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Bounds on the Decay of Correlations for $\lambda(\nabla\phi)^4$ Models*

Jean-Raymond Fontaine

Institut de Physique Théorique, Université Catholique de Louvain, Chemin du Cyclotron, 2, B-1348 Louvain-la-Neuve, Belgium

Abstract. We consider models, with an abelian continuous group of symmetry, of the type:

$$H=\sum_x\left[\frac{1}{2}(\nabla_x\phi)^2+\frac{\lambda}{4}(\nabla_x\phi)^4\right].$$

We generalize Brascamp-Lieb inequalities to get (λ-independent) bounds on the low momentum behaviour of general correlation functions when these are truncated into two clusters. We then use this result to derive an asymptotic expansion (up the second order in λ) of the dielectric constant of this system.

I. Introduction

In this paper we consider perturbations of the massless Gaussian lattice field which preserve its abelian continuous group of symmetry. Our results mainly concern the following model of an anharmonic crystal (defined on $\mathbb{Z}^d$)

$$H=1/2\sum_x(\nabla_x\phi)^2+\lambda/4(\nabla_x\phi)^4.$$

The main particularity of this model is that its correlations, e.g. $\langle\nabla_0\phi\nabla_x\phi\rangle$, are nonintegrable ($d>1$). A lot of work is now being done in order to understand critical properties of classical lattice systems [1–6]. This model, one of the simplest nonexplicitly soluble critical models, has been investigated quite a lot. Two approaches have been developed so far. One is based on a rigorous version of renormalization group ideas and has been considered by [3]. The results produced mainly concern weak coupling (λ small). Another approach based on nonperturbative methods, producing therefore results which are not sensitive to the strength of λ, was proposed in [4–6]. In this note we want to develop further the second approach to get more detailed information about the decay of correlations.

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Before describing the results in detail, we start by defining precisely the system we consider.

II. Definition and Properties of the Model

At each point $i \in \mathbb{Z}^d$, is associated a random variable $\phi_i \in \mathbb{R}$. The finite volume Hamiltonian H_Λ (Λ is a parallelipiped) is given by:

$$H_\Lambda = \tfrac{1}{2} \sum_{\langle i,j\rangle \subset \Lambda} (\phi_i - \phi_j)^2 + \lambda/4 \sum_{\langle i,j\rangle \subset \Lambda} (\phi_i - \phi_j)^4 ; \tag{1}$$

$\langle i,j\rangle$ means that i and j are nearest neighbor in $\mathbb{Z}^d$, or that i and j are at the opposite ends of Λ (periodic boundary conditions).

We also consider

$$H_{\Lambda,m} = H_\Lambda + m^2/2 \sum_{i \in \Lambda} \phi_i^2 .$$

It is convenient to introduce difference variables: Let e_α, $\alpha = 1, \ldots, d$, be a basis of $\mathbb{Z}^d$ given by $e_\alpha = \delta_{i\alpha}$; $\phi_i - \phi_{i+e_\alpha}$ will be denoted by $\nabla_i^{e_\alpha}\phi$; $\sum\limits_{e_\alpha, \alpha=1,\ldots,d}$ will be denoted by $\sum\limits_e$ or $\sum\limits_\xi$. For $m \neq 0$, expectation values of functions of the type $\prod\limits_{i \in A} \phi_i$ are defined via:

$$\langle f \rangle_{\Lambda,m} = Z_{\Lambda,m}^{-1} \int_{\mathbb{R}^{|\Lambda|}} f \exp - H_{\Lambda,m} \prod_{i \in \Lambda} d\phi_i ,$$

$$Z_\Lambda = \int_{\mathbb{R}^{|\Lambda|}} \exp - H_{\Lambda,m} \prod_{i \in \Lambda} d\phi_i .$$

Here A is a "subset" of Λ; in order to avoid exponents we allow repetitions of the same element in A, which is why we used the word subset in quotation marks.

As in [4] we can use Brascamp-Lieb inequalities [7, 8] to take the limits (possibly via subsequences):

$$\lim_{m \downarrow 0} \lim_{\Lambda \to \infty} \langle f \rangle_{\Lambda,m} \quad \text{if} \quad d \geq 3 .$$

For $d < 3$ those limits can be obtained if f is of the type $\prod\limits_{(i,\xi) \in B} \nabla_i^\xi \phi$, where B is a "subset" of $\Lambda x \{e_1, \ldots, e_d\}$. All results we are going to describe are true for any limiting state $\langle\ \rangle$ we choose.

If f is a real L^2-function defined on $\mathbb{Z}^d$, its Fourier transform is defined by:

$$\hat{f}(p) = \sum_x f(x) \exp(ip \cdot x) .$$

The Fourier transform of the gradient-gradient two point function $\langle \nabla_0^e \phi \nabla_x^e \phi \rangle$, $S^{ee}(p)$, obeys the following bounds:

$$(2 - 2\cos p_e)\left[(1 + 3\lambda\langle(\nabla_0^e\phi)^2\rangle)\left(2\sum_e (1 - \cos p_e)\right)\right]^{-1}$$
$$\leqq S^{ee}(p) \leqq (2 - 2\cos p_e)\left[2\sum_e (1 - \cos p_e)\right]^{-1} . \tag{2}$$

The lower bound results from a Mermin-Wagner argument [9, 10] and the upper bound is the Brascamp-Lieb inequality. As remarked by [10, 11] these bounds imply that $S^{ee}(p)$ is not continuous at the origin and therefore that $\langle \nabla_0^e \phi \nabla_x^e \phi \rangle$ cannot be absolutely integrable. This is the main interest of the model. The bound (2) shows that the singularity for small p of the two point function $S(p) = \langle \phi_0 \phi_x \rangle\hat{}(p)$ $(d \geqq 3)$ is exactly the same as in the purely Gaussian case: it is given by $\varepsilon \left[2 \sum_e (1 - \cos p_e)\right]^{-1}$, where the dielectric constant ε obeys the bounds

$$[1 + 3\lambda \langle (\nabla_0^e \phi)^2 \rangle]^{-1} \leqq \varepsilon \leqq 1 .$$

This suggests that the decay of $\langle \phi_0 \phi_x \rangle$ should be like $\frac{\varepsilon}{|x|}$. For technical reasons, we could only prove in [4, II] the bound $|\langle \phi_0 \phi_x \rangle| \leqq \text{const} \frac{\log |x|}{|x|}$. In order to avoid difficulties in translating momentum space bounds into "x-space" bounds, all the results we are going to describe are expressed in p-space.

III. The Results

As suggested by renormalization group arguments, we expect the large distance behaviour of general correlation functions to be Gaussian. We therefore expect bounds of the type (2) to be true for general correlation functions. In other words, given a correlation function, we expect that its Fourier transform is bounded from above and from below by its value computed into two Gaussian theories with different dielectric constants. Our first theorem goes in that direction: it gives momentum space bounds on general correlation functions when these are truncated into two clusters.

Our second result is valid for weak coupling only. We derive the first correction in λ to the Gaussian singularity of $S(p)$ at $p=0$. This may be viewed as an asymptotic expansion (up the second order in λ), of the dielectric constant ε.

We now introduce the notations: if A is a "subset" of Λ, and B is a "subset" of $\Lambda x \{e_1, \ldots, e_d\}$,

$$A + x \equiv \{ i + x | i \in A \} \quad \text{and} \quad B + x = \{ (i + x, \zeta) | (i, \zeta) \in B \} ,$$

δ will denote an integral operator on $L^2(\mathbb{R}^d)$ $(d > 3)$ of kernel:

$$(2\pi)^{-3} \int \exp(ip \cdot x) \mu(p) d^d p ,$$

$$\mu(p) = (-\Delta)^{-1}(p) * (-\Delta)^{-1}(p), \ * \text{means convolution product} ,$$

$$(-\Delta)^{-1}(p) = \left[2 \sum_e (1 - \cos p_e)\right]^{-1} ,$$

$$\langle A ; B \rangle \equiv \langle AB \rangle - \langle A \rangle \langle B \rangle .$$

In what follows c will denote a constant which can take different values at different places.

Theorem 1. *For any f in $L^2(\mathbb{Z}^d)$ we have:*

a) $0 \leqq \sum_{x,y} f(x) f(y) \left\langle \prod_{i \in A + x} \phi_i ; \prod_{j \in A + y} \phi_j \right\rangle$

$\leqq c(f, Df)$,

where $D=(-\Delta)^{-1}$ *if* $|A|$ *(the cardinality of A) is odd and* $D=\delta$ *if* $|A|$ *is even;* $d\geqq 3$.

b) $0\leqq \sum_{x,y} f(x)f(y)\left\langle \prod_{(i,\xi)\in B+x} (\nabla_i^\xi\phi); \prod_{(i,\xi)\in B+y} (\nabla_i^\xi\phi)\right\rangle$

$\leqq c\|f\|_{L^2}$.

Theorem 2. *In any dimension d,*

$$\langle \nabla_0^e\phi\nabla_x^e\phi\rangle\hat{}(p)=2(1-\cos p_e)\left[2\sum_e(1-\cos p_e)\right]^{-1}$$
$$\cdot[1-3\lambda w+\lambda^2 c(p,\lambda)],$$

where

$$w=\langle(\nabla_0^e\phi)^2\rangle_{\lambda=0} \quad \text{and} \quad c(p,\lambda)\leqq c$$

uniformly in λ *and p.*

Remarks. 1. Theorem 1 implies the bound

$$0\leqq\langle\phi_0^{2n+1}\phi_x^{2n+1}\rangle\hat{}(p)\leqq c_1\left[2\sum_e(1-\cos p_e)\right]^{-1}.$$

On the other hand, a Mermin-Wagner argument (see [11, 4, II]) gives

$$\langle\phi_0^{2n+1}\phi_x^{2n+1}\rangle\hat{}(p)\geqq c_2\left[2\sum_e(1-\cos p_e)\right]^{-1}.$$

We therefore know exactly the low momentum behavior of those correlations. A Mermin-Wagner argument should also give

$$\langle\phi_0^{2n};\phi_x^{2n}\rangle\hat{}(p)\geqq c_2\mu(p),$$

but we have not checked this in detail.

2. All our results are also true in the case where $\lambda/4(\nabla\phi)^4$ is replaced by an even polynomial in $(\nabla\phi)$ with positive coefficients. In that case, w in Theorem 2 equals $\langle P''(\nabla\phi)\rangle_{\lambda=0}$.

3. For $d\geqq 3$ Theorem 2 implies

$$\langle\phi_0\phi_x\rangle\hat{}(p)=\left[2\sum_e(1-\cos p_e)\right]^{-1}[1-3\lambda w+\lambda^2 c(p,\lambda)].$$

IV. The Proofs

The proof of Theorem 1 is by induction on $|A|$ or $|B|$. The only ingredients of the induction step are Brascamp-Lieb inequalities and Schwartz' inequality. We also make use of duplicate variables, so let us introduce some notations.

Consider the unnormalized density

$$\exp\sum_{i,\xi}[1/2(\nabla_i^\xi\phi)^2-\lambda/4(\nabla_i^\xi\phi)^4-1/2(\nabla_i^\xi\phi')^2-\lambda/4(\nabla_i^\xi\phi')^4], \tag{3}$$

where ϕ' is a duplication of ϕ. Using the variables:

$$\psi_i^+=1/2(\phi_i+\phi_i'),$$
$$\psi_i^-=1/2(\phi_i-\phi_i'),$$

(3) becomes:

$$G(\psi^+,\psi^-)=\exp 2\sum_{i,\xi}[-1/2(\nabla_i^\xi\psi^+)^2-\lambda/4(\nabla_i^\xi\psi^+)^4 \\ -3/2\lambda(\nabla_i^\xi\psi^+)^2(\nabla_i^\xi\psi^-)^2-1/2(\nabla_i^\xi\psi^-)^2-\lambda/4(\nabla_i^\xi\psi^-)^4]. \tag{4}$$

Now $\langle\ \rangle$ will also denote the normalized measure associated with (4). We also use the notations:

$$H(\psi^+,\psi^-)=\exp 2\sum_{i,\xi}[-1/2(\nabla_i^\xi\psi^-)^2-\lambda/4(\nabla_i^\xi\psi^-)^4-3/2\lambda(\nabla_i^\xi\psi^-)^2(\nabla_i^\xi\psi^+)^2],$$

$$\prod_{i\in A}\phi_{i+x}\equiv\phi_x^A,\quad \prod_{(i,\xi)\in B}\nabla_{i+x}^\xi\phi\equiv\nabla_x\phi^B,$$

$$\prod_{i\in A}\psi_{i+x}^\pm\equiv\psi_x^{\pm A},\quad \prod_{(i,\xi)\in B}\nabla_{i+x}^\xi\psi^\pm\equiv\nabla_x\psi^{\pm B}.$$

We now start by proving Theorem 1b). The proof will follow from three lemmas.

Lemma 1.

$$0\leqq\sum_{x,y}f(x)f(y)\langle\nabla_x\psi^{+B}\nabla_y\psi^{+B}\nabla_{j+x}^e\psi^-\nabla_{j+y}^e\psi^-\rangle \\ \leqq\sum_{x,y}f(x)f(y)\nabla_x^e\nabla_y^e C(x,y)\langle\nabla_x\psi^{+B}\nabla_y\psi^{+B}\rangle, \tag{5}$$

where

$$\nabla_x^e\nabla_y^e C(x,y)=\nabla_x^e\nabla_y^e(-\Delta)^{-1}(x,y).$$

Proof. The positivity is immediate because the left hand side of (5) equals

$$\left\langle\left[\sum_x f(x)\nabla_x\psi^{+B}\nabla_{j+x}^e\psi^-\right]^2\right\rangle.$$

The left hand side of (5) can be written as:

$$\sum_{x,y}f(x)f(y)\int d\psi^+\nabla_x\psi^{+B}\nabla_y\psi^{+B}\{\int d\psi^-\nabla_{j+x}^e\psi^-\nabla_{j+y}^e\psi^- \\ \cdot H(\psi^+\psi^-)(\int d\psi^-H(\psi^+\psi^-))^{-1}\}\int d\psi^-G(\psi^+\psi^-)Z_\Lambda^{-1}. \tag{6}$$

We used the notation: $d\psi^\pm\equiv\prod_{i\in\Lambda}d\psi_i^\pm$, and $Z_\Lambda=\int d\psi^-d\psi^+G(\psi^+,\psi^-)$. Now for any configuration of ψ^+,

$$2\sum_{i,\xi}[\lambda/4(\nabla_i^\xi\psi^-)^4+3/2\lambda(\nabla_i^\xi\psi^+)^2(\nabla_i^\xi\psi^-)^2]$$

is convex. Therefore, using Brascamp-Lieb inequalities [7, 8], we have:

$$(6)\ \leqq\sum_{x,y}f(x)f(y)\nabla_x^e\nabla_y^e C(x,y)\int d\psi^+\nabla_x\psi^{+B}\nabla_y\psi^{+B} \\ \cdot\int d\psi^-G(\psi^+\psi^-)Z_\Lambda^{-1},$$

and this is the right hand side of (5). □

Lemma 2.

$$\langle\nabla_x\psi^{+B}\nabla_y\psi^{+B}\nabla_{j+x}^e\psi^-\nabla_{j+y}^e\psi^-\rangle \tag{7}$$

can be written as a sum of truncated correlation functions of the type:

$$\langle\nabla_x\phi^{B_0};\nabla_y\phi^{B_1}\rangle\langle\nabla_x\phi^{B_2};\nabla_y\phi^{B_3}\rangle$$

or

$$\langle \nabla_x \phi^{B_0}; \nabla_y \phi^{B_1} \rangle \langle \nabla_0 \phi^{B_2} \rangle \langle \nabla_0 \phi^{B_3} \rangle .$$

Proof. Going back to the ϕ, ϕ' variables, (7) will be a sum of terms of the type:

$$\langle \nabla_x \phi^C \nabla_x \phi'^D \nabla_y \phi^E \nabla_y \phi'^F \rangle , \tag{8}$$

where $C+D=B+(j,\xi)=E+F$. Exchanging $\nabla^\eta_{i+x}\phi$ and $\nabla^\eta_{i+x}\phi'$ in (7) for each $(i,\eta)\in B+(j,\xi)$ will produce, up to a minus sign, the same expansion in terms of the ϕ' and ϕ variables. Therefore if (8) belongs to the expansion of (7), then that expansion also contains:

$$-\langle \nabla_x \phi'^C \nabla_x \phi^D \nabla_y \phi^E \nabla_y \phi'^F \rangle . \tag{9}$$

Using the factorization of the measure (3), (8), and (9) equals:

$$\langle \nabla_x \phi^C \nabla_y \phi^E \rangle \langle \nabla_x \phi^D \nabla_y \phi^F \rangle - \langle \nabla_x \phi^D \nabla_y \phi^E \rangle \langle \nabla_x \phi^C \nabla_y \phi^F \rangle ,$$

and this can be rewritten as

$$\langle \nabla_x \phi^C \nabla_y \phi^E \rangle \langle \nabla_x \phi^D ; \nabla_y \phi^F \rangle + \langle \nabla_x \phi^D \rangle \langle \nabla_y \phi^F \rangle \langle \nabla_x \phi^C ; \nabla_y \phi^E \rangle$$
$$- \langle \nabla_x \phi^D \rangle \langle \nabla_y \phi^E \rangle \langle \nabla_x \phi^C ; \nabla_y \phi^F \rangle - \langle \nabla_x \phi^C \nabla_y \phi^F \rangle \langle \nabla_x \phi^D ; \nabla_y \phi^E \rangle .$$

Finally if we write

$$\langle \nabla_x \phi^C \nabla_y \phi^E \rangle = \langle \nabla_x \phi^C ; \nabla_y \phi^E \rangle + \langle \nabla_x \phi^C \rangle \langle \nabla_y \phi^E \rangle$$

and

$$\langle \nabla_x \phi^C \nabla_y \phi^F \rangle = \langle \nabla_x \phi^C ; \nabla_y \phi^F \rangle + \langle \nabla_x \phi^C \rangle \langle \nabla_y \phi^F \rangle ,$$

the lemma is proven. □

Lemma 3. *The Induction Step. Assume the validity of Theorem 1b) for all B with $|B|\leqq n$, then Theorem 1b) is true for all B with $|B|\leqq n+1$.*

Proof. We shall use the rotation $B(l)$ for a "subset" of $\Lambda\times\{e_1,\dots,e_d\}$ of cardinality l. If we rewrite (5) using the ϕ and ϕ' variables, the factorization of the measure associated with (3) and Lemma 2 imply:

$$\begin{aligned}
&\sum_{x,y} f(x)f(y)\Bigg\{\langle \nabla_x \phi^{B(n+1)}; \nabla_y \phi^{B(n+1)}\rangle + \sum_{l=1}^{n} \sum_{B(l)\subset B(n+1)} b(B(l))\langle \nabla_x \phi^{B(n+1)}; \nabla_y \phi^{B(l)}\rangle \\
&\quad + \sum_{l,m=1}^{n} \sum_{B(l),B(m)\subset B(n+1)} b(B(l),B(m))\langle \nabla_x \phi^{B(l)}; \nabla_y \phi^{B(m)}\rangle \\
&\quad + \sum_{l,m,s,t=1}^{n} \sum_{B(l),B(m),B(s),B(t)\subset B(n+1)} b(B(l),B(m),B(s),B(t)) \\
&\quad \cdot \langle \nabla_x \phi^{B(l)}; \nabla_y \phi^{B(m)}\rangle \langle \nabla_x \phi^{B(s)}; \nabla_y \phi^{B(t)}\rangle \Bigg\} \\
&\leqq \sum_{x,y} f(x)f(y)\nabla_x^e \nabla_y^e C(x,y)\Bigg\{ \sum_{l,m=1}^{n} \sum_{B(l),B(m)\subset B(n)} c(B(l),B(m)) \\
&\quad \cdot \langle \nabla_x \phi^{B(l)}; \nabla_y \phi^{B(m)}\rangle + \sum_{l,m,s,t=1}^{n-1} \sum_{B(l),B(m),B(s),B(t)\subset B(n)} c(B(l),B(m),B(s),B(t)) \\
&\quad \cdot \langle \nabla_x \phi^{B(l)}; \nabla_y \phi^{B(m)}\rangle \langle \nabla_x \phi^{B(s)}; \nabla_y \phi^{B(t)}\rangle + c\Bigg\},
\end{aligned} \tag{10}$$

where the coefficients $b(\ )$ and $c(\ )$ are numerical factors multiplied by x and y independent quantities of the type: $\langle \nabla\phi^C\rangle\langle \nabla\phi^D\rangle\langle \nabla\phi^E\rangle\langle \nabla\phi^F\rangle$, with $|C+D+E+F|\leqq n+1$.

The right hand side of (10) is a trivial rewriting of the right hand side of (5). We used:

$$\langle \nabla\phi^C \nabla\phi^D\rangle = \langle \nabla\phi^C ; \nabla\phi^D\rangle + \langle \nabla\phi^C\rangle\langle \nabla\phi^D\rangle .$$

By Brascamp-Lieb inequalities, $\exists$ a λ-independent constant $c(n+1)$ such that all the coefficients $b(\ )$ and $c(\ )$ are bounded by $c(n+1)$. Before going further we make three remarks.

R.1. By Schwartz inequality,

$$\Big|\sum_{x,y} f(x)f(y)\langle \nabla_x\phi^D ; \nabla_y\phi^E\rangle\Big|$$
$$\leqq \Big[\sum_{x,y} f(x)f(y)\langle \nabla_x\phi^D ; \nabla_y\phi^D\rangle\Big]^{1/2}\Big[\sum_{x,y} f(x)f(y)\langle \nabla_x\phi^E ; \nabla_y\phi^E\rangle\Big]^{1/2} .$$

R.2.

$$\sum_{x,y} f(x)f(y)h(x-y)g(x-y) = \int |\hat{f}(k)|^2(\hat{h}*\hat{g})(k)d^dk$$

R.3.

$$0\leqq \langle \nabla_0\phi^D ; \nabla_x\phi^D\rangle\hat{}(p)\leqq \text{const} \qquad \text{if } |D|\leqq n$$

by the induction hypothesis.

Using the induction hypothesis and R.1, R.2, and R.3, (10) becomes:

$$\begin{aligned}&\sum_{x,y} f(x)f(y)\langle \nabla_x\phi^{B(n+1)} ; \nabla_y\phi^{B(n+1)}\rangle \\ &\quad - \Big[\sum_{x,y} f(x)f(y)\langle \nabla_x\phi^{B(n+1)} ; \nabla_y\phi^{B(n+1)}\rangle\Big]^{1/2} c_1 \|f\|_{L_2} \\ &\leqq c_2\|f\|_{L_2}^2 ,\end{aligned} \tag{11}$$

where c_1 and c_2 are two λ-independent positive constants. Writing

$$\alpha^2 = \sum_{x,y} f(x)f(y)\langle \nabla_x\phi^{B(n+1)} ; \nabla_y\phi^{B(n+1)}\rangle ,$$

$$(11) \Rightarrow \alpha^2 - c_1\|f\|_{L_2}\alpha - c_2\|f\|_{L_2}^2 \leqq 0 .$$

This implies that $\alpha\|f\|_{L_2}^{-1}\leqq$ const, and this is the induction hypothesis for $B(n+1)$. □

Since the induction hypothesis is true for $B(1)$ (because it is Brascamp-Lieb inequality) Theorem 1b) is proven.

Proof of Theorem 1a). The proof of Theorem 1a) follows the lines of the proof of Theorem 1b). Lemmas 1 and 2 remain unchanged up to the fact that everywhere $\nabla\phi$ is replaced by ϕ and that $\nabla_x^e\nabla_y^e C(x,y)$ is replaced by $C(x,y)$ in (5). In R.1, $\nabla\phi$ is replaced by ϕ everywhere and R.3 becomes:

R.3'.

$$0\leqq \langle \phi_0^D\phi_x^D\rangle\hat{}(p)\leqq \text{const}\Big[2\sum_e(1-\cos p_e)\Big]^{-1} \quad \text{if } |D| \text{ is odd,}$$

and

$$0 \leqq \langle \phi_0^D ; \phi_x^D \rangle\hat{}(p) \leqq \text{const}\ \mu(p) \quad \text{if } |D| \text{ is even}.$$

Now to prove Lemma 3 in this case, that is to prove the validity of the induction hypothesis for $|A| \leqq n+1$ assuming it for all A with $|A| \leqq n$, we have to distinguish between $n+1$ even or odd.

a) Assume $n+1$ odd. Using Lemmas 1 and 2 and Remarks R.1, R.2, and R.3′, we have

$$\sum_{x,y} f(x)f(y)\langle \phi_x^{A(n+1)} \phi_y^{A(n+1)} \rangle - \left[\sum_{x,y} f(x)f(y)\langle \phi_x^{A(n+1)} \phi_y^{A(n+1)} \rangle\right]^{1/2} c_1(f,(-\Delta)^{-1}f)^{1/2}$$
$$\leqq c_2(f,(-\Delta)^{-1}f).$$

As in the proof of Lemma 3, this implies

$$\sum_{x,y} f(x)f(y)\langle \phi_x^{A(n+1)} \phi_y^{A(n+1)} \rangle \leqq \text{const}\,(f,(-\Delta)^{-1}f),$$

which is the induction hypothesis for $|A| = n+1$.

b) Assume $n+1$ even. Since $n+1$ is even, using $\langle \phi^{2l+1} \rangle = 0$, $b(B(l)) = 0$ for l odd, $b(B(l), B(m)) = 0$ for l or m odd, $c = 0$ in (10). Therefore using this and R.1, R.2, and R.3′, (10) becomes:

$$\sum_{x,y} f(x)f(y)\langle \phi_x^{A(n+1)} ; \phi_y^{A(n+1)} \rangle - \left[\sum_{x,y} f(x)f(y)\langle \phi_x^{A(n+1)} ; \phi_y^{A(n+1)} \rangle\right]^{1/2} c_1(f,\delta f)^{1/2}$$
$$\leqq c_2(f,\delta f).$$

Again this implies:

$$\sum_{x,y} f(x)f(y)\langle \phi_x^{A(n+1)} ; \phi_y^{A(n+1)} \rangle \leqq \text{const}\,(f,\delta f),$$

which is the induction hypothesis for A $(n+1)$. □

Proof of Theorem 2. It combines the integration by parts formula [12, 4] with Theorem 1b). The integration by parts formula gives:

$$\begin{aligned}\langle \nabla_0^e \phi \nabla_x^e \phi \rangle &= \nabla_0^e \nabla_x^e C(0,x) - \lambda \sum_{i,\xi} \nabla_0^e \nabla_i^\xi C(0,i) \langle \nabla_x^e \phi (\nabla_i^\xi \phi)^3 \rangle = \nabla_0^e \nabla_x^e C(0,x) \\ &\quad - 3\lambda \sum_{i,\xi} \nabla_0^e \nabla_i^\xi C(0,i) \nabla_x^e \nabla_i^\xi C(x,i) \langle (\nabla_i^\xi \phi)^2 \rangle \\ &\quad + \lambda^2 \sum_{i,\xi} \sum_{j,\eta} \nabla_0^e \nabla_i^\xi C(0,i) \nabla_x^e \nabla_j^\eta C(x,j) \langle (\nabla_i^\xi \phi)^3 (\nabla_j^\eta \phi)^3 \rangle . \end{aligned} \tag{12}$$

By integration by parts,

$$\begin{aligned}&\sum_{i,\xi} \nabla_0^e \nabla_i^\xi C(0,i) \nabla_x^e \nabla_i^\xi C(x,i) \\ &\quad = \sum_i (-\Delta(\nabla_0^e C(0,i))) \nabla_x^e C(x,i) \\ &\quad = \sum_i (\delta_{0i} - \delta_{ei}) \nabla_x^e C(x,i) \\ &\quad = \nabla_0^e \nabla_x^e C(0,x). \end{aligned} \tag{13a}$$

Using (13) and taking the Fourier transform of both sides of (12) we get:

$$S^{ee}(p)=[1-3\lambda\langle(\nabla_0^e\phi)^2\rangle][2-2\cos p_e]\left[2\sum_e(1-\cos p_e)\right]^{-1}$$
$$+\lambda^2\sum_{\xi,\eta}(\exp ip_\xi-1)(\exp ip_\eta-1)\left[2\sum_e(1-\cos p_e)\right]^{-1}B^{\xi,\eta}(p,\lambda), \qquad (13b)$$

where $B^{\xi,\eta}(p,\lambda)$ is the Fourier transform of $\langle(\nabla_0^\xi\phi)^3(\nabla_x^\eta\phi)^3\rangle$, which is well defined by Theorem 1b).

Now

$$(\exp ip_\xi-1)(\exp ip_\eta-1)\left[2\sum_e(1-\cos p_e)\right]^{-1}\leqq c,$$

and by Theorem 1b) and Schwartz' inequality,

$$B(p_\xi,p_\eta,\lambda)\leqq\text{const uniformly in } \lambda \text{ and } p.$$

Finally $\langle(\nabla\phi)^2\rangle$ can be replaced by $\langle(\nabla\phi)^2\rangle_{\lambda=0}$ because perturbation theory is asymptotic [4]. □

Remark. It is easy to generalize Theorems 1 and 2 to the case where $(\nabla\phi)^4$ is replaced by an even polynomial in $\nabla\phi$ with positive coefficients. The only change in the proof of Theorem 1 arises in the proof of Lemma 1. Assume

$$\lambda P(\nabla\phi)=\lambda\sum_{p=1}^{n}a_p(\nabla\phi)^{2p}a_p\geqq 0 \;\;\forall p.$$

Then $H(\psi^+,\psi^-)$ will be defined by

$$\exp-\lambda\sum_{i,\xi}K(\nabla_i^\xi\psi^+,\nabla_i^\xi\psi^-)$$

and

$$K=(\nabla_i^\xi\psi^-)^2+2\sum_p a_p\sum_{q=0}^{2p}\binom{2p}{q}(\nabla\psi^+)^{2p-q}(\nabla\psi^-)^q. \qquad (14)$$

By symmetry it is easy to check that only q even enters in (14). Therefore for any configuration of ψ^+, $\sum_{i,\xi}[K-(\nabla_i^\xi\psi^-)^2]$ is convex, and the rest of the proof of Lemma 3 is unchanged.

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