \sigma^\beta \ln \sigma\right)^n$ plus terms of lower order in $\sigma^\beta \ln \sigma$.

In Sect. 6, a class of Bethe-Salpeter type kernels, depending on the choice of an analytic cut off factor φ in the definition of the Feynman-type operation $\circ$, is introduced in the first approach of the simplified theory, through the equation:

$$F = G + F \circ G\,. \tag{4}$$

The mathematical analysis of [14] (together with the algebraic argument of the Appendix: see below) allows one to show that these kernels are in general well defined in terms of F and, like the kernels U, are uniform around $s = (m\mu)^2$, with possibly a singularity at $s = (m\mu)^2$ and a discrete set of poles in k that may accumulate near $k^2 = (m\mu)^2$. We shall here assume, as in the Bethe-Salpeter type approach, that these singularities are excluded locally, for an adequate choice of φ, the corresponding kernel G being thus irreducible. If F_λ is defined in terms of G (see [13]) through the equation:

$$F_\lambda = G + \lambda F_\lambda \circ G \tag{4'}$$

(with $F_\lambda \equiv F$ at $\lambda = 1$), the following local expansion, which is the Neumann series of F_λ and is convergent at small λ, occurs naturally in this framework:

$$F_\lambda = \sum_{n=0}^{\infty} \lambda^n G^{\circ(n+1)}. \tag{5}$$

Each term $G^{\circ(n+1)}$ is here an actual Feynman-type integral associated with the graph $G_n^{(m)}$ of Eq. (3), in which each vertex represents a kernel G. In contrast to the expansion (2), the series (5) is not necessarily convergent at the physical value $\lambda = 1$, and the terms $G^{\circ(n+1)}$ do not have the well specified behaviour in $(\sigma^\beta \ln \sigma)^n$ [for $(m-1)(v-1)$ even] of the corresponding terms in (2): as explained in Sect. 7, they are rather, like $U^{\circ(n+1)}$, combinations of various contributions in $(\sigma^\beta \ln \sigma)^p$, $o \leqq p \leqq n$, with locally analytic coefficients.

In Sect. 7, we then exhibit the links between the kernels G and U (in both parity cases), and explain how the expansion (2) of F can be recovered from the expansion (5) by regrouping together all terms with common powers of $\sigma^\beta \ln \sigma$ (and then by analytic continuation in λ up to $\lambda = 1$).

Once their analyticity or meromorphy around $k^2 = (m\mu)^2$ has been established by the methods of Sect. 2 and of [14] respectively, the uniformity of the kernels U in Sect. 5 and G in Sect. 6 is based on an algebraic argument inspired by a method of [5] at $m = 2$ and used previously in [12, 13] in the discussion of the on-shell kernel U and of the kernel G respectively. This argument is presented in the Appendix, where a general class of uniform or irreducible kernels, including the kernels G and U, is also introduced in a qualitative manner on this basis.

Lorentz invariance is unessential in the present work and is therefore not assumed.

2. Mathematical Extension of Some Results of Fredhom Theory

We consider below kernels $A(k; z, z')$, where z, z' are real or complex n-dimensional variables and k is a real or complex v-dimensional parameter, and a composition operation $*$ defined by the formula:

$$(A*B)(k, z, z') = \int_S A(k; z, z(k, t)) B(k; z(k, t), z') \alpha(k, t) dt, \tag{6}$$

where S is a given r-dimensional real compact set, $r < n$, e.g. the unit spere S_r and $t \to z(k, t)$ is, for each k, a mapping from S to a submanifold $S(k)$ of real dimension r in z-space; $z(k, t)$ and $\alpha(k, t)$ will have continuity or analyticity properties in t and k specified later. We are then interested in the solutions R_λ of the equation:

$$R_\lambda - A = \lambda R_\lambda * A, \lambda \in \mathbb{C}. \tag{7}$$

Equation (7) is formally identical to a Fredholm resolvent equation, but differs from it in two respects which are here mixed together: the composition operation

$*$ denotes integration over a submanifold in z-space and the kernels considered, as also the submanifold $S(k)$ and the integration measure, depend on the parameter k.

Theorem 1. *Let $A(k; z z')$ be analytic in k, z, z' in a domain $\mathscr{W} \times \mathscr{D} \times \mathscr{D}$ where $\mathscr{D}$ is a domain of $\mathbb{C}^n$ and $\mathscr{W}$ is a domain of $\mathbb{C}^\nu$ or of a Riemann surface over $\mathbb{C}^\nu$, such that:*

$$S(k) \subset \mathscr{D} \quad \forall k \in \mathscr{W}, \tag{8}$$

and let $z(k,t)$, $\alpha(k,t)$ be continuous in t and analytic in k in $\mathscr{W}, \forall t \in S$.

Then, except for a discrete set Λ of values of λ, Eq. (7) admits a unique solution $R_\lambda(k, z, z')$, analytic in $\mathscr{W} \times \mathscr{D} \times \mathscr{D}$ apart possibly from a discrete set Σ_λ of polar submanifolds (of complex dimension $\nu - 1$) in k. More precisely,

$$R_\lambda(k; z, z') = \frac{N(\lambda, k, z, z')}{D(\lambda, k)}, \tag{9}$$

where N is analytic in λ, k, z, z' in $C \times \mathscr{W} \times \mathscr{D} \times \mathscr{D}$ and D is analytic in λ, k in $C \times \mathscr{W}$, Λ is the set of values of λ such that $D(\lambda, k)$ is identically zero and Σ_λ is the set of zeroes of $D(\lambda, k)$ for λ outside Λ.

R_λ satisfies the equalities:

$$R_\lambda * A = A * R_\lambda, \tag{10}$$

$$R_\lambda - R_\mu = (\lambda - \mu) R_\lambda * R_\mu, \tag{11}$$

$$R_\lambda * R_\mu = R_\mu * R_\lambda. \tag{12}$$

Proof. Let $a\,(k; t, t') = A(k; z(k,t), z(k,t'))$; a is a continuous function of t, t' in $S \times S$ and is analytic in k in $\mathscr{W}$ in view of the assumptions made on A. Fredholm theory, applied to the Fredholm kernels $a(k, t, t')$, considered as kernels of operators in t-space depending analytically on the complex parameter k (see e.g. [13]), ensures the existence for almost all λ (see below) of a unique solution of the resolvent equation

$$r_\lambda - a = \lambda r_\lambda * a, \tag{13}$$

where $*$ is defined by:

$$(a * b)(k, t, t') = \int_S a(k; t, t'') b(k; t'', t') \alpha(k, t'') dt''. \tag{14}$$

The solution r_λ is of the form:

$$r_\lambda(k, t, t') = \frac{n(\lambda, k, t, t')}{d(\lambda, k)}, \tag{15}$$

where n and d are the standard Fredholm series, and depend here analytically on k in $\mathscr{W}$. The solution does not exist at the discrete set Λ of values of λ such that $d(\lambda, k)$ vanishes identically in k. For other values of λ, it is well defined as a meromorphic function in k in $\mathscr{W}$, with poles in k at the zeroes of $d(\lambda, k)$. The relation:

$$r_\lambda * a = a * r_\lambda \tag{16}$$

follows from Fredholm theory.

Any solution R_λ of Eq. (7), if it exists, satisfies, by restriction to $z = z(k, t)$, $z' = z(k, t')$, the relation:

$$R_\lambda(k, z(k,t), z(k,t')) \equiv r_\lambda(k, t, t'). \tag{17}$$

We show below that it must satisfy on the other hand the commutation rule:

$$R_\lambda * A = A * R_\lambda\,, \tag{18}$$

and then construct explicitly a unique solution of Eqs. (7) and (18).

Proof of (18). We first prove (18) when $z = z(k,t), t \in S$. Any solution R_λ of Eq. (7), must satisfy, in view of (17):

$$R_\lambda(k; z(k,t), z') = A(k; z(k,t), z') + \lambda \int r_\lambda(k,t,t') A(k, z(k,t'), z') \alpha(k,t) dt'\,. \tag{19}$$

Thus:

$$\begin{aligned}(A * R_\lambda)(k, z(k,t), z') &= \int a(k,t,t') A(k, z(k,t'), z') \alpha(k,t') dt' \\ &\quad + \lambda \int a(k,t,t'') r_\lambda(k,t'',t') A(k, z(k,t'), z') \alpha(k,t'') \alpha(k,t') dt'' dt' \\ &= \int r_\lambda(k,t,t') A(k, z(k,t'), z') \alpha(k,t') dt' \\ &\equiv (R_\lambda * A)(k, z(k,t), z')\,,\end{aligned} \tag{20}$$

where the second and third equalities follow from (13), (16), and (17) respectively.

We now remove the constraint $z = z(k,t)$. If R_λ satisfies Eq. (7), one can write:

$$R_\lambda * A = R_\lambda * R_\lambda - \lambda R_\lambda * (R_\lambda * A) = R_\lambda * R_\lambda - \lambda R_\lambda * (A * R_\lambda)\,, \tag{21}$$

where the equality $R_\lambda * (R_\lambda * A) = R_\lambda * (A * R_\lambda)$ follows from the previous result (20). In view of the associativity of $*$, one thus has:

$$R_\lambda * A = R_\lambda * R_\lambda - \lambda (R_\lambda * A) * R_\lambda = (R_\lambda - \lambda R_\lambda * A) * R_\lambda \equiv A * R_\lambda\,. \tag{22}$$

Proof of Theorem 1 (continued). Let $H_\lambda(k,t,z')$ denote the right hand side of Eq. (19), which is well defined in terms of A and r_λ. If R_λ is to be solution of Eq. (7), it must also satisfy in view of (18) the equation:

$$R_\lambda - A = \lambda A * R_\lambda\,. \tag{23}$$

Hence, in view of (19), it must be equal to:

$$R_\lambda(k,z,z') = A(k,z,z') + \lambda \int A(k,z,z(k,t)) H_\lambda(k,t,z') \alpha(k,t) dt\,. \tag{24}$$

By an argument similar to that used in the proof of Eq. (18), one checks that this unique solution of Eq. (23) satisfies the commutation rule $A * R_\lambda = R_\lambda * A$, and hence Eq. (7). It satisfies on the other hand the relation [derived from (24) and the definition of H_λ]:

$$\begin{aligned}R_\lambda(k,z,z') &= A(k,z,z') + \lambda A * A(k,z,z') \\ &\quad + \lambda^2 \int A(k,z,z(k,t)) r_\lambda(k,t,t') A(k, z(k,t'), z') \alpha(k,t) \alpha(k,t') dt dt'\,,\end{aligned} \tag{25}$$

from which the expression (9) of R_λ follows with:

$$D(\lambda,k) \equiv d(\lambda,k)\,, \tag{26}$$

$$\begin{aligned}N_\lambda(k,z,z') &= d(\lambda,k)\,[A(k,z,z') + \lambda A * A(k,z,z')] + \lambda^2 \int A(k,z,z(k,t)) n_\lambda(k,t,t') \\ &\quad \cdot A(k, z(k,t'), z') \alpha(k,t) \alpha(k,t') dt\, dt'\,.\end{aligned} \tag{27}$$

The analyticity properties of N, D, and R_λ follow from (26) and (27).

Finally, Eq. (11) is proved by an algebraic argument analogous to that used in standard Fredholm theory. Namely, in view of Eqs. (7) and (10) applied to λ and μ:

$$\begin{aligned}R_\lambda - R_\mu &= \lambda R_\lambda * A - \mu A * R_\mu \\ &= \lambda R_\lambda * [A + \mu A * R_\mu] - \mu[\lambda R_\lambda * A + A] * R_\mu \\ &= (\lambda - \mu) R_\lambda * R_\mu .\end{aligned} \tag{28}$$

Similarly $R_\mu - R_\lambda = (\mu - \lambda) R_\mu * R_\lambda$. Equation (12) is then proved by comparison with Eq. (11). Q.E.D.

The following complement to Theorem 1 will also be useful.

Theorem 2. *Let $\mathscr{W}$ admit a real domain I in its closure, let $\mathscr{D}$ contain a real domain $\mathscr{D}_r$, and let $A(k,z,z')$, $z(k,t)$, $\alpha(k,t)$ admit continuous limits $A_0(k,z,z')$, $z_0(k,t)$, $\alpha_0(k,t)$ in k,z,z' and in k,t respectively when $k \in I$, $z,z' \in \mathscr{D}_r$, $t \in S$, with $S(k) \subset \mathscr{D}_r$, $\forall k \in I$.*

Then the functions N and D of Eq. (9) admit continuous limits N_0, D_0 in k,z,z', which are entire functions of λ. For any λ outside Λ, the equation:

$$R_\lambda - A_0 = \lambda R_\lambda * A_0 \tag{29}$$

admits the unique solution $\dfrac{N_0}{D_0}$ in this limit, outside the set Σ'_λ of points $k \in I$ such that $D_0(\lambda,k)=0$. This set is not dense in I.

If for a given value λ_0 of λ, Eq. (29) is known to admit a (continuous) solution for k,z,z' real, $k \in I'$, $z,z' \in \mathscr{D}'_r$ where I' and $\mathscr{D}'_r$ are subdomains of I and $\mathscr{D}_r$ respectively, λ_0 does not belong to Λ.

Proof. As in Theorem 1, the result first holds for the resolvent r_λ in view of Fredholm theory, the functions n and d having continuous limits in k,z,z' which are entire in λ. The extension to R_λ is then made by the same methods as in Theorem 1, and is thus omitted.

The set Σ'_λ of zeroes of $D_0(\lambda,k)$ for a given $\lambda \notin \Lambda$ cannot be dense in I since otherwise the continuous function $D_0(\lambda,k)$ would vanish in I: the function $D(\lambda,k)$ would thus vanish identically in $\mathscr{W}$ by the edge-of-the-wedge theorem, which is contrary to the assumption $\lambda \notin \Lambda$.

Finally, if a solution R_λ of Eq. (29) is known for $k \in I'$, $z,z' \in \mathscr{D}'_r$, the function $D_0(\lambda,k)$ is not identically zero for k in I' and thus $D(\lambda_0,k)$ is not identically zero in $\mathscr{W}$.

3. The Simplified Theory: Preliminary Assumptions and Notations

The initial and final, real or complex, energy-momenta variables of the $m \to m$ process considered are denoted $p_1, \dots, p_m$ and $p'_1, \dots, p'_m$ respectively; they are always restricted to the subspace $C^{(2m-1)\nu}$ of points $(p_1, \dots, p_m, p'_1, \dots, p'_m)$ satisfying $\sum_{i=1}^{m} p_i = \sum_{i=1}^{m} p'_i$ (energy-momentum conservation). The *complex m-particle mass-shell* is the set of points $(k_1, \dots, k_m)$, $k_i \in C^\nu$, $i=1,\dots,m$ satisfying $k_i^2 = \mu^2$, $i=1,\dots,m$ ($k_i^2 = k_{i,0}^2 - \mathbf{k}_i^2$ where $k_{i,0}$ and $\mathbf{k}_i$ are the energy and $(\nu-1)$-dimensional momentum components of k_i). The *real m-particle mass-shell* is defined by the further constraints k_i real, $k_{i,0}>0$, $i=1,\dots,m$. The *complex, respectively real, mass-shell of the $m\to m$ process* is the set of points $(p_1,\dots,p_m,p'_1,\dots,p'_m)$ such that $(p_1,\dots,p_m)$ and $(p'_1,\dots,p'_m)$ both belong to the complex, respectively real, m-particle mass-shell (and satisfy energy-momentum conservation).

The m-particle threshold of the process is the subset of the real mass-shell defined by the condition $p_1=\dots=p_m=p'_1=\dots=p'_m$, which belongs to the surface $s=(m\mu)^2$, where $s=\left(\sum_{i=1}^{m} p_i\right)^2$.

It will be convenient to use the variables k, z, z', $k\in C^\nu$, $z\in\mathbb{C}^{(m-1)\nu}$, $z'=\mathbb{C}^{(m-1)\nu}$, $z=(z_1,\dots,z_m;\ z_i\in\mathbb{C}^\nu,\ i=1,\dots,m,\ \sum_{i=1}^{m} z_i=0)$ defined through the relations:

$$k=\sum_{i=1}^{m} p_i\left(=\sum_{i=1}^{m} p'_i\right), \tag{30}$$

$$z_i=p_i-k/m \text{ (respectively } z'_i=p'_i-k/m), \qquad i=1,\dots,m. \tag{31}$$

In these variables, the m-particle threshold is the set of points $(K,0,0)$ such that $K\subset H^+_{m\mu}$, where $H^+_{m\mu}=\{K, K\in\mathbb{R}^\nu,\ K^2=K_0^2-\vec{K}^2=(m\mu)^2,\ K_0>0\}$.

The following assumption on the $2m$-point function F in the first approach of the simplified theory (= analyticity in a domain containing a cut neighborhood of the threshold and off-shell unitarity) will be made:

Assumption 1. Being given any real point $K\in H^+_{m\mu}(K^2=(m\mu)^2,\ K_0>0)$ there exist a complex neighborhood $\mathscr{W}_K$ of K in k-space and a domain $\mathscr{D}_K$ containing a complex neighborhood of the origin in z-space such that:

(i) *F is analytic in a domain containing the union $U_K\Delta_K^{(\mathrm{cut})}$, where*

$$\Delta_K^{(\mathrm{cut})}=\{k,z,z';\ k\in\mathscr{W}_K^{(\mathrm{cut})},\ z\in\mathscr{D}_K,\ z'\in\mathscr{D}_K\}, \tag{32}$$

$$\mathscr{W}_K^{(\mathrm{cut})}=\mathscr{W}_K\backslash\{k^2\geqq(m\mu)^2(k^2 \text{ real})\}, \tag{33}$$

and admits continuous boundary values F_0, F_1 at $p_1,\dots,p_m,p'_1,\dots,p'_m$ real ($=k,z,z'$ real) on the boundary $s=k^2>(m\mu)^2$ of $\Delta_K^{(\mathrm{cut})}$ from the respective sides $\operatorname{Im}s>0$ *and* $\operatorname{Im}s<0$.

(ii) *F_0 and F_1 satisfy the off shell unitarity-type equation:*

$$\begin{aligned}&F_0(p_1,\dots,p_m,p'_1,\dots,p'_m)-F_1(p_1,\dots,p_m,p'_1,\dots,p'_m)\\&=\int F_0(p_1,\dots,p_m,k_1,\dots,k_m)F_1(k_1,\dots,k_m,p'_1,\dots,p'_m)\frac{1}{m!}\delta^\nu\left(\sum_{i=1}^{m}k_i-\sum_{i=1}^{m}p_i\right)\\&\quad\cdot\prod_{i=1}^{m}\delta(k_i^2-\mu^2)\theta(k_{i,0})d^\nu k_i\end{aligned} \tag{34}$$

in the region $p_1,\dots,p_m,p'_1,\dots,p'_m$ real, $s>(m\mu)^2$, $k,z,z'\in\mathscr{W}_K\times\mathscr{D}_K\times\mathscr{D}_K$.

In a large part of the results, Assumption 1 will be completed by the following condition (local boundedness of F near the threshold):

Assumption 1'. For each real point $K\in H^+_{m\mu}$, $|F(k,z,z')|$ is uniformly bounded by a constant C_K in a complex cut neighborhood $\Delta_K^{\prime(\mathrm{cut})}$ of the threshold point $(K,0,0)$.

Here $\Delta_K^{\prime(\mathrm{cut})}$ is defined in the same way as $\Delta_K^{(\mathrm{cut})}$, $\mathscr{W}_K$ and $\mathscr{D}_K$ being replaced by complex neighborhoods $\mathscr{W}'_K \subset \mathscr{W}_K$ and $\mathscr{D}'_K \subset \mathscr{D}_K$ of K and of $z=0$ respectively.

In order to exploit the off-shell unitarity equation (34), we shall use the local parametrization given in [12] of the real m-particle mass-shell by the variable $k=\sum_{i=1}^{m} k_i$ varying in the set k real, $k^2 \geqq (m\mu)^2$, $k_0>0$ and angular variables Ω varying in the real $[(m-1)\nu-m]$-dimensional unit sphere S. Using this parametrization, Eq. (34) becomes for k, z, z' in a real neighborhood of the threshold, with $k^2>(m\mu)^2$:

$$F_0(k,z,z')-F_1(k,z,z')=(\sigma^\beta)_0 \int_S F_0(k,z,z_0(k,\Omega))F_1(k,z_0(k,\Omega),z')\hat{\alpha}_0(k,\Omega)d\Omega\,, \tag{35}$$

where $\sigma=(m\mu)^2-k^2$, $\beta=\dfrac{(m-1)\nu-m-1}{2}$, $(\sigma^\beta)_0$ is the restriction to $k^2>(m\mu)^2$ of the function σ^β defined e.g. by the condition $\sigma^\beta>0$ for $\sigma>0$, $k\in\mathscr{W}_K^{(\mathrm{cut})}$, if $(m-1)(\nu-1)$ is odd (β half-integer), and $\hat{\alpha}_0(k,\Omega)$, $z_0(k,\Omega)$ are the restrictions to k real, $k^2>(m\mu)^2$ of a function $\hat{\alpha}(k,\Omega)$ and a mapping $z(k,\Omega)$ satisfying for each real $K\in H^+_{m\mu}(K^2=(m\mu)^2,\ K_0>0)$, the following properties:

(i) $\hat{\alpha}(k,\Omega)$ is analytic in k in a complex neighborhood $\mathscr{W}''_K$ of K, is continuous in Ω in S and satisfies the relation:

$$\hat{\alpha}(k,\Omega)=\hat{\alpha}(k,-\Omega)\,. \tag{36}$$

(ii) The mapping $z(k,\Omega)=\{z_i(k,\Omega),\ i=1,\dots,m,\ \sum_{i=1}^{m} z_i(k,\Omega)=0\}$ is continuous in Ω and is analytic in k in a two-sheeted Riemann surface around $k^2=(m\mu)^2$, which is a covering of $\mathscr{W}''_K\backslash\{k^2=(m\mu)^2\}$: it is in fact (see [12]) the restriction to $x=(k^2-(m\mu)^2)^{1/2}\Omega$ of a mapping $\varphi(k,x)$ analytic in k in a complex neighborhood of K and in x in a complex neighborhood of the origin and such that $\varphi(k,0)=0$. It thus satisfies the relations:

$$z_1(k,\Omega)=z_0(k,-\Omega)\,, \tag{37}$$

where z_0 and z_1 are the two determinations of $z(k,\Omega)$ obtained at k real, $k^2>(m\mu)^2$, with z_0 corresponding e.g. to $(k^2-(m\mu)^2)^{1/2}>0$, and:

$$\lim_{k^2\to(m\mu)^2} z(k,\Omega)=0\,. \tag{38}$$

The domain $\mathscr{W}''_K$ will be chosen below, such that:

$$\mathscr{W}''_K\subset\mathscr{W}_K\,, \tag{39}$$

and as always possible in view of (38):

$$\operatorname*{Max}_{k\in\mathscr{W}''_K}\int_S|\hat{\alpha}(k,\Omega)|d\Omega=C'_K<\infty\,, \tag{39'}$$

$$S(k)\subset\mathscr{D}_K,\ \forall k\in\mathscr{W}''_K\,, \tag{39''}$$

where $S(k)$ is the subset of the complex m-particle mass-shell such that $\sum_{i=1}^{m} k_i=k$ and $z=\{k_i-k/m, i=1,\dots,m\}=z(k,\Omega)$ for some Ω in S [$z(k,\Omega)$ is here any one of the two determinations of the mapping z in its two sheets: $S(k)$ is unchanged in view of (37)].

Equations (34) or (35) will be written for brevity in the form:

$$F_0 - F_1 = F_0 *_0 F_1, \tag{40}$$

the operation $*_r$ between kernels $A(k,z,z')$, $B(k,z,z')$ defined at k real, $k^2 > (m\mu)^2$ being generally defined for any $r \in \mathbb{Z}$ by:

$$(A *_r B)(k,z,z') = (\sigma^\beta)_r \int_S A(k,z,z_r(k,\Omega))B(k,z_r(k,\Omega),z')\hat{\alpha}_r(k,\Omega)d\Omega\,, \tag{41}$$

where $z_r(k,\Omega) = z_0(k,(-1)^r\Omega)$, $(\sigma^\beta)_r = (-1)^{2\beta r}(\sigma^\beta)_0$ and $\hat{\alpha}_r(k,\Omega) = \hat{\alpha}_0(k,\Omega)$ are the determinations of $z(k,\Omega)$, $\hat{\alpha}(k,\Omega)$ and σ^β obtained at k real, $k^2 > (m\mu)^2$ after r (anticlockwise) turns around $k^2 = (m\mu)^2$.

For kernels A, B depending analytically (or meromorphically) on k around $k^2 = (m\mu)^2$, possibly in a multisheeted domain, the operation $*$ is defined by:

$$(A * B)(k,z,z') = \sigma^\beta \int_S A(k,z,z(k,\Omega))B(k,z(k,\Omega),z')\hat{\alpha}(k,\Omega)d\Omega\,. \tag{42}$$

Finally, if we consider kernels A, B defined in a given sheet, i.e. in a given cut neighborhood of the threshold [with the cut along $k^2 > (m\mu)^2$], the operation $\underline{*}_r$ is defined by:

$$(A \underline{*}_r B)(k,z,z') = (\underline{\sigma}^\beta)_r \int A(k,z,\underline{z}_r(k,\Omega))B(k,\underline{z}_r(k,\Omega),z')\underline{\hat{\alpha}}_r(k,\Omega)d\Omega\,, \tag{43}$$

where $\underline{z}_r(k,\Omega) = \underline{z}_0(k,(-1)^r\Omega)$, $(\underline{\sigma}^\beta)_r = (-1)^{2\beta r}(\underline{\sigma}^\beta)_0$ and $\underline{\hat{\alpha}}_r(k,\Omega) = \underline{\hat{\alpha}}_0(k,\Omega)$ are the determinations of $z(k,\Omega)$, σ^β, $\hat{\alpha}(k,\Omega)$ in the sheet r obtained after r turns around $k^2 = (m\mu)^2$, $z_0(k,\Omega)$ and $(\sigma^\beta)_0$ being the boundary values of $\underline{z}_0(k,\Omega)$ and $(\underline{\sigma}^\beta)_0$ from the directions $\mathrm{Im}\, s > 0$.

We note that the determinations $z_r(k,\Omega)$ and $\underline{z}_r(k,\Omega)$ in the right hand side of Eqs. (41) and (43) can be replaced there by $z_0(k,\Omega)$ and $\underline{z}_0(k,\Omega)$ in view of (36) and (37).

4. Off-Shell Unitarity and Local Maximal Analyticity of F

Theorem 3. *If F satisfies Assumption 1 of Sect. 3, it admits, for each real K, $K^2 = (m\mu)^2$, $K_0 > 0$, a multisheeted analytic continuation in $\hat{\mathscr{W}}''_K \times \mathscr{D}_K \times \mathscr{D}_K$, where $\hat{\mathscr{W}}''_K$ is the (universal) covering of $\mathscr{W}''_K \backslash \{k^2 = (m\mu)^2\}$, apart possibly from a discrete set of polar manifolds in k. The Riemann surface of F has two sheets for $(m-1)(v-1)$ odd, respectively an infinite number of sheets for $(m-1)(-1)$ even, if $F_0 *_0 F_0$ is not identically zero.*

If F satisfies Assumptions 1, 1′ and if $\beta > 0$, F is analytic in each sheet (i.e. has no poles) in a sufficiently small complex neighborhood of K.

The set $\mathscr{W}''_K$ in this statement is the complex neighborhood of K introduced in Sect. 3 [and satisfying conditions (39) and (39″)].

Proof of Theorem 3. Let $\underline{F}_0(k,z,z')$ denote the determination of F initially given in $\Delta_K^{(\mathrm{cut})}$ and let us restrict our attention to points k in $\mathscr{W}''_K$. According to Theorems 1 and 2 of Sect. 2 (applied with A, t and α replaced by $\underline{F}_0$, Ω and $-\sigma^\beta\hat{\alpha}$ respectively), a unique function $\underline{F}_1(k,z,z')$, which is analytic in $\mathscr{W}_K''^{(\mathrm{cut})}\times\mathscr{D}_K\times\mathscr{D}_K$, $\mathscr{W}_K''^{(\mathrm{cut})}=\mathscr{W}_K''\backslash\{k^2\geqq(m\mu)^2\}$, apart possibly from a discrete set of poles in k, is defined by the equation:

$$\underline{F}_0-\underline{F}_1=\underline{F}_0\underline{*}_0\underline{F}_1\,. \tag{44}$$

In fact, the value $\lambda=1$ does not belong to the set Λ since a (continuous) solution of Eq. (44), namely F_1, is known to exist in the limit $\mathrm{Im}\,s\to 0$, from the directions $\mathrm{Im}\,s>0$, at k,z,z' real, $k^2>(m\mu)^2$ [Eq. (40)]. Moreover F_1 is known from this analysis to be the boundary value of $\underline{F}_1$ from the directions $\mathrm{Im}\,s>0$. Since it is also, by Assumption 1, the boundary value of $\underline{F}_0$ from the directions $\mathrm{Im}\,s<0$, the edge-of-the-wedge theorem guarantees that F_1 is analytic and that $\underline{F}_1$ is an analytic continuation of $\underline{F}_0$ in a new sheet $r=1$.

Also $\underline{F}_1$ admits a boundary value $F_{1,-}$ at k,z,z' real, $k^2>(m\mu)^2$ from the directions $\mathrm{Im}\,s<0$ of $\mathscr{W}_K''^{(\mathrm{cut})}$, which, in view of the analyticity of F_1 previously established (and of a new application of Theorem 1) is analytic apart possibly from a discrete set of poles in k, and satisfies (outside this set) the equation:

$$F_1-F_{1,-}=F_1 *_1 F_{1,-}\,. \tag{45}$$

If $(m-1)(\nu-1)$ is odd, $*_1=-*_0$ in view of the relation $(\sigma^\beta)_1=-(\sigma^\beta)_0$. A comparison of Eqs. (45) and (40) and the unicity of the solution in Theorems 1 and 2 entail that $F_{1,-}\equiv F_0$, i.e. the two-sheeted structure of F is proved.

If $(m-1)(\nu-1)$ is even, $*_1\equiv *_0$ and similarly $*_r\equiv *_0$, $\underline{*}_r\equiv\underline{*}_0$, $\forall r\in\mathbb{Z}$. Equations (40) and (45) entail [by the same algebraic argument as that used in the proof of Eq. (11)] that:

$$F_0-F_{1,-}=2F_0 *_0 F_{1,-}\,. \tag{46}$$

By the same arguments as above, this allows one in turn to define an analytic continuation $\underline{F}_2$ of $\underline{F}_0$ in a sheet $r=2$, with possibly a discrete set of poles in k, as the (unique) solution of the equation:

$$\underline{F}_0-\underline{F}_2=2\underline{F}_0\underline{*}_0\underline{F}_2\,. \tag{47}$$

More generally, one defines by induction a multisheeted analytic continuation of F, with possibly a discrete set of poles in k, in the covering of $\mathscr{W}_K''\backslash\{k^2=(m\mu)^2\}$, this continuation being analytic in z, z' in $\mathscr{D}_K\times\mathscr{D}_K$. The determination $\underline{F}_r$ of F in each sheet r satisfies the relation:

$$\underline{F}_0-\underline{F}_r=r\underline{F}_0\underline{*}_0\underline{F}_r,\ \forall r\in\mathbb{Z}\,. \tag{48}$$

If $\underline{F}_r\equiv\underline{F}_0$ for some $r\in\mathbb{Z}$, $r\neq 0$, then in view of (48), $\underline{F}_0\underline{*}_0\underline{F}_0\equiv 0$ and similarly $F_0 *_0 F_0\equiv 0$. Apart from this case, an infinite number of sheets is therefore obtained around $k^2=(m\mu)^2$. This ends the proof of the first part of the theorem.

For the proof of the second part, we consider the Neumann series $\sum_{n=0}^{\infty}(-r)^n\underline{F}_0^{\underline{*}_0(n+1)}$ of $\underline{F}_r$ in Eq. (44) if $r=1$, or in Eq. (48) ($\forall r\in\mathbb{Z}$) for $(m-1)(\nu-1)$ even. This series is absolutely convergent in the region $(k,z,z')\in\Delta_K'^{(\mathrm{cut})}$, $|\sigma|^\beta<d_r=(rC_KC_K')^{-1}$, where C_K is the bound on $|\underline{F}_0|$ given in Assumption 1′, C_K' is

defined in Eq. (39′) and $\mathscr{W}'_K$, $\mathscr{D}'_K$ are chosen such that $\mathscr{W}'_K \subset \mathscr{W}''_K$, $S(k) \subset \mathscr{D}'_K$, $\forall k \in \mathscr{W}'_K$, as always possible. The convergence of the series then follows from the bounds:

$$|\underline{F}_0^{*\circ(n+1)}| < C_K [C_K C'_K |\sigma|^\beta]^n, \tag{49}$$

which are easily derived from the definition of $\underline{*}_0$ in the region $\Delta_K'^{(\mathrm{cut})}$.

Thus $\underline{F}_r$, which is equal to the sum of this series (unicity of the solution) is bounded in modulus in the region $(k, z, z') \in \Delta_K'^{(\mathrm{cut})}$, $|\sigma|^\beta < d_r$, and therefore cannot have poles in k.

Remark. It has been shown in [12, 13] that, for $(m-1)(\nu-1)$ even, the on-shell restriction

$$T(k, \Omega, \Omega') = F(k, z(k, \Omega), z(k, \Omega')) \tag{50}$$

of F is nonholonomic at $s = (m\mu)^2$ unless $T_0^{*(q)} \equiv 0$ for some $q > 0$. More precisely, the vector space generated by the successive determinations of T in its various sheets is not finite-dimensional. The same result trivially follows for F.

5. The Off-Shell Irreducible Kernels U and Related Results

Theorem 4 (definition and uniformity of a class of kernels U). *Let F satisfy Assumption 1, let g be a function of k analytic in a complex neighborhood $\mathscr{N} = U_K \mathscr{N}_K$ of $H^+_{m\mu} = \{K; K \in R^\nu, K^2 = (m\mu)^2, K_0 > 0\}$, and let:*

$$\chi(k) = \tfrac{1}{2}\sigma^\beta + g(k) \qquad \text{if} \quad (m-1)(\nu-1) \text{ is odd, } (\beta \text{ half-integer}), \tag{51}$$

$$\chi(k) = \frac{i}{2\pi}\sigma^\beta \ln\sigma + g(k) \quad \text{if} \quad (m-1)(\nu-1) \text{ is even, } (\beta \text{ integer}), \tag{52}$$

with e.g. $\ln\sigma$ *real at $\sigma > 0$ ($k^2 < (m\mu)^2$) in the sheet $r = 0$ where F is originally defined.*

Then, if $\chi(k)$ is not an eigenvalue function of $T(k, \Omega, \Omega') = F(k, z(k, \Omega), z(k, \Omega'))$ with respect to the integration measure $\hat{\alpha}(k, \Omega')d\Omega'$, there exists a unique solution $U(k, z, z')$ of the equation

$$F - U = \chi(k) F \hat{*} U, \qquad (* = \sigma^\beta \hat{*}), \tag{53}$$

and U is analytic in k, z, z' in $(\mathscr{W}''_K \cap \mathscr{N}_K) \backslash \{k^2 = (m\mu)^2\} \times \mathscr{D}_K \times \mathscr{D}_K$, apart possibly from a discrete set of polar manifolds in k.

Proof. Theorem 1 of Sect. 2, applied with A, t and α replaced by $\underline{F}_0$, Ω and $-\chi(k)\hat{\alpha}(k, \Omega)$ ensures the existence of a unique solution $\underline{U}_0(k, z, z')$, analytic in the cut domain $(\mathscr{W}''_K \cap \mathscr{N}_K) \backslash \{k^2 \geqq (m\mu)^2\} \times \mathscr{D}_K \times \mathscr{D}_K$ apart possibly from a discrete set of poles in k, of the equation:

$$\underline{F}_0 - \underline{U}_0 = \underline{F}_0(\chi(k)*)\underline{U}_0. \tag{54}$$

In fact, the assumption that χ is not an eigenvalue function of $T(k, \Omega, \Omega')$ with respect to the measure $\hat{\alpha}(k, \Omega)d\Omega$ ensures (by a standard argument of Fredholm theory) that the function $d(\lambda, k)$ does not vanish identically at the value $\lambda = 1$, when the operation $*$ of Theorem 1 is the operation $\chi(k)\hat{*}$.

The kernel $\underline{U}_0$, on the other hand, admits boundary values U_0, U_1 at k, z, z' real, $k^2 > (m\mu)^2$, from the respective directions $\mathrm{Im}\, s > 0$ and $\mathrm{Im}\, s < 0$ which in view of the known analyticity of F_0, F_1 (Sect. 4) and of a new application of Theorem 1, are analytic apart possibly from a discrete set of poles in k.

The operation $\times = \chi(k)\hat{*}$ satisfies in each parity case (in view of the analyticity of the functions g and $\hat{\alpha}$ in k) the relation:

$$\times_0 - \times_1 = *_0 (\equiv (\sigma^\beta)_0 \hat{*}_0), \tag{55}$$

where $\times_0 = (\chi(k))_0 \hat{*}_0$, $\times_1 = (\chi(k))_1 \hat{*}_1$. The algebraic argument described in the Appendix thus ensures that:

$$U_0 = U_1, \tag{56}$$

i.e. the uniformity of U around $k^2 = (m\mu)^2$. Theorem 4 is therefore proved.

We next state:

Theorem 5 (irreducibility of U and local expansion of F). *Let F satisfy Assumptions 1, 1′ and let β be >0. Then being given any function g of k analytic in a complex neighborhood $\mathcal{N}$ of $H^+_{m\mu}$ and satisfying $g(k)=0$ at $k^2 = (m\mu)^2$, there exists a unique solution $U(k, z, z')$ of Eq. (53). This kernel is analytic in k, z, z' in $(\mathcal{W}''_K \cap \mathcal{N}_K) \times \mathcal{D}_K \times \mathcal{D}_K$ apart possibly from a discrete set of polar manifolds in k which all lie outside a complex neighborhood of $H^+_{m\mu}$.*

The following convergent expansion of F holds in any sheet when k lies in a sufficiently small complex neighborhood of K and $(z, z') \in \mathcal{D}_K \times \mathcal{D}_K$:

$$F(k, z, z') = \sum_{n=0}^{\infty} (\chi(k))^n U^{\hat{*}(n+1)}(k, z, z'), \tag{57}$$

where all terms $U^{\hat{*}(n+1)}$ are, like U, analytic in k in a common complex neighborhood of K and in z, z' in $\mathcal{D}_K \times \mathcal{D}_K$.

The proof is given below. We first note that a natural choice of g under the assumptions of Theorem 5 is $g \equiv 0$, in which case U is defined by the equation:

$$F - U = f(\sigma) F * U, \tag{58}$$

where

$$f(\sigma) = \tfrac{1}{2} \quad \text{if} \quad (m-1)(\nu-1) \text{ is odd}, \tag{59}$$

$$f(\sigma) = \frac{i}{2\pi} \ln \sigma \quad \text{if} \quad (m-1)(\nu-1) \text{ is even}, \tag{60}$$

$\mathcal{W}''_K \cap \mathcal{N}_K \equiv \mathcal{W}''_K$, and the expansion (57) of F becomes the expansion (2) given in Sect. 1.

Proof of Theorem 5. $\mathcal{W}'_K$ and $\mathcal{D}'_K$ are chosen in Assumption 1′, such that $\mathcal{W}'_K \subset \mathcal{W}''_K \cap \mathcal{N}_K$, $S(k) \subset \mathcal{D}'_K$, $\forall k \in \mathcal{W}'_K$. The bounds

$$|\underline{F}_0^{\hat{*}o(n+1)}| < C_K (C_K C'_K)^n \tag{61}$$

in the region $\Delta_K'^{(\mathrm{cut})}$ [see Eq. (49)] and the fact that the function g considered is of the form $\sigma^{\beta'} g'(k)$, where β' is a >0 integer and g' is analytic in $\mathcal{N}_K$, ensure the absolute convergence of the Neumann series $\sum_{n=0}^{\infty}(-1)^n(\chi(k))^n \underline{F}_0^{\hat{*}\circ(n+1)}$ of $\underline{U}_0$ in terms of $\underline{F}_0$ in Eq. (54) for $|\sigma|$ sufficiently small and k, z, z' in $\Delta_K'^{(\mathrm{cut})}$. Thus $\underline{U}_0$, which is then equal to the sum of this series, is bounded in that region. Therefore U cannot have poles in k for k in a sufficiently small complex neighborhood of K. A singularity of U at $k^2=(m\mu)^2$ is excluded similarly, first when $z, z' \in \mathscr{D}_K'$ and therefrom for $z, z' \in \mathscr{D}_K$.

The first part of Theorem 5 is therefore proved. On the other hand, the expansion (57) is the Neumann series of F in terms of U in Eq. (53). This series is absolutely convergent in each sheet for k in a sufficiently small complex neighborhood of K which, like $\chi(k)$, depends in the case $(m-1)(\nu-1)$ even on the sheet considered. This follows from the analyticity of U previously established and the corresponding bounds:

$$|U^{\hat{*}(n+1)}(k, z, z')| \leqq \mathrm{Max}_{\Omega\in S}|U(k, z, z(k, \Omega))| \times \mathrm{Max}_{\Omega\in S}|U(k, z(k, \Omega), z')| C_K'(C_K' D_K)^{n-1} \tag{62}$$

which hold, for each K, when $z, z' \in \mathscr{D}_K$ and k lies in a given complex neighborhood of K, D_K being a bound on $|U(k, z(k, \Omega), z(k, \Omega')|$ independent of Ω, $\Omega' \in S$. This proves the second part of the theorem. Q.E.D.

Remark. The kernel $u(k, \Omega, \Omega') = U(k, z(k, \Omega), z(k, \Omega'))$, which is the mass-shell restriction of U (for Ω, Ω' real) is not necessarily analytic in k at $k^2=(m\mu)^2$, but is in general two-sheeted around $k^2=(m\mu)^2$ and satisfies the relation:

$$u_1(k, \Omega, \Omega') = u_0(k, -\Omega, -\Omega'), \tag{63}$$

which follows from the uniformity of U and the equality (37). The uniformity of this kernel in [12] is there a consequence of the symmetry assumption made on T ($T(k, \Omega, \Omega') = T(k, -\Omega, -\Omega')$).

We below restrict our attention to the kernel U obtained from Theorem 5 in the case when $g \equiv 0$.

We shall consider in Sect. 6 cases when the domain $\mathscr{D}_K$ in z-space is not bounded and contains in fact a neighborhood of the euclidean space $\mathscr{E}_K$ (see Sect. 6). In this case, it can be checked, e.g. from the formula:

$$U = F - F \times F + F \times U \times F, \tag{64}$$

where $\times = f(\sigma)*$ and in particular:

$$\begin{aligned} F \times U \times F(k, z, z') = (f(\sigma)\sigma^{\beta})^2 \textstyle\int F(k, z, z(k, \Omega))\, U(k, z(k, \Omega), z(k, \Omega')) \\ \cdot F(k, z(k, \Omega'), z')\hat{\alpha}(k, \Omega)\hat{\alpha}(k, \Omega')\, d\Omega d\Omega', \end{aligned} \tag{65}$$

that $U(k, z, z')$ has the same type of decrease, or increase, as F itself in euclidean directions. [Note in fact that $z(k, \Omega)$ and $z(k, \Omega')$ remain in a bounded complex neighborhood of $z=0$ when $k \in \mathscr{W}_K''$.]

If the analytic cut off factor φ in the definition of the Feynman type operator (see Sect. 6) has a sufficient decrease at infinity, $U^{\circ(n+1)}(k, z, z')$ is then well defined for each $n \in \mathbb{N}$, and can be written for the same reasons as $G^{\circ(n+1)}$ in Lemma 4 of Sect. 7 in the form:

$$U^{\circ(n+1)}(k,z,z') = \sum_{p=0}^{n} U_p^{(n)}(k,z,z')\,(\sigma^{\beta} f(\sigma))^p, \tag{66}$$

where the coefficients $U_p^{(n)}$ are analytic in the same domain as U and satisfy Eq. (82) of Sect. 7 with G replaced by U. In particular:

$$U_n^{(n)}(k,z,z') = U^{\hat{*}(n+1)}(k,z,z'). \tag{67}$$

The "leading" or "dominant" contribution to $U^{\circ(n+1)}$ is by definition the term of maximal degree $r=n$ in the decomposition (66), which in the case $(m-1)(v-1)$ even is the term of maximal power in $\ln\sigma$. The expansion (2) of F established in Theorem 5 thus appears, as announced in the Introduction, as a sum of dominant contributions to the Feynman-type integrals $U^{\circ(n+1)}$ associated with the graphs (3).

6. Bethe-Salpeter Type Kernels

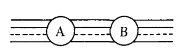
We first discuss below the first approach of the simplified theory in which F satisfies Assumption 1. We shall assume moreover that for each K, the domain $\mathscr{D}_K$ contains a neighborhood of the euclidean space $\mathscr{E}_K$ ($\operatorname{Im}\vec{z}_i = \operatorname{Re} z_{i,0} = 0$, $i=1, \ldots, m$, in a Lorentz frame where $\vec{K}=0$, $K_0 = m\mu$), and that F is bounded, or has at most a polynomial or exponential increase at infinity in euclidean directions.

For each choice of an analytic cut-off factor φ, equal to one on the mass-shell, e.g.:

$$\varphi(k) = \exp\left(-\frac{(k^2-\mu^2)^2}{\alpha^2}\right), \tag{68}$$

the Feynman-type operation $\circ$ between kernels $A(k,z,z')$, $B(k,z,z')$ [with e.g. at most exponential increase at infinity of a given order in euclidean directions if φ has the form (68)] is defined by the formula (see details in [13]):

$$A \circ B(k,z,zz') = \int_{\Gamma(k)} \frac{A(k,z,z'')\,B(k,z'',z')}{\prod_{i=1}^{m} [(k_i(k,z''))^2 - \mu^2]} \prod_{i=1}^{m} \varphi(k_i(k,z''))\,d\mu(z''), \tag{69}$$

where $k_i(k,z'') = z_i'' + \dfrac{k}{m}$ is an energy-momentum variable attached to each internal line $i=1,\ldots,m$ of the diagram  , $\Gamma(k)$ is an integration contour of real dimension $(m-1)v$ obtained by a continuous local distortion of the euclidean space when k^2 turns around $(m\mu)^2$, and $d\mu(z'') = dz_1'' \ldots dz_{m-1}''$ if z'' is represented e.g. by the set $z_1'', \ldots, z_{m-1}''$.

The mathematical analysis of [14], which includes an adequate extension of results of Fredholm theory, is applied in [13] to the second approach of the simplified theory in which G is given and is assumed to be irreducible and in which properties of F_λ or F are derived in turn from Eq. (4). It can be equally applied to the first approach. Together with results of Sect. 4 (analyticity of F_0, F_1), it then provides (as explained below) the following result: if φ has a sufficient decrease at infinity and apart possibly from particular choices of φ, there is a unique solution G of Eq. (4), which is analytic in a domain containing, for each K real, $K^2=(m\mu)^2$, $K_0>0$, the domain $\mathscr{W}_K\backslash\{k^2=(m\mu)^2\}\times\mathscr{D}_K\times\mathscr{D}_K$ apart possibly from a discrete set of polar manifolds in k, and satisfies $F\circ G=G\circ F$.

The uniformity of $G(G_0=G_1)$ around $k^2=(m\mu)^2$ follows here from the algebraic argument of the appendix, and the formula:

$$_0 - \;_1 = *_0 \,. \tag{70}$$

The latter is a consequence (see [14]) of Picard-Lefschetz theory and Leray's several dimensional residue formula; $_0$ and $_1$ are the limits of at k, z, z' real, $k^2>(m\mu)^2$ from the respective sides $\operatorname{Im} s>0$ and $\operatorname{Im} s<0$ of the physical sheet $r=0$ [integration being made over the corresponding contours $\Gamma_0(k)$ and $\Gamma_1(k)$].

The result on G just described is analogous to that obtained on the kernels U in Theorem 4 of Sect. 5, with the following differences:

(i) The meromorphy domain of G in k is larger than that obtained for U. For each K real, $K^2=(m\mu)^2$, $K_0>0$, it includes the full domain $\mathscr{W}_K\backslash\{k^2=(m\mu)^2\}$, whereas the corresponding domain obtained for U is restricted to $\mathscr{W}_K''\backslash\{k^2=(m\mu)^2\}$ as a consequence of the condition $S(k)\subset\mathscr{D}_K$.

(ii) Under Assumptions 1, 1′ and for $\beta>0$, it was shown in Sect. 5 that the kernels U are well defined and irreducible for any choice of the analytic function g such that $g=0$ at $k^2=(m\mu)^2$ (and in particular for $g\equiv 0$). A similar analysis has not yet been achieved for the Bethe-Salpeter type kernels G. (In this connection, see note added at the end.)

We shall below *assume*, as in Bethe-Salpeter type approach of the simplified theory, that G is indeed irreducible for an adequate choice of the operation $\circ$, and more precisely that G is analytic in a domain containing, for each K, a set of the form $\mathscr{V}_K\times\mathscr{D}_K\times\mathscr{D}_K$, where $\mathscr{V}_K$ is a complex neighborhood of K ($\mathscr{V}_K\subset\mathscr{W}_K$). For the simplicity of the following discussion, we moreover assume that $|G|$ is uniformly bounded in $\mathscr{V}_K\times\mathscr{D}_K\times\mathscr{D}_K$, although similar results would hold equally for kernels G with e.g. polynomial or exponential increase in euclidean directions, for adequate analytic cut-off factors φ.

Assumption 2. $G(k,z,z')$ is analytic and uniformly bounded in modulus in a domain of the form $\mathscr{V}_K\times\mathscr{D}_K\times\mathscr{D}_K$, for each K real, $K^2=(m\mu)^2$, $K_0>0$, where $\mathscr{V}_K$ is a complex neighborhood of K and $\mathscr{D}_K$ is a domain containing the euclidean space $\mathscr{E}_K$.

The following lemma, based on results of [14], then holds.

Lemma 1. *For any $n\in\mathbb{N}$, $n\geqq 1$, $G^{\circ(n+1)}$ is well defined and analytic in the same physical sheet domain as G, in particular in $\mathscr{V}_K^{(\mathrm{cut})}\times\mathscr{D}_K\times\mathscr{D}_K$ ($\forall K$ real, $K^2=(m\mu)^2$, $K_0>0$) and satisfies there the bounds:*

$$|G^{\circ(n+1)}(k,z,z')|<C_1C_2^n\,, \tag{71}$$

where C_1, C_2 are constants independent of k,z,z' (which may depend on K).

$G^{\circ(n+1)}$ admits an analytic continuation in the domain $\widehat{\mathscr{W}_K''\cap\mathscr{V}_K}\times\mathscr{D}_K\times\mathscr{D}_K$, where $\hat{\mathscr{W}}$ denotes the universal covering of $\mathscr{W}\backslash\{k^2=(m\mu)^2\}$ and $\mathscr{W}_K''$ is the domain

introduced in Sect. 3. It is two-sheeted if $(m-1)(\nu-1)$ *is odd, or infinitely sheeted if* $(m-1)(\nu-1)$ *is even, (unless* $G_0 *_0 G_0 \equiv 0$*), and satisfies in each sheet* r *(*$r \in \mathbb{Z}$ *if* $(m-1)(\nu-1)$ *is even,* $r=1$ *if* $(m-1(\nu-1)$ *is odd) the bounds:*

$$|G^{o(n+1)}(k,z,z')| < C_1(C_2 + rC_1 C'_K|\sigma|^{\beta})^n, \tag{72}$$

where $C'_K = \operatorname*{Max}_{k \in \mathscr{W}''_K} \int |\hat{\alpha}(k,\Omega)| d\Omega$.

The following convergent expansion of F_λ *holds correspondingly in the physical sheet if* $|\lambda| < 1/C_2$*, and in other sheets for* $|\lambda|$ *sufficiently small:*

$$F_\lambda(k,z,z') = \sum_{n=0}^{\infty} \lambda^n G^{o(n+1)}(k,z,z'). \tag{73}$$

Proof. The first, nontrivial part, including the bounds (71), is contained in [14]. The second part, including the bounds (72) is a consequence of formula (70). In fact, the latter entails that the restriction $\underline{\ }_r$ of to the sheet r satisfies:

$$\underline{\circ}_r = \underline{\circ}_0 - r\underline{*}_0 \qquad \text{if} \quad (m-1)(\nu-1) \quad \text{is even}, \tag{74}$$

$$\underline{\ }_1 = \underline{\ }_0 - \underline{*}_0,\ \underline{\ }_2 = \underline{\ }_0 \qquad \text{if} \quad (m-1)(\nu-1) \quad \text{is odd}. \tag{75}$$

It thus yields, in the sheet r ($r=1$ if $(m-1)(\nu-1)$ is odd, $r \in \mathbb{Z}$ otherwise):

$$G^{o(n+1)} = \sum_{p=0}^{n} (-1)^p \sum_{\substack{n_1,\dots,n_{p+1} \\ \Sigma n_i = n-p}} G^{\underline{\circ}o(n_1+1)}(r\underline{*}_0) G^{\underline{\circ}o(n_2+1)}(r\underline{*}_0) \dots (r\underline{*}_0) G^{\underline{\circ}o(n_{p+1}+1)}. \tag{76}$$

The bounds (72) follow from the previous bounds (71) in the physical sheet and from the properties of the operation $\underline{*}_0$ (see Sect. 3).

Finally, the expansion (73) follows from the convergence of the series at the right hand side of (73), which is the Neumann series of F_λ in Eq. (4′), and from the unicity of the solution of the latter established in [14]. (Note that F_λ is well defined in terms of G through Eq. (4′) in view of the results of [14, 13], outside possibly a discrete set of values of λ.)

7. Irreducible Kernels and Local Expansions of F or F_λ

Lemma 2. *The operations and* $\times = f(\sigma)*$ *satisfy the relation:*

$$\circ = \times + \Delta, \tag{77}$$

where Δ *is uniform around* $k^2 = (m\mu)^2$ *(*$\Delta_1 = \Delta_0$*, where* $\Delta_r = {}_r - \times_r$*).*

Formula (77), in which the uniformity of Δ directly follows from formulae (55) and (70), thus exhibits the operation as a sum of the uniform operation Δ and of the nonuniform operation $f(\sigma)\sigma^\beta \hat{*}$: the factor $f(\sigma)\sigma^\beta$ has in fact either a square root singularity at $\sigma = 0$ (arising from σ^β) if $(m-1)(\nu-1)$ is odd, or a logarithmic singularity [arising from $f(\sigma)$] if $(m-1)(\nu-1)$ is even.

Lemma 3 (link between the kernels U and G). *The kernels U and G satisfy the integral relation.*

$$U - G = U\Delta G = G\Delta U. \tag{78}$$

Proof. The relations

$$F - U = F \times U = U \times F\,,$$
$$F - G = F \circ G = G \circ F\,,$$

yield:

$$\begin{aligned} U - G &= G \circ F - F \times U \\ &= G \circ (F - F \times U) + (G \circ F - F) \times U \\ &= G \circ U - G \times U \equiv G \Delta U\,. \end{aligned} \tag{79}$$

Similarly

$$U - G = U \Delta G. \quad \text{Q.E.D.} \tag{80}$$

In (79) and (80), the uniformity of the operation Δ is fully consistent with the fact that both U and G are themselves uniform.

We next state the following decomposition of each term $G^{o(n+1)}$ in powers of $\sigma^\beta f(\sigma)$ with well specified locally analytic coefficients:

Lemma 4.

$$G^{o(n+1)}(k, z, z') = \sum_{p=0}^{n} A_p^{(n)}(k, z, z')\,(\sigma^\beta f(\sigma))^p\,, \tag{81}$$

where the functions $A_p^{(n)}$ *are equal to:*

$$A_p^{(n)} = \sum_{\substack{n_1 \ldots n_{p+1} \\ n_i \geqq 0,\, \Sigma n_i = n - p}} G^{\Delta(n_1+1)} \hat{*} G^{\Delta(n_2+1)} \ldots \hat{*}^{\Delta(n_{p+1}+1)}\,, \tag{82}$$

and are analytic in $(\mathscr{V}_K \cap \mathscr{W}_K'') \times \mathscr{D}_K \times \mathscr{D}_K (\forall K$ *real*, $K^2 = (m\mu)^2$, $K_0 > 0)$.

Proof. The decomposition (81) and (82) of $G^{o(n+1)}$ is a direct consequence of formula (77). The uniformity of the functions $A_p^{(n)}$ follows from the uniformity of G and of the operations Δ and $\hat{*}$. As a byproduct of Lemma 5 below, each term $G^{\Delta(n_i+1)}$ is bounded in modulus, and is thus analytic in $(\mathscr{V}_K \cap \mathscr{W}_K'') \times \mathscr{D}_K \times \mathscr{D}_K$. The same result holds in turn for each term in the sum $\sum$ at the right hand side of (82) in view of the properties of the operation $\hat{*}$, and therefore for $A_p^{(n)}$ itself. Q.E.D.

Lemma 5.

$$|G^{\Delta(n+1)}(k, z, z')| < C_1 (C_2 + C_1 C_K' |\sigma^\beta f(\sigma)|)^n \tag{83}$$

in $(\mathscr{V}_K \cap \mathscr{W}_K'') \times \mathscr{D}_K \times \mathscr{D}_K$.

The proof is the same as that of the bounds (72) in Lemma 1 of Sect. 6, the relations (74) and (75) being replaced here by $\Delta \equiv \Delta_0 = \varrho_0 - \times_0$. [The uniformity of Δ and Eq. (77) have been used.]

In view of Lemmas 4 and 5 the expansion (73) of F_λ in terms of G then directly provides:

Lemma 6. *The following convergent expansion of* F_λ *holds at small* λ:

$$F_\lambda(k, z, z') = \sum_{n=0}^{\infty} U_\lambda^{\hat{*}(n+1)} (\lambda \sigma^\beta f(\sigma))^n\,, \tag{84}$$

where

$$U_\lambda(k,z,z')=\sum_{n=0}^{\infty}\lambda^n G^{\Delta(n+1)}(k,z,z')\,. \tag{85}$$

The kernel U_λ is well defined at small λ by Eq. (85) as an analytic function of k,z,z' in $(\mathscr{V}_K\cap\mathscr{W}_K'')\times\mathscr{D}_K\times\mathscr{D}_K$, in view of Lemma 5.

Proof of Lemma 6. The expansion (73) of F_λ and Lemma 4 yield:

$$F_\lambda(k,z,z')=\lim_{N\to\infty}\sum_{p=0}^{N}(\lambda\sigma^\beta f(\sigma))^p X_{p,N}(\lambda,k,z,z')\,, \tag{86}$$

where

$$X_{p,N}(\lambda,k,z,z')=\sum_{n=p}^{N}\sum_{\substack{n_1\dots n_{p+1}\\ \Sigma n_i=n-p}}(\lambda^{n_1}G^{\Delta(n_1+1)})\hat{*}\dots\hat{*}(\lambda^{n_{p+1}}G^{\Delta(n_{p+1}+1)})\,. \tag{87}$$

As easily checked, $\lim\limits_{N\to\infty}X_{p,N}=U_\lambda^{\hat{*}(p+1)}$ and the following more precise bounds are obtained (see below):

$$|X_{p,N}-U_\lambda^{\hat{*}(p+1)}|<D_1D_2^p(\tfrac{1}{2})^N\,, \tag{88}$$

with constants D_1,D_2 independent of λ,k,z,z' in the region $|\lambda|<\dfrac{1}{4D_2'}$, $k,z,z'\in(\mathscr{V}_K\cap\mathscr{W}_K'')\times\mathscr{D}_K\times\mathscr{D}_K$, $D_2'=(C_2+C_1C_K\,\mathrm{Max}|\sigma^\beta f(\sigma)|)$. In fact, in view of (85) and Lemma 5:

$$U_\lambda^{\hat{*}(p+1)}=\sum_{n=p}^{\infty}\sum_{\substack{n_1\dots n_{p+1}\\ \Sigma n_i=n-p}}(\lambda^{n_1}G^{\Delta(n_1+1)})\hat{*}\dots\hat{*}(\lambda^{n_{p+1}}G^{\Delta(n_{p+1}+1)}) \tag{89}$$

for $|\lambda|<\dfrac{1}{D_2'}$.

The bound (88) follows, with $D_1=2C_1$, $D_2=2C_1C_K'$.

$$\left(\text{Note that }\sum_{\substack{n_1\dots n_{p+1}\\ \Sigma n_i=n-p}}1\leqq 2^n.\right)$$

The expansion (84) of F_λ is obtained in turn at small λ. Q.E.D.

The kernel U_λ defined by Eq. (85) at small λ clearly satisfies the integral equation:

$$U_\lambda-G=\lambda U_\lambda\Delta G=\lambda G\Delta U_\lambda\,. \tag{90}$$

By combining the methods of Sect. 2 and those of [14, 13], one can check that Eq. (90) defines in fact U_λ in terms of G for more general values of λ (outside a discrete set). The kernel U of Sect. 5 then appears in this framework as the analytic continuation in λ of U_λ up to the value $\lambda=1$, at which Eqs. (90), (78) coincide. The expansion (84) of F_λ can similarly be extended analytically up to $\lambda=1$, under some technical conditions not discussed here.

Note. In the first part of [10], it will be established that at $m=2$ (v arbitrary) the Bethe-Salpeter kernel G is indeed irreducible (i.e. analytic at $s=4\mu^2$) if F satisfies off-shell unitarity (= asymptotic completeness), is bounded near $s=4\mu^2$ (Assumption 1′), if β is >0 and if the cut-off factor φ is of the form (68) with α sufficiently small. Under similar conditions, the expansion (73) of F_λ is also valid at $\lambda=1$ $\left(\text{i.e. the series } \sum_{n=0}^{\infty} G^{o(n+1)} \text{ is convergent and is equal to } F\right)$ for $|\sigma|$ sufficiently small. The method used indicates that the same result is probably true also for arbitrary values of m in the simplified theory.

We note that, as in the present work, kernels G and $U(=K$ at $v=4)$ are still expected, at $m=2$, to have poles in s below the next threshold [$s=(3\mu)^2$, or $s=(4\mu)^2$ in an even theory] and possibly on the real axis, even if F_0 and F_1 remain continuous and satisfy off-shell unitarity.

Appendix: A General Class of Irreducible Kernels: Heuristic Discussion

We consider here the approach of the simplified theory in which F is given and satisfies Assumptions 1 and possibly 1′. In both cases of the kernels U and G, a kernel V is introduced through an equation of the form:

$$F=V+F\times V, \tag{A.1}$$

where $\times$ is a convolution operation of the form:

$$(A\times B)(k,z,z')=\int_{z''\in\Sigma(k)} A(k,z,z'')B(k,z'',z')d\mu(k,z''), \tag{A.2}$$

with an integration measure $d\mu(k,z'')$ depending analytically on k when k^2 turns around $(m\mu)^2$ and an integration set $\Sigma(k)$ depending continuously on k. [We recall that the dimension of $\Sigma(k)$ is not the same in the case of the kernels U and G.] The kernels G or U also satisfy the relation

$$F\times V=V\times F. \tag{A.3}$$

Let us consider a general class of convolution operations $\times$ of the form (A.2), or linear combinations of such operations, such that V is well defined, as in the previous cases, as an analytic or meromophic function around $k^2=(m\mu)^2$ and satisfies (A.3). (We do not construct explicitly this class here.) The relations:

$$F_0-V_0=V_0\times{}_0F_0=F_0\times{}_0V_0, \tag{A.4}$$

$$F_1-V_1=V_1\times{}_1F_1=F_1\times{}_1V_1, \tag{A.5}$$

obtained as the limits of (A.1) and (A.3) from the respective directions $\operatorname{Im}s>0$ and $\operatorname{Im}s<0$ of the physical sheet, then provide the algebraic relation:

$$(F_0-F_1-F_0*_0F_1)\times{}_1(\mathbb{1}_1-V_1)-(\mathbb{1}_0+F_0)\times{}_0(V_0-V_1)=F_0(\times{}_0-\times{}_1-*_0)V_1, \tag{A.6}$$

easily checked from the associativity of the operations $*_0$, $\times_0$, $\times_1$ and the relation $F_1\times{}_1(\mathbb{1}_1-V_1)=V_1$ ($\mathbb{1}_1$ and $\mathbb{1}_0$ denote here the unit operators with respect to $\times_1$ and $\times_0$, respectively).

If off-shell unitarity $(F_0 - F_1 - F_0 *_0 F_1 = 0)$ is assumed, (A.6) thus entails the uniformity of V around $k^2 = (m\mu)^2$ $(V_0 = V_1)$, provided that:

$$\times_0 - \times_1 = *_0 . \tag{A.7}$$

[In fact, (A.6) entails under this condition that $(\mathbb{1}_0 + F_0) \times_0 (V_0 - V_1) = 0$; by "multiplication" on the left by $(\mathbb{1}_0 - V_0) \times_0$ and use of (A.4), it provides in turn $V_0 - V_1 = 0$.] Conversely, Eq. (A.7) is needed if V is to be uniform, apart from exceptional cases.

The uniformity of V entails its irreducibility, e.g. if the Neumann series $\sum_{n=0}^{\infty} V^{\times(n+1)}$ is absolutely convergent in some complex neighborhood of the threshold.

As we have seen the operations $\times$ and $\circ$ that give rise to the kernels U and G satisfy (A.7). On the other hand, any operation $\times$ satisfying (A.7) is equal, locally, as $\circ$ in Sect. 7, to:

$$\times = f(\sigma) * + \mathcal{V}, \tag{A.8}$$

where $\mathcal{V}$ is uniform $(\mathcal{V}_0 = \mathcal{V}_1)$ and kernels V, V' satisfying Eqs. (A.1) and (A.3) for respective operations $\times$, $\times'$, satisfy as in Sect. 6 the relation:

$$V - V' = V(\times - \times')V' = V'(\times - \times')V, \tag{A.9}$$

where $\times - \times' = \mathcal{V} - \mathcal{V}'$ is uniform around $k^2 = (m\mu)^2$. [The proofs of (A.8) and (A.9) are the same as those given for G and U in Sect. 6.]

Acknowledgements. This work has largely benefitted from my previous collaboration with J. Bros (see [11, 12]) and further discussions with him.

References

1. Zimmermann, W.: Nuovo Cimento **21**, 249 (1961)
   Oehme, R.: Phys. Rev. **121**, 1840 (1961)
2. Martin, A.: Scattering theory: Unitarity, analyticity, and crossing. In: Lecture Notes in Physics. Berlin, Heidelberg, New York: Springer 1970
3. Eden, R.J., Landshoff, P.V., Olive, D.I., Polkinghorne, J.C.: The analytic S-matrix. Cambridge: Cambridge University Press 1966 and references therein
4. Epstein, H., Glaser, V., Iagolnitzer, D.I.: Commun. Math. Phys. **80**, 99 (1981)
5. Bros, J.: Analytic methods in mathematical physics. New York: Gordon and Breach 1970, p. 85
   Bros, J., Lassalle, M.: Commun. Math. Phys. **43**, 279 (1975), **54**, 33 (1977)
6. Iagolnitzer, D.: Commun. Math. Phys. **77**, 251 (1980)
7. Kawai, T., Stapp, H.P.: Publ. RIMS, Kyoto Univ. **12**, Suppl. 155 (1977)
8. Iagolnitzer, D.: Acta Phys. Austr. Suppl. **XXIII**, 235 (1981)
9. Bros, J.: Complex analysis, microlocal calculus and relativistic quantum theory. In: Lecture Notes in Physics, Vol. 126, p. 254 Berlin, Heidelberg, New York: Springer 1980, Acta Phys. Austr. Suppl. **XXIII**, 329 (1981)

10. Bros, J.: In preparation
11. Bros, J., Iagolnitzer, D., Pesenti, D.: Nonholonomic singularities of the S-matrix and Green functions. Saclay DPh-T report 8118 (Feb. 1981)
Bros, J., Iagolnitzer, D.: Structure of scattering functions at m-particle thresholds in a simplified theory and non holonomic character of the S-matrix and Green functions. Saclay, DPh-T preprint 8213, Phys. Rev. (to be published)
12. Bros, J., Iagolnitzer, D.: Unitarity equation and structure of the S-matrix at the m-particle threshold in a theory with pure $m \to m$ interaction. Commun. Math. Phys. **85**, 197 (1982)
13. Bros, J., Pesenti, D.: Fredholm resolvents of meromorphic kernels with complex parameters: a Landau singularity and the associated equations of type U in a non holonomic case. Dept. of Mathematics, Paris-Sud preprint. J. Math. Pures Appl. (to be published)
14. Bros, J., Pesenti, D.: J. Math. Pures Appl. **58**, 375 (1980)

Communicated by K. Osterwalder

Received July 2, 1982