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Supersymmetric Two-Dimensional Toda Lattice

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Abstract. The two-dimensional Toda lattice connected with contragradient Lie superalgebras is studied. The systems of linear equations associated with the models for which the inverse scattering method is applicable are written down. The reduction group is calculated.

1. Introduction

Many papers in the last few years considerable discuss the hidden symmetries of integrable systems. Bogoyavlensky discovered that the classical Toda lattice (TL) is connected with the simple Lie algebras [1]. Then Leznov and Saveliev showed [2] that the periodic TL corresponds to contragradient Lie algebras (Kac-Moody algebras). This connection enables investigation of the systems by means of the Inverse Scattering Method (ISM) (see also [3]). On the other hand the simplest cases of the two-dimensional Toda lattice (TTL) – the Sinh-Gordon equation and the Liouville equation have supersymmetric extensions, while the integration by the ISM is applicable as before [4, 5]. In the present work we construct a supersymmetric version of TTL and discover the connection of these systems with the Contragradient Lie Superalgebras (CLS) classified by Kac [7]. This connection allows us to write down the equations of the systems as "the zero-curvature conditions" (Zakharov-Shabat equations) and, in principle, to apply the ISM.

The systems under consideration are described by the action

$$S=\int d^2x d\theta_1 d\theta_2\left(-i\sum_{j=1}^{h}\Phi^j D_1 D_2 \Phi^j - U(\Phi)\right), \qquad (1.1)$$

where $x=(x_0, x_1)$, θ_1, θ_2 are scalar, and the Grassmann superspace parameters, D_1 and D_2, are supersymmetric covariant derivatives, and $\Phi=(\Phi^1, \dots, \Phi^n)$ is a multiplet of the scalar bosonic superfields.

The potential $U(\Phi)$ is connected with the root systems of the CLS. Let $\{\alpha_s\}$ be a set of simple roots of CLS ($s \in I = (0, 1, \ldots, n)$), and τ be a subset ($\tau \subset I$) corresponding to the odd ones. Then $U(\Phi)$ takes the form

$$U(\Phi) = \sum_{s \in \tau} \frac{1}{(\alpha_s, \alpha_s)} \exp 2(\Phi, \alpha_s) - i\theta_1\theta_2 \sum_{s \bar{\in} \tau} \frac{1}{(\alpha_s, \alpha_s)} \exp 2(\Phi, \alpha_s) . \tag{1.2}$$

We classify and investigate all classes of Lagrangians with the positivly defined kinetic energy of bosonic components, corresponding to CLS.

It turns out that excluding the algebras $B(0, 1)$ and $C^{(2)}(2)$ [1] corresponding to the Liouville and Sinh-Gordon equations, the subset of odd roots does not cover the set of simple roots. Thus the second sum in (1.2) does not vanish. Because of these terms in (1.1), the supersymmetry is broken in contrast with the other supersymmetric models connected with Lie superalgebras [14]. However, it is worth mentioning that a similar phenomenon takes place in some realistic physical models[2].

The contents of the paper are as follows. In Sect. 2 we derive the systems and equations of motion. A complete classification of the systems contains five infinite series and four special systems. In Sect. 3 we construct associate linear sets. All classes of models are explicitly worked out. Section 4 is devoted to the definition and the calculation of the group of reduction from complete CLS. In conclusion, in Sect. 5 we discuss some unsolved problems. Some facts of CLS theory based on [6–8] are presented in the appendix.

2. Description of Systems

Let ξ and η be the light cone coordinates $\left(\xi = \dfrac{x_1 - x_0}{z}, \eta = \dfrac{x_1 + x_0}{2}\right)$ and let θ_1 and θ_2 be the elements of the Grassmann algebra. Supersymmetric covariant derivatives are

$$D_1 = -\partial_{\theta_2} + i\theta_2\partial_\eta, \qquad D_2 = \partial_{\theta_1} + i\theta_1\partial_\xi . \tag{2.1}$$

Consider the multiplet of bosonic superfields $\Phi = (\Phi^1, \ldots, \Phi^n)$

$$\Phi^k = \varphi^k(\xi, \eta) + i\bar{\theta}\psi^k(\xi, \eta) + \tfrac{1}{2}\bar{\theta}\theta F^k(\xi, \eta) . \tag{2.2}$$

In this formula φ^k and F^k are scalar fields, $\psi^k = \begin{pmatrix} \psi_1^k \\ \psi_2^k \end{pmatrix}$ is a two-component column Majorana spinor, $\bar{\theta} = \theta^T\gamma_0$, $\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$.

The general expressions for the action and the potentials of supersymmetric TTL are given in (1.1) and (1.2). From the action (1.1), one obtains the equations of motion

$$D_1 D_2 \Phi^j = i \sum_{s \in \tau} b^j_{\alpha_s} \exp 2(\varphi, \alpha_s) + \theta_1\theta_2 \sum_{s \bar{\in} \tau} b^j_{\alpha_s} \exp 2(\Phi, \alpha_s) , \tag{2.3}$$

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1 The notations of CLS are explained in the appendix

2 See for example [9], where the supersymmetry version of the grand unified theory was considered

where $b^j_{\alpha_s} = \frac{\alpha^j_s}{(\alpha_s, \alpha_s)}$, α^j_s is the j component of α_s. Using (2.2) one can rewrite Eq. (2.3) in components

$$\begin{aligned}
\partial^2_{\xi\eta}\varphi^j &= 2\sum_{s\in\tau} b^j_{\alpha_s} e^{2(\varphi,\alpha_s)}(e^{2(\varphi,\alpha_s)} + 2i(\psi_1,\alpha_s)(\psi_2,\alpha_s)) + \sum_{s\bar{\in}\tau} b^j_{\alpha_s} e^{2(\varphi,\alpha_s)}\,,\\
\partial_\xi \psi^j_1 &= -2\sum_{s\in\tau} b^j_{\alpha_s}(\psi^j_2,\alpha_s)e^{2(\varphi,\alpha_s)}\,,\\
\partial_\eta \psi^j_2 &= -2\sum_{s\in\tau} b^j_{\alpha_s}(\psi^j_1,\alpha_s)e^{2(\varphi,\alpha_s)}\,.
\end{aligned} \tag{2.4}$$

We specify now the type of potentials, introduçing the following notation:

$$V_k = -\frac{i}{2}\theta_1\theta_2 \sum_{j=1}^{k-1} \exp 2(\Phi^j - \Phi^{j+1})\,. \tag{2.5}$$

The data of the type of potentials corresponding to the list of superalgebras from Table 2 in appendix are summarized in Table 1.

Table 1

Types of superalgebras	Bosonic limit	$U(\Phi)$
1. $B(0,n)$	$B_n(n>1)$ $A_1(n=1)$	$V_n(\Phi) + \exp 2\Phi^n$
2. $B^{(1)}(0,n)$ $(n>1)$	$A^{(2)}_{2n}$	$V_n(\Phi) - \frac{i}{4}\theta_1\theta_2\exp(-4\Phi^1) + \exp 2\Phi^n$
3. $B^{(1)}(0,1)$	$A^{(1)}_1$	$-\frac{i}{4}\theta_1\theta_2\exp(-4\Phi) + \exp 2\Phi$
4. $A^{(2)}(0,2n-1)$ $(n>2)$	$A^{(2)}_{2n-1}$	$V_n(\Phi) - \frac{i}{2}\theta_1\theta_2\exp(-2(\Phi^1+\Phi^2)) + \exp 2\Phi^n$
5. $A^{(2)}(0,3)$	$D^{(2)}_2$	$-\frac{i}{4}\theta_1\theta_2(\exp(-\Phi^1-\Phi^2) + \exp(\Phi^1-\Phi^2)) + \frac{1}{2}\exp 2(\Phi^2)$
6. $C^{(2)}(n+1)$ $(n>1)$	$C^{(2)}_n$	$V_n(\Phi) + \exp 2\Phi^n + \exp(-2\Phi^1)$
7. $C^{(2)}(2)$	$A^{(1)}_1$	$\exp 2\Phi + \exp(-2\Phi)$
8. $A^{(4)}(0,2h)$ $(n>1)$	$C^{(1)}_n$	$V_n(\Phi) - i\theta_1\theta_2\exp(-2\Phi^1) + \exp 2\Phi^n$
9. $A^{(4)}(0,2)$	$A^{(2)}_2$	$\exp 2\Phi - i\theta_1\theta_2\exp(-2\Phi)$

All potentials except those corresponding to CLS $B(0,1)$ and $C^{(2)}(2)$ contain the terms with a $\theta_1\theta_2$ factor. As it follows from (2.2), the corresponding superfields Φ^k have only bosonic components φ^k. Thus the systems have only one superfield with a fermionic component. It corresponds to the unique odd root of CLS (see Table 2). The system $C^{(2)}(n+1)$ has two odd roots and consequently two superfields with a fermionic component.

The types of CLS defining $U(\Phi)$ are presented in the first column of Table 1. The second column contains the types of contragradient Lie algebras which are obtained in the bosonic limit. We use here the notations of [7].

It is worthwhile to note that the list of potentials corresponds to CLS with the positively defined bilinear form. Thus the kinetic term in (1.1) is positive.

Let us consider the systems with one and two superfields in terms of component fields:

1) $\boldsymbol{B(0,1)}$ – supersymmetric Liouville equation:

$$\partial^2_{\xi\eta}\varphi = 2e^{2\varphi}(e^{2\varphi}+2i\psi_1\psi_2)\,, \qquad \partial_\xi\psi_1 = -2\psi_2 e^{2\varphi}\,, \qquad \partial_\eta\psi_2 = -2e^{2\varphi}\psi_1\,,$$

2) $\boldsymbol{B^{(1)}(0,1)}$:

$$\partial^2_{\xi\eta}\varphi = 2e^{2\varphi}(e^{2\varphi}+2i\psi_1\psi_2) - \tfrac{1}{2}e^{-4\varphi}\,, \qquad \partial_\xi\psi_1 = -2\psi_2 e^{2\varphi}\,, \qquad \partial_2\psi_2 = -2\psi_1 e^{2\varphi}\,,$$

3) $\boldsymbol{C^{(2)}(2)}$ – supersymmetric Sinh-Gordon equation:

$$\partial^2_{\xi\eta}\varphi = 8\sinh 4\varphi - 16i\sinh -2\varphi(V)\psi_1\psi_2 \quad \begin{aligned} \partial_\xi\psi_1 &= -4\psi_2\cos\varphi \\ \partial_\eta\psi_2 &= -4\psi_1\cos\varphi\,, \end{aligned}$$

4) $\boldsymbol{A^{(4)}(0,2)}$:

$$\begin{aligned} &\partial^2_{\xi\eta}\varphi = 2e^{2\varphi}(e^{2\varphi}+2i\psi_1\psi_2) - e^{-2\varphi}\,, \\ &\partial_\xi\psi_1 = -2e^{2\varphi}_{\psi_2}\,, \qquad \partial_\eta\psi_2 = -2e^{2\varphi}\psi_1\,. \end{aligned}$$

The bosonic limit coincides with the so-called Bullough-Dodd equation [10]

5) $\boldsymbol{B^{(1)}(0,2)}$:

$$\begin{aligned} &\partial^2_{\xi\eta}\varphi^1 = \tfrac{1}{2}(e^{2(\varphi^1-\varphi^2)} - \tfrac{1}{4}e^{-4\varphi^1})\,, \qquad \partial_\xi\psi_1 = -2\psi_2 e^{2\varphi^2}\,, \qquad \partial_\eta\psi_2 = -2\psi_1 e^{2\varphi^2}\,, \\ &\partial^2_{\xi\eta}\varphi^2 = 2e^{2\varphi^2}(e^{2\varphi^2}+2i\psi_1\psi_2) - \tfrac{1}{2}e^{2(\varphi^1-\varphi^2)}\,, \end{aligned}$$

6) $\boldsymbol{A^{(2)}(0,3)}$:

$$\begin{aligned} \partial^2_{\xi\eta}\varphi^1 &= e^{2(\varphi^1-\varphi^2)}(e^{2(\varphi^1-\varphi^2)} + 2i\psi_1\psi_2) - \tfrac{1}{4}e^{-4\varphi^1}\,, \\ \partial^2_{\xi\eta}\varphi^2 &= -e^{2(\varphi^1-\varphi^2)}(e^{2(\varphi^1-\varphi^2)} + 2i\psi_1\psi_2) + \tfrac{1}{4}e^{4\varphi^2}\,, \\ \partial_\xi\psi_1 &= -2\psi_2 e^{2(\varphi^1-\varphi^2)}\,, \\ \partial_\eta\psi_2 &= -2\psi_1 e^{2(\varphi^1-\varphi^2)}\,, \end{aligned}$$

7) $\boldsymbol{C^{(2)}(3)}$:

$$\begin{aligned} &\partial^2_{\xi\eta}\varphi^1 = -2e^{-2\varphi^1}(e^{-2\varphi^1} + 2i\psi_1\psi_2) + \tfrac{1}{2}e^{2(\varphi^1-\varphi^2)}\,, \\ &\partial^2_{\xi\eta}\varphi^2 = 2e^{2\varphi^2}(e^{2\varphi^2}+2i\psi_1\psi_2) - \tfrac{1}{2}e^{2(\varphi^1-\varphi^2)}\,, \\ &\partial_\xi\psi^1_1 = -2\psi^1_2 e^{-2\varphi^1}\,, \qquad \partial_\xi\psi^2_1 = 2\psi^2_2 e^{2\varphi^2}\,, \\ &\partial_\eta\psi^1_2 = -2\psi^1_1 e^{-2\varphi^1}\,, \qquad \partial_\eta\psi^2_2 = 2\psi^2_1 e^{2\varphi^2}\,, \end{aligned}$$

8) $\boldsymbol{A^{(4)}(0,4)}$:

$$\begin{aligned} &\partial^2_{\xi\eta}\varphi^1 = \tfrac{1}{2}e^{2(\varphi^1-\varphi^2)} - e^{-2\varphi^1}\,, \\ &\partial^2_{\xi\eta}\varphi^2 = 2e^{2\varphi^2}(e^{2\varphi^2}+2i\psi_1\psi_2) - \tfrac{1}{2}e^{2(\varphi^1-\varphi^2)}\,, \\ &\partial_\xi\psi_1 = -2\psi_2 e^{2\varphi^2}\,, \\ &\partial_\eta\psi_2 = -2\psi_1 e^{2\varphi^2}\,. \end{aligned}$$

3. The Associated Linear Problem

According to the ISM it is necessary to represent the equations of motion (2.3) as an integrability condition for the system of linear equations

$$\begin{aligned} D_1\chi &= U\chi\,, \\ D_2\chi &= V\chi\,, \end{aligned} \tag{3.1}$$

where U and V are operators of a finite-dimensional representation π of $G^{(k)}$ connected with the system. For the sake of simplicity we shall denote the element of CLS and the operators of its representation by the same letters. Since D_1 and D_2 are odd, U and V must be also odd.

Because of the oddness of U, V the compatibility condition of (3.1) takes the form

$$D_2U+D_1V-\{U,V\}=0\,. \tag{3.2}$$

Proposition 1. *The equations of motion* (2.3) *are equal to the linear problem* (3.1), *where*

$$\begin{aligned} U &= U_0+\lambda U_1\,, \\ V &= \lambda^{-1}V_{-1}\,. \end{aligned} \tag{3.3}$$

(λ is the Lorentz spectral parameter)

$$\begin{aligned} U_0 &= -2(D_1\varrho^j)h_j\,, \\ U_1 &= -\theta_2\sum_{j\bar{\in}\tau} e_j+i\sum_{j\in\tau} e_j\,, \\ V_{-1} &= \theta_1\sum_{j\bar{\in}\tau}\exp(2\varrho^k a_{kj})f_j+\sum_{j\in\tau}\exp(2\varrho^k a_{kj})f_j\,. \end{aligned} \tag{3.4}$$

Here the superfield ϱ^k is related to Φ^k

$$\varrho^k a_{kj}=(\Phi,\alpha_j)\,, \tag{3.5}$$

h_j, e_j, f_j – are generators of CLS $G^{(k)}$ *with the Cartan matrix $\{a_{jk}\}$.*

Proof. The integrability condition (3.2) splits into two equations

$$D_1V_{-1}-\{U_0,V_{-1}\}=0\,, \tag{3.6}$$

$$D_2U_0-\{U_1,V_{-1}\}=0\,. \tag{3.7}$$

It follows from commutation rules (A.3) that (3.6) is identically satisfied, while (3.7) takes the form

$$2D_1D_2\varrho^j=\begin{cases}\theta_1\theta_2\exp(2\varrho^k a_{kj}) & j\bar{\in}\tau\\ -i\exp(2\varrho^k a_{kj}) & j\in\tau\,.\end{cases} \tag{3.8}$$

The equivalence (3.8) to (2.3) is provided by (3.5).

We shall now give the explicit forms of U and V using the representations of the superalgebras sl(m,n), osp$(1,2n)$ and osp$(2,2n)$ given in appendix and the data from Table 2. Note that U and V belong to the superalgebras pointed in brackets. These superalgebras are equal to those from the last column of Table 2.

1. $\boldsymbol{B(0,n)}$ **(osp(1, 2n)):**

$$U_0 = -2D_1 \operatorname{diag}(\Phi^1, \ldots, \Phi^n, 0, -\Phi^n, \ldots, -\Phi^1)$$

$$U_1 = \begin{pmatrix} A & \tilde{B} & 0 \\ 0 & 0 & B \\ 0 & 0 & -\tilde{A} \end{pmatrix} \quad \begin{array}{l} B = (\sqrt{2}i, 0 \ldots 0) \quad \tilde{B} = JB^T \\ J = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix} (A)_{jk} = -\delta_{j+1}, k\theta_2 \end{array}$$

$$\tilde{A} = JA^TJ \tag{3.9}$$

$$V_{-1} = \begin{pmatrix} A_1 & 0 & 0 \\ B_1 & 0 & 0 \\ 0 & -\tilde{B}_1 & -\tilde{A}_1 \end{pmatrix} \quad \begin{array}{l} (A_1)_{jk} = \theta_1 \delta_{j-1,k} \exp 2(\Phi^k - \Phi^{k+1}) \\ B_1 = (0, \ldots, 0, \sqrt{2} \exp 2\Phi^n)\,. \end{array}$$

2. $\boldsymbol{B^{(1)}(0,n)}$ **(osp(1, 2n)):**

$$U_0 = -2D_1 \operatorname{diag}(\Phi^1, \ldots, \Phi^n, 0, -\Phi^n, \ldots, -\Phi^1)$$

$$U_1 = \begin{pmatrix} A & \tilde{B} & 0 \\ 0 & 0 & B \\ C & 0 & -\tilde{A} \end{pmatrix} \quad V_{-1} = \begin{pmatrix} A_1 & 0 & C_1 \\ B_1 & 0 & 0 \\ 0 & -\tilde{B}_1 & -\tilde{A}_1 \end{pmatrix}$$

[A, B, A_1, B_1 see (3.9)]

$$(C)_{jk} = \theta_2 \delta_{j-n+1,k}\,, \qquad (C_1)_{jk} = \theta_1 \delta_{j,k-n+1} \exp(-4\Phi^1)\,.$$

3. $\boldsymbol{B^{(1)}(0,1)}$ **(osp(1, 2)):**

$$U_0 = -2D_1 \operatorname{diag}(\Phi, 0, -\Phi)$$

$$U_1 = \begin{pmatrix} 0 & \sqrt{2}i & 0 \\ 0 & 0 & \sqrt{2}i \\ \theta_2 & 0 & 0 \end{pmatrix} \quad V_{-1} = \begin{pmatrix} 0 & 0 & \theta_1 e^{-4\Phi} \\ \sqrt{2}e^{2\Phi} & 0 & 0 \\ 0 & \sqrt{2}e^{2\Phi} & 0 \end{pmatrix}.$$

4. $\boldsymbol{A^{(2)}(0,2n-1)}$ **(sl(1, 2n)):**

$$U_0 = -2D_1 \operatorname{diag}(\Phi^1, \ldots, \Phi^n, 0, -\Phi^n, \ldots, -\Phi^1)$$

$$U_1 = \begin{pmatrix} A & \tilde{B} & 0 \\ 0 & 0 & -B \\ C & 0 & -\tilde{A} \end{pmatrix} \quad (C)_{jk} = \delta_{j-n+2,k}\theta_2(-1)^k$$

$$V_{-1} = \begin{pmatrix} A_1 & 0 & C_1 \\ B_1 & 0 & 0 \\ 0 & \tilde{B}_1 & 0 \end{pmatrix} \quad (C_1)_{jk} = \theta_1 \delta_{j,k-n+2} \exp(-2(\Phi^1 + \Phi^2))(-1)^{j-1}$$

[A, B, A_1, B_1 see (3.9)].

5. $A^{(2)}(0,3)\,(\mathrm{sl}(1,4))$:

$$U_0=-2D_1\,\mathrm{diag}(\Phi^1,\Phi^2,0,-\Phi^2,-\Phi^1)$$

$$U_1=\begin{pmatrix}0 & -\theta_2 & 0 & 0 & 0\\ 0 & 0 & 2i & 0 & 0\\ 0 & 0 & 0 & 2i & 0\\ \theta_1 & 0 & 0 & 0 & \theta_2\\ 0 & -\theta_1 & 0 & 0 & 0\end{pmatrix}$$

$$V_{-1}=\begin{pmatrix}0 & 0 & 0 & \theta_1-e^{2(\Phi^1+\Phi^2)} & 0\\ \theta_1 e^{2(\Phi^1-\Phi^2)} & 0 & 0 & 0 & -\theta_1-e^{2(\Phi^1+\Phi^2)}\\ 0 & e^{2\Phi^2} & 0 & 0 & 0\\ 0 & 0 & -e^{2\Phi^2} & 0 & 0\\ 0 & 0 & 0 & -\theta_1 e^{2(\Phi^1-\Phi^2)} & 0\end{pmatrix}.$$

6. $C^{(2)}(n+1)\,(\mathrm{osp}(2,2n))$:

$$U_0=-2D_1\,\mathrm{diag}(\Phi^1,\ldots,\Phi^n,0,0-\Phi^n,\ldots,-\Phi^1)$$

$$U_1=\begin{pmatrix}A & \tilde{B} & \tilde{B} & 0\\ C & 0 & 0 & B\\ -C & 0 & 0 & B\\ 0 & \tilde{C} & -\tilde{C} & -\tilde{A}\end{pmatrix}$$

$$V_{-1}=\begin{pmatrix}A_1 & \tilde{B}_1 & -\tilde{B}_1 & 0\\ C_1 & 0 & 0 & -B_1\\ C_1 & 0 & 0 & B_1\\ 0 & -\tilde{C}_1 & -\tilde{C}_1 & -\tilde{A}_1\end{pmatrix} \tag{3.10}$$

$B=(i,0\ldots0)$, $\quad C=(-i,0\ldots0)$,

$B_1=(0\ldots0,e^{-2\Phi^1})\quad C_1=(0,\ldots,0,e^{2\Phi^n})$

[A, A_1 see (3.9)].

7. $C^{(2)}(2)$ $(\mathrm{osp}(2,2))$:

$$U_0=-2D_1\,\mathrm{diag}(\Phi,0,0,-\Phi)$$

$$U_1=-\begin{pmatrix}0 & i & i & 0\\ -i & 0 & 0 & i\\ i & 0 & 0 & i\\ 0 & -i & i & 0\end{pmatrix},\quad V_{-1}=\begin{pmatrix}0 & e^{-2\Phi} & -e^{-2\Phi} & 0\\ e^{2\Phi} & 0 & 0 & -e^{-2\Phi}\\ e^{2\Phi} & 0 & 0 & e^{-2\Phi}\\ 0 & -e^{2\Phi} & -e^{2\Phi} & 0\end{pmatrix}.$$

The associated linear equations for the Sinh-Gordon equation presented here differ from those in [5] where U and V are the 3×3 matrices.

8. $A^{(4)}(0,2n)$ $(\mathrm{sl}(1,2n+1))$:

The matrices U_0, U_1, V_{-1} are equal to (3.10)

$$C=(-\theta_2,0,\ldots,0),\quad B_1=(0,\ldots,0,\theta_1 e^{-2\Phi^1}),\quad \tilde{C}\to-\tilde{C}.$$

9. $A^{(4)}(0,2)$ **(sl(1,3)):**

$$U_0 = -2D_1 \operatorname{diag}(\Phi, 0, -\Phi, 0),$$

$$U_1 = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & \theta_1 \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & \theta_1 & 0 \end{pmatrix}, \quad V_{-1} = \sqrt{2}\begin{pmatrix} 0 & e^{2\Phi} & 0 & 0 \\ 0 & 0 & e^{2\Phi} & 0 \\ 0 & 0 & 0 & \theta_1 e^{-2\Phi} \\ \theta_1 e^{-2\Phi} & 0 & 0 & 0 \end{pmatrix}.$$

4. Reduction Group

1. The elements U and V of CLS defining the system have the very specific form (3.4). Therefore Eq. (3.2), which is equal to (2.3), allows a reduction to a subclass. In general we define reduction as the imposition of constraints on U and V which are compatible with Eq. (3.2). The main point is that reductions are defined by the group G_R of automorphisms of the algebra (see [11]). It should be noted that the information on the reduction group G_R is interesting for a number of reasons. First, the equations of motion can be derived directly from the whole superalgebra. We shall demonstrate it in our particular case. Second, the explicit calculations of conservation laws are simplified when one uses the reduction group [11]. Third, the reduction group imposes the limitation on the scattering data in the ISM [11].

Our purpose is to find the reduction group for the supersymmetric TTL. For the bosonic case it was calculated in [3].

Let h be a number defined by numerical marks γ_j on a Dynkin diagram of $G^{(k)}$

$$h = \sum_{j=0}^{n} \gamma_j, \quad [3] \tag{4.1}$$

and $\hat{Q}$ be an automorphism of $G^{(k)}$ which on generators $\{h_j, e_j, f_j\}$, $j=1, \ldots, n$, acts as follows

$$\begin{aligned} \hat{Q}(e_j) &= q^{-1} e_j, \\ \hat{Q}(h_j) &= h_j \quad \left(q = \exp\frac{2\pi i}{h}\right), \\ \hat{Q}(f_i) &= q f_j. \end{aligned} \tag{4.2}$$

Because of (A.8) and (A.11) the relations (4.2) are valid for h_0, e_0, and f_0. It is obvious that $\hat{Q}$ generates the cyclic group $\mathbb{Z}_h$.

Let U and V be arbitrary elements of the real Grassman envelope of $G^{(k)}$. We derive the system (3.2) using the constraints.

$$\begin{aligned} \hat{Q}U(\lambda) &= U(q\lambda), \\ \hat{Q}V(\lambda) &= V(q\lambda). \end{aligned} \tag{4.3}$$

It is worthwhile to notice that (4.3) is compatible with the linear equations (3.1). The following proposition generalizes the results of [3] to superalgebras.

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3 h is called the Coxeter number for $G^{(1)}$

Proposition 2. *Let U and V have the form* (3.3). *Then the relations* (4.3) *determine Eq.* (2.3) *of supersymmetric* TTL.

Proof. Because of (3.3) the action of $\hat{Q}$ (4.3) takes the form

$$\hat{Q}U_0 = U_0\,,$$
$$\hat{Q}U_1 = U_1 q^{-1}\,,$$
$$\hat{Q}V_{-1} = V_{-1} q\,.$$

In view of (4.2) U_0, U_1, and V_{-1} can be decomposed as follows:

$$U_0 = \sum_{j=0}^{n} A_j(\xi,\eta) h_j\,,$$

$$U_1 = \sum_{j=0}^{n} B_j(\xi,\eta) e_j\,,$$

$$V_{-1} = \sum_{j=0}^{n} C_j(\xi,\eta) f_j\,.$$

Substituting these expressions in (3.6) and (3.7) one gets $2n$ equations for $3n$ unknown functions.[4] The indeterminacy of the system is connected with the gauge freedom, preserving the form of the equations [11]. Fixing the gauge we put $B_j = \begin{cases} i & j \in \tau \\ -\theta_2 & j \bar{\in} \tau \end{cases}$, $A_j(\xi,\eta) = -2D_1 \varrho^j(\xi,\eta)$, and then arrive immediately at Eq. (3.8) which are equal to (2.3).

2. We shall discuss now the connection between the spectral parameter λ and the indeterminate x in the representation of CLS $G^{(k)}$ (A.7). Let v be an automorphism of $G^{(k)}$ which acts as (4.2) and q is equal to λ. By means of v one can define the transformation

$$\begin{aligned} &g(x) = \sum x^j L_j \to \tilde{g}(\lambda) = \sum \lambda^j U_j\,, \\ &\tilde{g}(\lambda) = v(g(\lambda^h))\,. \end{aligned} \tag{4.4}$$

Let $\{e_{\bar{\alpha}}\}$ be root spaces (A.10). Then in view of (4.2), the elements U_j in (4.4) are determined by the condition

$$U_j = \left\{ e_{\bar{\alpha}} | \bar{\alpha} = \sum_{\bar{\alpha}_s \in \Pi} k_s \bar{\alpha}_s,\ \sum k_s = j \right\}. \tag{4.5}$$

The decomposition (4.4) is connected with the $\mathbb{Z}$-grading of $G^{(k)}$ which is equal to the so-called $(1, \ldots, 1)$-grading in [7]. Thus the reduction group G_R determines homogeneous elements of $G^{(k)}$ in the sense of this grading.

5. Concluding Remarks

1. It follows from the definition of $U(\Phi)$ (1.2) that in all systems except $B(0,n)$ the potential energy of the bosonic part takes a minimal value. Let us consider the

4 It is necessary to take into account the linear dependence of the basis $\{h_j, j = 0, \ldots, n\}$. Thus in spite of the $3n+3$ functions $A_0, A_1 \ldots C_n$ there are only $3n$ linear independent ones

problem of calculation the mass spectrum of the vacuum excitations, i.e. the states near the minimum. In the bosonic (nonsupersymmetric) case for algebras with the trivial automorphism v ($k=1$) [see (A.6)], this problem has been solved in [3]. Because of the interaction between the bosonic and the fermionic components of superfields (2.4), formulae for the vacuum states of the bosonic components are more complicated in comparison with the bosonic case. Nevertheless the mass spectrum $\{m_j^2\}$ of bosonic components is described by the same formulae as in the bosonic case [3]. Namely m_j^2 are equal to the eigenvalues of the matrix

$$\tilde{\Omega}=2N\Omega\,, \tag{5.1}$$

where $N=\prod_{s\in I}(\gamma_s)^{-\frac{\gamma_s}{h}}$ and

$$\Omega_{jk}=\sum_{s\in I}\gamma_s\alpha_s^j\alpha_s^k\,. \tag{5.2}$$

These formulae are valid for all CLS.

It is remarkable that Ω is defined by the Dynkin diagram of the corresponding CLS. Comparing the Dynkin diagrams considered in [3] and those in Table 2, one gets the bosonic mass spectrum for the $A^{(2)}(0,2n-1)$ system

$$m_j^b=(2N\mu_j)^{1/2}\,,\qquad \mu_j=8\sin^2\left(\frac{2\pi}{n}j\right),\qquad j=1,\ldots,n-1\,;\qquad \mu_n=2\,.$$

It is natural to suppose that the eigenvalues of Ω have simple dependence on the exponents of CLS $G^{(k)}$ [5], the order k and h.

As it follows from (2.4) the vacuum states of fermionic components are equal to zero, while their masses are determined by the vacuum states $\bar{\varphi}$ of the bosonic components. For example, the mass of the fermionic component for the systems $B^{(1)}(0,1)$ and $A^{(4)}(0,2)$ is equal to $2\exp(2\bar{\varphi})$.

2. The recurrent procedure for the conservation laws calculation of the bosonic TTL was proposed in [3]. It can be directly generalized to the supersymmetric TTL. The conserved supercurrents have been calculated for the supersymmetric Sinh-Gordon equation [17]. It is worthwhile to notice that there are gaps in the sequence of conservation laws. The invariant meaning of the gaps for the generalized K-dV equations connected with contragradient Lie algebras had been explained in [12]. For the systems considered here we propose the following conjecture. The gaps are periodic with the period $k\cdot h$, where k is the order of the automorphism v. The orders of the nontrivial conserved supercurrents are equal to exponents of $G^{(k)}+k\cdot h\cdot n$ ($n\in\mathbb{Z}$).

3. Assuming in (2.4) $\xi=-\eta$, one gets ordinary differential equations. The system is described by the action which can be obtained from S (1.1) after the integration over the Grassmann variables θ_1,θ_2

$$S=\int dt\left[i\sum_{j\in I}\varphi^j\ddot{\varphi}^j+\sum_{j\in\tau}(\psi_1^j\dot{\psi}_1^j+\psi_2^j\dot{\psi}_2^j)+U(\varphi,\psi)\right],$$

where the dot stands for the time derivative. It is a particular case of systems considered in [13]. The system is the one-dimensional TL connected with CLS. In

5 The definition of exponents is given in [7]

contrast with the bosonic case, the particles corresponding to the odd roots acquire the spin degrees of freedom (ψ_1^j, ψ_2^j).

Nevertheless it is natural to conjecture that a number of well-known features of bosonic TL can be extended to supersymmetric TL. It implies the existence of additional integrals of motion both in the classical and quantum cases, the interpretation of phase superspaces as "superorbits", the explicit form of solutions for the nonperiodic system $B(0,n)$ and its connection with geodesic flows on a supermanifold.

Appendix. Contragradient Lie Superalgebras

1. A superalgebra Lie G is a $\mathbb{Z}_2$ graded linear space $G = G_{\bar{0}} + G_{\bar{1}}$. The elements of $G_{\bar{0}}$ are called even, those of $G_{\bar{1}}$ are called odd. The commutator in G satisfies the following axioms:

$$[a,b] = -(-1)^{\deg a \cdot \deg b}[b,a],$$
$$[a,[b,c]] = [[a,b],c] + (-1)^{\deg a \cdot \deg b}[b[a,c]],$$

where $\deg a = 0$ if $a \in G_{\bar{0}}$ and $\deg a = 1$ if $a \in G_{\bar{1}}$.

Two important examples are of interest.

a) The superalgebra $l(m,n)$ of the linear transformation of a linear $\mathbb{Z}_2$ graded space $V = V_{\bar{0}} + V_{\bar{1}}$ $(\dim V_{\bar{0}} = m, \dim V_{\bar{1}} = n)$. The elements of $l(m,n)$ are $(m+n)\times(m+n)$ matrices which can be written in the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $G_{\bar{0}} = \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right\}$ are even elements and $G_{\bar{1}} = \left\{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\right\}$ are odd ones. The superalgebra $\mathrm{sl}(m,n)$ is the subset of $l(m,n)$ satisfying the condition $\mathrm{tr}\, a = \mathrm{tr}\, d$.

b) The orthogonal-symplectic superalgebra $\mathrm{osp}(m,2n)$ is a subalgebra of $l(m,2n)$, which can be identified with the superalgebra of the canonical transformation of superspace V. The even part $G_{\bar{0}}$ which is equal to $\mathrm{so}(m) \oplus \mathrm{sp}(2n)$ defines the canonical transformation in the even and the odd subspaces of V separately. The odd part $G_{\bar{1}}$ mixes the even and the odd variables of V. We need in what follows the explicit forms of $\mathrm{osp}(1,2n)$ and $\mathrm{osp}(2,2n)$. Let J be the $(n \times n)$ matrix of the form $\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$ and $\tilde{A} = JA^T J$. There is a basis in V in which the superalgebras have the form

$\mathrm{osp}(1,2n)$

$$\begin{pmatrix} A & \tilde{f} & B \\ g & 0 & f \\ C & -\tilde{g} & -\tilde{A} \end{pmatrix} \quad \begin{matrix} B = \tilde{B} \quad C = \tilde{C} \\ f = (f_1, \ldots, f_n) \quad g = (g_1, \ldots, g_n), \end{matrix}$$
$$G_{\bar{0}} = \left\{\begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ C & 0 & -\tilde{A} \end{pmatrix}\right\} \quad G_{\bar{1}} = \left\{\begin{pmatrix} 0 & \tilde{f} & 0 \\ g & 0 & f \\ 0 & -\tilde{g} & 0 \end{pmatrix}\right\}, \tag{A.1}$$

and osp(2, 2n)

$$\begin{pmatrix} A & \tilde{X} & \tilde{X}_1 & B \\ Y_1 & \alpha & 0 & X_1 \\ Y & 0 & -\alpha & X \\ C & -Y & -\tilde{Y}_1 & -\tilde{A} \end{pmatrix} \qquad \begin{aligned} &B=\tilde{B}, \quad C=\tilde{C} \\ &X=(x^1,\dots,x^n) \\ &Y=(y^1,\dots,y^n), \end{aligned}$$

$$G_0=\left\{\begin{pmatrix} A & 0 & 0 & B \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ C & 0 & 0 & -\tilde{A} \end{pmatrix}\right\} \quad G_{\bar{1}}=\left\{\begin{pmatrix} 0 & X & \tilde{X}_1 & 0 \\ Y_1 & 0 & 0 & X_1 \\ Y & 0 & 0 & X \\ 0 & -\tilde{Y} & -\tilde{Y}_1 & 0 \end{pmatrix}\right\}. \tag{A.2}$$

2. A contragradient Lie superalgebra (CLS) $G(A,\tau)$ is defined by means of generators $(h_j, e_j, f_j) j\in I=(0,1,\dots,n)$. Let τ be a subset of I. The even generators are $(h_j, j\in I)(e_j, f_j, j\bar{\in}\tau)$ and the odd ones are $(e_j, f_j, j\in\tau)$. The defining relations in $G(A,\tau)$ are described by means of the Cartan matrix $A=(a_{ij})$ $(i,j\in I)$

$$\begin{aligned} &[e_j, f_k]=\delta_{jk}h_j, \quad [h_j, h_k]=0, \\ &[h_i, e_j]=a_{ij}e_j, \quad [h_i, f_j]=-a_{ij}f_j, \\ &(\mathrm{ad}\, e_i)^{1-a_{ij}}e_j=\underbrace{[e_i[e_i\dots[e_i, e_j]]}_{1-a_{ij}}=0\,(\mathrm{ad}\, f_i)^{1-a_{ij}}f_j=0\,. \end{aligned} \tag{A.3}$$

The Cartan matrix has the following properties

$$\begin{aligned} &a_{ij}\in\mathbb{Z}, \quad a_{ii}=2, \\ &a_{ij}\leqq 0 \quad (i\neq j), \quad a_{ij}=0\rightarrow a_{ji}=0. \end{aligned} \tag{A.4}$$

The simple roots $\bar{\alpha}$ of the superalgebra $G(A,\tau)$ are linear forms on the linear span H of the vectors h_i $(i\in I)$ defined by the relations

$$\bar{\alpha}_i(h_j)=a_{ji}\,. \tag{A.5}$$

The set of simple roots Π is the union of the subset of the even roots $\Pi_{\bar{0}}=\{\bar{\alpha}_i, i\bar{\in}\tau\}$ and the odd roots $\Pi_{\bar{1}}=\{\bar{\alpha}_i, i\in\tau\}$.

The CLS can be described by its Dynkin diagram consisting of $n+1$ nodes corresponding to the simple roots $\{\bar{\alpha}_j\}$. The j^{th} node is white if α_j is even and black if α_j is odd. The i^{th} and j^{th} nodes are connected by $\max(|a_{ij}|, |a_{ji}|)$ segments. If $|a_{ij}| > |a_{ji}|$, the segment has an arrow pointing toward the i^{th} node.

If $\tau=\emptyset$ we have the affine Lie algebras [8] connected with the usual TTL. In Table 2 the Dynkin diagrams of CLS $(\tau\neq\emptyset)$ are represented. This list of superalgebras contains only those which have a non-negative invariant bilinear form (see [7, Proposition 1.5, IV]).

The numerical marks γ_j near nodes are coefficients of linear dependence between the corresponding columns of the Cartan matrix A. Note that the set $C=\left\{\sum_{j\in I}\gamma_j h_j\right\}$ is the center of $G(A,\tau)$. The CLS $B(0,n)=\mathrm{osp}(1,2n)$ is finite-dimensional with a trivial center, while the others are infinite-dimensional. The factors $G^{(k)} = G(A, \tau)/C$ can be derived as follows.

Table 2

	Notation	Dynkin diagrams	L_0	L
1	2	3	4	5
1.	$B(0, n)$	$\circ - \circ - \dots - \circ \Rightarrow \bullet$		osp $(1, 2n)$
2.	$B^{(1)}(0, n)$ $(n > 1)$	$\overset{1}{\circ} \Rightarrow \overset{2}{\circ} - \dots \overset{2}{\circ} \Rightarrow \overset{2}{\bullet}$	osp $(1, 2n)$	osp $(1, 2n)$
3.	$B^{(1)}(0, 1)$	$\overset{1}{\circ} \Rrightarrow \overset{2}{\bullet}$	osp $(1, 2)$	osp $(1, 2)$
4.	$A^{(2)}(0, 2n-1)$ $(n > 2)$	$\begin{smallmatrix} \circ^{1} \\ \circ_{1} \end{smallmatrix} > \overset{2}{\circ} - \dots \overset{2}{\circ} \Rightarrow \overset{2}{\bullet}$	osp $(1, 2n)$	sl $(1, 2n)$
5.	$A^{(2)}(0, 3)$	$\overset{1}{\circ} \Rightarrow \overset{2}{\bullet} \Leftarrow \overset{1}{\circ}$	osp $(1, 2)$	sl $(1, 4)$
6.	$C^{(2)}(n+1)$ $(n > 1)$	$\overset{1}{\bullet} \Leftarrow \overset{1}{\circ} - \dots \overset{1}{\circ} \Rightarrow \overset{1}{\bullet}$	osp $(1, 2n)$	osp $(2, 2n)$
7.	$C^{(2)}(2)$	$\overset{1}{\circ} = \overset{1}{\bullet}$	osp $(1, 2)$	osp $(2, 2)$
8.	$A^{(4)}(0, 2n)$ $(n > 1)$	$\overset{1}{\circ} \Leftarrow \overset{1}{\circ} - \dots \overset{1}{\circ} \Rightarrow \overset{1}{\bullet}$	osp $(1, 2n)$	sl $(1, 2n+1)$
9.	$A^{(4)}(0, 2)$	$\overset{1}{\circ} = \overset{1}{\bullet}$	osp $(1, 2)$	sl $(1, 3)$

Let L be a superalgebra in the last column in Table 1. It has the automorphism v induced by an isometry of order k of its Dynkin diagram. Then we have the direct decomposition

$$L = \bigoplus_{j=0}^{k-1} L_j, \tag{A.6}$$

where L_j is the eigenspace of v. Note that L_0 is the subalgebra of L. The Dynkin diagram of L_0 coincides with the one of $G(A, \tau)$ in Table 1 without the left node. The factor algebra $G^{(k)} = G(A, \tau)/C$ can be represented as the superalgebra of Laurent polynomials in the even indeterminate x with coefficients in L:

$$G^{(k)} = \{\sum x^j L_{j(\mathrm{mod}\, k)}\}. \tag{A.7}$$

Let $\Pi' = \{\alpha_j, j = 1, \dots, n\}$ be a set of simple roots of L_0. We put

$$\theta = \sum_{j=1}^{n} \gamma_j \alpha_j. \tag{A.8}$$

The set of pairs

$$\{(-\theta, 1), (\alpha_1, 0), \dots, (\alpha_n, 0)\} \tag{A.9}$$

is equivalent to the set of simple roots; Π has been defined earlier (A.5). An arbitrary root of $G^{(k)}$ is a pair $\bar{\alpha}=(\alpha, m)$, where $m\in\mathbb{Z}$. This definition is compatible with the one of the root subspace $e_{\bar{\alpha}}\subset G^{(k)}$. Let H_0 be a Cartan subalgebra of L_0. Then

$$e_{\bar{\alpha}}=\{X\in L_{m(\mathrm{mod}\,k)}|[h,X]=\alpha(h)X, h\in H_0\}. \tag{A.10}$$

In particular the images in $G^{(k)}$ of the generators f_0 and e_0 have the form

$$e_0\to e_{(-\theta,1)}=xe_{-\theta}; \quad f_0\to e_{(\theta,-1)}=x^{-1}e_\theta. \tag{A.11}$$

As was mentioned above, there is a bilinear, invariant, non-negative form on $G(A,\tau)$ which is nondegenerate on $G^{(k)}$. [Its kernel coincides with the center $C\subset G(A,\tau)$.] By means of this form, the Cartan matrix can be represented as follows:

$$A=\left\{a_{ij}=\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}\right\}. \tag{A.12}$$

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References

1. Bogoyavlensky, O.I.: On perturbations of the periodic toda lattice. Commun. Math. Phys. **51**, 201–209 (1976)
2. Leznov, A.N., Saveliev, M.A.: Representation of zero curvature for the system of nonlinear partial differential equations $x_{\alpha,z,\bar{z}}=\exp(kx)_\alpha$ and its integrability. Lett. Math. Phys. **3**, 489–494 (1979)
3. Mikhailov, A.V., Olshanetsky, M.A., Perelomov, A.M.: Two-dimensional generalized toda lattice. Commun. Math. Phys. **79**, 473–488 (1981)
4. Girardello, L., Sciuto, S.: Inverse scattering-like problem for supersymmetric models. Phys. Lett. **77**B, 267–269 (1978)
5. Chaichian, M., Kulish, P.: On the method of inverse scattering problem and Bäcklund transformations for supersymmetric equations. Phys. Lett. **78**B, 413–416 (1978)
6. Kac, V.G.: Adv. Math. **26**, 8–96 (1977)
7. Kac, V.G.: Adv. Math. **30**, 85–136 (1978)
8. Kac, V.G.: Math. USSR Izv. **2**, 1271–1311 (1968)
9. Dimopoulos, S., Georgi, H.: Softly broken supersymmetry and SU(5). Nucl. Phys. **193**B, 150–162 (1981)
10. Dodd, R.K., Bullough, R.K.: Polynomial conserved densities for sine-Gordon equations. Proc. R. Soc. London A**352**, 481–490 (1977)
11. Mikhailov, A.V.: The reduction problem and the inverse scattering method. Report at the Soviet-American Symposium on Soliton Theory (Kiev, September, 1979). Physica **3**D, 73–117 (1981)
12. Drinfeld, V.G., Sokolov, V.V.: DAN **281**, 11–16 (1981)
13. Casalbuoni, R.: The classical mechanics for Bose-Fermi systems. Nuovo Cimento **33**A, 389–431 (1976)
14. D'Auria, R., Sciuto, S.: Group theoretical construction of two-dimensional supersymmetry models. Nucl. Phys. **171**B, 189–210 (1980)
15. Ferrara, S., Girardello, L., Sciuto, S.: Preprint TH 2474-CERN 1978

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