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Linking Numbers of Closed Manifolds at Random in $\mathbb{R}^n$, Inductances and Contacts

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Abstract. We give here new results of topology and integral geometry concerning the Gauss linking number I of closed manifolds in n-dimensional space. The rigid manifolds have arbitrary shapes and dimensions, and are statistically at random positions in $\mathbb{R}^n$. Generalizing Pohl's work, for two closed manifolds $\mathscr{C}^r_1$, $\mathscr{C}^s_2$, of respective dimensions r and s, with $0 \leqq r \leqq n-1$, and $r+s+1=n$, we consider the "kinematic linking integral" $\mathscr{I} = \langle \int I^2(\mathbf{x}, \mathscr{O}) d^n x \rangle$, of the square linking number I of $\mathscr{C}^r_1$ and $\mathscr{C}^s_2$, over the group of Euclidean motions of one manifold (translations $\mathbf{x}$, rotations $\mathscr{O}$). Introducing a new tensorial method, and using group theory, we show quite generally that $\mathscr{I} = \text{num. fact.} \int_0^\infty d\varrho [\mathscr{A}_1(\varrho)\mathscr{A}_2(\varrho) + \delta_{r,s}\mathscr{B}_1(\varrho)\mathscr{B}_2(\varrho)]$, where ϱ is a length variable and where $\mathscr{A}_\alpha$, $\mathscr{B}_\alpha (\alpha = 1, 2)$ are characteristic functions associated with the manifold $\mathscr{C}_\alpha$ only. We study functions $\mathscr{A}$ and $\mathscr{B}$ of a manifold $\mathscr{C}^r$, of dimension r, in all cases $0 \leqq r \leqq n-1$. $\mathscr{A}$ always exists. $\mathscr{A}(0)$ gives $\mathscr{C}$'s area, whereas $\int_0^\infty \mathscr{A}(\varrho) d\varrho$ equals the interior volume of a hypersurface $\mathscr{C}$. $\mathscr{B}$ is found to exist and not to vanish only if $2 \dim \mathscr{C} + 1 = n$ and $n = 3 + 4q = 3, 7, 11 \ldots$ $\mathscr{A}$ and $\mathscr{B}$ are explicitly calculated for segments and r-spheres S^r. As an application the topological excluded volume of a gas of nonlinked spheres S^r moving in $\mathbb{R}^{2r+1}$ is calculated. We generalize to N manifolds $\mathscr{C}_\alpha$, $\alpha = 1, \ldots, N$, linked successively to each other and forming a ring. The cyclic product of their linking numbers is integrated over the group of motions of the manifolds. It is shown to factorize completely in Fourier space, with special algebraic rules, over the set of $2N$ characteristic functions $\mathscr{A}_\alpha$, $\mathscr{B}_\alpha$, associated with the $\mathscr{C}_\alpha$'s. The same algebra of characteristic functions is shown to describe a larger class of topology and electromagnetism properties: a new theorem is given for a family of Euclidean group integrals involving the random linking numbers, mutual inductances and contact distributions of N manifolds.

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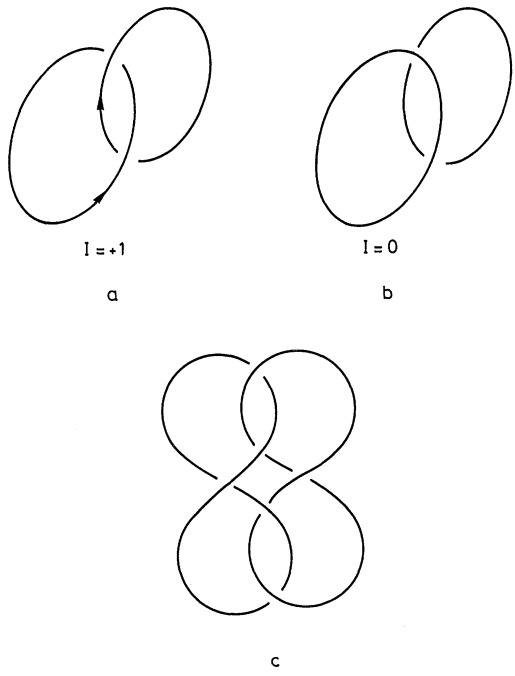

Fig. 1a–c. Linking number I of two closed curves in $\mathbb{R}^3$; **a** $I=+1$ because each curve crosses the interior surface of the other from the negative to the positive side; **b** two unlinked curves; **c** the two "eights" are topologically linked, nevertheless $I=0$ (taken from [5])

1. Introduction

1.1. General Background

Topological constraints can exist in a statistical system and bring in new physical effects [1]. An example can be found in polymer theory. A set of closed polymer rings not linked together have a phase space restricted by that topological constraint, and the osmotic pressure deviates from that of an ideal solution [2]. Topology of links and topological invariants involve very interesting mathematical problems. In this article, we study remarkable integral properties of the Gauss linking numbers of manifolds in $\mathbb{R}^n$. The Gauss linking number I of two closed manifolds is a topological invariant counting the number of times one closed manifold winds around the other one, both being oriented. Figure 1 gives examples of linking numbers of curves in $\mathbb{R}^3$. Two curves can be topologically linked, their algebraic linking number being nevertheless equal to zero (Fig. 1c). The analytic formula for I was given by Gauss [3, 4]

$$I=\frac{1}{4\pi}\int_{\mathscr{C}_1\times\mathscr{C}_2}\left(\nabla_g\frac{1}{\|\mathbf{f}-\mathbf{g}\|},d\mathbf{f},d\mathbf{g}\right), \tag{1.1}$$

where $\mathbf{f}$ and $\mathbf{g}$ stand for the positions in $\mathbb{R}^3$ of two generic points on $\mathscr{C}_1$ and $\mathscr{C}_2$ respectively, and where $d\mathbf{f}$, $d\mathbf{g}$ are the corresponding differential vectors along the

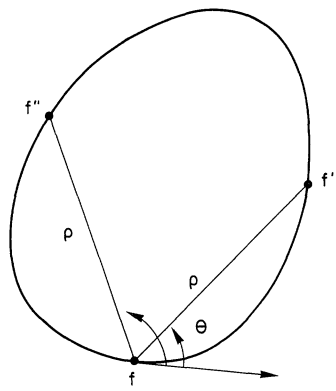

Fig. 2. Cords of length ϱ contributing to $\mathscr{A}(\varrho)$

oriented curves. ∇_g represents the gradient with respect to point **g**. As indicated by Maxwell [5], Gauss' formula is closely related to electromagnetism: "It was the discovery by Gauss of this very integral, expressing the work done on a magnetic pole while describing a closed curve in presence of a closed electric current, and indicating the geometrical connexion between the two closed curves, that led him to lament the small progress made in the Geometry of Position since the time of Leibnitz, Euler and Vandermonde...." A general Gauss' formula exists for closed manifolds embedded in $\mathbb{R}^n$ [4] (Sect. 2). A quite interesting mathematical property of the linking number of curves in space has been very recently discovered by Pohl [6]. He considered the integral

$$\mathscr{I} = \left\langle \int_{\mathbb{R}^3} I^2(\mathbf{x}, \mathcal{O}) d^3x \right\rangle \tag{1.2}$$

of I^2 over the translations **x** and the rotations $\mathcal{O}$ of one *rigid* curve with respect to the other. One notices that, for two plane convex curves, $I = \begin{cases} 0 \\ \pm 1 \end{cases}$ and $\mathscr{I} = V$, where V is the *topological excluded volume* between the two unlinked curves. For nonplane curves $\mathscr{I}$ gives only a approximation of V. Pohl proved that for two plane convex curves $\mathscr{C}_1$ and $\mathscr{C}_2$, $\mathscr{I}$ can be transformed into a single integral

$$\mathscr{I} = \frac{1}{8\pi} \int_0^\infty \mathscr{A}_1(\varrho) \mathscr{A}_2(\varrho) d\varrho, \tag{1.3}$$

where ϱ is a variable having the dimension of a length. The $\mathscr{A}$'s are functions characteristic of each manifold respectively. For a given curve $\mathscr{C}$ the $\mathscr{A}$ function reads [6]:

$$\mathscr{A}(\varrho) = \int ds |\cos\theta|, \tag{1.4}$$

where s is the curvilinear abscissa of a point f along $\mathscr{C}$; θ is the angle between the tangent vector at f and the points f' on $\mathscr{C}$ such that $\|\mathbf{f} - \mathbf{f}'\| = \varrho$ (Fig. 2).

Pohl's theorem was generalized recently to nonplane closed curves in $\mathbb{R}^3$ by Des Cloizeaux and Ball [7], and by Duplantier [8], using various methods. For each nonplane curve, besides $\mathscr{A}$, a second characteristic function $\mathscr{B}$ appears, which

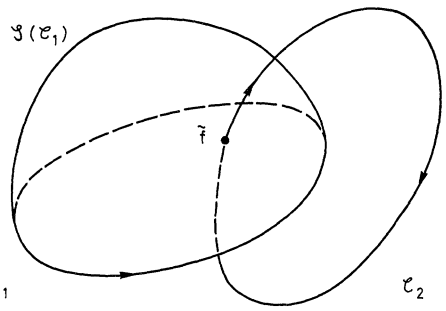

Fig. 3. The manifold $\mathscr{C}_2$ crosses the Seifert manifold $\mathscr{S}(\mathscr{C}_1)$ at discrete points $\tilde{f}$

modifies (1.3). New further generalizations of Pohl's theorem were also found by the author [8]. For a set of N curves, the cyclic product $I_{12}I_{23}...I_{N1}$ was integrated over the group of rigid motions of the curves. It was shown to *factorize* over a special algebra involving the characteristic functions $\mathscr{A}$ and $\mathscr{B}$, associated with each curve separately. Besides linking numbers, mutual inductances and contact distributions were also studied [8].

The aim of this article is to study quite generally the linking numbers of closed manifolds in $\mathbb{R}^n$, and their linking integrals similar to (1.2). It is quite general because the dimensions of the manifolds can be arbitrary, going from zero dimensional set of points to hypersurfaces in $\mathbb{R}^n$. We give a set of factorization theorems, similar to Pohl's theorem (1.3). Because of the factorization property [see (1.3)], these theorems yield a solution to the further problems of averaging over the deformations of the manifolds. For a given statistical weight, one has simply to average the characteristic functions *independently*. For proving these theorems, we introduce a new mathematical object, characteristic of a manifold, its inductance tensor $\mathbf{\Gamma}$ (Sect. 3), in terms of which all quantities can be expressed. We believe this tensor to be quite basic for such studies of integral geometry. The linking integrals of two manifolds and of multiple manifolds are studied. Mutual inductances and contacts are also treated. Topology, potential theory and electromagnetism in $\mathbb{R}^n$ are used here. Some results have been recently published elsewhere [10].

1.2. Linking Numbers in $\mathbb{R}^n$

Consider two orientable differentiable manifolds $\mathscr{C}^r_1$ and $\mathscr{C}^s_2$, of dimensions r and s, embedded in $\mathbb{R}^n$. They are given by parametric equations in $\mathbb{R}^n$:

$$\begin{matrix} f \in \mathscr{C}^r_1 \,;\, \mathbf{f} : f_i(u_1, \ldots, u_r) \\ g \in \mathscr{C}^s_2 \,;\, \mathbf{g} : g_i(v_1, \ldots, v_s) \end{matrix} \qquad i = 1, \ldots, n, \tag{1.5}$$

where the u_a, $a = 1, \ldots, r$; v_b, $b = 1, \ldots, s$ are parameters, and $\mathbf{f}$, $\mathbf{g}$ are differentiable. The orientations are given by the natural orders $a = 1, \ldots, r$ and $b = 1, \ldots, s$. The manifolds are *closed*, i.e. they have no boundary $\partial : \partial \mathscr{C} = 0$. For a closed and orientable manifold $\mathscr{C}^r_1$, there exists a Seifert manifold $\mathscr{S}(\mathscr{C}^r_1)$, the boundary of which is $\mathscr{C}^r_1$ (Fig. 3). The dimension of $\mathscr{S}(\mathscr{C}^r_1)$ is $r+1$, and its points $\tilde{f}$ are given by

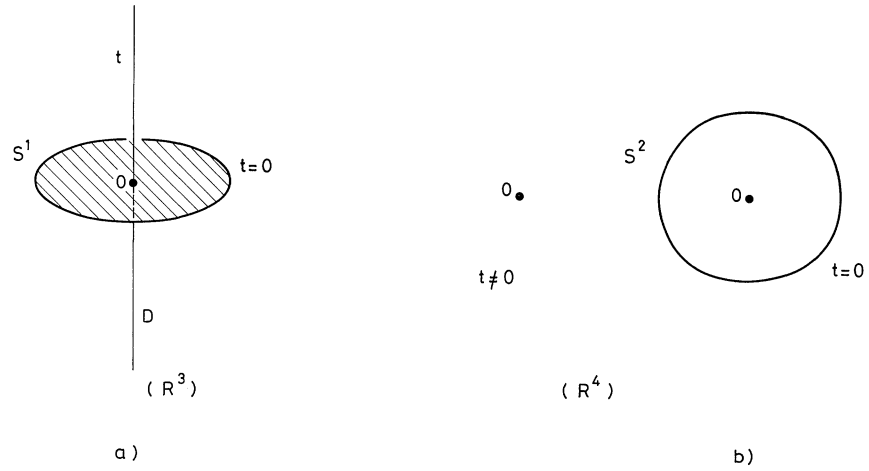

Fig. 4. a The straight line $D(x, y=0, t\in\mathbb{R})$ and the circle $S^1(x^2+y^2=1, t=0)$ are linked in $\mathbb{R}^3$; and **b** the corresponding situation in $\mathbb{R}^4$. The line $(x, y, z=0, t\in\mathbb{R})$ and the sphere $S^2(x^2+y^2+z^2=1, t=0)$ form a link in $\mathbb{R}^4$

parametric coordinates $\tilde{\mathbf{f}}(w_1, \ldots, w_{r+1})$, such that $\mathscr{S}(\mathscr{C}_1^r)$ is coherently oriented with respect to its boundary $\mathscr{C}_1^r$ [11]. We suppose that the dimensions of $\mathscr{C}_1^r$ and $\mathscr{C}_2^s$ satisfy

$$r+s+1=n. \tag{1.6}$$

Then the intersection $\mathscr{S}(\mathscr{C}_1^r)\cap\mathscr{C}_2^s$ is a set of points P, possibly empty. The linking number of $\mathscr{C}_1^r$ with $\mathscr{C}_2^s$ is defined by

$$I_{12}=\sum_{P\in\mathscr{S}(\mathscr{C}_1^r)\cap\mathscr{C}_2^s}\varepsilon(P), \tag{1.7}$$

where $\varepsilon(P)=\pm 1$ is the orientation of the local basis of $\mathbb{R}^n$

$$\{\partial\tilde{\mathbf{f}}/\partial w_1, \ldots, \partial\tilde{\mathbf{f}}/\partial w_{r+1}, \partial\mathbf{g}/\partial v_1, \ldots, \partial\mathbf{g}/\partial v_s\}$$

with respect to a canonical basis. I_{12} is a relative integer, topologically invariant. A simple example in $\mathbb{R}^4$ is given in Fig. 4.

1.3. Summary

In Sect. 2, generalizing to $\mathbb{R}^n$, we consider the "kinematic linking integral"

$$\mathscr{I}=\langle\int I^2(\mathbf{x}, \mathscr{O})d^n x\rangle \tag{1.8}$$

over the group of relative rigid motions $(\mathbf{x}, \mathscr{O})$ of two manifolds. Using a generalized Gauss' integral formula for the linking number, we introduce a tensorial formalism, which enables us to calculate (1.8). We introduced the same formalism in [8] in the simpler case of curves in $\mathbb{R}^3$.

Section 3 deals with the properties of the "inductance tensor" of a manifold. It is calculated with the help of group theory [12]. In Sect. 4, $\mathscr{I}$ (1.8) for two manifolds is calculated explicitly. We prove that two characteristic functions $\mathscr{A}$

and $\mathscr{B}$ exist for each manifold, in terms of which (1.8) can be "factorized." The end of Sect. 4 is devoted to the geometrical properties of the functions $\mathscr{A}$ and $\mathscr{B}$.

Section 5 is concerned with the calculation of the "kinematic linking integral" of multiple manifolds, which generalizes (1.8). We use the Fourier space and study the characteristic functions $\mathscr{A}$ and $\mathscr{B}$ in the momentum representation. We give a general *factorization theorem* for the multiple linking integral.

Section 6 deals with a further generalization. For manifolds having the same dimension, we define the mutual inductance M [13] and the contact distribution C. Both M and C are related to linking number I via potential theory. We consider cyclic products containing either the linking numbers I, or the inductances M or the contacts C of the successive manifolds. A general factorization theorem gives the integral of these products on the group of motions of the manifolds.

Finally, in Sect. 7, some particular geometrical cases are studied. We calculate the $\mathscr{A}$ function of a zero dimensional manifold made of two points, and the $\mathscr{A}$ function of the r-sphere S^r. We finally compute the topological second virial coefficient of a set of r-spheres S^r moving in $\mathbb{R}^{2r+1}$.

2. Linking Numbers and Euclidean Group of Motion

2.1. Gauss' Integral

The algebraic linking number (1.7) in $\mathbb{R}^n$, is given by the Gauss' integral, for $n \geqq 1$:

$$I(\mathscr{C}_1, \mathscr{C}_2) = (S_{n-1}(n-2)r!s!)^{-1} \int_{\mathscr{C}_1 \times \mathscr{C}_2} \det(\nabla_g \|\mathbf{f}-\mathbf{g}\|^{-(n-2)}, d\mathscr{F}_r, d\mathscr{G}_s), \quad (2.1)$$

where $f \in \mathscr{C}_1$, $g \in \mathscr{C}_2$, and where $\| \ \|$ denotes the Euclidean norm [1]. S_{n-1} is the area of the unit sphere S^{n-1} of $\mathbb{R}^n$

$$S_{n-1} = 2\pi^{n/2}/\Gamma(n/2); \quad (2.2)$$

$d\mathscr{F}_r$ is the *r-volume form* associated with manifold $\mathscr{C}_1$ at point $\mathbf{f}$ [11]:

$$d\mathscr{F}_r|_{i_1 \ldots i_r} = df_{i_1} \wedge \ldots \wedge df_{i_r}. \quad (2.3)$$

This tensorial r-volume form possesses r indices i_a, $a = 1, \ldots, r$, taking their values in the set $\{1, \ldots, n\}$. The s-volume form $d\mathscr{G}_s$ is defined in the same way at point g of manifold $\mathscr{C}_2$. Equation (2.1) reads explicitly

$$I(\mathscr{C}_1, \mathscr{C}_2) = [S_{n-1}(n-2)r!s!]^{-1} \int_{\mathscr{C}_1 \times \mathscr{C}_2} \varepsilon^{li_1 \ldots i_r j_1 \ldots j_s} \frac{\partial}{\partial g_l} \frac{1}{\|\mathbf{f}-\mathbf{g}\|^{n-2}} (df_{i_1} \wedge \ldots \wedge df_{i_r}) \cdot (dg_{j_1} \wedge \ldots \wedge dg_{j_s}), \quad (2.4)$$

where ε represents the *totally antisymmetric tensor of rank n* in $\mathbb{R}^n$:

$$\varepsilon^{\sigma(1) \ldots \sigma(n)} = \varepsilon(\sigma)$$

for any permutation σ of $\{1, \ldots, n\}$, its signature being $\varepsilon(\sigma)$. Otherwise, as soon as two indices coincide: $\varepsilon^{i \ldots i \ldots} \equiv 0$. The identity of the Gauss' integral (2.4) with the

1 Expanding the gradient ∇_g shows that (2.1) is also defined for $n = 2$

linking number (1.7) is proved in Appendix A. Owing to (2.4), the linking number I satisfies

$$I_{12}=(-1)^{rs+1}I_{21}. \tag{2.5}$$

Equation (2.1) gives the intrinsic formula for the Gauss linking number. For a particular parametrization (1.5) of the manifolds, we have, according to the theory of differential exterior calculus [11]:

$$d\mathscr{F}_r|_{i_1,\ldots,i_r}=\frac{D(f_{i_1},\ldots,f_{i_r})}{D(u_1,\ldots,u_r)}du_1\wedge\ldots\wedge du_r, \tag{2.6}$$

$\frac{D(f\ldots)}{D(u\ldots)}$ being the partial Jacobian of coordinates f_{i_a} with respect to parameters $u_{a'}$, with $a,a'=1,\ldots,r$.

It will be very convenient in the following to use the generalized Kronecker delta tensor, defined as the $r\times r$ determinant [14]:

$$\delta^{l_1\ldots l_r}_{i_1\ldots i_r}=\begin{vmatrix}\delta^{l_1}_{i_1} & \delta^{l_1}_{i_2}\ldots & \delta^{l_1}_{i_r}\\ \delta^{l_2}_{i_1} & & \vdots\\ \vdots & & \\ \delta^{l_r}_{i_1} & \ldots\ldots & \delta^{l_r}_{i_r}\end{vmatrix}. \tag{2.7}$$

The tensor (2.7) possesses an equal number of indices i_a, l_a with $a=1,\ldots,r$, taking their values in $\mathbb{R}^n$ in $\{1,\ldots,n\}$. The tensor (2.7) is obviously *skew-symmetric* under interchange of any two of the indices of the set $I=\{i_1,\ldots,i_r\}$ and under interchange of any two of the indices of the set $L=\{l_1,\ldots,l_r\}$. Furthermore δ differs from zero only if the sets I and L are identical, up to a permutation. For any totally *skew-symmetric* quantity $A_{l_1\ldots l_r}$, depending on r indices

$$\delta^{l_1\ldots l_r}_{i_1\ldots i_r}A_{l_1\ldots l_r}=r!\,A_{i_1\ldots i_r}. \tag{2.8}$$

The partial traces of tensor δ are equal to

$$\delta^{l_1\ldots l_t h_{t+1}\ldots h_r}_{i_1\ldots i_t h_{t+1}\ldots h_r}=\frac{(n-t)!}{(n-r)!}\delta^{l_1\ldots l_t}_{i_1\ldots i_t}, \tag{2.9}$$

where repeated indices h_a are summed over. Using (2.7), the Jacobian (2.6) reads

$$d\mathscr{F}_r|_{i_1\ldots i_r}=\delta^{l_1\ldots l_r}_{i_1\ldots i_r}\frac{\partial f_{l_1}}{\partial u_1}\ldots\frac{\partial f_{l_r}}{\partial u_r}(du_1\wedge\ldots\wedge du_r). \tag{2.10}$$

Substitution of (2.10) into (2.4) and use of property (2.8) finally give a parametric expression of the linking number:

$$I_{12}=(S_{n-1})^{-1}\int\det\left(\frac{\mathbf{f}-\mathbf{g}}{\|\mathbf{f}-\mathbf{g}\|^n},\frac{\partial\mathbf{f}}{\partial u_1},\ldots,\frac{\partial\mathbf{f}}{\partial u_r},\frac{\partial\mathbf{g}}{\partial v_1},\ldots,\frac{\partial\mathbf{g}}{\partial v_s}\right)du_1\ldots du_r dv_1\ldots dv_s$$

in agreement with that of [4]. However, it will be more convenient in the following to use differential exterior calculus and the intrinsic Gauss' integral (2.1). To

facilitate calculations, we define the tensor

$$\boldsymbol{\varepsilon}_{sr}(\mathbf{x})|_{j_1\ldots j_s,i_1\ldots i_r}=\frac{1}{s!}x_l\varepsilon^{li_1\ldots i_rj_1\ldots j_s} \tag{2.11}$$

which depends on one vector $\mathbf{x}$ of $\mathbb{R}^n$ and bears r free *right* indices i_a, and s free *left* indices j_b. One notices the conventional inversion of order for s and r between the two sides of Eq. (2.11). Owing to (2.11), the Gauss' integral (2.4) reads:

$$I_{12}=[S_{n-1}(n-2)r!]^{-1}\int\limits_{\mathscr{C}_1\times\mathscr{C}_2}[d\mathscr{G}_s\cdot\boldsymbol{\varepsilon}_{sr}(V_g)\cdot d\mathscr{F}_r]\frac{1}{\|\mathbf{f}-\mathbf{g}\|^{n-2}}, \tag{2.12}$$

where the dots represent the *ordered contractions* of the r indices of the volume form $d\mathscr{F}_r$ with the r right indices of $\boldsymbol{\varepsilon}_{sr}$, and the contraction of the s indices of $d\mathscr{G}_s$ with the s left indices of $\boldsymbol{\varepsilon}_{sr}$[2].

2.2. *Euclidean Group of Motions and Kinematic Integrals*

Consider first a manifold $\mathscr{C}$ in $\mathbb{R}^n$ as a rigid body. The position in $\mathbb{R}^n$ of $\mathscr{C}$ is defined by that of an origin O on $\mathscr{C}$ and by the set of the Euler angles $\mathscr{O}$ of $\mathscr{C}$. Let Ω_n be the measure of the group of rotations of a solid in $\mathbb{R}^n$:

$$\Omega_n\equiv\int d\mathscr{O}. \tag{2.13}$$

The angular position $\mathscr{O}$ of $\mathscr{C}$ in $\mathbb{R}^n$ is completely defined by the choice of an axis rigidly attached to $\mathscr{C}$, and by the rotation of this solid $\mathscr{C}$ in a hyperplane of dimension $n-1$ around the axis. Thus we have $\Omega_n=S_{n-1}\Omega_{n-1}$, where S_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, and

$$\Omega_n=S_{n-1}S_{n-2}\ldots S_1. \tag{2.14}$$

In particular, for $n=3$: $\Omega_3=S_2S_1=8\pi^2$. The linking number of two manifolds $\mathscr{C}_1$ and $\mathscr{C}_2$ depends on the translation vector $\mathbf{x}=O_1O_2$ joining the two origins O_1 and O_2 of $\mathscr{C}_1$ and $\mathscr{C}_2$, and on the relative angular position $\mathscr{O}$ of $\mathscr{C}_2$ with respect to $\mathscr{C}_1$. It writes $I_{12}(\mathbf{x},\mathscr{O})$. Generalizing the work of W. Pohl, we define the "kinematic linking integral" $\mathscr{I}$

$$\mathscr{I}(\mathscr{C}_1,\mathscr{C}_2)=\left\langle\int\limits_{\mathbb{R}^n}d^nxI_{12}(\mathbf{x},\mathscr{O})I_{21}(-\mathbf{x},\mathscr{O})\right\rangle, \tag{2.15}$$

where the brackets represent the average over the set of angular orientations $\mathscr{O}$:

$$\langle(\ldots)\rangle\equiv\int\frac{d\mathscr{O}}{\Omega_n}(\ldots). \tag{2.16}$$

This integral reads also, owing to (2.5)

$$\mathscr{I}=(-1)^{rs+1}\left\langle\int\limits_{\mathbb{R}^n}d^nxI^2(\mathbf{x},\mathscr{O})\right\rangle, \tag{2.17}$$

where $I=|I_{12}|$.

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2 One has explicitly:

$$d\mathscr{G}_s\cdot\boldsymbol{\varepsilon}_{sr}(\mathbf{x})\cdot d\mathscr{F}_r=\frac{1}{s!}x_l\varepsilon^{li_1\ldots i_rj_1\ldots j_s}d\mathscr{F}_r|_{i_1\ldots i_r}d\mathscr{G}_s|_{j_1\ldots j_s}$$

Successive Links of N Manifolds. Consider now a set of N differentiable and orientable closed manifolds in $\mathbb{R}^n$:

$$\{\mathscr{C}_\alpha, \alpha = 1, \ldots, N\}.$$

This set is ordered. Let us define $I_{\alpha\alpha+1}$ as the linking number of two successive manifolds $\mathscr{C}_\alpha$, $\mathscr{C}_{\alpha+1}$. By convention we set for $\alpha = N$:

$$I_{NN+1} \equiv I_{N1}.$$

The linking number of manifolds $\mathscr{C}_\alpha$ and $\mathscr{C}_{\alpha+1}$ can be defined only if $\dim\mathscr{C}_\alpha + \dim\mathscr{C}_{\alpha+1} = n-1$. Thus, if we denote by r the dimension of the first manifold $\mathscr{C}_1$, and set $s = n-1-r$, we must have the sequence of dimensions:

$$\begin{aligned} \dim\mathscr{C}_{2\beta-1} &= r, \\ \dim\mathscr{C}_{2\beta} &= s. \end{aligned} \tag{2.18}$$

Two different cases therefore appear.

a) $r \neq s$

The existence of the whole set $\{I_{\alpha\alpha+1}, \alpha = 1, \ldots, N\}$, together with condition (2.18) obviously requires the number N of manifolds $\mathscr{C}_\alpha$ to be *even.*

b) $r = s$

This case occurs if $r = s = n-1-r$, that is for:

$$n = 2r+1.$$

The dimension n of space is therefore *odd.* All $\mathscr{C}_\alpha$'s have the same dimension. N is arbitrary. This corresponds for instance to closed curves of dimension $r=1$ embedded in $\mathbb{R}^3$, the case considered in [6–8].

In the following, we shall denote the dimensions by r and s quite generally, and distinguish cases a) and b) only when necessary.

Let us write $\mathbf{x}_\alpha$ the vector joining the origin O_α of $\mathscr{C}_\alpha$ to origin $O_{\alpha+1}$ of $\mathscr{C}_{\alpha+1}$ (Fig. 5). The set of translation vectors $\mathbf{x}_\alpha$ forms a closed polygon:

$$\sum_{\alpha=1}^{N} \mathbf{x}_\alpha = 0. \tag{2.19}$$

Each linking number $I_{\alpha\alpha+1}(\mathbf{x}_\alpha, \mathscr{O}_\alpha)$ also depends on the set $\mathscr{O}_\alpha$ of Euler angles of $\mathscr{C}_{\alpha+1}$ with respect to $\mathscr{C}_\alpha$. The kinematic linking integral $\mathscr{I}$ of an ordered set of N closed manifolds is defined as

$$\mathscr{I}(\mathscr{C}_1, \ldots, \mathscr{C}_N) = \left\langle \int_{\mathbb{R}^n \times \ldots \times \mathbb{R}^n} d^n x_1 \ldots d^n x_N \delta\left(\sum_{\alpha=1}^{N} \mathbf{x}_\alpha\right) \prod_{\alpha=1}^{N} I_{\alpha\alpha+1}(\mathbf{x}_\alpha, \mathscr{O}_\alpha) \right\rangle, \tag{2.20}$$

where the brackets represent the average

$$\langle(\ldots)\rangle = (\Omega_n)^{-(N-1)} \int \prod_{\alpha=1}^{N-1} d\mathscr{O}_\alpha (\ldots). \tag{2.20a}$$

For $N=2$, (2.20) coincides with (2.15). Integral $\mathscr{I}$ represents a measure of the successive links made by N manifolds over the group of rigid motions of these manifolds. The integrand of (2.20) is nonvanishing only for the ring configurations

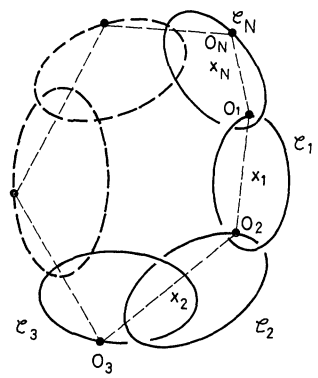

Fig. 5. N manifolds $\mathcal{C}_\alpha$, $\alpha=1,\ldots,N$, successively linked to each other. The set of translation vectors $\mathbf{x}_\alpha$ forms a closed polygon: $\mathbf{x}_1+\ldots+\mathbf{x}_N=0$

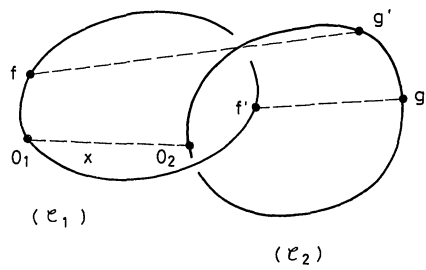

Fig. 6. The generic points on two manifolds used for calculating the square linking number $I^2(\mathbf{x})$, with $\mathbf{x}=O_1O_2$

of the N manifolds, in which all successive couples $\mathcal{C}_\alpha$, $\mathcal{C}_{\alpha+1}$ are linked (Fig. 5). We shall in the next sections show how linking integrals (2.15) and (2.20) can be "factorized" into a product of terms associated with each manifold.

2.3. Tensorial Factorization Method

We generalize here in $\mathbb{R}^n$ a method previously introduced for closed curves in $\mathbb{R}^3$ [8]. For the sake of simplicity, we make it explicit only for two manifolds, the extension to N manifolds being straightforward.

Replacing in (2.15) each linking number by a Gauss' integral gives

$$I_{12}(\mathbf{x},\mathcal{O})I_{21}(-\mathbf{x},\mathcal{O})=[S_{n-1}(n-2)]^{-2}(r!s!)^{-1}\int_{\mathcal{C}_1\times\mathcal{C}_2} d\mathcal{G}'_s\cdot\boldsymbol{\varepsilon}_{sr}(V_{g'})\cdot d\mathcal{F}_r\frac{1}{\|\mathbf{f}-\mathbf{g}'-\mathbf{x}\|^{n-2}}$$
$$\cdot\int_{\mathcal{C}_2\times\mathcal{C}_1} d\mathcal{F}'_r\cdot\boldsymbol{\varepsilon}_{rs}(V_{f'})\cdot d\mathcal{G}_s\frac{1}{\|\mathbf{g}-\mathbf{f}'+\mathbf{x}\|^{n-2}}, \tag{2.21}$$

where generic points (f,g') and (f',g) have been related to each other for building up linking numbers I_{12} and I_{21} respectively (Fig. 6). Here, vectors $\mathbf{f}$, $\mathbf{f}'$ measure positions with respect to the origin O_1, and $\mathbf{g}$, $\mathbf{g}'$ with respect to the origin O_2 (Fig. 6). Inserting (2.21) in integral (2.15), we use Fubini's theorem and exchange

the order of integrations over $(\mathscr{C}_1 \times \mathscr{C}_2, \mathscr{C}_2 \times \mathscr{C}_1)$ and $\mathbf{x}$. We then perform the change of variable $\mathbf{x} \to \mathbf{y}$:

$$\mathbf{x} = \mathbf{y} + \mathbf{f}' - \mathbf{g}'. \tag{2.22}$$

The Jacobian of this *local* change of variable trivially equals one. Then we have

$$\begin{aligned} \mathbf{f} - \mathbf{g}' - \mathbf{x} &= \mathbf{f} - \mathbf{f}' - \mathbf{y}, \\ \mathbf{g} - \mathbf{f}' + \mathbf{x} &= \mathbf{g} - \mathbf{g}' + \mathbf{y}. \end{aligned} \tag{2.23}$$

The gradient operators transform, according to (2.23), into:

$$\nabla_{g'} = \nabla_y, \nabla_{f'} = \nabla_{-y}. \tag{2.24}$$

We notice that the change of variable $\mathbf{x} \to \mathbf{y}$ has separated contributions of manifolds $\mathscr{C}_1$ and $\mathscr{C}_2$ in (2.23). It remains for us to factorize $\mathscr{C}_1$ and $\mathscr{C}_2$ in (2.21) with respect to their own differential volume-forms. We use the trivial algebraic identity:

$$(d\mathscr{G}_s' \cdot \boldsymbol{\varepsilon}_{sr} \cdot d\mathscr{F}_r)(d\mathscr{F}_r' \cdot \boldsymbol{\varepsilon}_{rs} \cdot d\mathscr{G}_s) = \mathrm{tr}[\{\boldsymbol{\varepsilon}_{sr} \cdot (d\mathscr{F}_r \otimes d\mathscr{F}_r')\}\{\boldsymbol{\varepsilon}_{rs} \cdot (d\mathscr{G}_s \otimes d\mathscr{G}_s')\}], \tag{2.25}$$

where the dots represent naturally the *ordered contraction* of adjacent sets of indices. The trace represents the ordered contraction of the external sets of s indices. Collecting Eqs. (2.22)–(2.25), we obtain for (2.15)

$$\mathscr{I} = \left\langle \int_{\mathbb{R}^n} d^n y I_{12} I_{21}[\mathbf{y}] \right\rangle, \tag{2.26}$$

where the integrand is the matricial trace

$$\begin{aligned} I_{12} I_{21}[\mathbf{y}] = [S_{n-1}(n-2)]^{-2}(r!s!)^{-1} \mathrm{tr}\left\{ \left[\int_{\mathscr{C}_1 \times \mathscr{C}_1} \boldsymbol{\varepsilon}_{sr}(\nabla_y) \cdot (d\mathscr{F}_r \otimes d\mathscr{F}_r') \frac{1}{\|\mathbf{f} - \mathbf{f}' - \mathbf{y}\|^{n-2}} \right] \right. \\ \left. \cdot \left[\int_{\mathscr{C}_2 \times \mathscr{C}_2} \boldsymbol{\varepsilon}_{rs}(\nabla_{-y}) \cdot (d\mathscr{G}_s \otimes d\mathscr{G}_s') \frac{1}{\|\mathbf{g} - \mathbf{g}' + \mathbf{y}\|^{n-2}} \right] \right\}. \end{aligned} \tag{2.27}$$

For a given manifold $\mathscr{C}$, of dimension r, we then define an associated tensor $\mathbf{C}$

$$\mathbf{C}(\mathbf{y}) = (S_{n-1}(n-2)r!)^{-1} \left\langle \int_{\mathscr{C} \times \mathscr{C}} \varepsilon_{sr}(\nabla_y) \cdot (d\mathscr{F}_r \otimes d\mathscr{F}_r') \frac{1}{\|\mathbf{f} - \mathbf{f}' - \mathbf{y}\|^{n-2}} \right\rangle. \tag{2.28}$$

Here the brackets represent the angular average (2.16) over the rotation group of $\mathscr{C}$. Owing to Definition (2.11) (up to an exchange of r and s), tensor $\mathbf{C}$ reads, with explicit indices:

$$\begin{aligned} \mathbf{C}(\mathbf{y})|_{i_1 \ldots i_s, j_1 \ldots j_r} = [S_{n-1}(n-2)(r!)^2]^{-1} \left\langle \int_{\mathscr{C} \times \mathscr{C}} \varepsilon^{ll_1 \ldots l_r i_1 \ldots i_s} \frac{\partial}{\partial y_l} \frac{1}{\|\mathbf{f} - \mathbf{f}' - \mathbf{y}\|^{n-2}} \right. \\ \left. \cdot (df_{l_1} \wedge \ldots \wedge df_{l_r})(df'_{j_1} \wedge \ldots \wedge df'_{j_r}) \right\rangle. \end{aligned} \tag{2.29}$$

$\mathbf{C}$ appears clearly as a tensor of rank $s + r = n - 1$. For a manifold $\mathscr{C}$ of dimension r, $\mathbf{C}$ possesses r free indices i on the left and s indices j on the right. After the angular average over rotations of manifold $\mathscr{C}$, the tensor $\mathbf{C}$ *depends only on vector* $\mathbf{y}$.

Using Definition (2.28), integral $\mathscr{I}$ (2.26) finally reads

$$\mathscr{I} = \int_{\mathbb{R}^n} d^n y \, \mathrm{tr}[\mathbf{C}_1(\mathbf{y}) \cdot \mathbf{C}_2(-\mathbf{y})], \tag{2.30}$$

where tensors $\mathbf{C}_1$ and $\mathbf{C}_2$ are respectively associated with manifolds $\mathscr{C}_1$ and $\mathscr{C}_2$[3]. We have thus achieved a first factorization of the integrand of $\mathscr{I}$. This method can be applied to general integral (2.20). Integral (2.15) and (2.20) have indeed the same *cyclic* structure. The result is the following. To each manifold $\mathscr{C}_\alpha$ is attached its characteristic tensor $\mathbf{C}_\alpha(\mathbf{y}_\alpha)$. This tensor depends on an arbitrary external vector $\mathbf{y}_\alpha$ of $\mathbb{R}^n$. The cyclic linking integral $\mathscr{I}$ (2.20) can then be written as the generalized trace:

$$\mathscr{I}(\mathscr{C}_1, \ldots, \mathscr{C}_N) = \int_{\mathbb{R}^n \times \ldots \times \mathbb{R}^n} d^n y_1 \ldots d^n y_N \delta\left(\sum_{\alpha=1}^{N} \mathbf{y}_\alpha\right) \mathrm{tr}\left\{\prod_{\alpha=1}^{N} \mathbf{C}_\alpha(\mathbf{y}_\alpha)\right\}. \tag{2.31}$$

The set of vectors $\mathbf{y}_\alpha$ is obtained from the set of translation vectors $\mathbf{x}_\alpha$ by a local change of variable generalizing (2.22). We refer to [8] where a similar algebraic result was obtained for the successive links of N closed curves in $\mathbb{R}^3$.

We have now to calculate, for a given manifold, its characteristic tensor $\mathbf{C}$.

3. Characteristic Tensor of a Manifold

3.1. Definition and Properties

Let us write tensor $\mathbf{C}$ (2.29) in the form

$$\mathbf{C}(\mathbf{y}) = \boldsymbol{\varepsilon}_{sr}(\nabla_y) \cdot \boldsymbol{\Gamma}(\mathbf{y}), \tag{3.1}$$

where *the characteristic tensor* $\boldsymbol{\Gamma}$ is defined by

$$\begin{aligned} \boldsymbol{\Gamma}(\mathbf{y}) &= a_0 \left\langle \int_{\mathscr{C} \times \mathscr{C}} \frac{d\mathscr{F}_r \otimes d\mathscr{F}'_r}{\|\mathbf{y} + \mathbf{f}' - \mathbf{f}\|^{n-2}} \right\rangle, \\ a_0 &= (S_{n-1}(n-2)r!)^{-1}. \end{aligned} \tag{3.2}$$

The tensor (3.2) generalizes to $\mathbb{R}^n$ a tensor previously introduced for closed curves in $\mathbb{R}^3$ [8]. In (3.2), $n=2$ is a special case for which the potential $\frac{1}{n-2} \|\mathbf{y} + \mathbf{f}' - \mathbf{f}\|^{-(n-2)}$ has to be replaced by the limit for $n \to 2$: $-\ln \|\mathbf{y} + \mathbf{f}' - \mathbf{f}\|$. We shall not distinguish this case in the following, all the results being regular for $n \to 2$. $\boldsymbol{\Gamma}$ is a basic mathematical object for our problem, and will also be useful later on, for describing the contact distributions and mutual inductances of closed manifolds. Definition (2.3) of exterior r-forms $d\mathscr{F}_r$, $d\mathscr{F}'_r$ shows that $\boldsymbol{\Gamma}$ is a tensor of rank $2r$, with an equal number r of left and right indices, reading explicitly:

$$\Gamma(\mathbf{y})|_{i_1 \ldots i_r, j_1 \ldots j_r} = a_0 \left\langle \int_{\mathscr{C} \times \mathscr{C}} \frac{(df_{i_1} \wedge \ldots \wedge df_{i_r})(df'_{j_1} \wedge \ldots \wedge df'_{j_r})}{\|\mathbf{y} + \mathbf{f}' - \mathbf{f}\|^{n-2}} \right\rangle. \tag{3.2a}$$

---

3 In (2.26) the angular average was performed on relative orientations $\mathscr{O}$. It is obviously equivalent to averaging over orientations of $\mathscr{C}_1$ and $\mathscr{C}_2$ separately, as in (2.29)

After averaging over angular orientations of $\mathscr{C}$, tensor $\mathbf{\Gamma}$ *depends on vector* $\mathbf{y}$ *only*. Moreover, tensor $\mathbf{\Gamma}$ is *completely skew-symmetric with respect to the set of left indices* $\{i_a, a=1, \ldots, r\}$ *and completely skew-symmetric with respect to the set of right indices* $\{j_a, a=1, \ldots, r\}$. $\mathbf{\Gamma}$ is not defined a priori, for $2r \leqq n-2$, on a set of vectors $\mathbf{y}$ of zero measure (i.e. $\mathbf{y}=\mathbf{f}-\mathbf{f}'$). It is regularized by averaging on rotations and $\mathbf{\Gamma}$ is then defined by continuity for all values of $\mathbf{y}$ (Sect. 3.4).

We define the *left divergence of* $\mathbf{\Gamma}$ *with respect to the* a^{th} *indice* i_a *by*

$$\operatorname{div}_{(i_a)} \Gamma = \frac{\partial}{\partial y_l} \Gamma_{i_1 \ldots i_{a-1} l i_{a+1} \ldots i_r, j_1 \ldots j_r} \tag{3.3}$$

with a summation over values l of i_a. The result of this divergence operation is naturally a tensor, the rank of which equals $2r-1$. A similar definition holds for the right divergences of $\mathbf{\Gamma}$ with respect to indices j_a, $a=1, \ldots, r$.

Lemma 1. *All the r left and r right divergences of tensor* $\mathbf{\Gamma}$ *vanish identically:*

$$\left.\begin{aligned} \operatorname{div}_{(i_a)} \mathbf{\Gamma} = 0 \\ \operatorname{div}_{(j_a)} \mathbf{\Gamma} = 0 \end{aligned}\right\} \quad \forall a,\ a=1, \ldots, r. \tag{3.4}$$

Let us prove this for instance for the first left divergence of $\mathbf{\Gamma}$. Defining the $(r-1)$-exterior form ω

$$\omega \equiv \|\mathbf{y}+\mathbf{f}'-\mathbf{f}\|^{-(n-2)} (df_{i_2} \wedge \ldots \wedge df_{i_r})$$

and using $\dfrac{\partial}{\partial y_{i_1}} = -\dfrac{\partial}{\partial f_{i_1}}$ we may write immediately the first left divergence (3.3) of (3.2a)

$$\operatorname{div}_{(i_1)} \mathbf{\Gamma} = -a_0 \left\langle \int_{f \in \mathscr{C}} \int_{f' \in \mathscr{C}} d_f \omega \otimes d\mathscr{F}'_r \right\rangle,$$

where $d_f \omega$ denotes the exterior derivative, or coboundary, of form ω with respect to vector f[4]. Keeping point f' fixed, we perform the integration over point f, $f \in \mathscr{C}$, of the coboundary form $d_f \omega$. *Stoke's theorem* [11] gives immediately

$$\int_{\mathscr{C}} d_f \omega = \int_{\partial \mathscr{C} = 0} \omega = 0 \tag{3.5}$$

where for a closed manifold $\mathscr{C}$: $\partial \mathscr{C} = 0$. Thus: $\operatorname{div}_{(i_1)} \mathbf{\Gamma} = 0$, Q.E.D. The same proof extends to the whole set (3.4) of divergences.

These rather simple properties of tensor $\mathbf{\Gamma}$ are sufficient to determine the a priori form of Γ, as will now be shown.

3.2. General Form of the Characteristic Tensor $\mathbf{\Gamma}$

We use the following key lemma:

Lemma 2. *Consider a tensor* $\Gamma_{i_1 \ldots i_r, j_1 \ldots j_r}(\mathbf{y})$, *depending on only one vector* $\mathbf{y}$, *vanishing at infinity, completely skew-symmetric with respect to indices*

4 By definition of the exterior derivative [5, 6], for a form

$$\omega = H(f) df_{i_2} \wedge \ldots \wedge df_{i_r},$$

where H is a function of vector f, one has

$$d\omega \equiv \frac{\partial H(f)}{\partial f_{i_1}} df_{i_1} \wedge df_{i_2} \wedge \ldots \wedge df_{i_r}$$

$\{i_a, a=1,\ldots,r\}$ and $\{j_b, b=1,\ldots,r\}$ separately, and the $2r$ divergences of which vanish. Then Γ is necessarily of the form

$$\Gamma_{i_1\ldots i_r, j_1\ldots j_r}(\mathbf{y}) = \delta^{li_1\ldots i_r}_{l' j_1\ldots j_r} \frac{\partial}{\partial y_l}\frac{\partial}{\partial y_{l'}} \Phi(\|y\|) + \delta_{2r+1,n}\varepsilon^{li_1\ldots i_r j_1\ldots j_r}\frac{\partial}{\partial y_l}\Psi(\|y\|), \tag{3.6}$$

where Φ and Ψ are two arbitrary scalar functions of the modulus $\|y\|$.

The Kronecker tensor δ (2.7) is here a determinant of order $r+1$. Because tensor ε in $\mathbb{R}^n$ possesses necessarily n indices, there is a factor $\delta_{2r+1,n}$ in front of the second term. Thus the function Ψ exists only for $\dim \mathscr{C} = r = (n-1)/2$, n being odd.

The proof of Lemma 2, given in Appendix B, uses main results of classical group theory [12], here for the orthogonal group $O(n)$. The condition that Γ vanishes at infinity is auxiliary [5]. Tensor Γ (3.2) vanishes at infinity like $\|y\|^{-(n-2)}$ for $n>2$. For $n=1,2$, a direct construction of Γ gives also the form (3.6) (Appendix B).

Let us introduce a more compact notation:

$$\boldsymbol{\delta}_{rr}(\mathbf{x}\otimes\mathbf{y})|_{i_1\ldots i_r, j_1\ldots j_r} = \frac{1}{r!} x_l y_{l'} \delta^{li_1\ldots i_r}_{l' j_1\ldots j_r}. \tag{3.7}$$

Using Definition (2.11) and (3.7) we finally may write Lemma 2 for tensor Γ in the form

$$\boxed{\boldsymbol{\Gamma}(\mathbf{y}) = \frac{r!\,s!}{(n-1)!}\{\boldsymbol{\delta}_{rr}(\nabla\otimes\nabla)\Phi(\|y\|) + (-1)^{rs}\delta_{2r+1,n}\boldsymbol{\varepsilon}_{rr}(\nabla)\Psi(\|y\|)\}}\ , \tag{3.8}$$

where we have introduced some numerical factors for simplifying forthcoming calculations. One must notice that *the Ψ term exists only for: $2r+1=n$, or equivalently: $r=s$.*

3.3. Some Identities on Tensors

For calculating Γ, we shall need the following identities relating the numerical tensors $\boldsymbol{\varepsilon}$ (2.11) and $\boldsymbol{\delta}$ (3.7).

$$\boldsymbol{\varepsilon}_{sr}(\mathbf{x})\cdot\boldsymbol{\varepsilon}_{rs}(\mathbf{y}) = (-1)^{rs}\delta_{ss}(\mathbf{x}\otimes\mathbf{y}), \tag{3.9}$$

$$\boldsymbol{\varepsilon}_{sr}(\mathbf{x})\cdot\boldsymbol{\delta}_{rr}(\mathbf{y}\otimes\mathbf{x}) = (\mathbf{x}\cdot\mathbf{y})\boldsymbol{\varepsilon}_{sr}(\mathbf{x}), \tag{3.10}$$

$$\delta_{rr}(\mathbf{x}\otimes\mathbf{y})\cdot\boldsymbol{\varepsilon}_{rs}(\mathbf{x}) = (\mathbf{x}\cdot\mathbf{y})\boldsymbol{\varepsilon}_{rs}(\mathbf{x}), \tag{3.11}$$

$$\delta_{rr}(\mathbf{x}\otimes\mathbf{y})\cdot\boldsymbol{\delta}_{rr}(\mathbf{x}\otimes\mathbf{y}) = (\mathbf{x}\cdot\mathbf{y})\boldsymbol{\delta}_{rr}(\mathbf{x}\otimes\mathbf{y}). \tag{3.12}$$

The dots represent the ordered contraction on the r internal indices common to both factors. These identities, proved in Appendix C, hold for any vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.

---

5 If the condition of vanishing at infinity is not fulfilled, one can have for $2r+1>n$, besides (3.6), an independent harmonic tensor $\mathbf{U}$, depending *linearly* on $\mathbf{y}$, and having the same properties as Γ (Appendix B)

Let us now consider the generalized matricial traces of tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\delta}$, defined by:

$$\mathrm{tr}\,\boldsymbol{\varepsilon}_{rr}(\mathbf{x}) = \frac{1}{r!} x_l \varepsilon^{li_1 \ldots i_r i_1 \ldots i_r},$$

$$\mathrm{tr}\,\boldsymbol{\delta}_{rr}(\mathbf{x}\otimes\mathbf{y}) = \frac{1}{r!} x_l y_{l'} \delta^{li_1 \ldots i_r}_{l' i_1 \ldots i_r}.$$

We thus have trivially

$$\mathrm{tr}\,\boldsymbol{\varepsilon}_{rr} = 0. \tag{3.13}$$

Owing to identity (2.9), we have:

$$\mathrm{tr}\,\boldsymbol{\delta}_{rr}(\mathbf{x}\otimes\mathbf{y}) = \frac{(n-1)!}{r!\,s!}(\mathbf{x}\cdot\mathbf{y}). \tag{3.14}$$

3.4. Potential Theory and Calculation of Tensor $\boldsymbol{\Gamma}$

Starting from the *a priori* form (3.8) of tensor $\boldsymbol{\Gamma}$, we determine now the two unknown functions Φ and Ψ. Let us first calculate the trace of tensor $\boldsymbol{\Gamma}$:

$$\mathrm{tr}\boldsymbol{\Gamma} \equiv \Gamma_{i_1 \ldots i_r, i_1 \ldots i_r}. \tag{3.15}$$

Equations (3.13) and (3.14) immediately give

$$\mathrm{tr}\boldsymbol{\Gamma}(\mathbf{y}) = \Delta\Phi(\|y\|), \tag{3.16}$$

where the n-dimensional Laplacian Δ acts on y. For determining the second function Ψ, we come back to tensor $\mathbf{C}$ (3.1) and calculate it with the help of Eqs. (3.9), (3.10), and (3.8). We find

$$\mathbf{C} = \frac{r!\,s!}{(n-1)!}\{\boldsymbol{\varepsilon}_{sr}(\nabla)\Delta\Phi + \delta_{2r+1,n}\boldsymbol{\delta}_{rr}(\nabla\otimes\nabla)\Psi\}. \tag{3.17}$$

For $r=s$, the trace of $\mathbf{C}$ can be defined and reads

$$\mathrm{tr}\mathbf{C}(\mathbf{y}) = \Delta\Psi(\|y\|). \tag{3.18}$$

On the other hand, the traces of $\boldsymbol{\Gamma}$ and $\mathbf{C}$ can be calculated directly from Eqs. (3.2) and (2.28):

$$\mathrm{tr}\boldsymbol{\Gamma} = a_0 \left\langle \int_{\mathscr{C}\times\mathscr{C}} \frac{\mathrm{tr}(d\mathscr{F}_r \otimes d\mathscr{F}'_r)}{\|\mathbf{y}+\mathbf{f}'-\mathbf{f}\|^{n-2}} \right\rangle \tag{3.19}$$

$$\mathrm{tr}\mathbf{C} = \delta_{2r+1,n} a_0 \left\langle \int_{\mathscr{C}\times\mathscr{C}} d\mathscr{F}'_r \cdot \boldsymbol{\varepsilon}_{rr}(\nabla) \cdot d\mathscr{F}_r \frac{1}{\|\mathbf{y}+\mathbf{f}'-\mathbf{f}\|^{n-2}} \right\rangle, \tag{3.20}$$

where $a_0 = [S_{n-1}(n-2)r!]^{-1}$. Consider first (3.19). It involves the trace $\mathrm{tr}(d\mathscr{F}_r \otimes d\mathscr{F}'_r) \equiv d\mathscr{F}_r \cdot d\mathscr{F}'_r$, which is the ordered scalar product of the differential exterior forms (2.3). Thus $d\mathscr{F}_r \cdot d\mathscr{F}'_r$ is *invariant with respect to rotations of* $\mathscr{C}$ *in* $\mathbb{R}^n$. Thus, the angular average in (3.19) can be inserted into the integral:

$$\Delta\Phi = \mathrm{tr}\boldsymbol{\Gamma} = a_0 \int_{\mathscr{C}\times\mathscr{C}} (d\mathscr{F}_r \cdot d\mathscr{F}'_r) \langle \|\mathbf{y}+\mathbf{f}'-\mathbf{f}\|^{-(n-2)} \rangle. \tag{3.21}$$

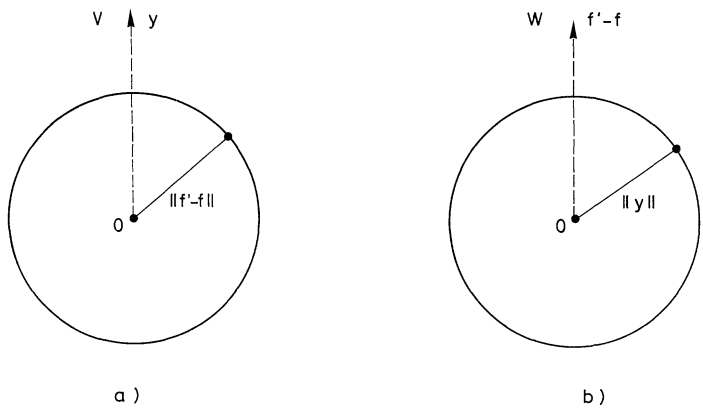

Fig. 7a and b. Newtonian potentials V and W are created by spherical distributions of respective radii $\|\mathbf{f}-\mathbf{f}'\|$ **a** and $\|\mathbf{y}\|$ **b**

Consider now (3.20). Factor $d\mathscr{F}_r' \cdot \boldsymbol{\varepsilon}_{rr}(V) \cdot d\mathscr{F}_r$ is clearly not invariant with respect to rotations of $\mathscr{C}$. However, $\mathrm{tr}\mathbf{C}$ is a scalar function of modulus $\|y\|$ only, and is therefore trivially equal to its average $\mathrm{tr}\mathbf{C}=(\mathrm{tr}\mathbf{C})$, where the *round* brackets represent the angular average

$$(A)=(S_{n-1})^{-1}\int_{S^{n-1}} d\hat{y}A(\mathbf{y}) \tag{3.22}$$

with $\hat{y}=\mathbf{y}/\|y\|$.

Thus, using $V_y \equiv V_{f'}$ in (3.20) we may write

$$\Delta\Psi=\mathrm{tr}\mathbf{C}=\delta_{2r+1,n}a_0\int_{\mathscr{C}\times\mathscr{C}}[d\mathscr{F}_r'\cdot\boldsymbol{\varepsilon}_{rr}(V_{f'})\cdot d\mathscr{F}_r](\|\mathbf{y}+\mathbf{f}'-\mathbf{f}\|^{-(n-2)}). \tag{3.23}$$

If we compare Eq. (3.23) for $\mathrm{tr}\mathbf{C}$ and the generalized Gauss' formula (2.12) for linking number, we remark that

$$\Delta\Psi(\|y\|)=\mathrm{tr}\mathbf{C}(\mathbf{y})=(S_{n-1})^{-1}\int_{S^{n-1}} du I(\mathscr{C},\tau_{\|y\|\mathbf{u}}\mathscr{C}), \tag{3.24}$$

where $\mathbf{u}, \mathbf{u}\in S^{n-1}$ is an arbitrary unit vector and where $\tau_{\|y\|\mathbf{u}}$ represents the translation of length $\|y\|$ in direction $\mathbf{u}$. Thus, $\mathrm{tr}\mathbf{C}(\|y\|)$ is exactly *the isotropic average of the linking number of manifold $\mathscr{C}$ with its translated images by a fixed distance $\|y\|$*.

We now face the simple task of calculating the two different angular averages $V=\langle\|\mathbf{y}+\mathbf{f}'-\mathbf{f}\|^{-(n-2)}\rangle$, and $W=(\|\mathbf{y}+\mathbf{f}'-\mathbf{f}\|^{-(n-2)})$. Quantity $\|\mathbf{y}+\mathbf{f}'-\mathbf{f}\|^{-(n-2)}$ is exactly, in $\mathbb{R}^n$, the *Newtonian potential* between points y and $f-f'$. It satisfies the Poisson equation [15]

$$\Delta\|\mathbf{y}+\mathbf{f}'-\mathbf{f}\|^{-(n-2)}=-(n-2)S_{n-1}\delta(\mathbf{y}+\mathbf{f}'-\mathbf{f}). \tag{3.25}$$

In V, rotations of manifold $\mathscr{C}$ reduce trivially to rotations in $\mathbb{R}^n$ of vector $\mathbf{f}'-\mathbf{f}$ rigidly attached to manifold $\mathscr{C}$ (Fig. 7a). Thus V represents the Newtonian potential created at point y by a uniform distribution of unit masses on the S^{n-1}-sphere of radius $\|\mathbf{f}'-\mathbf{f}\|$. In the same way, W is the Newtonian potential at point

$f-f'$ created by a source sphere of radius $\|y\|$ (Fig. 7b). These two Newtonian potentials V and W are *equal*[6] and read:

$$V=W=\theta(\|y\|-\|\mathbf{f}-\mathbf{f}'\|)\|y\|^{-(n-2)}+\theta(\|\mathbf{f}-\mathbf{f}'\|-\|y\|)\|\mathbf{f}-\mathbf{f}'\|^{-(n-2)}, \quad (3.26)$$

where θ is the step distribution $\theta(\zeta)=1$ for $\zeta>0$, $\theta(\zeta)=0$ for $\theta<0$. We shall need in the following the Newtonian fields associated with these spherical sources:

$$\begin{aligned} \nabla_y V &= -(n-2)\theta(\|y\|-\|\mathbf{f}-\mathbf{f}'\|)\mathbf{y}/\|y\|^n \\ \nabla_{f'} V &= -(n-2)\theta(\|\mathbf{f}-\mathbf{f}'\|-\|y\|)\frac{\mathbf{f}'-\mathbf{f}}{\|\mathbf{f}'-\mathbf{f}\|^n}, \end{aligned} \quad (3.27)$$

which vanish inside their own spherical source.

At this stage, we have all the equations determining $\Delta\Phi$ and $\Delta\Psi$. We shall use in the following $\nabla(\Delta\Phi)$ and $\Delta\Psi$. Equations (3.21), (3.23), and (3.27) give

$$\nabla(\Delta\Phi)(\mathbf{y})=-(S_{n-1}r!)^{-1}\frac{\mathbf{y}}{\|y\|^n}\int_{\mathscr{C}\times\mathscr{C}}\theta(\|y\|-\|\mathbf{f}-\mathbf{f}'\|)(d\mathscr{F}_r\cdot d\mathscr{F}_r') \quad (3.28)$$

and

$$\Delta\Psi(\|y\|)=-\delta_{2r+1,n}(S_{n-1}r!)^{-1}\int_{\mathscr{C}\times\mathscr{C}}\theta(\|\mathbf{f}-\mathbf{f}'\|-\|y\|)\frac{1}{\|\mathbf{f}'-\mathbf{f}\|^n}[d\mathscr{F}_r'\cdot\boldsymbol{\varepsilon}_{rr}(\mathbf{f}'-\mathbf{f})\cdot d\mathscr{F}_r]. \quad (3.29)$$

4. Kinematic Linking Integral of Two Manifolds

4.1. Factorization Theorem

We can now give an explicit formula for the kinematic linking integral $\mathscr{I}$ (2.15) of two manifolds $\mathscr{C}_1^r$ and $\mathscr{C}_2^s$, of dimensions r,s, with $r+s=n-1$. Both possess an associated tensor $\mathbf{C}$, which is decomposed into two parts containing function Φ and Ψ. Using factorization formula (2.30) and formula (3.17), we may write

$$\begin{aligned} \mathscr{I}(\mathscr{C}_1,\mathscr{C}_2)=\left(\frac{r!\,s!}{(n-1)!}\right)^2\int_{\mathbb{R}^n}d^n y\,\mathrm{tr}\{&[\boldsymbol{\varepsilon}_{sr}(\nabla)\Delta\Phi_1+\delta_{r,s}\boldsymbol{\delta}_{rr}(\nabla\otimes\nabla)\Psi_1] \\ &\cdot[-\boldsymbol{\varepsilon}_{rs}(\nabla)\Delta\Phi_2+\delta_{r,s}\boldsymbol{\delta}_{rr}(\nabla\otimes\nabla)\Psi_2]\}, \end{aligned} \quad (4.1)$$

where we used $\delta_{2r+1,n}\equiv\delta_{r,s}$ and the parity property $\nabla\Phi(-\mathbf{y})=-\nabla\Phi(\mathbf{y})$ in order to take all the functions in the integrand of (4.1) at point y. Using algebraic rules (3.9)–(3.14), we obtain, after simplifications:

$$\mathscr{I}(\mathscr{C}_1,\mathscr{C}_2)=\frac{r!\,s!}{(n-1)!}\int_{\mathbb{R}^n}d^n y\{(-1)^{rs+1}\nabla(\Delta\Phi_1)\cdot\nabla(\Delta\Phi_2)+\delta_{r,s}\Delta\Psi_1\Delta\Psi_2\}. \quad (4.2)$$

One must notice the vanishing in (4.2) of cross terms involving products $\Phi\Psi$, due to the opposite symmetry properties of tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\delta}$. The integrand of (4.2) is rotationally invariant. Passing then to spherical coordinates

$$\varrho=\|y\|,\ \hat{y}=\mathbf{y}/\|y\|,\ d^n y=\varrho^{n-1}d\varrho d\hat{y}, \quad (4.3)$$

---

6 The fact that $V\equiv W$ is directly related to the *covariant transformation properties of Newtonian potential in an inversion with respect to a sphere* [16]. The inversion with respect to a sphere of radius $R=(\|f-f'\|\,\|y\|)^{1/2}$ indeed transforms Fig. 7a into Fig. 7b

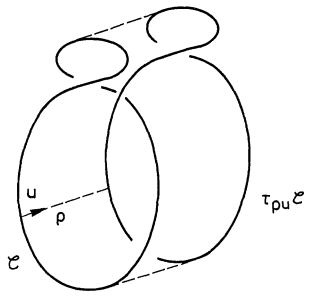

Fig. 8. An example of a twisted curve $\mathscr{C}$ making a link with its translated image $\tau_{\varrho u}\mathscr{C}$, such that $I(\mathscr{C}, \tau_{\varrho u}\mathscr{C}) \neq 0$

and defining two characteristic functions for each manifold

$$\mathscr{A}(\varrho) = S_{n-1}\varrho^{n-r-1}\left(-\frac{d}{d\varrho}\right)\Delta\Phi(\varrho), \tag{4.4a}$$

$$\mathscr{B}(\varrho) = S_{n-1}\varrho^{r}\Delta\Psi(\varrho), \tag{4.4b}$$

we obtain $\mathscr{I}$ as a single integral

$$\mathscr{I}(\mathscr{C}_1, \mathscr{C}_2) = r!\,s!/(n-1)!\,S_{n-1}\int_0^\infty d\varrho[(-1)^{rs+1}\mathscr{A}_1(\varrho)\mathscr{A}_2(\varrho) + \delta_{r,s}\mathscr{B}_1(\varrho)\mathscr{B}_2(\varrho)]. \tag{4.5}$$

In this "kinematic linking formula," characteristic functions $\mathscr{A}_\alpha$, $\mathscr{B}_\alpha (\alpha = 1, 2)$ depend only on the manifold $\mathscr{C}_\alpha$ with which they are associated. Using the explicit forms (3.28) and (3.29) of $\nabla(\Delta\Phi)$ and $\Delta\Psi$ in Definitions (4.4a) and (4.4b), we find the formulae for $\mathscr{A}$ and $\mathscr{B}$:

$$\mathscr{A}(\varrho) = \frac{1}{r!}\varrho^{-r}\int_{\mathscr{C}\times\mathscr{C}} \theta(\varrho - \|\mathbf{f}-\mathbf{f}'\|)(d\mathscr{F}_r \cdot d\mathscr{F}_r'), \tag{4.6}$$

and

$$\mathscr{B}(\varrho) = \delta_{2r+1,n}(n-2)^{-1}(r!)^{-2}\varrho^r \int_{\mathscr{C}\times\mathscr{C}} \theta(\|\mathbf{f}-\mathbf{f}'\| - \varrho)\det(\nabla_{f'}\|\mathbf{f}-\mathbf{f}'\|^{-(n-2)}, d\mathscr{F}_r, d\mathscr{F}_r'). \tag{4.7}$$

Function $\mathscr{A}$ exists for any manifold, and is given by the integral of the generalized scalar product of volume forms $d\mathscr{F}$ and $d\mathscr{F}'$. On the contrary, the $\mathscr{B}$-function exists only for a manifold $\mathscr{C}$ having a dimension $r = (n-1)/2$ (with n odd). $\mathscr{C}$ must indeed be able to link with its own translated image. According to Eqs. (4.4b) and (3.24), we have:

$$\mathscr{B}(\varrho) = \varrho^r \int_{S^{n-1}} du I(\mathscr{C}, \tau_{\varrho\mathbf{u}}\mathscr{C}); \tag{4.8}$$

$\mathscr{B}$ is thus proportional to the isotropic average of the linking number of $\mathscr{C}$ with its *translated image* $\tau_{\varrho\mathbf{u}}\mathscr{C}$ at a distance ϱ (Fig. 8). $\mathscr{C}$ and $\tau\mathscr{C}$ have the same dimension r and can be linked if and only if: $2r+1 = n$, as expected. Equation (4.7) gives the explicit form of the angular average (4.8). Because Gauss linking number I satisfies (2.5), function $\mathscr{B}$ (4.8) verifies $\mathscr{B} = (-1)^{r^2+1}\mathscr{B}$ which can also be checked

directly on (4.7). Therefore for r^2 even, that is for r itself even, $\mathscr{B}$ vanishes identically. Finally, function $\mathscr{B}$ is *nonvanishing* only for

$$\begin{cases}\dim \mathscr{C} = 2q+1, \; q \in \mathbb{N} \\ n = 2\dim \mathscr{C} + 1 = 4q+3 . \end{cases} \tag{4.9}$$

Using Eq. (2.17), we may also write (4.5) in a final form [7].

Theorem 1.

$$\left\langle \int_{\mathbb{R}^n} I^2(\mathbf{x}) d^n x \right\rangle = \frac{r!\,s!}{(n-1)!\,S_{n-1}} \int_0^\infty d\varrho [\mathscr{A}_1(\varrho)\mathscr{A}_2(\varrho) + \delta_{r,s}\mathscr{B}_1(\varrho)\mathscr{B}_2(\varrho)] . \tag{4.10}$$

This formula (4.10) generalizes, for arbitrary closed manifolds in $\mathbb{R}^n$, the factorization property (1.3) discovered by Pohl [6] for curves in $\mathbb{R}^3$. The simple case of two closed curves ($r=s=1$, $n=3$) corresponds in (4.9) to the first value $q=0$. Then, one can expect the presence of two terms in integral $\mathscr{I}$. Pohl's original proof [6] was given in fact only for convex plane curves, for which $\mathscr{B}$ vanishes identically. The existence of a characteristic $\mathscr{B}$ function was then established for general skew curves in $\mathbb{R}^3$, by Des Cloizeaux and Ball [7], using a method based on Fourier transforms, and by Duplantier [8], by a direct method, which is the origin of the method used here.

In summary, we notice that functions $\mathscr{A}$ and $\mathscr{B}$ are directly related to the functions Φ and Ψ appearing in the fundamental lemma (3.6). The existence of two functions $\mathscr{A}$ and $\mathscr{B}$ is thus a nontrivial result of orthogonal group theory, which gives the decomposition of $\mathbf{\Gamma}$ into two irreducible components.

4.2. Properties of Functions $\mathscr{A}$ and $\mathscr{B}$

4.2a. Parametric Forms. Consider a particular parametrization $\{u_a, a=1,\ldots,r\}$ of a manifold $\mathscr{C}$. The ordered scalar product $d\mathscr{F}\cdot d\mathscr{F}'$ of volume forms can be transformed with the help of Eqs. (2.10) and (2.7):

$$\frac{1}{r!} d\mathscr{F}_r \cdot d\mathscr{F}'_r = \det_{(a,b)} \left[\frac{\partial \mathbf{f}}{\partial u_a} \cdot \frac{\partial \mathbf{f}'}{\partial u'_b} \right] (du_1 \wedge \ldots \wedge du_r)(du'_1 \wedge \ldots \wedge du'_r), \tag{4.11}$$

$\dfrac{\partial \mathbf{f}}{\partial u_a} \cdot \dfrac{\partial \mathbf{f}'}{\partial u'_b}$ is the scalar product of the two tangent vectors along lines u_a, u'_b at f and f'. The determinant is, as it must be, invariant with respect to rotations of $\mathscr{C}$. The characteristic function $\mathscr{A}$ (4.6) has the parametrized form:

$$\mathscr{A}(\varrho) = \varrho^{-r} \int \theta(\varrho - \|\mathbf{f}-\mathbf{f}'\|) \det_{(a,b)} \left(\frac{\partial \mathbf{f}}{\partial u_a} \cdot \frac{\partial \mathbf{f}'}{\partial u'_b} \right) du_1 \ldots du_r du'_1 \ldots du'_r . \tag{4.12}$$

In a similar way, substituting Eq. (2.10) into (4.7), and using (2.8), gives

$$\mathscr{B}(\varrho) = \delta_{2r+1,n}(n-2)^{-1}\varrho^r \int \theta(\|\mathbf{f}-\mathbf{f}'\|-\varrho) \cdot \det \left(\nabla_{f'} \|\mathbf{f}-\mathbf{f}'\|^{-(n-2)}, \right.$$
$$\left. \cdot \frac{\partial \mathbf{f}}{\partial u_1}, \ldots, \frac{\partial \mathbf{f}}{\partial u_r}, \frac{\partial \mathbf{f}'}{\partial u'_1}, \ldots, \frac{\partial \mathbf{f}'}{\partial u'_r} \right) du_1 \ldots du_r du'_1 \ldots du'_r . \tag{4.13}$$

7 Owing to (4.9), the $\mathscr{B}_1\mathscr{B}_2$ term exists only for $r=s=2q+1$ and thus $(-1)^{rs+1} \equiv 1$

4.2b. Limit $\varrho \to 0$. In this limit, $\mathscr{A}$ (4.6) depends only on the local properties of $\mathscr{C}$, whereas $\mathscr{B}$ involves the whole structure. The measure $d^r f$ of an infinitesimal r-volume element or "area" of $\mathscr{C}$ reads, for parameters u_a:

$$d^r f = \left[\det_{(a,b)}\left(\frac{\partial \mathbf{f}}{\partial u_a}\cdot\frac{\partial \mathbf{f}}{\partial u_b}\right)\right]^{1/2} du_1 \ldots du_r, \tag{4.14}$$

and the total *r-volume* or "area" of $\mathscr{C}$ then reads

$$S \equiv \int_{\mathscr{C}} d^r f. \tag{4.15}$$

In the integrand of $\mathscr{A}$, one has $\|\mathbf{f}-\mathbf{f}'\| \leqq \varrho$ and for $\varrho \to 0$, $f \approx f'$. Thus, we can write in this limit

$$\det\left(\frac{\partial \mathbf{f}}{\partial u_a}\cdot\frac{\partial \mathbf{f}'}{\partial u'_b}\right) \simeq \left|\det\left(\frac{\partial \mathbf{f}}{\partial u_a}\cdot\frac{\partial \mathbf{f}}{\partial u_b}\right)\right|^{1/2}\left|\det\left(\frac{\partial \mathbf{f}'}{\partial u'_a}\cdot\frac{\partial \mathbf{f}'}{\partial u'_b}\right)\right|^{1/2}.$$

Thus, using Definition (4.14), we find for $\mathscr{A}$:

$$\mathscr{A}(\varrho) \underset{\varrho\to 0}{\approx} \varrho^{-r} \int_{\mathscr{C}\times\mathscr{C}} \theta(\varrho - \|\mathbf{f}-\mathbf{f}'\|) d^r f d^r f'.$$

For $\varrho \to 0$, the neighbourhood of f on $\mathscr{C}^r$ is locally flat and tangent to $\mathbb{R}^r$, and integration of f' gives the volume $\frac{S_{r-1}}{r}\varrho^r$ of a ball of radius ϱ. We finally find

$$\mathscr{A}(0) = \frac{S_{r-1}}{r} S, \tag{4.16}$$

where S is the total r-volume (4.15) of $\mathscr{C}$. For a closed curve of length L in $\mathbb{R}^3$, $\mathscr{A}(0) = 2L (S_0 = 2)$, in agreement with previous results [7].

For $\varrho \to 0$, in (4.8), $\tau_{\varrho u}\mathscr{C} = \mathscr{C}$ and $\mathscr{B}$ reads [10]:

$$\mathscr{B}(\varrho) \underset{\varrho\to 0}{\approx} S_{n-1}\varrho^{(n-1)/2} I_{\text{self}}, \tag{4.17}$$

where I_{self} is the *self-linking number* of manifold $\mathscr{C}$, formally obtained from Gauss' integral (2.1) by making the two integration points describe the same manifold $\mathscr{C}$. It is known that this Gauss self-linking number is not a topological invariant counting the number of knots made by the manifold with itself [17]. For instance, it differs from zero for a skew closed polygon in $\mathbb{R}^3$, even if this polygon has no knot [18].

4.2c. Chords of Constant Length. It is possible to write expressions of functions $\mathscr{A}$ and $\mathscr{B}$ uniquely in terms of the *chords of manifold* $\mathscr{C}$ *of constant length* ϱ. Such is the case for the original result (1.4) of Pohl for a plane closed curve [6] in $\mathbb{R}^3$. The calculation is made in Appendix D. We find

$$\mathscr{A}(\varrho) = \frac{1}{r!}\varrho^{1-r} \int_{\mathscr{C}\times\mathscr{C}} \delta(\varrho - \|\mathbf{f}-\mathbf{f}'\|) \operatorname{tr}[(\hat{f}\hat{f}'\cdot d\mathscr{F}_r)\otimes(\hat{f}\hat{f}'\cdot d\mathscr{F}'_r)], \tag{4.18}$$

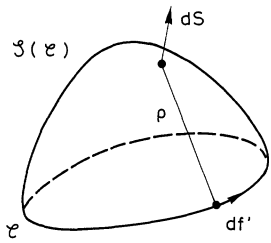

Fig. 9. Differential vectors building $\mathscr{B}(\varrho)$ for a curve in $\mathbb{R}^3$

where $\hat{ff}' = (\mathbf{f}-\mathbf{f}')/\|\mathbf{f}-\mathbf{f}'\|$ is the unit vector joining $\mathbf{f}'$ to $\mathbf{f}$. The trace reads explicitly

$$\mathrm{tr}[(\hat{ff}' \cdot d\mathscr{F}_r) \otimes (\hat{ff}' \cdot d\mathscr{F}_r')] = \sum_{i,j,i_2,\ldots,i_r} (\hat{ff}')_i (df_i \wedge df_{i_2} \wedge \ldots \wedge df_{i_r})(\hat{ff}')_j \cdot (df_j' \wedge df_{i_2}' \wedge \ldots \wedge df_{i_r}').$$

Thus $\mathscr{A}$ is obtained by sweeping manifold $\mathscr{C}$ with a needle of length ϱ, the extremities of which are constantly in contact with $\mathscr{C}$. For function $\mathscr{B}$, we find (Appendix D)

$$\mathscr{B}(\varrho) = \delta_{2r+1,n} \frac{1}{r!} \frac{1}{(r+1)!} \varrho^{-r} \int_{f \in \mathscr{S}(\mathscr{C})} \int_{f' \in \mathscr{C}} \delta(\varrho - \|\mathbf{f}-\mathbf{f}'\|) \det(d\mathscr{F}_{r+1}, d\mathscr{F}_r'), \quad (4.19)$$

where $d\mathscr{F}_{r+1}$ denotes the differential $(r+1)$-volume form attached to a Seifert-manifold $\mathscr{S}(\mathscr{C})$ of manifold $\mathscr{C}$. An obvious consequence of the preceding formulae is

$$\mathscr{A}(\varrho) = 0, \mathscr{B}(\varrho) = 0 \quad \text{for} \quad \varrho > \text{diameter}(\mathscr{C}). \quad (4.20)$$

Therefore $\mathscr{A}$ and $\mathscr{B}$ have *bounded supports.* Specifying these results for a closed curve $\mathscr{C}$ in $\mathbb{R}^3$, with $r=1$, Eq. (4.18) gives Pohl's form (1.4) [7]. For $\mathscr{B}$, we can obtain an interesting formula. We define at a point f on the two-dimensional surface $\mathscr{S}(\mathscr{C})$ the normal vector $d\mathbf{S} : dS_k = \frac{1}{2}\varepsilon^{kij} df_i \wedge df_j$. We have then trivially in (4.19) $\det(d\mathscr{F}_2, d\mathscr{F}_1') = 2d\mathbf{S} \cdot d\mathbf{f}'$ and $\mathscr{B}$ reads

$$\mathscr{B}(\varrho) = \varrho^{-1} \int \delta(\varrho - \|\mathbf{f}-\mathbf{f}'\|) d\mathbf{S} \cdot d\mathbf{f}'. \quad (4.21)$$

This gives a simple geometric interpretation for $\mathscr{B}$ (Fig. 9).

Hyperplane Manifold

Definition. A manifold $\mathscr{C}^r$ in $\mathbb{R}^n$, is *hyperplane*, if, having the dimension r, it can be embedded in a subspace $\mathbb{R}^{r+1}$ of $\mathbb{R}^n$ $(r+1 \leqq n)$. This generalizes the notion of a *plane* curve $(r=1)$ in $\mathbb{R}^3$, which can be embedded in $\mathbb{R}^2$. The coordinates of a point of $\mathscr{C}$ are

$$\mathbf{f} = \{f_l, l = 1, \ldots, n\} \quad \forall l > r+1 : f_l = 0. \quad (4.22)$$

For such a hyperplane manifold, *considered as embedded in* $\mathbb{R}^{r+1}$, one can define a normal vector

$$dS_i \equiv \frac{1}{r!} \varepsilon^{il_1 \ldots l_r} df_{l_1} \wedge \ldots \wedge df_{l_r}, \quad (4.23)$$

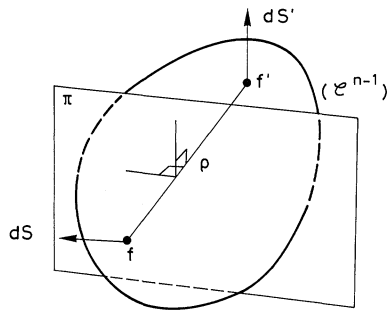

Fig. 10. Construction of $\mathscr{A}(\varrho)$ for a hyperplane manifold $\mathscr{C}^{n-1}$. (Π) is the hyperplane orthogonal to segment ff'

where the indices $i, l_a (a=1,\ldots,r)$, take their values in set $\{1,\ldots,r+1\}$ only. It is actually also possible to express volume form $d\mathscr{F}_r$ in function of dS:

$$d\mathscr{F}_r|_{j_1\ldots j_r} = \varepsilon^{ij_1\ldots j_r} dS_i, \tag{4.24}$$

a formula which follows from the general identity [14]:

$$\varepsilon^{j_1\ldots j_{r+1}}\varepsilon^{l_1\ldots l_{r+1}} \equiv \delta^{l_1\ldots l_{r+1}}_{j_1\ldots j_{r+1}}$$

valid for $2(r+1)$ indices j, l taking their values in the set $\{1,\ldots,r+1\}$. Calculating $(dS)^2$, one can check that the Euclidean norm of dS gives the elementary r-volume (4.14) of the manifold: $\|dS\| = d^r f$. Substituting (4.24) into function $\mathscr{A}$ (4.18) and using (2.9), we finally find for a hyperplane manifold:

$$\mathscr{A}(\varrho) = \frac{1}{r}\varrho^{1-r} \int_{\mathscr{C}\times\mathscr{C}} \delta(\varrho - \|\mathbf{f}-\mathbf{f}'\|)\{d\mathbf{S}\cdot(\mathbb{1} - \hat{ff}'\otimes\hat{ff}')\cdot d\mathbf{S}'\}, \tag{4.25}$$

where $\mathbb{1}$ denotes the usual unit tensor of $\mathbb{R}^{r+1}$: $\mathbb{1}_{ij} = \delta_{ij} \cdot \mathbb{\Pi} = \mathbb{1} - \hat{ff}'\otimes\hat{ff}'$ projects therefore onto the hyperplane (Π) orthogonal to the direction of $\hat{ff}'$ (Fig. 10). The remarkably simple formula (4.25) allows practical calculations of $\mathscr{A}$ for hyperplane manifolds (Sect. 7). *The function $\mathscr{B}$ of a hyperplane manifold vanishes identically.* This is obvious from formula (4.13) where the $2r+1=n$ vectors of the determinant all lie in $\mathbb{R}^{r+1}$.

5. Kinematic Linking Integral of N Manifolds

5.1. Fourier Transforms

The aim of this section is to evaluate the general linking integral $\mathscr{I}$ (2.20), (2.31), for an arbitrary number N of manifolds. We shall give a factorization theorem analogous to Theorem (4.10). The convolution integral (2.31) is best evaluated by means of Fourier transforms. We define the Fourier transform $\hat{h}$ of any function h by

$$\hat{h}(\mathbf{p}) = \int d^n y e^{i\mathbf{p}\cdot\mathbf{y}} h(\mathbf{y}), \tag{5.1}$$

where $\mathbf{p}$ and $\mathbf{y}$ belong to vectorial space $\mathbb{R}^n$.

Using the Fourier representation

$$\delta(\mathbf{y}_1 + \ldots + \mathbf{y}_N) = \frac{1}{(2\pi)^n} \int d^n p e^{i\mathbf{p}\cdot(\mathbf{y}_1 + \ldots + \mathbf{y}_N)}, \tag{5.2}$$

we obtain immediately (2.31) as a Fourier integral

$$\mathscr{I}(\mathscr{C}_1, \ldots, \mathscr{C}_N) = \frac{1}{(2\pi)^n} \int d^n p \, \mathrm{tr}\left[\prod_{\alpha=1}^{N} \hat{\mathbf{C}}_\alpha(\mathbf{p})\right], \tag{5.3}$$

where the tensors $\hat{\mathbf{C}}$ are defined by:

$$\hat{\mathbf{C}}(\mathbf{p}) = \int d^n y e^{i\mathbf{p}\cdot\mathbf{y}} \mathbf{C}(\mathbf{y}).$$

We have obtained in (3.17) the general form of tensor $\mathbf{C}$. The decomposition of its Fourier transform $\hat{\mathbf{C}}$ follows immediately

$$\hat{\mathbf{C}}(\mathbf{p}) = \frac{r!s!}{(n-1)!} \{\boldsymbol{\varepsilon}_{sr}[(\nabla(\Delta\Phi))\hat{}(\mathbf{p})] + \delta_{2r+1,n}\boldsymbol{\delta}_{rr}[((\nabla\otimes\nabla)\Psi)\hat{}(\mathbf{p})]\}. \tag{5.4}$$

The Fourier transforms are related to those of functions $\mathscr{A}$ and $\mathscr{B}$. Defining the angular average

$$Q_n(\|p\|\varrho) = \frac{1}{S_{n-1}} \int_{S^{n-1}} d\hat{y} e^{i\mathbf{p}\cdot\mathbf{y}} \tag{5.5}$$

and using Eqs. (4.4a) and (4.4b) and standard properties of Fourier transforms, we obtain

$$\begin{aligned} \frac{r!s!}{(n-1)!}(\nabla(\Delta\Phi))\hat{}(\mathbf{p}) &= -i\hat{p}\mathscr{A}[p], \\ \frac{(r!)^2}{(n-1)!}(\nabla\otimes\nabla\Psi)\hat{}(\mathbf{p}) &= \hat{p}\otimes\hat{p}\mathscr{B}[p], \end{aligned} \tag{5.6}$$

where $p = \|p\|$ and $\hat{p} = \mathbf{p}/p$, and the functions $\mathscr{A}$ and $\mathscr{B}$ read in wave vector space

$$\mathscr{A}[p] = -\frac{r!s!}{(n-1)!} \frac{d}{dp} \int_0^\infty d\varrho Q_n(p\varrho)\varrho^{r-1}\mathscr{A}(\varrho), \tag{5.7a}$$

$$\mathscr{B}[p] = \frac{(r!)^2}{(n-1)!} \int_0^\infty d\varrho Q_n(p\varrho)\varrho^r \mathscr{B}(\varrho). \tag{5.7b}$$

Thus (5.4) becomes

$$\hat{\mathbf{C}}(\mathbf{p}) = -i\boldsymbol{\varepsilon}_{sr}(\hat{p})\mathscr{A}[p] + \delta_{2r+1,n}\boldsymbol{\delta}_{rr}(\hat{p}\otimes\hat{p})\mathscr{B}[p]. \tag{5.8}$$

Defining an angle θ by $\hat{p}\cdot\hat{y} = \cos\theta$, we find $Q_n(p\varrho) = S_{n-2}\int_0^\pi e^{ip\varrho\cos\theta}(\sin\theta)^{n-2}d\theta$, where factor S_{n-2} corresponds to $O(n-2)$ rotations about $\hat{p}$. In terms of Bessel functions [19]:

$$Q_n(x) = \frac{(2\pi)^{n/2}}{S_{n-1}} J_{\frac{n}{2}-1}(x)/x^{\frac{n}{2}-1}, \tag{5.9}$$

where J_ν is the standard Bessel function of index ν[8].

8 Its series expansion is:

$$J_\nu(x) \equiv \frac{x^\nu}{2^\nu} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{2k}k!\Gamma(\nu+k+1)}$$

5.2. Factorization Theorem

We have to calculate the trace of the product

$$\mathbf{P}(\mathbf{p}) \equiv \prod_{\alpha=1}^{N} \hat{\mathbf{C}}_{\alpha}(\mathbf{p}), \tag{5.10}$$

where each tensor $\hat{\mathbf{C}}$ is given by (5.8). As already discussed in Sect. (2.2) the dimensions r and s of the manifolds are supposed to satisfy conditions (2.18). We then distinguish two cases.

a) $r \neq s$.

In this case $\left.\begin{matrix} 2r+1 \\ 2s+1 \end{matrix}\right\} \neq n$ and only $\mathscr{A}$ functions exist. Moreover the number of manifolds N is *even*. Product $\mathbf{P}$ (5.10) involves then a tensorial factor [see (5.8)] $[\boldsymbol{\varepsilon}_{sr}(\hat{p}) \cdot \boldsymbol{\varepsilon}_{rs}(\hat{p})]^{N/2}$, which, according to rules (3.9)–(3.12) for unit vector $\hat{p}$, reads simply $\boldsymbol{\delta}_{ss}(\hat{p} \otimes \hat{p})$. Taking the trace gives therefore trivially [use (3.14)]:

$$\mathrm{tr}\,\mathbf{P}(\mathbf{p}) = (-1)^{(rs+1)N/2} \frac{(n-1)!}{r!s!} \prod_{\alpha=1}^{N} \mathscr{A}_{\alpha}[p], \tag{5.11}$$

where N is necessarily even.

b) $r = s$, $2r+1 = n$.

In this case, all the manifolds have the same dimension r and their number N is arbitrary. Owing to Eqs. (3.9)–(3.12) the tensors $\boldsymbol{\varepsilon}_{rr}(\hat{p})$ and $\boldsymbol{\delta}_{rr}(\hat{p} \otimes \hat{p})$ obey very simple multiplication rules. We then define the isomorphism

$$\begin{aligned} -i\boldsymbol{\varepsilon}_{rr}(\hat{p}) &\to \varepsilon, \\ \boldsymbol{\delta}_{rr}(\hat{p} \otimes \hat{p}) &\to 1, \end{aligned} \tag{5.12}$$

where ε, 1 are two objects, which, according to Eqs. (3.9)–(3.12), obey the algebraic rules:

$$\varepsilon^2 = (-1)^{r^2+1}, \qquad \varepsilon \cdot 1 = 1 \cdot \varepsilon = \varepsilon, \qquad 1^2 = 1. \tag{5.13}$$

The image of tensor $\hat{\mathbf{C}}(\mathbf{p})$ by this isomorphism is $\mathscr{C}[p]$

$$\mathscr{C}[p] = \varepsilon \mathscr{A}[p] + \mathscr{B}[p]. \tag{5.14}$$

The image of product $\mathbf{P}(\mathbf{p})$ is accordingly the product

$$\mathscr{P}[p] = \prod_{\alpha=1}^{N} (\varepsilon \mathscr{A}[p] + \mathscr{B}[p]). \tag{5.15}$$

According to rules (5.13), product $\mathscr{P}$ can always be written as a linear combination of 1 and ε:

$$\mathscr{P}[p] = \mathscr{X}[p] + \varepsilon \mathscr{Y}[p]. \tag{5.16}$$

Inverting the isomorphism (5.12), tensor $\mathbf{P}$ is obtained as

$$\mathbf{P}(\mathbf{p}) = \boldsymbol{\delta}_{rr}(\hat{p} \otimes \hat{p}) \mathscr{X}[p] - i\boldsymbol{\varepsilon}_{rr}(\hat{p}) \mathscr{Y}[p], \tag{5.17}$$

and its trace reads

$$\mathrm{tr}\,\mathbf{P}(\mathbf{p}) = \frac{(n-1)!}{r!s!}\mathscr{X}[p]. \tag{5.18}$$

At this stage, we see that the case a) $r \neq s$, can be eventually treated by the same method. For $r \neq s$, N even, we take the rule

$$\varepsilon^2 = (-1)^{rs+1}. \tag{5.19}$$

No terms $\mathscr{B}$ exist and Eqs. (5.15)–(5.19) give together the result (5.11) we want.

Substituting Eqs. (5.14)–(5.18) into integral (5.3), and performing a partial integration on the angular variables, we obtain the theorem:

Theorem II. *The integral $\mathscr{I}$ of the cyclic product of the successive linking numbers of N closed manifolds $\mathscr{C}_\alpha$, $\alpha = 1, \ldots, N$, with alternate dimensions r and s, $r+s=n-1$, (N being even if $r \neq s$) over the group of motions of the manifolds, can be written as the single integral*

$$\mathscr{I}(\mathscr{C}_1, \ldots, \mathscr{C}_N) = j \int_0^\infty dp\, p^{n-1}\mathscr{X}[p], \tag{5.20}$$

where $j = \dfrac{1}{(2\pi)^n} S_{n-1} \dfrac{(n-1)!}{r!s!}$ *and where function $\mathscr{X}$ is the even part in ε of the algebraic product*

$$\mathscr{P}(p) \equiv \prod_{\alpha=1}^{N} (\varepsilon \mathscr{A}_\alpha[p] + \mathscr{B}_\alpha[p])$$
$$= \mathscr{X}[p] + \varepsilon \mathscr{Y}[p]$$

calculated with the rule: $\varepsilon^2 = (-1)^{rs+1}$. $\mathscr{A}_\alpha, \mathscr{B}_\alpha$ are characteristic functions associated with each manifold $\mathscr{C}_\alpha$. The function $\mathscr{B}_\alpha$ exists only for manifolds $\mathscr{C}_\alpha$ such that $2\dim \mathscr{C}_\alpha + 1 = n$, n being odd. In this case, $\mathscr{B}_\alpha$ differs from zero only if $\dim \mathscr{C}_\alpha$ is odd.

This theorem extends to $\mathbb{R}^n$, and to arbitrary closed manifolds, a theorem proven in [8] for closed curves in $\mathbb{R}^3$. The latter case corresponds to $\dim \mathscr{C}_\alpha = 1$, $\forall \alpha$, and $r = s = 1$, $n = 3$. According to (4.9), product $\mathscr{P}$ is complete only for a set of manifolds such that

$$\begin{cases} \dim \mathscr{C}_\alpha = r = 2q+1 & \forall \alpha = 1, \ldots, N, \quad q \in \mathbb{N} \\ n = 2r+1 = 4q+3. \end{cases} \tag{5.21}$$

This gives space dimensions $n = 3, 7, 11 \ldots !$. Then the algebraic rule (5.19) reduces to $\varepsilon^2 = (-1)^{(2q+1)^2+1} \equiv 1$. In all other cases, only $\mathscr{A}$ functions exist. Then the integrand $\mathscr{X}$ is a simple product $\prod_{\alpha=1}^{N} \mathscr{A}_\alpha$. If all the manifolds have the same dimension r such that $2r+1 = n$ (r being even) their number N can be arbitrary. If this number N is odd, $\mathscr{X}$ vanishes identically and so does $\mathscr{I}$. If the dimensions of the manifolds are r and s alternatively, $r \neq s$, then their number N must be even.

We finally remark that the product $\mathscr{P}$ is *Abelian*. This means that one can exchange the order of the manifolds in $\mathscr{I}$. However, if $r \neq s$, one must respect the sequence of dimensions $r, s, r \ldots$ for defining linking numbers. We therefore obtain the nontrivial corollary:

Corollary. *The cyclic linking integral* $\mathscr{I}(\mathscr{C}_1, \ldots, \mathscr{C}_\alpha, \ldots, \mathscr{C}_\beta, \ldots, \mathscr{C}_N)$ *is invariant under any transposition of two manifolds* $\mathscr{C}_\alpha$ *and* $\mathscr{C}_\beta$ *having the same dimension.*

5.3. Properties of $\mathscr{A}$ *and* $\mathscr{B}$ *in Wave Vector Space*

The calculation of $\mathscr{A}[p]$, $\mathscr{B}[p]$ defined in (5.7a) and (5.7b) is given in Appendix E. We find the very simple expression:

$$\mathscr{A}[p] = \frac{(n-r-1)!}{(n-1)!} p^{-1} \int_{\mathscr{C}\times\mathscr{C}} \langle e^{i\mathbf{p}\cdot(\mathbf{f}-\mathbf{f}')} \rangle_{S^{n-1}} (d\mathscr{F}_r \cdot d\mathscr{F}'_r), \tag{5.22}$$

or using the explicit form (5.9) of the angular average (5.5), we have

$$\mathscr{A}[p] = \frac{(n-r-1)!}{(n-1)!} \frac{(2\pi)^{n/2}}{S_{n-1}} p^{-n/2} \int_{\mathscr{C}\times\mathscr{C}} \zeta^{1-n/2} J_{\frac{n}{2}-1}(p\zeta)(d\mathscr{F}_r \cdot d\mathscr{F}'_r), \tag{5.23}$$

where $\zeta \equiv \|\mathbf{f}-\mathbf{f}'\|$. In a similar way, $\mathscr{B}[p]$ is given by

$$\mathscr{B}[p] = \delta_{2r+1,n} (2\pi)^{n/2} \frac{1}{(n-1)! S_{n-1}} \int_{\mathscr{C}\times\mathscr{C}} (p\zeta)^{-n/2} J_{n/2}(p\zeta) \det(\mathbf{f}-\mathbf{f}', d\mathscr{F}_r, d\mathscr{F}'_r). \tag{5.24}$$

Recalling Eqs. (5.5) and (5.9), it is possible to express formally $\mathscr{B}$ as an angular average over the unit sphere S^{n+1} in $\mathbb{R}^{n+2}$:

$$\mathscr{B}[p] = \frac{1}{n!} \delta_{2r+1,n} \int_{\mathscr{C}\times\mathscr{C}} \langle e^{i\mathbf{p}\cdot(\mathbf{f}-\mathbf{f}')} \rangle_{S^{n+1}} \det(\mathbf{f}-\mathbf{f}', d\mathscr{F}_r, d\mathscr{F}'_r), \tag{5.25}$$

where $\mathbf{p} \in \mathbb{R}^{n+2}$. The characteristic functions $\mathscr{A}$ and $\mathscr{B}$, in momentum space, may be viewed as exploring, at wavelength $2\pi/p$, the geometrical properties of the manifold. Function $\mathscr{A}$ involves the invariant scalar product of volume-forms, whereas function $\mathscr{B}$ reads as the interference superposition of infinitesimal volume elements.

Asymptotic Limits. The linking integral $\mathscr{I}$ (2.20) is necessarily convergent, because linking numbers are finite integers, and because the volume of the group of motions, where finite manifolds are linked together, is finite. We check the convergence of the momentum integral (5.20) with the help of the asymptotic behaviours of $\mathscr{A}$ and $\mathscr{B}$.

Limits $p \to 0$. They are trivial. Using Eqs. (5.7a) and (5.7b) and expanding Q_n (5.9) for small p, we find

$$\underset{p\to 0}{\mathscr{A}[p]} = p\mathscr{A}_0 + \ldots, \qquad \mathscr{B}[p] = \mathscr{B}_0 + \ldots, \tag{5.26}$$

$\mathscr{A}_0$ and $\mathscr{B}_0$ are constants proportional to r^{th} moments of $\mathscr{A}(\varrho)$ and $\mathscr{B}(\varrho)$ respectively, which exist since $\mathscr{A}$ and $\mathscr{B}$ have finite supports in direct space.

Limits $p \to \infty$. For studying the large p limit of $\mathscr{A}[p]$ the asymptotic form of J_ν [20]

$$J_\nu(x) \underset{x\to\infty}{\approx} \left(\frac{2}{\pi x}\right)^{1/2} \cos(x - \nu\pi/2 - \pi/4) \tag{5.27}$$

does not rapidly decrease, and large p limits cannot be trivially extracted from (5.23). We find in Appendix F:

$$\mathscr{A}[p] \underset{p\to\infty}{\approx} c\mathscr{A}(0) p^{-(r+1)}, \tag{5.28}$$

where c is a numerical constant and where $\mathscr{A}$ ($\varrho = 0$) is the value (4.16). Inserting (5.27) into the explicit integral (5.24), we can majorize $\mathscr{B}$ by the integral of the modulus and get immediately

$$|\mathscr{B}[p]| < \mathscr{B}_\infty p^{-(n+1)/2}, \tag{5.29}$$

where $\mathscr{B}_\infty$ is a constant. As $2r+1=n$, for $\mathscr{B}$ to exist, both power law decays for $\mathscr{A}$ and $\mathscr{B}$ coincide. Coming back to the momentum integral $\mathscr{I}$, we may evaluate the product $\mathscr{P}$ for large p:

$$|\mathscr{X}|, |\mathscr{Y}| \underset{p\to\infty}{\propto} p^{-\frac{N}{2}(n+1)}. \tag{5.30}$$

Furthermore, there are no divergences at the origin. Thus, as expected, we check that, for $N \geqq 2$, Fourier integral $\mathscr{I}$ (5.20) is *absolutely convergent.*

6. Mutual Inductances and Contacts of Manifolds

In Theorem (5.20), use was only made of the even part $\mathscr{X}$ of the product $\mathscr{P}$. A geometrical interpretation can also be given to the odd part $\mathscr{Y}$. To find this interpretation, it is necessary to consider not only linking numbers of manifolds, but also their mutual inductances and contacts, which are defined below.

6.1. Definitions

We consider in this section manifolds having the same dimension r. The mutual inductance M of two such manifolds $\mathscr{C}^r_1$, $\mathscr{C}^r_2$, embedded in space $\mathbb{R}^n$, is defined by

$$M = [S_{n-1}(n-2)r!]^{-1} \int_{\mathscr{C}_1 \times \mathscr{C}_2} \frac{d\mathscr{F}_r \cdot d\mathscr{G}_r}{\|\mathbf{f}-\mathbf{g}\|^{n-2}}, \tag{6.1}$$

where $d\mathscr{F}_r$, $d\mathscr{G}_r$ are the r-volume forms (2.3) of $\mathscr{C}_1$ and $\mathscr{C}_2$.

This definition is the generalization to $\mathbb{R}^n$ of the notion of mutual inductance of two circuits, in electromagnetism in $\mathbb{R}^3$ [13]. For two closed curves, ($n=3$, $r=1$) the mutual inductance is given by Neumann's formula

$$M = \frac{1}{4\pi} \int_{\mathscr{C}_1 \times \mathscr{C}_2} \frac{d\mathbf{f} \cdot d\mathbf{g}}{\|\mathbf{f}-\mathbf{g}\|}. \tag{6.2}$$

In Eq. (6.1), the scalar product of tangent vectors appearing in (6.2) becomes the scalar product of differential forms, given by (4.11). The *contact distribution* C of two closed manifolds $\mathscr{C}_1^r$ and $\mathscr{C}_2^r$ is defined in a similar way by

$$C=(r!)^{-1}\int\limits_{\mathscr{C}_1\times\mathscr{C}_2}\delta(\mathbf{f}-\mathbf{g})(d\mathscr{F}_r\cdot d\mathscr{G}_r), \tag{6.3}$$

where C is a distribution of dimension $2r-n$ in length units, as shown by a trivial dimensional analysis on (2.3). This generalizes to $\mathbb{R}^n$ the contact distribution introduced in [8] for closed curves in $\mathbb{R}^3$. Let us now make clear the relation existing between linking number I, mutual inductance M and contact distribution C. The translation $\mathbf{x}$ of $\mathscr{C}_2$ with respect to $\mathscr{C}_1$ reads

$$\mathbf{f}-\mathbf{g}\to\mathbf{f}-\mathbf{g}-\mathbf{x} \tag{6.4}$$

and substitution of (6.4) into Eqs. (6.1) and (6.3) gives two functions $M(\mathbf{x})$ and $C(\mathbf{x})$. Defining the "*mutual inductance tensor*" $\mathbf{\Gamma}_{12}$[9]:

$$\mathbf{\Gamma}_{12}(\mathbf{x})=(S_{n-1}(n-2)r!)^{-1}\int\limits_{\mathscr{C}_1\times\mathscr{C}_2}\frac{d\mathscr{F}_r\otimes d\mathscr{G}_r}{\|\mathbf{f}-\mathbf{g}-\mathbf{x}\|^{n-2}}, \tag{6.5}$$

we may write I, M, C solely in terms of this tensor:

$$I(\mathbf{x})=\mathrm{tr}\{\boldsymbol{\varepsilon}_{rr}(\nabla)\cdot\mathbf{\Gamma}_{12}(\mathbf{x})\}\ \text{(a)},\qquad M(\mathbf{x})=\mathrm{tr}\,\mathbf{\Gamma}_{12}(\mathbf{x})\ \text{(b)},\qquad C(\mathbf{x})=-\Delta\,\mathrm{tr}\,\mathbf{\Gamma}_{12}(\mathbf{x})\ \text{(c)}, \tag{6.6}$$

where ∇ and Δ act on variable $\mathbf{x}$. Note that a linking number and a mutual inductance can be simultaneously defined only if both manifolds have the same dimension

$$r=(n-1)/2, \tag{6.7}$$

n being *odd*. The process could be iterated by considering higher derivatives $(\boldsymbol{\varepsilon}(\nabla))^p$ and primitives $(\boldsymbol{\varepsilon}(\nabla))^{-p}$ of $\mathbf{\Gamma}_{12}(\mathbf{x})$. Differentiation gives local contact distributions of higher order, whereas integration would give "mutual inductances" of higher order describing long range influences.

Let us now consider a set of N closed manifolds $\mathscr{C}_\alpha$, $\alpha=1,\ldots,N$ having all the same dimension (6.7). To these manifolds we may associate a cyclic product, the factors of which are quantities I, M or C:

$$(\ldots I_{\alpha\alpha+1}\ldots M_{\beta\beta+1}\ldots C_{\gamma\gamma+1}\ldots). \tag{6.8}$$

This cyclic product contains N such factors, with indices running from 1 to $N+1\equiv 1$. This product depends on the relative positions in space $\mathbb{R}^n$ of the manifolds $\mathscr{C}_\alpha$. As in (2.20), we consider the integral over the group G of motions of the manifolds

$$\mathscr{I}=\int\limits_{g\in G_2\times\ldots\times G_N}(\ldots I_{\alpha\alpha+1}\ldots M_{\beta\beta+1}\ldots C_{\gamma\gamma+1}\ldots)dg. \tag{6.9}$$

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9 $\mathbf{\Gamma}_{12}(\mathbf{x})$ relates two manifolds. The tensor $\mathbf{\Gamma}(\mathbf{y})$ (3.2) relates a single manifold and its own translated image by vector $\mathbf{y}$

The measure over the group of relative motions is, as in (2.20), (2.20a):

$$d\mathscr{g} = \prod_{\alpha=1}^{N} d^n x_\alpha \delta\left(\sum_{\alpha=1}^{N} \mathbf{x}_\alpha\right) \prod_{\alpha=1}^{N} \frac{d\mathscr{O}_\alpha}{\Omega_n}.$$

We shall now calculate $\mathscr{I}$.

6.2. Factorization Theorem

We have shown in Sect. 2 that the linking integral (2.20) can be expressed by Eq. (2.31). The rule is very simple: each factor $I_{\alpha\alpha+1}$ has to be replaced by tensor $\mathbf{C}_\alpha$. One may notice that tensor $\mathbf{C}$ is related to $\mathbf{\Gamma}$ by (3.1) $\mathbf{C} = \boldsymbol{\varepsilon}_{rr}(V) \cdot \mathbf{\Gamma}$, an equation very similar in structure to (6.6a). We notice, at this stage, the formal analogy between this last relation and the first Eq. (6.10) giving I. For the general cyclic integral (6.9), the whole argument, given in Sect. 2, can be repeated step by step. One obtains new substitution rules, which can be immediately guessed by inspection of Eq. (6.6a–c):

$$I_{\alpha\alpha+1} \to \mathbf{C}_\alpha = \boldsymbol{\varepsilon}(V) \cdot \mathbf{\Gamma}_\alpha, \qquad M_{\alpha\alpha+1} \to \mathbf{\Gamma}_\alpha, \qquad C_{\alpha\alpha+1} \to -\Delta \mathbf{\Gamma}_\alpha. \tag{6.10}$$

As already stressed, these quantities I, M, C relate two successive manifolds $\mathscr{C}_\alpha, \mathscr{C}_{\alpha+1}$, whereas the image by (6.10) is associated with manifold $\mathscr{C}_\alpha$ only. Inserting (6.10) into (6.9) gives the tensorial factorization generalizing (2.31):

$$\mathscr{I} = \int_{\mathbb{R}^n \times \ldots \times \mathbb{R}^n} \prod_{\alpha'=1}^{N} dy_{\alpha'} \delta\left(\sum_{\alpha'=1}^{N} \mathbf{y}_{\alpha'}\right) \mathrm{tr}\{\ldots \mathbf{C}_\alpha(\mathbf{y}_\alpha) \ldots \mathbf{\Gamma}_\beta(\mathbf{y}_\beta) \ldots (-\Delta)\mathbf{\Gamma}_\gamma(\mathbf{y}_\gamma) \ldots\}. \tag{6.11}$$

In Fourier representation (5.1), (6.11) reads immediately

$$\mathscr{I} = (2\pi)^{-n} \int_{\mathbb{R}^n} d^n p \, \mathrm{tr}\{\ldots \hat{\mathbf{C}}_\alpha(\mathbf{p}) \ldots \hat{\mathbf{\Gamma}}_\beta(\mathbf{p}) \ldots (-)\Delta\hat{\mathbf{\Gamma}}_\gamma(\mathbf{p}) \ldots\}. \tag{6.12}$$

We have to calculate the above Fourier transforms. Using Eq. (3.1) and well-known properties of Fourier transforms, we have

$$\hat{\mathbf{C}}(\mathbf{p}) = -ip\boldsymbol{\varepsilon}_{rr}(\hat{p}) \cdot \hat{\Gamma}(\mathbf{p}); \qquad \Delta\hat{\mathbf{\Gamma}} = -p^2\hat{\mathbf{\Gamma}}. \tag{6.13}$$

Inverting (6.13) by using (5.8) for $r = s$, and tensorial identities (3.9)–(3.12), we find

$$\hat{\Gamma}(p) = p^{-1}\{\boldsymbol{\delta}_{rr}(\hat{p} \otimes \hat{p}) \mathscr{A}[p] + (-1)^{r^2+1} \delta_{2r+1,n}(-i)\boldsymbol{\varepsilon}_{rr}(\hat{p}) \mathscr{B}[p]\}. \tag{6.14}$$

Using isomorphism (5.13) gives the rules[10]

$$\hat{\Gamma} \to p^{-1}(\mathscr{A} + \varepsilon\mathscr{B}), \qquad \Delta\hat{\Gamma} \to p(\mathscr{A} + \varepsilon\mathscr{B}).$$

Using this isomorphism in (6.12), as in the previous section, and taking care of the numerical factors, we find the theorem:

Theorem III. *Consider in* $\mathbb{R}^n$, *N manifolds* $\mathscr{C}_\alpha$ *of dimensions* $r = (n-1)/2$, *n being odd. The integral*

$$\mathscr{I} = \int_{G \text{ motions}} (\ldots I_{\alpha\alpha+1} \ldots M_{\beta\beta+1} \ldots C_{\gamma\gamma+1} \ldots) d\mathscr{g}$$

10 $(-1)^{r^2+1}$ preceding $\mathscr{B}$ has been dropped, because Eq. (5.21) gives $(-1)^{r^2+1} \equiv 1$

of the cyclic product of linking numbers $I_{\alpha\alpha+1}$, *of mutual inductances* $M_{\beta\beta+1}$ *and of contact distributions* $C_{\gamma\gamma+1}$, *over the group of Euclidean motions (translations and rotations) of the manifolds, can be factorized into the single integral*

$$\mathscr{I} = j \int_0^\infty dp p^{n-1} \mathscr{P}_{\text{even}}[p], \tag{6.15}$$

where $j = \dfrac{S_{n-1}}{(2\pi)^n} \dfrac{(n-1)!}{(r!)^2}$, *and* $\mathscr{P}_{\text{even}}$ *is the even part in* ε *of the algebraic product*

$$\begin{aligned} \mathscr{P} &\equiv [\ldots(\varepsilon\mathscr{A}_\alpha + \mathscr{B}_\alpha)\ldots p^{-1}(\mathscr{A}_\beta + \varepsilon\mathscr{B}_\beta)\ldots p(\mathscr{A}_\gamma + \varepsilon\mathscr{B}_\gamma)\ldots] \\ &\equiv \mathscr{P}_{\text{even}} + \varepsilon\mathscr{P}_{\text{odd}} \end{aligned} \tag{6.16}$$

calculated with the rule $\varepsilon^2 = (-1)^{r^2+1}$ $(\equiv(-1)^{r+1})$, *and built with obvious correspondence rules.*

Theorem III generalizes Theorem (6.18) of [8]. It yields in particular the interpretation for the *odd* part $\mathscr{Y}$ of product (5.15). One checks indeed that $p^{-1}\mathscr{Y}$ (respectively $p\mathscr{Y}$) gives the *even* part of (6.16) for the product of $N-1$ linking numbers and of one single mutual inductance (respectively one contact). More generally, we note that a linking number I on one hand, and an inductance M or a contact C on the other, have *opposite parities with respect* to ε. Furthermore two factors M and C annihilate each other according to the equivalence rule [see (6.16)]:

$$MC \leftrightarrow (-1)^{r^2+1} I^2. \tag{6.17}$$

Until now, dimension r of the manifolds was fixed at $2r+1=n$, so that I *and* (M, C) were simultaneously defined. If we restrict ourselves to inductances and contacts, the common dimension r can be arbitrary and differ from $(n-1)/2$. Only $\mathscr{A}$ functions exist in this case. $\mathscr{I}$ reads for instance for N inductances:

$$\int_{G_2 \times \ldots \times G_N} \prod_{\alpha=1}^{N} M_{\alpha\alpha+1} dg = j \int_0^\infty dp p^{n-1-N} \prod_{\alpha=1}^{N} \mathscr{A}_\alpha. \tag{6.18}$$

If one considers two manifolds only, Theorem III can be also given in direct space. The integral (6.11) can indeed be calculated in direct space for $N=2$, as it has been done in Sect. 4 for the kinematic linking integral of two manifolds. We refer to [8] where a similar calculation has been done for 1-curves. The kinematic integrals $\int M^2$, $\int C^2$, $\int IM$, $\int IC$, $\int MC \equiv \int I^2$ (provided they exist) are given by factorization theorems analogous to Theorem I, where the characteristic functions $\mathscr{A}(\varrho), \mathscr{B}(\varrho)$ are replaced by some of their primitives or derivatives [8]. Similar results hold here and can be obtained by starting either from Eqs. (6.11), (3.8), and (3.17), or from Theorem III by inverting the Fourier transforms.

Convergence. The general integral $\mathscr{I}$ (6.9) can be divergent. Inductances bring in long distance divergences ($p \to 0$), while contacts bring in short distance divergences ($p \to \infty$) [rule (6.16)]. The convergence must be checked in each case, with the help of the asymptotic behaviours of $\mathscr{A}$ and $\mathscr{B}$ (Sect. 5). For instance, integral (6.18) is defined for $N \leqq n-1$, being otherwise infrared divergent at $p=0$, if functions $\mathscr{B}$ appear.

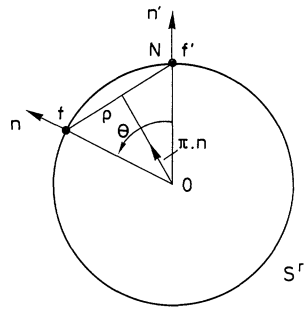

Fig. 11. Coordinates on sphere S^r

7. Particular Geometrical Shapes

7.1. The Spheres

We consider the sphere S^r, of dimension r and radius R, embedded in $\mathbb{R}^n$ ($r \leqq n-1$):

$$\begin{cases} x_1^2 + \ldots + x_{r+1}^2 = R^2 \\ x_j = 0, \quad r+2 \leqq j \leqq n. \end{cases} \tag{7.1}$$

Leaving aside the x_j's, for $j \geqq r+2$, the sphere S^r, considered as embedded in a subspace $\mathbb{R}^{r+1}$, is a hyperplane manifold (Sect. 4). Thus $\mathscr{B}=0$ and function $\mathscr{A}$ (4.25) reads

$$\mathscr{A}(\varrho) = r^{-1}\varrho^{1-r}\int \delta(\varrho - \|\mathbf{f}-\mathbf{f}'\|)(\mathbf{n}\cdot\mathbb{\Pi}\cdot\mathbf{n}')dS_r dS_r', \tag{7.2}$$

where $\mathbf{n}$ and $\mathbf{n}'$ are the normal unit vectors at f and f', and where $\mathbb{\Pi} \equiv \mathbb{1} - \hat{ff}' \otimes \hat{ff}'$ is the projector orthogonal to $\mathbf{f}-\mathbf{f}'$.

Let f' be the North pole of S^r and θ be the cone angle $(0f', 0f)$ (Fig. 11):

$$\begin{aligned} \mathbf{n}\cdot\mathbf{n}' &= \cos\theta, \qquad \theta \in [0, \pi] \\ \mathbf{n}\cdot\mathbb{\Pi}\cdot\mathbf{n}' &= \cos^2\theta/2, \qquad \|\mathbf{f}-\mathbf{f}'\| = 2R\sin\theta/2. \end{aligned} \tag{7.3}$$

Besides θ, the position of point f is given by $r-1$ other spherical angles belonging to the unit sphere S^{r-1} and the measure on S^r reads $dS_r = R^r(\sin\theta)^{r-1}d\theta dS_{r-1}$. This, together with (7.3), gives

$$\begin{aligned} \mathscr{A}(\varrho) &= r^{-1}\varrho^{1-r}S_r S_{r-1}\int_0^\pi \delta(\varrho - 2R\sin\theta/2)\cos^2\theta/2\,(\sin\theta)^{r-1}d\theta \\ &= \frac{S_{r-1}}{r} S_r R^r (1-\varrho^2/4R^2)^{r/2}\theta(2R-\varrho). \end{aligned} \tag{7.4}$$

For a circle S^1 ($r=1$), (7.4) coincides with the result of Pohl [6]. As expected from (4.16): $\mathscr{A}(0) = \dfrac{S_{r-1}}{r} S$, where $S = S_r R^r$ is the area of sphere S^r. As expected also, the finite support of $\mathscr{A}$ ends at the diameter $2R$.

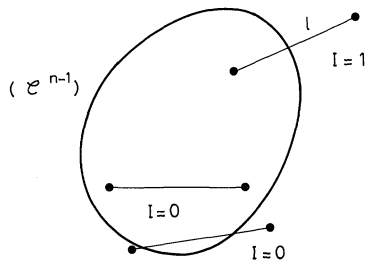

Fig. 12. Linking number of two points at a fixed distance l, with a hypersurface $\mathscr{C}^{n-1}$

7.2. *Zero Dimensional Case*

As a limiting case, let us consider a manifold made of two points: $\mathscr{C}_0 = \{\mathbf{g}_0, \mathbf{g}'_0\}$ in $\mathbb{R}^n$, separated by a fixed distance l. $\mathscr{C}_0$ is zero dimensional ($r=0$) and thus it can be linked with any closed hypersurface $\mathscr{C}^{n-1}$ of $\mathbb{R}^n$. $\mathscr{C}^{n-1}$ divides space $\mathbb{R}^n$ into interior $\mathscr{C}^+$ and exterior $\mathscr{C}^-$. The linking number $I(\mathscr{C}^{n-1}, \mathscr{C}_0)$ is then defined by $I=+1$ if $g_0 \in \mathscr{C}^+$ and $g'_0 \in \mathscr{C}^-$, $I=-1$ if $g'_0 \in \mathscr{C}^+$ and $g_0 \in \mathscr{C}^-$, and $I=0$ otherwise. $\mathscr{C}_0$ and $\mathscr{C}^{n-1}$ are thus linked whenever g_0 and g'_0 stay on opposite sides of boundary surface $\mathscr{C}^{n-1}$ (Fig. 12).

The set $\mathscr{C}_0$ can be considered as the limiting case of a zero dimensional sphere S^0 of radius R, $2R=l$. Thus, taking the limit $r \to 0$ in (7.4) gives immediately the $\mathscr{A}$-function of $\mathscr{C}_0$:

$$\mathscr{A}_0(\varrho) = 2\theta(l-\varrho), \tag{7.5}$$

which can also be calculated from (4.6). In Fourier space $\mathscr{A}_0$ (5.7a) reads accordingly

$$\mathscr{A}_0[p] = -\frac{(2\pi)^{n/2}}{S_{n-1}} p^{-1} J_{\frac{n}{2}-1}(pl)/(pl)^{\frac{n}{2}-1} \quad {}^{11}.$$

The function $\mathscr{B}$ of $\mathscr{C}_0$ vanishes for $n=2r+1=1$, and does not exist for $n \geqq 2$.

The linking integral (4.10) of $\mathscr{C}_0$ and of a hypersurface $\mathscr{C}^{n-1}$ reads therefore

$$\langle \int I^2 d^n x \rangle = \frac{2}{S_{n-1}} \int_0^l d\varrho \mathscr{A}(\varrho), \tag{7.6}$$

where $\mathscr{A}$ is associated with closed hypersurface $\mathscr{C}^{n-1}$. This formula gives the overlap probability for a segment and a closed hypersurface. This generalizes a result by Pohl [6] for segments and curves in plane $\mathbb{R}^2$.

For $l \to \infty$, one of the points at most lies inside $\mathscr{C}^{n-1}$, and $\langle \int I^2 \rangle$ equals two times the interior volume $V_{\text{int}}(\mathscr{C}^{n-1})$ of $\mathscr{C}^{n-1}$. Thus we get the simple formula

$$V_{\text{int}}(\mathscr{C}) = \frac{1}{S_{n-1}} \int_0^\infty d\varrho \mathscr{A}(\varrho), \tag{7.7}$$

11 For $p \to \infty$, $\mathscr{A}_0[p]$ is not given by (5.28), which holds only for $r>0$ (see Appendix F)

valid for any closed hypersurface $\mathscr{C}$ in $\mathbb{R}^n$. This can be checked for instance for the sphere S^r considered as a hypersurface in $\mathbb{R}^{r+1}$. Using (7.4) and properties of the Euler Γ-function, we obtain $S_r^{-1}\int_0^\infty \mathscr{A}_{S^r}(\varrho)d\varrho = \frac{S_r}{r+1}R^{r+1}$, which is exactly the $(r+1)$-volume of the interior of sphere S^r in space $\mathbb{R}^{r+1}$.

7.3. Topological Second Virial Coefficient for Spheres

Consider a gas of spheres S^r of radius R moving in space $\mathbb{R}^n$ with

$$n=2r+1. \tag{7.8}$$

Their phase space is restricted by the fact that they make no links together. For two spheres the linking number I can only be equal to 0 or ± 1. Therefore $\mathscr{I}$ reads

$$|\mathscr{I}| = \left\langle \int_{\substack{\text{top. excluded}\\ \text{volume}}} d^n x \right\rangle \tag{7.9}$$

and measures the volume in space of translations x, averaged over angular motions, over which the spheres are linked. This is exactly the topological excluded volume. Then the virial expansion of the pressure Π [9] of the gas of spheres at concentration $\mathbb{C}$, reads

$$\beta\Pi|_{\text{topol.}} = \mathbb{C} + \tfrac{1}{2}|\mathscr{I}|\mathbb{C}^2 + \ldots. \tag{7.10}$$

The topological excluded volume $\mathscr{I}$ for two r-spheres is thus given by Theorem (4.10): $|\mathscr{I}| = \frac{(r!)^2}{(2r)!}S_{2r}^{-1}\int_0^\infty d\varrho\,\mathscr{A}_{S^r}^2(\varrho)$. We get by using (7.4) and properties of Euler Γ-functions:

$$|\mathscr{I}| = \frac{(S_{r-1}S_r)^2}{S_{2r}}\frac{[(r-1)!]^2}{(2r)!}R^{2r}\int_0^{2R}(1-\varrho^2/4R^2)^r d\varrho = \frac{2}{2r+1}S_{2r}R^{2r+1}. \tag{7.11}$$

One must notice, according to (7.8) that $|\mathscr{I}| = 2V_n(R)$, where $V_n(R) \equiv \frac{S_{n-1}}{n}R^n$ is the interior volume in $\mathbb{R}^n$ of the $2r$-sphere S^{2r}, generated by the rotations in $\mathbb{R}^n$ of a sphere S^r about its centre. Thus, we finally have

$$\beta\Pi|_{\text{topol.}} = \mathbb{C} + V_n(R)\mathbb{C}^2 + \ldots, \tag{7.12}$$

for a gas of *nonlinked* S^r-spheres.

This can be compared with the pressure of a gas of *hard* spheres S^{n-1} in $\mathbb{R}^n$. In this case, the excluded volume is obviously that of a S^{n-1}-sphere of radius $2R: V_n(2R) = 2^n V_n(R)$, leading to the virial expansion:

$$\beta\Pi|_{\text{hard sph.}} = \mathbb{C} + 2^{n-1}V_n(R)\mathbb{C}^2 + \ldots. \tag{7.13}$$

The topological excluded volume is evidently smaller than that of hard spheres. Both are equal only for $n=1$. This corresponds to the obvious fact that, in one dimension, the notions of link and overlap of two intervals S^0 coincide.

Conclusion

In this article, we considered the Gauss linking number I in $\mathbb{R}^n$ of two closed manifolds of dimensions r and s, with $r+s+1=n$. We also defined the mutual inductances M and contact distributions C of closed manifolds, which are closely related to I. This study is quite general. The physical cases $n=1$ to 3 are contained in it. For general n, all the cases $r+s+1=n$, $0 \leqq r \leqq n-1$ are treated, going from sets of points to hypersurfaces.

We have introduced a new, compact tensorial formalism, which underlies the calculation of scalar quantities like I, M, and C. The new mathematical tool we introduced is the inductance tensor Γ, a $2r$ tensor associated with any manifold of dimension r. We believe that such a tensor Γ, and objects similar to it, are important and quite basic for such studies of topology, potential theory and electromagnetism, and provide a generic method. Using group theory, we proved a key lemma, which shows that Γ has two and only two distinct parts, each of which generates characteristic functions $\mathscr{A}$ and $\mathscr{B}$ of a manifold $\mathscr{C}$. The calculation of $\mathscr{A}$ and $\mathscr{B}$ was reduced by this method to a direct application of the well-known Gauss theorem of electromagnetism. This completely elucidates the origin and the form of $\mathscr{A}$ and $\mathscr{B}$. The set of values $\mathscr{A}(\varrho)$ yields a scanning of the geometrical shapes of manifold $\mathscr{C}$ at distance ϱ. $\mathscr{A}$ interpolates in particular between the area of $\mathscr{C}$ and its inner volume, when $\mathscr{C}$ is a hypersurface. $\mathscr{B}$, on the other hand, is quite peculiar, being directly proportional to the average linking number of $\mathscr{C}$ with its own translated images. As a consequence, we have found that $\mathscr{B}$ appears only for a manifold of dimension $(n-1)/2$ with $n=3+4q=3, 7, 11, \ldots$! Luckily enough, $\mathbb{R}^3$ is generic. We calculated the kinematic linking integral $\mathscr{I}=\langle \int I^2 d^n x \rangle$ over the group of motions of two manifolds $\mathscr{C}_1$ and $\mathscr{C}_2$. We established a general theorem giving $\mathscr{I}$ as the single integral of products $\mathscr{A}_1\mathscr{A}_2+\mathscr{B}_1\mathscr{B}_2$, factorized over manifolds 1 and 2. As an application, the topological excluded volume $V \equiv \mathscr{I}$ of hyperspheres was calculated. In general, $\mathscr{I}$ can provide useful numerical information on the relevance of topological constraints. In particular, the problem of a further statistical average over the deformations of the manifolds is solved by the factorization theorem. One has simply to *average independently* the characteristic functions.

More generally, the theory of rings (like polymer rings), or closed surfaces, with topological constraints, could be tentatively described by a partition function

$$Z = \mathrm{Tr} \exp\{-\beta \mathscr{H} - g I^2\},$$

where $\mathscr{H}$ is the Hamiltonian. The limit of large values of g, $g \to \infty$, would select configurations such that $I=0$ (alas not necessarily unlinked). Thus, expanding in powers of g, the integral $\mathscr{I}=\langle \int I^2 \rangle$ gives the first moment of Z. It would then be necessary, but difficult, to calculate the higher order moments for studying the large g limit of the theory.

We gave in this article a generalization in another direction by considering N manifolds, and the cyclic product of their successive linking numbers. The integral of this linking product over the group of motions of the N manifolds, has been shown to factorize completely in Fourier space. The factors read $\varepsilon\mathscr{A}+\mathscr{B}$, and are associated with one manifold only. The objects $(1, \varepsilon)$ obey special algebraic rules.

The same algebra $\{\mathscr{A}, \mathscr{B}\}$ with coefficients $(1, \varepsilon)$ has been shown to describe a quite large class of topological and electromagnetic quantities. Defining mutual inductances M and contacts C of manifolds in $\mathbb{R}^n$, we considered cyclic products, the factors of which are quantities I, M or C. We established a general theorem giving the integral of these products over the group of motions of the manifolds. We showed that this integral factorizes over the same set of characteristic functions $\mathscr{A}$ and $\mathscr{B}$ of the manifolds. These theorems allow practical or numerical calculations of these integrals, which are reduced by the theory to integrals over one single momentum variable.

All the quantities considered here were arranged in a cycle, going from manifolds 1 to N. Work remains to be done for calculating kinematic integrals involving multiple crossed terms, as in usual cluster expansions in statistical mechanics. It would be an important step towards a general theory of topological constraints.

Hopefully, these mathematical methods and results could be useful in topology, electromagnetism and in statistical mechanics.

Appendix A

Proof of Gauss' Formula

Equation (2.4) can be written $I = \int_{\mathscr{C}_1 \times \mathscr{C}_2} \omega$, where ω is the mixed $(r+s)$-form:

$$\omega = H_k(\mathbf{f}-\mathbf{g})\varepsilon^{ki_1 \ldots i_r j_1 \ldots j_s}(df_{i_1} \wedge \ldots \wedge df_{i_r})(dg_{j_1} \wedge \ldots \wedge dg_{j_s}) \tag{A.1}$$

with

$$H_k \equiv -[S_{n-1}(n-2)r!s!]^{-1} \frac{\partial}{\partial f_k} \|\mathbf{f}-\mathbf{g}\|^{-(n-2)}. \tag{A.2}$$

Stoke's theorem gives

$$I = \int_{\mathscr{S}(\mathscr{C}_1) \times \mathscr{C}_2} d_f \omega, \tag{A.3}$$

where $\mathscr{S}(\mathscr{C}_1)$ is a Seifert-surface of $\mathscr{C}_1$ such that $\partial \mathscr{S}(\mathscr{C}_1) = \mathscr{C}_1$. The exterior derivative $d_f\omega$ of ω with respect to f is by definition

$$d_f \omega \equiv \partial_l H_k \varepsilon^{ki_1 \ldots i_r j_1 \ldots j_s}(df_l \wedge df_{i_1} \wedge \ldots \wedge df_{i_r})(dg_{j_1} \wedge \ldots \wedge dg_{j_s}). \tag{A.4}$$

Substituting the trivial identity $df_l \wedge df_{i_1} \wedge \ldots \wedge df_{i_r} = \frac{1}{(r+1)!} \frac{1}{s!} \varepsilon^{li_1 \ldots i_r l_1 \ldots l_s} \varepsilon^{l' k_1 \ldots k_r l_1 \ldots l_s}(df_{l'} \wedge df_{k_1} \wedge \ldots \wedge df_{k_r})$ into (A.4) and summing over indices i_1 to i_r, we get:

$$d_f \omega = \partial_l H_k \frac{1}{r+1} \frac{1}{s!} \delta^{kj_1 \ldots j_s}_{ll_1 \ldots l_s} \varepsilon^{l' k_1 \ldots k_r l_1 \ldots l_s}(df_{l'} \wedge df_{k_1} \wedge \ldots \wedge df_{k_r})(dg_{j_1} \wedge \ldots \wedge dg_{j_s}).$$

Then, expanding the generalized Kronecker delta with respect to indice l [14]:

$$\delta^{kj_1 \ldots j_s}_{ll_1 \ldots l_s} = \delta^k_l \delta^{j_1 \ldots j_s}_{l_1 \ldots l_s} - \delta^{j_1}_l \delta^{kj_2 \ldots j_s}_{l_1 \ldots l_s} + \ldots, \tag{A.5}$$

we see that $d_f\omega$ can be written as the sum of two terms:

$$d_f\omega = \partial_l H_l \frac{1}{r+1} \varepsilon^{l'k_1\dots k_r j_1\dots j_s}(df_{l'} \wedge df_{k_1} \wedge \dots \wedge df_{k_r})(dg_{j_1} \wedge \dots \wedge dg_{j_s}) + d_g\omega',$$

where $d_g\omega'$ is the exterior derivative with respect to g of a second form ω' coming from the second term of (A.5) $\left(\text{use Definition (A.4) and } \frac{\partial H}{\partial f_j} = -\frac{\partial H}{\partial g_j}\right)$. Thus we find the exterior derivative of a form of type (A.1):

$$\omega = \det(H, d\mathscr{F}_r, d\mathscr{G}_s)$$

$$d_f\omega = \frac{1}{r+1}(\operatorname{div} H)\det(d\mathscr{F}_{r+1}, d\mathscr{G}_s) + d_g\omega'. \tag{A.6}$$

Inserting (A.6) into (A.3) gives

$$I = \int_{\mathscr{S}(\mathscr{C}_1)\times\mathscr{C}_2} \frac{1}{r+1}(\operatorname{div} H)\det(d\mathscr{F}_{r+1}, d\mathscr{G}_s),$$

by using $\int_{\mathscr{C}_2} d_g\omega' = \int_{\partial\mathscr{C}_2=0} \omega' = 0$. Using Poisson equation for H (A.2) yields $\operatorname{div} H = \frac{1}{r!s!}\delta(\mathbf{f}-\mathbf{g})$, and finally

$$I = \frac{1}{(r+1)!s!}\int_{\mathscr{S}(\mathscr{C}_1)\times\mathscr{C}_2} \delta(\mathbf{f}-\mathbf{g})\det(d\mathscr{F}_{r+1}, d\mathscr{G}_s)\,(\in\mathbb{Z}). \tag{A.7}$$

When explicit variables are introduced on both manifolds [see (2.10)], (A.7) reads:

$$I = \int \delta(\mathbf{f}-\mathbf{g})\det\left(\frac{\partial \mathbf{f}}{\partial w_1}, \dots, \frac{\partial \mathbf{f}}{\partial w_{r+1}}, \frac{\partial \mathbf{g}}{\partial v_1}, \dots, \frac{\partial \mathbf{g}}{\partial v_s}\right)(dw_1 \wedge \dots \wedge dw_{r+1})(dv_1 \wedge \dots \wedge dv_s), \tag{A.8}$$

which corresponds to the geometrical definition (1.7) Q.E.D.

Appendix B

Proof of Lemma (3.6) *for Tensor* Γ

Consider a tensor $\Gamma(\mathbf{y})|_{i_1\dots i_r, j_1\dots j_r}$, which has the following properties:

B.1 Γ depends on only one vector $\mathbf{y}$ of $\mathbb{R}^n$.

B.2 Γ is skew-symmetric with respect to the set of indices i_a, $a=1,\dots,r$ and separately skew-symmetric with respect to the set of indices j_b, $b=1,\dots,r$.

B.3 All its divergences (in number $2r$) vanish.

Then we want to prove that Γ has necessarily the form

$$\Gamma(\mathbf{y})|_{i_1\dots i_r, j_1\dots j_r} = \delta^{i_1\dots i_r l}_{j_1\dots j_r l'}\partial_l\partial_{l'}\Phi(\|y\|) + \delta_{2r+1,n}\varepsilon^{i_1\dots i_r j_1\dots j_r l} \cdot \partial_l\Psi(\|y\|) + U^{i_1\dots i_r}_{j_1\dots j_r}, \tag{B.4}$$

where Φ and Ψ are two arbitrary functions, and $\partial_l \equiv \frac{\partial}{\partial y_l}$. **U** is a harmonic tensor linear in **y** satisfying (B.2) and (B.3)[12]. Conversely, all tensors (B.4) satisfy conditions (B.1)–(B.3). Property (B.1) means that the transformation properties of Γ are given by those of vector **y**. Then we use a theorem analogous to a main theorem on the invariants of orthogonal group O(n), which can be found in Weyl's book [12]. Tensor Γ is built with the Kronecker delta $\delta_{ll'}$, the components y_l of **y** and the completely skew-symmetric n-tensor $\varepsilon^{l_1 \cdots l_n}$. The other factors will be invariants, i.e. functions of modulus $\|y\|$. It is equivalent to consider, instead of components y_l, the dual differentiation operators $\partial_l \equiv \frac{\partial}{\partial y_l}$ as elementary bricks for building Γ. The equivalence can be seen in Fourier space, where ∂_l generates components of the momentum variable. Using differentiation signs is more convenient for dealing with scalar divergence properties. These differentiation operators act on functions of the modulus $\|y\|$.

Consider then a component $\Gamma_{i_1 \ldots i_r, j_1 \ldots j_r}$. Because of the skew-symmetry property (B.2), no symmetric terms $\delta_{i_a i_{a'}}$, $\delta_{j_b j_{b'}}$, and $\delta_{i_a} \partial_{i_{a'}}$ or $\partial_{j_b} \partial_{j_{b'}}$ appear. One can have only terms:

$$\delta_{ij}, \quad \partial_i, \quad \partial_j \quad \text{and} \quad \partial_i \partial_j . \tag{B.5}$$

No monomials of higher degree in ∂ appear, because they would involve symmetrically at least two indices i or two j's.

Consider now the skew-symmetric tensor ε. It can bear some indices i and some j's: $\varepsilon^{i_a \cdots j_b \cdots}$. *Let us first assume that*

$$2r \leqq n-1 . \tag{B.6}$$

Then the set $i_1 \ldots i_r$, $j_1 \ldots r_r$ will not saturate the n indices of tensor ε. The $n-2r$ remaining indices l of ε must necessarily be contracted with components ∂_l of the gradient. If

$$2r < n-1 , \tag{B.7}$$

at least two dummy indices l, l' appear and $\varepsilon^{i_a \cdots j_b \cdots l l'} \partial_l \partial_{l'} = 0$. Thus, *if* $2r < n-1$, *no term involving tensor* ε *can appear in* Γ.

If $2r = n-1$, the only possibility is given by a term

$$\varepsilon^{i_1 \ldots i_r j_1 \ldots j_r l} \partial_l , \tag{B.8}$$

which satisfies properties (B.1) and (B.2).

If condition (B.6) is not satisfied, the number of indices $2r \geqq n$ allows much more possibilities. For instance, a term $\varepsilon^{i_1 \cdots i_p j_1 \cdots j_p} \delta_{i_{p+1} j_{p+1}} \ldots \delta_{i_r j_r}$, with $2p = n$, and antisymmetrized, could appear in Γ. However, we show later that a *duality* property exists for tensor Γ, because of (B.3), which relates tensor Γ with a large number of indices $2r \geqq n-1$ to a dual tensor with a small number of indices $2s \leqq n-1 (r+s+1=n)$. We thus prove first the lemma in case (B.6).

12 We give first the full group theoretic argument. The fact that $\mathbf{U} = 0$ for the actual Γ (3.2) is left to the end

I. Case $2r \leqq n-1$. $\Gamma_{i_1 \ldots i_r, j_1 \ldots j_r}$ reads, according to (B.5) and (B.8)

$$\sum_{\sigma} \delta_{i_1 j_{\sigma(1)}} \ldots \delta_{i_r j_{\sigma(r)}} \varphi_\sigma(\|y\|) + \sum_{a=1}^{r} \sum_{\sigma} \delta_{i_1 j_{\sigma(1)}} \ldots \hat{\delta}_{i_a j_{\sigma(a)}} \ldots \delta_{i_r j_{\sigma(r)}}$$
$$\cdot \partial_{i_a} \partial_{j_{\sigma(a)}} \varphi_\sigma^{(a)}(\|y\|) + \delta_{2r+1,n} \varepsilon^{i_1 \ldots i_r j_1 \ldots j_r l} \partial_l \Psi(\|y\|), \tag{B.9}$$

where σ is a permutation of $1, \ldots, r$. The φ's and Ψ are arbitrary functions. The notation $\hat{\delta}_{i_a j_{\sigma(a)}}$ means that this factor is missing.

First of all, one must remark that the last term of (B.9)

$$\delta_{2r+1,n} \varepsilon^{i_1 \ldots i_r j_1 \ldots j_r l} \partial_l \Psi \tag{B.10}$$

satisfies trivially properties (B.1) and (B.2) and also the divergence property (B.3). Thus this term is always present in $\mathbf{\Gamma}$, provided that $2r+1, = n$, and we drop it from now on. Now, the skew-symmetry property (B.2) gives immediately

$$\varphi_\sigma = \varepsilon(\sigma) \varphi_{\mathrm{id}}, \qquad \varphi_\sigma^{(a)} = \varepsilon(\sigma) \varphi_{\mathrm{id}}^{(a)}, \tag{B.11}$$

where $\varepsilon(\sigma)$ is the signature of σ, and id the identity permutation. Removing the subscript id, the unknown functions are φ, $\varphi^{(1)}, \ldots, \varphi^{(r)}$. We introduce the delta tensor (2.7), and tensor $\mathbf{\Gamma}$ (B.9) reads exactly, owing to (B.11):

$$\Gamma_{i_1 \ldots i_r, j_1 \ldots j_r}(\mathbf{y}) = \delta^{i_1 \ldots i_r}_{j_1 \ldots j_r} \varphi(\|y\|) + \sum_{a=1}^{r} \delta^{i_1 \ldots l \ldots i_r}_{j_1 \ldots j_a \ldots j_r} \partial_l \partial_{i_a} \varphi^{(a)}(\|y\|). \tag{B.12}$$

Let us fully exploit the skew-symmetry with respect to indices i_a. Consider two particular indices i_a and i_b. The only terms of (B.12) which are not obviously skew-symmetric under exchange of i_a and i_b are $\delta^{i_1 \ldots l \ldots i_r}_{j_1 \ldots j_a \ldots j_r} \partial_l \partial_{i_a} \varphi^{(a)} + \delta^{i_1 \ldots l \ldots i_r}_{j_1 \ldots j_b \ldots j_r} \partial_l \partial_{i_b} \varphi^{(b)}$. Taking the partial trace $i_{a'} = j_{a'}$ for $a' \neq (a, b)$ we get the object

$$\mathcal{O}^{i_a i_b}_{j_a j_b} = \delta^{l i_b}_{j_a j_b} \partial_l \partial_{i_a} \varphi^{(a)} + \delta^{i_a l}_{j_a j_b} \partial_l \partial_{i_b} \varphi^{(b)},$$

which must satisfy $\mathcal{O}^{i_a i_b}_{j_a j_b} = -\mathcal{O}^{i_b i_a}_{j_a j_b}$. Expanding the $\delta^{li}_{jj'}$, we find:

$$\partial_i \partial_j \varphi^{(a)} = \partial_i \partial_j \varphi^{(b)} + c \delta^i_j \quad \forall i, j, \tag{B.13}$$

where c is a constant. The contribution of $c\delta^{i_a}_l$ to (B.12) can be absorbed in the first term of (B.12) and we can set in (B.12):

$$\forall a, \qquad \varphi^{(a)} = -\Phi, \tag{B.14}$$

where Φ is an arbitrary function of $\|y\|$.

Using the expansion with respect to l':

$$\delta^{i_1 \ldots i_r l}_{j_1 \ldots j_r l'} = \delta^{i_1 \ldots i_r}_{j_1 \ldots j_r} \delta^l_{l'} - \sum_{a=1}^{r} \delta^{i_a}_{l'} \delta^{i_1 \ldots l \ldots i_r}_{j_1 \ldots j_a \ldots j_r} \tag{B.15}$$

and using (B.14), we may write (B.12) as

$$\Gamma^{i_1 \ldots i_r}_{j_1 \ldots j_r} = \delta^{i_1 \ldots i_r}_{j_1 \ldots j_r} (\varphi - \Delta\Phi) + \delta^{i_1 \ldots i_r l}_{j_1 \ldots j_r l'} \partial_l \partial_{l'} \Phi. \tag{B.16}$$

This is the general form of a tensor satisfying to properties (B.1) and (B.2) and with $2r < n-1$. Let us now exploit the property (B.3). The divergences of the second term of (B.16) all vanish because by skew-symmetry $\delta^{\ldots i_a \ldots l}_{\ldots \ldots} \partial_{i_a} \partial_l = 0$. The first term

of (B.16) is divergence-free if $\nabla(\varphi - \Delta\Phi) = 0$. Thus: $\varphi - \Delta\Phi = C$, where C is a constant. This constant term can in fact be absorbed in the second term of (B.16), by remarking trivially that $\delta^{i_1 \dots i_r}_{j_1 \dots j_r} C = \delta^{i_1 \dots i_r l}_{j_1 \dots j_r l'} \partial_l \partial_{l'} \left[\frac{C}{2(n-r)} y^2 \right]$. Remembering that a term (B.10) also exists, we thus have proved that tensor $\mathbf{\Gamma}$, for $2r \leqq n-1$, has the form

$$\Gamma = \delta^{\dots l}_{\dots l'} \partial_l \partial_{l'} \Phi + \delta_{2r+1,n} \varepsilon^{\dots l} \partial_l \Psi . \quad \text{Q.E.D.} \tag{B.16a}$$

II. Case $2r > n-1$. Here, the number of indices $2r$ is great enough to allow *a priori* much more possibilities than in (B.9). Here the vanishing divergence property (B.3) plays a crucial role. We define the *dual tensor* of $\mathbf{\Gamma}$:

$$K^{l_1 \dots l_s}_{k_1 \dots k_s} \equiv \varepsilon^{l_1 \dots l_s l i_1 \dots i_r} \varepsilon_{k_1 \dots k_s l j_1 \dots j_r} \Gamma^{i_1 \dots i_r}_{j_1 \dots j_r} , \tag{B.17}$$

The number s satisfies, as before $r+s+1=n$ and $2r > n-1$ implies

$$2s < n-1 . \tag{B.18}$$

Thus, for tensor $\mathbf{K}$, we are in case *I*. The dual tensor $\mathbf{K}$ satisfies trivially (B.1) (B.2) but not (B.3). However, we have proved above that the general form of a tensor (B.1) and (B.2) is given by (B.16):

$$K^{l_1 \dots l_s}_{k_1 \dots k_s} = \delta^{l_1 \dots l_s}_{k_1 \dots k_s} \tilde{\varphi} + \delta^{l_1 \dots l_s l'}_{k_1 \dots k_s l'} \partial_l \partial_{l'} \tilde{\Phi} , \tag{B.19}$$

where $\tilde{\varphi}(\|y\|)$, $\tilde{\Phi}(\|y\|)$ are arbitrary. The tensor $\mathbf{\Gamma}$ can be obtained back from $\mathbf{K}$.

We have indeed

$$\begin{aligned} (\varepsilon\partial)^2 \mathbf{K} &\equiv \varepsilon^{l_1 \dots l_s l' i'_1 \dots i'_r} \varepsilon_{k_1 \dots k_s l'' j'_1 \dots j'_r} \partial_{l'} \partial_{l''} K^{l_1 \dots l_s}_{k_1 \dots k_s} \\ &= (s!r!)^2 \Delta \Gamma^{i'_1 \dots i'_r}_{j'_1 \dots j'_r} . \end{aligned} \tag{B.20}$$

For proving it, one substitutes (B.17) into (B.20) and finds

$$(\varepsilon\partial)^2 \mathbf{K} = (s!)^2 \delta^{l' i'_1 \dots i'_r}_{l i_1 \dots i_r} \delta^{l'' j'_1 \dots j'_r}_{l j_1 \dots j_r} \partial_{l'} \partial_{l''} \Gamma^{i_1 \dots i_r}_{j_1 \dots j_r} .$$

If $l' = i_a$ a term $\partial_{i_a} \Gamma^{\dots i_a \dots} = 0$ appears [use (B.3)]. Likewise, $l'' = j_b$ gives $\partial_{j_b} \Gamma_{\dots j_b \dots} = 0$.

Thus $l' = l'' = l$ and $(\varepsilon\partial)^2 \mathbf{K}$ reads $(s!)^2 \delta^{i'_1 \dots i'_r}_{i_1 \dots i_r} \delta^{j'_1 \dots j'_r}_{j_1 \dots j_r} \Delta \Gamma^{i_1 \dots i_r}_{j_1 \dots j_r}$, which gives (B.20). Property (B.3) is crucial for (B.20) to hold. Substituting (B.19) into $(\varepsilon\partial)^2 \mathbf{K}$ (B.20), and making some manipulations on numerical tensors [use (2.8) (2.9)] finally give

$$[(\varepsilon\partial)^2 K]^{i'_1 \dots i'_r}_{j'_1 \dots j'_r} = (s!)^2 \delta^{i'_1 \dots i'_r l'}_{j'_1 \dots j'_r l''} \partial_{l'} \partial_{l''} (\tilde{\varphi} + \Delta \tilde{\Phi}) .$$

Therefore, owing to (B.20), we get the Poisson equation

$$\Delta \Gamma^{i_1 \dots i_r}_{j_1 \dots j_r} = \delta^{i_1 \dots i_r l}_{j_1 \dots j_r l'} \partial_l \partial_{l'} \varphi \tag{B.21}$$

with $\varphi = (r!)^{-2} (\tilde{\varphi} + \Delta \tilde{\Phi})$. Integrating (B.21) gives:

$$\mathbf{\Gamma} = \delta^{\dots l}_{\dots l'} \partial_l \partial_{l'} \Phi + \mathbf{U} , \tag{B.22}$$

where $\Phi(\|y\|)$ is a spherically invariant function such that $\Delta\Phi = \varphi$. $\mathbf{U}(\mathbf{y})$ is a *harmonic* tensor such that $\Delta \mathbf{U} = 0$, which satisfies also (B.1)–(B.3). If $\mathbf{\Gamma}$ vanishes at infinity, (for $n > 2$), then $\mathbf{U} \equiv 0$. It is sufficient to consider a tensor $\mathbf{U}$ independent of

the first term of (B.22). Then $\mathbf{U}$ vanishes at infinity, and being harmonic, vanishes everywhere. Then for $2r+1>n$ (and $n>2$) we obtain the form of Γ

$$\mathbf{\Gamma}=\delta_{\cdots l'}^{\cdots l}\partial_l\partial_{l'}\Phi\,, \tag{B.23}$$

which achieves the proof of Lemma (3.6) for $2r+1>n$, $n>2$. Q.E.D.

For $n=1,2$, Lemma (3.6) is also true. For $n=1$, $r=0$ and for $r=0$, Γ is a scalar function of $\|y\|$, in agreement with (B.23). For $n=2$, $r=0,1$. For $r=1$, $\mathbf{\Gamma}$ is a 2 indices tensor and has been constructed directly in [8] in the form (3.6). This, together with (B.16a) achieves the proof of Lemma (3.6) for tensor $\mathbf{\Gamma}$ (3.2). Q.E.D.

In the absence of boundary conditions, one can show that $\mathbf{U}$ is necessarily a tensor *linear* in $\mathbf{y}$. Owing to the first results of this appendix, a typical term of $U_{j_1\cdots j_r}^{i_1\cdots i_r}$ satisfying (B.1) and (B.2) depends on $\mathbf{y}$ in one of the following ways (apart from *numerical* tensors in factor)

$$\begin{gathered}\varphi_1(1);\quad \partial_{i_a}\partial_{j_b}\varphi_2(2);\quad \varepsilon^{l\cdots}\partial_l\varphi_3(3);\quad \varepsilon^{l\cdots}\partial_l\partial_{i_a}\varphi_4(4);\\ \varepsilon^{l\cdots}\partial_l\partial_{j_b}\varphi_5(5);\quad \varepsilon^{l\cdots}\partial_l\partial_{i_a}\partial_{j_b}\varphi_6(6)\,.\end{gathered} \tag{B.24}$$

Then $\Delta\mathbf{U}=0$ gives

$$\begin{gathered}\Delta\varphi_1=0\,;\quad \alpha=2,4,5\quad \forall i,j\quad \partial_i\partial_j(\Delta\varphi_\alpha)=0\,;\\ \forall l\partial_l(\Delta\varphi_3)=0;\quad \forall i,j,l\quad \partial_l\partial_i\partial_j(\Delta\varphi_6)=0\,.\end{gathered}$$

Spherically invariant functions $f(\|y\|)$ such that $\partial_i f=0$ or $\partial_i\partial_j f=0$ or $\partial_i\partial_j\partial_l f=0$, are necessarily *constant*. Therefore $\forall\alpha=1,\ldots,6$; $\Delta\varphi_\alpha=C_\alpha(C_1=0)$, C_α being constant. By integrating: $\varphi_\alpha=\dfrac{C_\alpha}{2n}\|y\|^2+C'_\alpha(C_1=0)$, and thus: $\partial_i\varphi_\alpha=\dfrac{C_\alpha}{n}y_i$, $\partial_i\partial_j\varphi_\alpha=\dfrac{C_\alpha}{n}\delta_{ij}$. This, inserted into (B.24), gives that all terms of (B.24) are *constant*, excepted (3) which reads $C_3\varepsilon^{l\cdots}y_l$. Therefore $\mathbf{U}$ has the linear form $\mathbf{U}_1+\mathbf{U}_2\cdot\mathbf{y}$, where $\mathbf{U}_1$ and $\mathbf{U}_2$ are constant tensors. This achieves the proof of the lemma in the form (B.4). Q.E.D.

Here again, the fact that $\mathbf{\Gamma}$ (3.2) vanishes at infinity (for $n>2$), gives $\mathbf{U}\equiv 0$ and (3.6) follows.

Appendix C

Multiplication of Tensors $\boldsymbol{\varepsilon}$ *and* $\boldsymbol{\delta}$

Let us prove Eqs. (3.9)–(3.12). Definition (2.11) gives for (3.9)

$$\boldsymbol{\varepsilon}_{sr}(\mathbf{x})\cdot\boldsymbol{\varepsilon}_{rs}(\mathbf{y})=\frac{1}{s!r!}x_l y_{l'}\varepsilon^{li_1\ldots i_r j_1\ldots j_s}\varepsilon^{l'j'_1\ldots j'_s i_1\ldots i_r}\,. \tag{C.1}$$

We then insert in (C.1) the general identity [14]

$$\varepsilon^{li_1\ldots i_r j_1\ldots j_s}\varepsilon^{l'i'_1\ldots i'_r j'_1\ldots j'_s}=\delta_{l'i'_1\ldots i'_r j'_1\ldots j'_s}^{li_1\ldots i_r j_1\ldots j_s}\,. \tag{C.2}$$

Permuting indices i and j in (C.1) and summing over indices i gives then

$$(-1)^{rs}\frac{1}{s!}x_l y_{l'}\delta_{l'j'_1\ldots j'_s}^{lj_1\ldots j_s}\equiv(-1)^{rs}\boldsymbol{\delta}_{ss}(\mathbf{x}\otimes\mathbf{y})\,.\quad \text{Q.E.D}$$

Equation (3.10) reads, with Definitions (2.11) and (3.7):

$$\boldsymbol{\varepsilon}_{sr}(\mathbf{x})\cdot\boldsymbol{\delta}_{rr}(\mathbf{y}\otimes\mathbf{x})=\frac{1}{s!r!}\,x_l\varepsilon^{li_1\dots i_r j_1\dots j_s}y_m x_{m'}\delta^{mi_1\dots i_r}_{m'i_1\dots i_r}. \tag{C.3}$$

We then use the expansion with respect to m'

$$\delta^{mi_1\dots i_r}_{m'i_1\dots i_r}=\delta^{m}_{m'}\delta^{i_1\dots i_r}_{i_1\dots i_r}-\delta^{i_1}_{m'}\delta^{mi_2\dots i_r}_{i_1\dots i_r}+\dots. \tag{C.4}$$

The second term of (C.4) (and the following ones) gives a contribution in (C.3)

$$x_l x_{i_1}\varepsilon^{li_1\cdots}=0.$$

Inserting only the first term of (C.4) into (C.3) and summing over the r indices i gives

$$(\mathbf{x}\cdot\mathbf{y})\frac{1}{s!}\,x_l\varepsilon^{li_1\dots i_r j_1\dots j_s}\equiv(\mathbf{x}\cdot\mathbf{y})\boldsymbol{\varepsilon}_{sr}(\mathbf{x}).\quad\text{Q.E.D.}$$

Equation (3.11) is proven in the same way. For Eq. (3.12), we have by definition

$$\boldsymbol{\delta}_{rr}(\mathbf{x}\otimes\mathbf{y})\cdot\boldsymbol{\delta}_{rr}(\mathbf{x}\otimes\mathbf{y})=\frac{1}{(r!)^2}\,x_l y_{l'}x_m y_{m'}\delta^{lj_1\dots j_r}_{l'i_1\dots i_r}\delta^{mi_1\dots i_r}_{m'i_1\dots i_r}, \tag{C.5}$$

and we use expansion (C.4) for the second δ. The second term of (C.4) (and the following ones) gives in (C.5) a vanishing factor of the type

$$y_{l'}y_{i_1}\delta^{\cdots}_{l'i_1\dots}=0.$$

Thus, keeping only the first term of (C.4) in (C.5) and summing over the indices i finally gives:

$$(\mathbf{x}\cdot\mathbf{y})\frac{1}{r!}\,x_l y_{l'}\delta^{lj_1\dots j_r}_{l'i_1\dots i_r}\equiv(\mathbf{x}\cdot\mathbf{y})\boldsymbol{\delta}_{rr}(\mathbf{x}\otimes\mathbf{y}).\quad\text{Q.E.D.}$$

Appendix D

Formulae with Chords of Constant Length

Proof of Eq. (4.18) *for* $\mathscr{A}$. We start from (4.6):

$$\mathscr{A}(\varrho)=\frac{1}{r!}\,\varrho^{-r}\int\theta(d\mathscr{F}_r\cdot d\mathscr{F}'_r). \tag{D.1}$$

Consider the exterior form:

$$\omega=\theta(\varrho-\|\mathbf{f}-\mathbf{f}'\|)(f-f')_{j_1}(df_{j_2}\wedge\dots\wedge df_{j_r}).$$

Its exterior derivative $d_f\omega$ is

$$d_f\omega=\theta\,d\mathscr{F}_r|_{j_1\dots j_r}-(f-f')_{j_1}\delta(\varrho-\|\mathbf{f}-\mathbf{f}'\|)(\hat{f}\hat{f}'\cdot d\mathscr{F}_r|_{j_2\dots j_r}).$$

Therefore, we have

$$\int\limits_{\mathscr{C}\times\mathscr{C}}\theta(d\mathscr{F}_r\cdot d\mathscr{F}'_r)=\int\limits_{\mathscr{C}\times\mathscr{C}}d_f\omega\cdot d\mathscr{F}'_r+\int\limits_{\mathscr{C}\times\mathscr{C}}\varrho\delta(\dots)\mathrm{tr}[(\hat{f}\hat{f}'\cdot d\mathscr{F}_r)\otimes(\hat{f}\hat{f}'\cdot d\mathscr{F}'_r)]. \tag{D.2}$$

The first term vanishes because of Stoke's theorem (3.5), and (D.2) inserted into (D.1) gives (4.18). Q.E.D.

Proof of Eq. (4.19) *for* $\mathscr{B}$. We start from Eq. (4.7) and consider the exterior form

$$\omega = \det(H, d\mathscr{F}_r, d\mathscr{F}'_r), \tag{D.3}$$

where

$$H \equiv -\theta(\|\mathbf{f}-\mathbf{f}'\|-\varrho)\nabla_f \|\mathbf{f}-\mathbf{f}'\|^{-(n-2)}. \tag{D.4}$$

Then

$$\mathscr{B} = \delta_{2r+1,n}(n-2)^{-1}(r!)^2 \varrho^r \int_{\mathscr{C}\times\mathscr{C}} \omega. \tag{D.5}$$

Stoke's theorem gives:

$$\int_{\mathscr{C}\times\mathscr{C}} \omega = \int_{\mathscr{S}(\mathscr{C})\times\mathscr{C}} d_f \omega. \tag{D.6}$$

Then the coboundary form $d_f\omega$ reads [use Eq. (A.6) of Appendix A]:

$$d_f\omega = \frac{1}{r+1}(\mathrm{div}_f H)\det(d\mathscr{F}_{r+1}, d\mathscr{F}'_r) + d_{f'}\omega'. \tag{D.7}$$

The second term of (D.7) gives no contribution to (D.6). Furthermore $\mathrm{div} H$ can be calculated easily with Eq. (3.25) and reads

$$\mathrm{div} H = (n-2)\delta(\varrho - \|\mathbf{f}-\mathbf{f}'\|)\varrho^{-(n-1)}. \tag{D.8}$$

Inserting (D.6)–(D.8) into (D.5) then gives Eq. (4.19). Q.E.D.

Appendix E

Functions $\mathscr{A}[p]$ *and* $\mathscr{B}[p]$

We perform the ϱ integrals in Eq. (5.7). Substituting (4.6) into (5.7a), we have

$$\mathscr{A}[p] = (r!)^{-1} \int_{\mathscr{C}\times\mathscr{C}} a[p]\,(d\mathscr{F}_r \cdot d\mathscr{F}'_r),$$

where $a[p] = -\dfrac{d}{dp}\displaystyle\int_0^\infty d\varrho\, \varrho^{-1}\theta(\varrho-\zeta)Q_n(p\varrho)$ and $\zeta \equiv \|\mathbf{f}-\mathbf{f}'\|$. Changing the integration variable ϱ into $x = p\varrho$, we find $a[p] = p^{-1}Q_n(p\zeta)$. Using the angular average (5.5), we find Eq. (5.22), while using Bessel functions (5.9), we get Eq. (5.23).

In the same way, substitution of (4.7) into (5.7b) gives for $\mathscr{B}[p]$:

$$\mathscr{B}[p] = \delta_{2r+1,n}(r!)^{-2} \int_{\mathscr{C}\times\mathscr{C}} b[p]\det(\mathbf{f}-\mathbf{f}', d\mathscr{F}_r, d\mathscr{F}'_r),$$

where $b[p] \equiv \zeta^{-n}\displaystyle\int_0^\infty d\varrho\,\theta(\zeta-\varrho)\varrho^{n-1}Q_n(p\varrho)$. For computing $b[p]$, we use (5.9) for Q_n and the identity [20]

$$x^{n/2}J_{\frac{n}{2}-1}(x) = \frac{d}{dx}[x^{n/2}J_{n/2}(x)].$$

Integrating by parts in $b[p]$ finally gives

$$b[p]=(2\pi p\zeta)^{-n/2}J_{n/2}(p\zeta),$$

and this yields Eq. (5.24) for $\mathscr{B}[p]$.

Appendix F

Large p Limit of $\mathscr{A}[p]$

$\mathscr{A}[p]$ can be written [Eqs. (5.7a) and (5.9)], with $x=p\varrho$

$$\mathscr{A}[p]=\frac{(2\pi)^{n/2}}{S_{n-1}}\left(-\frac{d}{dp}\right)p^{-r}\int_0^\infty dx\,x^{r-n/2}J_{\frac{n}{2}-1}(x)\mathscr{A}(x/p). \tag{F.1}$$

We use the identity

$$\left(-\frac{d}{xdx}\right)^q\frac{J_{\nu-q}(x)}{x^{\nu-q}}=\frac{J_\nu(x)}{x^\nu} \tag{F.2}$$

valid for integer q. Inserting (F.2) for $q-1$ into (F.1), we get

$$\mathscr{A}[p]=\frac{(2\pi)^{n/2}}{S_{n-1}}(-1)^q p\left(\frac{d}{pdp}\right)^q p^{2q-2-r}\int_0^\infty dx\,x^{r-\frac{n}{2}+1-q}J_{\frac{n}{2}-q}(x)\mathscr{A}(x/p). \tag{F.3}$$

If the integral

$$\kappa=\int_0^\infty dx\,x^\mu J_\nu(x)\begin{cases}\mu=r+1-\dfrac{n}{2}-q\\ \nu=\dfrac{n}{2}-q\end{cases}, \tag{F.4}$$

is convergent, then $\mathscr{A}(x/p)$ can be replaced for large p by $\mathscr{A}(0)$ in (F.3). Here κ converges and is equal to [20]

$$\kappa=2^\mu\frac{\Gamma[(1+\nu+\mu)/2]}{\Gamma[(1+\nu-\mu)/2]} \tag{F.5}$$

for

$$-(\nu+1)<\mu<\tfrac{1}{2}. \tag{F.6}$$

(F.6) applied to (F.4) gives the condition

$$r-(n-1)/2<q<(r+2)/2. \tag{F.7}$$

The difference between the two bounds in (F.7) is equal to $1+(n-r-1)/2>1$ and thus the integer q does exist. Therefore integral κ converges. On the contrary, for $q=1$, which corresponds to the initial form (F.1), $\mu=r-n/2$ does not necessarily satisfy (F.6). Inserting (F.4) and (F.5) into (F.3) finally gives the asymptotic behaviour:

$$\mathscr{A}[p]\underset{p\to\infty}{\approx}\frac{(2\pi)^{n/2}}{S_{n-1}}\mathscr{A}(0)\frac{1}{p^{r+1}}\frac{2^{r+1-n/2}\Gamma\left(\frac{r}{2}+1\right)}{\Gamma[(n-r)/2]},$$

where, of course, terms involving the auxiliary q have automatically cancelled each other. One must notice that this does not hold for $r=0$ because (F.7) gives then $q<1$, and we used (F.2) for $q-1$ which must be a positive integer.

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