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Stationary Solutions of the Bogoliubov Hierarchy Equations in Classical Statistical Mechanics. 4

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Abstract. This is the forth and final paper of a series in which we investigate the stationary solutions of the BBGKY equations. Herein we prove a lemma which forms the basic step in the proof of our Main Theorem characterizing the stationary solutions of these equations which was stated in the first of this series.

Contents

0. Introduction

0.1. This is the final paper of a series, all bearing the same title (see [1–3]), devoted to characterizing the stationary solutions of the BBGKY hierarchy equations. Our Main Theorem, stated in [1], deals with states of an infinite system of classical particles in R^ν, $\nu \geqq 1$. It asserts that the set of those states (within a certain class of states) which correspond to stationary solutions of the BBGKY hierarchy coincides with the set of equilibrium states. As our class of states we take the Gibbs (DLR) states which correspond to potentials (in our terminology, "generating functions") of a general type (many-body and depending not only on coordinates but on particle velocities as well) which satisfy conditions $(G_1, \mathbf{1}) - (G_6, \mathbf{1})$[1]. The hypotheses of our Main Theorem require that the pair interaction potential in the hierarchy satisfies conditions $(I_1, \mathbf{1}) - (I_4, \mathbf{1})$. The condition $(I_4, \mathbf{1})$ restricts the interaction potential to a finite range.

1 Instead of writing: condition (G_1) from [1], formula (4.1) from [3], etc., we shall write: condition $(G_1, \mathbf{1})$, formula (4.1,**3**), etc. This convention was adopted in [2] and [3]

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The proof of our Main Theorem was divided into two parts. In the first part we showed that the generating function of a Gibbs state corresponding to a stationary solution satisfies Eq. (2.8,**1**). This equation is the dual of the BBGKY hierarchy. This part was presented in [1] (see, in particular, Theorem 1,**1**). In the second part of the proof we show that any function satisfying Eq. (2.8,**1**) and conditions $(G_1, \mathbf{1})$ $-(G_6, \mathbf{1})$ is of the form (2.7,**1**), i.e., is the generating function of an equilibrium state. This was asserted in Theorem 2,**1**.

The proof of Theorem 2,**1** was started in [2], continued in [3] and will be completed in the present paper.

We will say a few words about the contents of [2] and [3] in order to make things easier. In [2] we proved Theorem 2,**2** from which a special case of Theorem 2,**1** immediately follows. In this special case the number n_0 which appears in condition $(G_3, \mathbf{1})$ is equal to 2, i.e., the generating function of the Gibbs state vanishes for all configurations consisting of more than two particles (for a configuration we write both the coordinates and velocities of the particles). In addition, Theorem 2,**2** is the first step of an inductive process by which we prove Theorem 2,**1** in the general case.

In [3] we show that the Theorem 2,**1** follows from Theorem 2,**2** and Theorem 0.1,**3**. Theorem 0.1,**3** asserts that if the generating function is 0 for all "admissible" configurations with more than n particles ($n \geqq 3$) it is 0 for all n-particle admissible configurations.

The proof of Theorem 0.1,**3** requires an additional inductive procedure which concerns itself with the geometrical characteristics of configurations. To every particle configuration we associate a graph in R^ν whose vertices coincide with the positions of particles and whose edges correspond to pairs of interacting particles. Associated with each graph is a triple of non-negative integers (n, m, k): the number, n, of vertices, the number, m, of edges (both of which are positive), and the order, k, of the graph (which may be 0). The order of the graph indicates, roughly speaking, the common length of the one-dimensional "tails" with the possible exclusion of a "chain" of maximal length (for a precise definition, see either [3] or a subsection 0.2 below).

We proved Theorem 0.1,**3** by separately proving the assertion of the theorem for the various subsets of configurations labeled by the triples (n, m, k). By this procedure the proof of Theorem 0.1,**3** is reduced to that of Theorem 1.2,**3**. The inductive procedure mentioned above tells us in which order we must take the triples (n, m, k) in order to prove Theorem 1.2,**3** (see [3, Sections 3–5]). The proof of Theorem 1.2,**3** is reduced in [3] to the proof of an auxiliary assertion – the Basic Lemma (see [3, Sect. 2]). The present paper is devoted entirely to the proof of the Basic Lemma.

Unfortunately, the Basic Lemma as stated in [3] needs to be modified. In this section we reformulate our Basic Lemma and indicate the changes which must be introduced into the proof of Theorem 1.2,**3** and Lemma 4.1,**3** which form the basis of the inductive procedure mentioned above.

0.2. New Formulation of the Basic Lemma. Assuming the notation of Sect. 1,**1** (see also Sect. 1,**2**), we first state conditions on the interaction potential U which we assume to hold throughout. In addition we will use (with few changes) the definiti ons put forth in [3][2].

The interaction potential U is assumed to be a real-valued function on the half-line $(d_0, +\infty)$ where $d_0 \geqq 0$. Assume that

$$(I_1'') \quad U \in C^3(d_0, +\infty).$$

Assume further that there exists $d_1 > d_0$ such that

$$(I_2'') \quad U \not\equiv 0 \quad \text{on} \quad (d_1 - \delta, d_1) \quad \text{for any} \quad \delta \in (0, d_1 - d_0),$$
$$(I_3'') \quad U \equiv 0 \quad \text{on} \quad [d_1, +\infty).$$

As usual, H will denote the Hamiltonian of a system of particles interacting via the pair potential U (the particle mass is set equal to 1):

$$H(\bar{x}) = 1/2 \sum_{v \in \bar{x}} \langle v, v \rangle + \sum_{q, q' \in \bar{x}, q \neq q'} U(|q - q'|), \quad x \in D^0. \tag{0.1}$$

The closure of the convex hull of a set $K \subset R^\nu$ will be denoted Conv (K).

For any $\bar{x} \in D^0$ let $\bar{q}(\bar{x}) = \{q \in R^\nu : (q, v) \in \bar{x}$ for some $v \in R^\nu\}$. We say that $x = (q, v) \in \bar{x}$ is an *external point* in $\bar{x}$ (or, q is an external point in $\bar{x}$) if q is an extremal point in Conv $(\bar{q}(\bar{x}))$.

Assume that q is an external point in $\bar{x}$, $n(\bar{x}) \geqq 2$. Let $B_q^e = B_q^e(\bar{x})$ be:

for $\nu \geqq 2$ – the open cone in R consisting of the open halflines originated at q which are normal to those supporting hyperplanes, P, of Conv$(\bar{q}(\bar{x}))$ for which $P \cap \text{Conv}(\bar{q}(\bar{x})) = q$,

for $\nu = 1$ – the open half-line which does not intersect $\bar{q}(\bar{x})$ and has q as a limit point.

It is not difficult to check that

1) $|q' - \tilde{q}| > |q - \tilde{q}| \quad$ for all $\quad q' \in B_q^e$, $\tilde{q} \in \bar{x}$, $\tilde{q} \neq q$,

2) every point $q' \in B_q^e$, for which $|q' - q| > d_0$, is an external point in $\bar{x} \cup x'$ where $x' = (q', v')$, $v' \in R^\nu$.

We say that $x = (q, v) \in \bar{x}$ is an *accessible point* in $\bar{x}$ (or q is an accessible point in $\bar{x}$) if there exists a non-empty open set $B_q^a = B_q^a(\bar{x}) \subset R^\nu$ such that $\tilde{q} \in B_q^a$ implies (1) $|\tilde{q} - q| > d_0$, (2) $U'(|\tilde{q} - q|) \neq 0$, and (3) $|\tilde{q} - q'| > d_1$ for all $q' \in \bar{x}$, $q' \neq q$.

The set B_q^a is not uniquely defined. We assume that it is fixed once and for all for all configurations $\bar{x}$ and all accessible points $q \in \bar{x}$.

Further, we assume that B_q^a is chosen in such a way that if q is an external point, then B_q^a consists of the points $\tilde{q} \in B_q^e$ for which conditions (1)–(3) above hold. Notice that in this case there is a point $\tilde{q} \in B_q^a$ such that[3]

$$|\tilde{q} - q| > d_1 | \sqrt{2}, \qquad U''(|\tilde{q} - q|) \neq 0 \neq \frac{U'(|\tilde{q} - q|)}{|\tilde{q} - q|}.$$

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2 In conformity with [1, 2] we will denote the conditions on U by (I_1''), (I_2''), (I_3'') and the conditions on the generating function f by (G_1''), (G_2''), (G_3'') (see *0.4*)

3 This assertion follows from conditions (I_1'')–(I_3'') on U and is proven in fact in the course of the proof of Proposition 2.2,**2**

We say that the point $x=(q,v)$ is an *endpoint* in $\bar{x}$ (or q is an endpoint in $\bar{x}$) if there exists a unique point $\tilde{q}\in\bar{x}$, $\tilde{q}\neq q$, for which $|q-\tilde{q}|\leqq d_1$.

We define a *chain* in $\bar{x}$ to be a collection of pairwise distinct points $x_1,\ldots,x_s\in\bar{x}$, where $x_i=(q_i,v_i)$, $i=1,\ldots,s$, $s\geqq 2$, such that

(a) $|q_i-q_j|\leqq d_1$ iff $|i-j|\leqq 1$, $1\leqq i,j\leqq s$;

(b) $|q_i-\tilde{q}|>d_1$ for all $i=1,\ldots,s$, $\tilde{q}\in\bar{x}\setminus(x_1\cup\ldots\cup x_s)$.

The points x_1, x_s (or q_1,q_s) are called the ends of the chain $x_1,\ldots,x_s$.

We say that $\bar{x}\in D^0$ has *order* 0 with respect to $q\in\bar{x}$ if one of the following three conditions hold

1^0 there are no endpoints in $\bar{x}$,

2^0 q is the only endpoint in $\bar{x}$,

3^0 $\bar{x}$ contains two endpoints both being the ends of a single chain and q is one of these endpoints.

We say that $\bar{x}\in D^0$ has *order* $k\geqq 1$ with respect to $q\in\bar{x}$, if

(a) the configuration $\bar{x}\setminus x'$ has order $\leqq k-1$ with respect to q for any point $x'=(q',v')\in\bar{x}$ such that i) $q'\neq q$; ii) q' is an endpoint of $\bar{x}$; iii) if q is an endpoint of a chain in $\bar{x}$, then q' does not coincide with the other end of this chain;

(b) there exists $x'=(q',v')\in\bar{x}$ with properties i)–iii) such that the configuration $\bar{x}\setminus x'$ has order $k-1$ with respect to q.

We denote the order of a configuration $\bar{x}$ with respect to $q\in\bar{x}$ by $k(\bar{x},q)$. The order, $k(\bar{x})$, of a configuration $\bar{x}$ is defined by $k(\bar{x})=\min k(\bar{x},q)$, where the minimum is taken over all external points $q\in\bar{x}$.

We say that $x=(q,v)\in\bar{x}$ is an *isolated point* in $\bar{x}$ (or q is an isolated point in $\bar{x}$) if $|q-q'|>d_1$ for all $q'\in\bar{x}$, $q'\neq q$.

Below, an important role will be played by configurations containing a chain, with ends consisting of an external and an accessible point. For this reason we give the following definition:

Let (n,m,k) be an admissible triple, $n\geqq 1$. Denote by $\mathscr{B}(n,m,k,s)$, $s\geqq 1$, the set of sequences of the form $(x_1,\ldots,x_s,\bar{y})$, where $x_i=(q_i,v_i)\in R^\nu\times R^\nu$, $i=1,\ldots,s$, $\bar{y}\in D^0$, such that

(a) $x_1\cup\ldots\cup x_s\cup\bar{y}$ is a configuration from D^0 of type (n,m,k);

(b) $x_1,\ldots,x_s$ is a chain in $x_1\cup\ldots\cup x_s\cup\bar{y}$ for $s\geqq 2$ and q_1 is an isolated point in $x_1\cup\bar{y}$ for $s=1$.

We call the points of $\mathscr{B}(n,m,k,s)$ *ordered* (n,m,k,s)-*configurations* or simply *configurations* when there is no danger of confusion. $\mathscr{B}(n,m,k,s)$ is endowed with the topology induced by the Euclidian topology in $(R^\nu\times R^\nu)^s\times(R^\nu\times R^\nu)^{n-s}$ under the "partial" symmetrization map $S_{n,n-s}\colon(x_1,\ldots,x_s,y_1,\ldots,y_{n-s})\mapsto(x_1,\ldots,x_s,\bar{y})$, where $\bar{y}=y_1\cup\ldots\cup y_{n-s}$. By $\mathscr{IB}(n,m,k,s)$ (respectively, $\mathscr{DB}(n,m,k,s)$) we denote the subset of $\mathscr{B}(n,m,k,s)$ consisting of such $(x_1,\ldots,x_s,\bar{y})$ that $|q-q'|\neq d_1$ for any $q,q'\in x_1\cup\ldots\cup x_s\cup\bar{y}$ (respectively, $|q-q'|=d_1$ for at least one pair $q,q'\in x_1\cup\ldots\cup x_s\cup\bar{y}$). Points of $\mathscr{IB}(n,m,k,s)$ [respectively, $\mathscr{DB}(n,m,k,s)$] are called *internal* (respectively, *boundary*) configurations. For any $\mathscr{C}\subset\mathscr{B}(n,m,k,s)$ we set $\mathscr{IC}=\mathscr{C}\cap\mathscr{IB}(n,m,k,s)$, $\mathscr{DC}=\mathscr{C}\cap\mathscr{DB}(n,m,k,s)$.

Using the map $S_{n,n-s}$ we can introduce in a natural way the notion of a function of class C^l (with values in R^μ, $\mu=1,2,\ldots$) at a point $(x_1,\ldots,x_s,\bar{y})\in\mathscr{IB}(n,m,k,s)$.

Further, we can define the gradients $\partial_{q_i} f$, $\partial_{v_i} f$, $i=1, \ldots, s$, and $\partial_q f$, $\partial_v f$, $(q,v) \in \bar{y}$, where $f \in C^1$. Let $f, g \in C^1$ at a point $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{IB}(n,m,k,s)$. We set

$$\begin{aligned} &\{f(x_1, \ldots, x_s, \bar{y}), g(x_1, \ldots, x_s, \bar{y})\} \\ &= \sum_{(q,v) \in x_1 \cup \ldots \cup x_s \cup \bar{y}} (\langle \partial_q f, \partial_v g \rangle - \langle \partial_v f, \partial_q g \rangle)(x_1, \ldots, x_s, \bar{y}) . \end{aligned}$$

A function f on $\mathscr{B}(n,m,k,s)$ is called *symmetric* if

$$f(x_1, \ldots, x_s, \bar{y}) = f(x_s, \ldots, x_1, \bar{y}), \quad (x_1, \ldots, x_s, \bar{y}) \in \mathscr{B}(n,m,k,s) .$$

Denote by $\mathscr{A}(n,m,k,s)$, $s \geqq 1$, the set of sequences $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{B}(n,m,k,s)$ satisfying the following conditions:

(c) q_1 is an external and q_s an accessible point in $\bar{x}$,

(d) $k(x_1 \cup \ldots \cup x_s \cup \bar{y}, q_1) = k$.

The quadruple (n,m,k,s), $s \geqq 1$, is said to be *admissible* if the set $\mathscr{A}(n,m,k,s)$ [or, equivalently, the set $\mathscr{B}(n,m,k,s)$] is non-empty.

Notice that if $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{IB}(n,m,k,s)$, $s \geqq 2$, then for $j=1, \ldots, s-1$:

(i) $(x_1, \ldots, x_{s-j}, \bar{y}) \in \mathscr{IB}(n-j, m-j, k, s-j)$,

(ii) $(x_{j+1}, \ldots, x_s, \bar{y}) \in \mathscr{IB}(n-j, m-j, k', s-j)$ for some k'.

We are now ready to reformulate our Basic Lemma.

Basic Lemma. *Let (n,m,k,s) be an admissible quadruple with $n \geqq 3$, $s \geqq 2$. Suppose that functions, f_s and f_{s-1}, are defined on $\mathscr{B}(n,m,k,s)$ and $\bigcup_{k_1} \mathscr{IB}(n-1, m-1, k_1, s-1)$ respectively, and that for $s \geqq 3$ a function, f_{s-2}, is defined on $\bigcup_{k_2} \mathscr{IB}(n-2, m-2, k_2, s-2)$ and that these functions satisfy the conditions*:

(1) *f_s is symmetric and continuous on $\mathscr{B}(n,m,k,s)$, $f_s \in C^2$ on $\mathscr{IB}(n,m,k,s)$ and $f_s \equiv 0$ on $\mathscr{DA}(n,m,k,s)$,*

(2) *$f_{s-1} \in C^2$ on $\bigcup_{k_1} \mathscr{IB}(n-1, m-1, k_1, s-1)$,*

(3) *for $s \geqq 3$, $f_{s-2} \in C^2$ on $\bigcup_{k_2} \mathscr{IB}(n-2, m-2, k_2, s-2)$,*

(4) *for $s=2$, $\lim_{|q| \to \infty} f_1((q,v), \bar{y}) = 0$ for any $v \in R^\nu$, $\bar{y} \in D^0$.*

For every point $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{IA}(n,m,k,s)$ we assume that

a) *the following equations hold*

$$\begin{aligned} &\{f_s(x_1, \ldots, x_s, \bar{y}), H(x_1, \ldots, x_s, \bar{y})\} + \{f_{s-1}(x_2, \ldots, x_s, \bar{y}), U(|q_1 - q_2|)\} \\ &\quad + \{f_{s-1}(x_1, \ldots, x_{s-1}, \bar{y}), U(|q_{s-1} - q_s|)\} = 0, \end{aligned} \tag{0.2}$$

$$\begin{aligned} &\{f_{s-1}(x_1, \ldots, x_{s-1}, \bar{y}), H(x_1, \ldots, x_{s-1}, \bar{y})\} + \{f_{s-2}(x_2, \ldots, x_{s-1}, \bar{y}), U(|q_1 - q_2|)\} \\ &\quad + \{f_{s-2}(x_1, \ldots, x_{s-2}, \bar{y}), U(|q_{s-2} - q_{s-1}|)\} = 0 \end{aligned} \tag{0.3}$$

(*if $s=2$*, Eq. (0.3) *reduces to* $\{f_1(x_1, \bar{y}), H(x_1, \bar{y})\} = 0$), *where* $H(x_1, \ldots, x_i, \bar{y}) = H(x_1 \cup \ldots \cup x_i \cup \bar{y})$;

b) *for any $x_0 = (q_0, v_0) \in B^a_{q_1} \times R^\nu$, $x_{s+1} = (q_{s+1}, v_{s+1}) \in B^a_{q_s} \times R^\nu$ the following equations hold*

$$\{f_s(x_1, \ldots, x_s, \bar{y}), U(|q_0 - q_1|)\} + \{f_s(x_0, \ldots, x_{s-1}, \bar{y}), U(|q_{s-1} - q_s|)\} = 0, \tag{0.4}$$

$$\{f_s(x_1, \ldots, x_s, \bar{y}), U(|q_s - q_{s+1}|)\} + \{f_s(x_2, \ldots, x_{s+1}, \bar{y}), U(|q_1 - q_2|)\} = 0. \tag{0.5}$$

Then $f_s \equiv 0$ *on* $\mathscr{A}(n, m, k, s)$.

Apart from some technical details, this formulation of the Basic Lemma differs from that given in [3] at two essential points. Firstly, the number of equations has been reduced by 1 [Eq. (2.1,d,**3**) has been dropped]. Secondly, we now require that the equations for f_s, f_{s-1}, f_{s-2} are satisfied on $\mathscr{I}\mathscr{A}(n, m, k, s)$, whereas in [3] we only required that these equations hold at some point $\bar{x} \in D^0$. Correspondingly, our present conclusion is that $f_s \equiv 0$ on $\mathscr{A}(n, m, k, s)$, whereas in [3] we concluded only that $f = 0$ at $\bar{x}$.

In the following we will call f_s, f_{s-1}, f_{s-2} higher, middle and lower functions respectively. Correspondingly, Eqs. (0.4), (0.5) are called higher equations, (0.2) and (0.3) middle and lower equations respectively.

0.3. We now make the necessary modifications in the proofs of Lemmas 3.1,**3** and 4.1,**3**. We first deal with the auxiliary propositions from Sect. 2,**3**. We will assume that the function $f: D^0 \to R^1$ appearing in the statements of these propositions is of class C^2 at every point $\bar{x} \in D^0$.

Proposition 0.1. (Corresponds to Proposition 2.1,**3**). *Let* $\bar{x} \in D^0$ *and* $n(\bar{x}) \geqq 2$. *Assume that there are points,* $x_i = (q_i, v_i) \in \bar{x}$, $i = 1, 2$, $x_1 \neq x_2$, *and that there is a non-empty open set,* $B \subset R^\nu$, *such that for any* $q_0 \in B$, $v \in R^\nu$:

a) $U'(|q_0 - q_1|) \neq 0$,

b) $$\{f((\bar{x} \setminus x_1) \cup (q_1, v)),\ U(|q_0 - q_1|)\} = 0, \tag{0.6}$$

$$\{f((\bar{x} \setminus x_1) \cup (q_1, v)),\ H((\bar{x} \setminus x_1) \cup (q_1, v))\} + \{f(\bar{x} \setminus x_1),\ U(|q_1 - q_2|)\} = 0. \tag{0.7}$$

Then $\partial_{q_1} f(\bar{x}) = 0$.

Proof. Equation (0.6) may be written as

$$\frac{U'(|q_0 - q_1|)}{|q_0 - q_1|} \langle \partial_v f((\bar{x} \setminus x_1) \cup (q_1, v)),\ q_0 - q_1 \rangle = 0,\ q_0 \in B,\ v \in R^\nu.$$

Due to condition a),

$$\langle \partial_v f((\bar{x} \setminus x_1) \cup (q_1, v)),\ q_0 - q_1 \rangle = 0,\ q_0 \in B,\ v \in R^\nu.$$

Since B is a non-empty open set, we conclude that

$$\partial_v f((\bar{x} \setminus x_1) \cup (q_1, v)) = 0,\ v \in R^\nu.$$

Taking this into account, we apply the operator ∂_v to (0.7). As a result, we find:

$$\partial_{q_1} f((\bar{x} \setminus x_1) \cup (q_1, v)) = 0,\ v \in R^\nu. \quad \square$$

Proposition 0.2 (Corresponds to Proposition 2.2.**3**). *Let* $\bar{z} \in D^0$ *and* $n(\bar{z}) \geqq 1$. *Assume that there are two points,* $z^{(i)} = (q^{(i)}, v^{(i)}) \in \bar{z}$, $i = 1, 2$, *and that there is a non-empty open set,* $B \subset R^\nu$, *such that for any* $z_0 = (q_0, v_0) \in B \times R^\nu$, $v \in R^\nu$:

(a) $U'(|q_0 - q^{(1)}|) \neq 0$,

(b) *there exists a point,* $q \in (\bar{z} \cup z_0) \setminus z^{(2)}$, *such that*:

$$\{f((\bar{z} \setminus z^{(2)}) \cup (q^{(2)}, v)),\ U(|q_0 - q^{(1)}|)\} + \{f((\bar{z} \cup z_0) \setminus z^{(2)}),\ U(|q - q^{(2)}|)\} = 0. \tag{0.8}$$

Then $\partial^2_{v^{(1)},v^{(2)}} f(\bar{z}) = 0$.

Proof. When $z^{(1)} \neq z^{(2)}$, the result of applying ∂_v to (0.8) is:

$$\frac{U'(|q_0 - q^{(1)}|)}{|q_0 - q^{(1)}|} \partial^2_{v^{(1)},v} f((\bar{z} \setminus z^{(2)}) \cup (q^{(2)}, v))(q_0 - q^{(1)}) = 0,$$
$$z_0 = (q_0, v_0) \in B \times R^\nu, \quad v \in R^\nu.$$

When $z^{(1)} = z^{(2)}$, the result is:

$$\frac{U'(|q_0 - q^{(1)}|)}{|q_0 - q^{(1)}|} \partial^2_{v,v} f((\bar{z} \setminus z^{(2)}) \cup (q^{(2)}, v))(q_0 - q^{(1)}) = 0,$$
$$z_0 = (q_0, v_0) \in B \times R^\nu, \; v \in R^\nu.$$

Because of (a) the scalar factor is non-zero. Since B is a non-empty open set, we conclude that for $z^{(1)} \neq z^{(2)}$

$$\partial^2_{v^{(1)},v} f((\bar{z} \setminus z^{(2)}) \cup (q^{(2)}, v)) = 0, \quad v \in R^\nu,$$

and for $z^{(1)} = z^{(2)}$

$$\partial^2_{v,v} f((\bar{z} \setminus z^{(2)}) \cup (q^{(2)}, v)) = 0, \quad v \in R^\nu.$$

Setting $v = v^{(2)}$, we obtain the required equality. □

Proposition 0.3 (Corresponds to Proposition 2.3,**3**). *Let* $\bar{x} \in D^0$ *and* $n(\bar{x}) \geqq 2$. *Assume that there are points,* $x_i = (q_i, v_i) \in \bar{x}$, $x^{(i)} = (q^{(i)}, v^{(i)}) \in \bar{x}$, $i = 1, 2$, $x_2 \neq x_1 \neq x^{(2)} \neq x^{(1)}$, *and that there is a non-empty open set,* $B \subset R^\nu$, *such that for any* $q_0 \in B$, $v \in R^\nu$,

(a) $U'(|q_0 - q_1|) \neq 0$,

(b) *the following equations hold*:

$$\{f((\bar{x} \setminus x_1) \cup (q_1, v)), \; U(|q_0 - q_1|)\} = 0, \tag{0.9}$$

$$\{f((\bar{x} \setminus x_1) \cup (q_1, v)), \; H((\bar{x} \setminus x_1) \cup (q_1, v))\} + \{f(\bar{x} \setminus x_1), \; U(|q_1 - q_2|)\}$$
$$+ \{f((\bar{x} \setminus (x_1 \cup x^{(2)})) \cup (q_1, v)), \; U(|q^{(1)} - q^{(2)}|)\} = 0, \tag{0.10}$$

$$\partial^2_{v,v^{(1)}} f((\bar{x} \setminus (x_1 \cup x^{(2)})) \cup (q_1, v)) = 0. \tag{0.11}$$

Then $\partial_{q_1} f(\bar{x}) = 0$.

Proof. As in the proof of Proposition 0.1, we deduce from condition (a) and Eq. (0.9) that

$$\partial_v f((\bar{x} \setminus x_1) \cup (q_1, v)) = 0, \qquad v \in R^\nu.$$

On account of this equality and Eq. (0.11), we obtain the required result by applying ∂_v to both sides of (0.10). □

0.4. We now turn to the proofs of Lemmas 3.1,**3** and 4.1,**3**. Since Lemma 3.1,**3** is a particular case of Lemma 4.1,**3**, we only consider the latter. Assume that $f: D^0 \to R^1$

is such that:

(G_1'') $f \in C^2$,

(G_2'') there exists n_0 such that $f(\bar{x})=0$ for $\bar{x} \in D^0$ and $n(\bar{x})>n_0$,

(G_3'') for all $\bar{x} \in D^0$ with $n(\bar{x}) \geqq 1$ and $v \in R^\nu$

$$\lim_{|q|\to\infty} f(\bar{x} \cup (q, v)) = 0.$$

Let (n, m, k) be an admissible triple with $n \geqq 3$ for which the conditions of Lemma 4.1,**3** hold. Let $D^0(n, m, k)$ be the set of configurations $\bar{x} \in D^0$ of type (n, m, k). We now define four subsets, $D^0_\cdot(n, m, k) \subseteq D^0(n, m, k)$, $\cdot = a, b, c, d$. A configuration $\bar{x} \in D^0(n, m, k)$ is an element of:

a) $D^0_a(n, m, k)$ if there exists an external, isolated point, $x=(q, v) \in \bar{x}$.

b) $D^0_b(n, m, k)$ if there exists an external point, $x=(q, v) \in \bar{x}$, which is neither an isolated nor an endpoint in $\bar{x}$,

c) $D^0_c(n, m, k)$ if there exists an external point $x=(q, v) \in \bar{x}$ which is an endpoint but is not an end of a chain in $\bar{x}$,

d) $D^0_d(n, m, k)$ if there exists an external point $x=(q, v) \in \bar{x}$ which is an end of a chain in $\bar{x}$.

The sets $D^0_\cdot(n, m, k)$, $\cdot = a, b, c, d$, cover $D^0(n, m, k)$.

The assertion of Lemma 4.1,**3** is that $f(\bar{x})=0$ for $\bar{x} \in D^0(n, m, k)$. Because of condition (G_2'') it is enough to prove this equality for internal configurations $\bar{x}$ of type (n, m, k), i.e., for $\bar{x} \in D^0(n, m, k)$ such that $|q-q'| \neq d_1$ for all $q, q' \in \bar{x}$. For $\bar{x} \in D^0_\cdot(n, m, k)$, $\cdot = a, b, c$, the proof in [3] holds (see Sects. 3,**3** and 4,**3**). Therefore, we have to show that $f(\bar{x})=0$ for all internal $\bar{x} \in D^0_d(n, m, k)$.

For every $s \geqq 2$ we introduce the set $D^0_d(n, m, k, s)$ consisting of configurations, $\bar{x} \in D^0_d(n, m, k)$, which satisfy the following condition: there exists a chain, $x_1, \ldots, x_s$, in $\bar{x}$ with ends q_1, q_s such that (i) q_1 is an external point in $\bar{x}$, (ii) $k(\bar{x}, q_1)=k$. Clearly, $\bigcup_s D^0_d(n, m, k, s) = D^0_d(n, m, k)$. From $D^0_d(n, m, k)$ we extract the subset $\tilde{D}^0(n, m, k)$ consisting of those internal configurations $\bar{x}$ which contain a chain $x_1, \ldots, x_s$ with ends q_1, q_s satisfying conditions (i), (ii) above and such that (iii) q_s is an accessible point in $\bar{x}$. We first show that $f=0$ on $\tilde{D}^0(n, m, k, s)$ for all $s \geqq 2$.

For fixed $s \geqq 2$, we define the functions, $f_s: \mathscr{B}(n, m, k, s) \to R^1$, $f_{s-1}: \bigcup_{k_1} \mathscr{I}\mathscr{B}(n-1, m-1, k_1, s-1) \to R^1$, and (when $s \geqq 3$) $f_{s-2}: \bigcup_{k_2} \mathscr{I}\mathscr{B}(n-2, m-2, k_2, s-2) \to R^1$ generated by f under the symmetrization maps $(x_1, \ldots, x_t, \bar{y}) \mapsto x_1 \cup \ldots \cup x_t \cup \bar{y}$, $t=s-2$, $s-1$, s. We should check that f_s, f_{s-1}, f_{s-2} satisfy the conditions of the Basic Lemma.

The continuity and smoothness of these functions follow from condition (G_1''). Now let $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{D}\mathscr{A}(n, m, k, s)$. We will show that $f_s(x_1, \ldots, x_s, \bar{y})=0$. It is enough to verify that $f(x_1 \cup \ldots \cup x_s \cup \bar{y})=0$. But this follows from the continuity of f and from the fact that we can approximate the configuration $x_1 \cup \ldots \cup x_s \cup \bar{y}$ with configurations of type (n, m', k'), $m' \leqq m-1$, for which $f=0$ by virtue of Lemma 4.1,**3**. We still have to check that for all $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}(n, m, k, s)$, conditions a) and b) of the Basic Lemma hold [i.e., Eqs. (0.2)–(0.5) are valid]. This can be done as in Sect. 4,**3**.

According to the Basic Lemma, $f_s=0$ on $\mathscr{A}(n,m,k,s)$. Since the image of the set $\mathscr{A}(n,m,k,s)$ under symmetrization covers $\tilde{D}_d^0(n,m,k,s)$, it follows that $f=0$ on $\tilde{D}_d^0(n,m,k,s)$. Since $\tilde{D}_d^0(n,m,k,s)$ is open, all gradients of f vanish on $\tilde{D}_d^0(n,m,k,s)$.

We now prove that $f(\bar{x})=0$ for any internal configuration $\bar{x}\in D_d^0(n,m,k,s)$, $s\geqq 2$. Let $\bar{x}$ be an internal configuration from $D_d^0(n,m,k,s)$ and $x_1,\ldots,x_s$ be a chain in $\bar{x}$ figuring in the definition of $D_d^0(n,m,k,s)$. There are two possibilities: 1) $U'(|q_{s-1}-q_s|)=0$ and 2) $U'(|q_{s-1}-q_s|)\neq 0$. It is not hard to check that in the first case $\bar{x}$ satisfies the conditions of Proposition 0.1 and hence $\partial_{q_1}f(\bar{x})=0$.

Consider the second case. Observe, that if $x_0=(q_0,v_0)\in B_{q_1}^a\times R^\nu$, then $(\bar{x}\setminus x_s)\cup x_0\in\tilde{D}_d^0(n,m,k,s)$ and, as we proved above, the function f together with all its gradients vanishes at the point $(\bar{x}\setminus x_s)\cup x_0$. Using this fact, it is not hard to check, that for $\bar{z}=\bar{x}\setminus x_s$, $z^{(1)}=x_1$, $z^{(2)}=x_{s-1}$, $B=B_{q_1}^a$ the conditions of Proposition 0.2 (with $q=q_{s-2}$) are satisfied. Whence, $\partial^2_{v_1,v_{s-1}}f(\bar{x}\setminus x_s)=0$.

Making use of the above results, we can verify that for $\bar{x}$ the assumptions of Proposition 0.3 are satisfied with $x^{(1)}=x_{s-1}$, $x^{(2)}=x_s$, $B=B_{q_1}^a$. Accordingly, $\partial_{q_1}f(\bar{x})=0$.

We have therefore proved that $\partial_{q_1}f(\bar{x})=0$ for any internal configuration $\bar{x}\in D_d^0(n,m,k,s)$ containing a chain, $x_1,\ldots,x_s$, where q_1 is an external point in $\bar{x}$ and $k(\bar{x},q_1)=k$. Notice that for any such configuration one can find a continuous curve, $q_1(t)$, $0\leqq t\leqq 1$, with the following properties: (i) $q_1(0)=q_1$, (ii) $(\bar{x}\setminus x_1)\cup(q_1(t),v_1)\in D_d^0(n,m,k,s)$ for $0\leqq t\leqq 1$, (iii) $(\bar{x}\setminus x_1)\cup(q_1(t),v_1)$ is an internal configuration for $0\leqq t<1$ and a boundary configuration for $t=1$. By what is proven above, $f((\bar{x}\setminus x_1)\cup(q_1(t),v_1))$ does not depend on t for $0\leqq t<1$. On the other hand, $f((\bar{x}\setminus x_1)\cup(q_1(1),v_1))=0$ since the configuration $(\bar{x}\setminus x_1)\cup(q_1(1),v_1)$ may be approximated by configurations, $\bar{x}'$, of type $(n,m-1,k')$ for which $f(\bar{x}')=0$ by virtue of Lemma 4.1,**3**. Consequently, $f(\bar{x})=0$. This concludes the proof of Lemma 4.1,**3**. □

0.5. The sections which follow are devoted to proving the Basic Lemma. We will assume, but will not repeatedly mention, that the quadruple (n,m,k,s) and the functions f_s, f_{s-1}, f_{s-2} satisfy the conditions of the Basic Lemma. To lighten the notation we will write $\mathscr{B}$ and $\mathscr{A}$ for $\mathscr{B}(n,m,k,s)$ and $\mathscr{A}(n,m,k,s)$.

1. Analysis of the Upper Function (An Application of the Upper and Middle Equations)

This section contains the first part of the proof of the Basic Lemma. Using Eqs. (0.2), (0.4), (0.5), we will obtain some information about f_s. Denote by $\mathscr{A}^{(i)}$, $i=1,\ldots,s-1$, the set of configurations $(x_1,\ldots,x_s,\bar{y})\in\mathscr{A}$ for which $U'(|q_i-q_{i+1}|)\neq 0$.

The results of this Section are summarized in the following theorem.

Theorem 1. (1) *Let* $\nu\geqq 2$ *and let* $(x_1,\ldots,x_s,\bar{y})\in\bigcap_{i=1}^{s-1}\mathscr{I}\mathscr{A}^{(i)}$. *Then*

$$\partial_{v_1}f_s(x_1,\ldots,x_s,\bar{y})=\partial_{v_s}f_s(x_1,\ldots,x_s,\bar{y})=0. \tag{1.1a}$$

(2) *Let $\nu \geqq 2$ and let the configuration $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A} \setminus \mathscr{A}^{(s-1)}$ be such that $|q_{s-1} - q_s|$ is a limit point of the set $\{r > d_0 : U'(r) = 0\}$. Then*

$$\partial_{q_1} f_s(x_1, \ldots, x_s, \bar{y}) = 0. \tag{1.1b}$$

We will start with the proof of the statement (2). Equation (0.4) takes the form

$$\langle \partial_{v_1} f_s(x_1, \ldots, x_s, \bar{y}), \partial_{q_1} U(|q_0 - q_1|) \rangle = 0, \; q_0 \in B^a_{q_1},$$

on $\mathscr{I}\mathscr{A} \setminus \mathscr{A}^{(s-1)}$. Cancelling $U'(|q_0 - q_1|)\, |q_0 - q_1|^{-1}$ we get

$$\langle \partial_{v_1} f_s(x_1, \ldots, x_s, \bar{y}), q_0 - q_1 \rangle = 0, \; q_0 \in B^a_{q_1}.$$

Since q_0 runs over an open set $B^a_{q_1}$ it follows that $\partial_{v_1} f_s(x_1, \ldots, x_s, \bar{y}) = 0$.

Now suppose $(x_1, \ldots, x_s, \bar{y})$ satisfies the conditions stated in (2) of Theorem 1. Then in any neighbourhood of $(x_1, \ldots, x_s, \bar{y})$ there is a configuration, $(x'_1, \ldots, x'_s, \bar{y}')$, from $\mathscr{I}\mathscr{A} \setminus \mathscr{A}^{(s-1)}$. But we have proved that $\partial_{v'_1} f_s(x'_1, \ldots, x'_s, \bar{y}') = 0$, where $x'_1 = (q'_1, v'_1)$. It follows that $\partial^2_{v_1, q} f_s(x_1, \ldots, x_s, \bar{y}) = \partial^2_{v_1, v} f_s(x_1, \ldots, x_s, \bar{y}) = 0$ for all $(q, v) \in x_1 \cup \ldots \cup x_s \cup \bar{y}$.

Applying ∂_{v_1} to (0.2) and using the above equalities we obtain (1.1b). Statement (2) of Theorem 1 is proven. □

We now pass to the proof of statement (1).

Proposition 1.1. *For $\nu \geqq 1$ the following formulas hold*

$$\partial_{v_1} f_s(x_1, \ldots, x_s, \bar{y}) = \partial_{q_s} U(|q_{s-1} - q_s|)\, A^{(1)}(x_1, \ldots, x_s, \bar{y}) \tag{1.2a}$$
$$(\text{when} \quad (x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}^{(s-1)}),$$

$$\partial_{v_s} f_s(x_1, \ldots, x_s, \bar{y}) = \partial_{q_1} U(|q_1 - q_2|)\, A^{(s)}(x_1, \ldots, x_s, \bar{y}) \tag{1.2b}$$
$$(\text{when} \quad (x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}^{(1)}),$$

where $A^{(1)}$, $A^{(s)}$ are matrix-valued functions on $\mathscr{I}\mathscr{A}^{(s-1)}$, $\mathscr{I}\mathscr{A}^{(1)}$, respectively, which are locally constant. That is, for any configuration $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}^{(s-1)}$ (respectively, $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}^{(1)}$) there exists a neighbourhood, $\mathcal{O}^{(s)} = \mathcal{O}^{(s)}(x_1, \ldots, x_s, \bar{y}) \subset R^\nu$, of q_s (respectively, a neighbourhood, $\mathcal{O}^{(1)} = \mathcal{O}^{(1)}(x_1, \ldots, x_s, \bar{y}) \subset R^\nu$, of q_1) such that for all $x'_s = (q'_s, v'_s) \in \mathcal{O}^{(s)} \times R^\nu$ (respectively, $x'_1 = (q'_1, v'_1) \in \mathcal{O}^{(1)} \times R^\nu$)

$$A^{(1)}(x_1, \ldots, x_{s-1}, x'_s, \bar{y}) = A^{(1)}(x_1, \ldots, x_{s-1}, x_s, \bar{y})$$

(respectively,

$$A^{(s)}(x'_1, x_2, \ldots, x_s, \bar{y}) = A^{(s)}(x_1, x_2, \ldots, x_s, \bar{y})).$$

In addition, if q_1 is the limit point for some arbitrary open half-line, $L \subset B^e_{q_1}$, then for all $(x'_1, x_2, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}^{(s-1)}$ with $x'_1 \in (L \cup q_1) \times R^\nu$, the neighbourhoods $\mathcal{O}^{(s)}(x'_1, x_2, \ldots, x_s, \bar{y})$ can be chosen to coincide.

Proof. We will only consider the derivation of formula (1.2a) and the properties of $A^{(1)}$. The derivation of (1.2b) and the proof of the fact that $A^{(s)}$ is locally constant involves a similar line of argument [we use (0.5) instead of (0.9) – see below]. In addition, we will assume that $s \geqq 3$ (for $s = 2$ the formulas below need to be modified in a superficial way).

Let $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}^{(s-1)}$ and let $L \in B^e_{q_1}(x_1 \cup \ldots \cup x_s \cup \bar{y})$ be an open half-line. Choose $\mathcal{O}^{(s)}$ to be a connected neighbourhood of q_s small enough so that the following conditions hold: a) $U'(|q_{s-1} - q'_s|) \neq 0$ for all $q'_s \in \mathcal{O}^{(s)}$, b) L consists of internal points from $\bigcap_{x'_s \in \mathcal{O}^{(s)} \times R^\nu} B^e_{q_1}(x_1 \cup \ldots \cup x'_s \cup \bar{y})$. Now let $(x'_1, x_2, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}^{(s-1)}$, where $x'_1 = (q'_1, v'_1) \in (L \cup q_1) \times R^\nu$. Using conditions a) and b) find a connected open set, $\mathcal{O} \subset B^a_{q'_1}(x'_1 \cup x_2 \cup \ldots \cup x_s \cup \bar{y})$, such that if $x'_s = (q'_s, v'_s) \in \mathcal{O}^{(s)} \times R^\nu$, $q'_0 \in \mathcal{O}$, then

$$(x'_1, x_2, \ldots, x_{s-1}, x'_s, \bar{y}) \in \mathscr{I}\mathscr{A}^{(s-1)}, \qquad q'_0 \in B^a_{q'_1}(x'_1 \cup x_2 \cup \ldots x_{s-1} \cup x'_s \cup \bar{y}). \tag{1.3}$$

Notice that Eq. (0.4) holds when x_0 is replaced by $x'_0 = (q'_0, v'_0) \in \mathcal{O} \times R^\nu$, x_s is replaced by $x'_s = (q'_s, v'_s) \in \mathcal{O}^{(s)} \times R^\nu$, and x_1 is replaced by x'_1. This equation has the form

$$\begin{aligned}&\langle \partial_{v'_1} f_s(x'_1, x_2, \ldots, x_{s-1}, x'_s, \bar{y}), \partial_{q'_1} U(|q'_0 - q'_1|)\rangle \\ &\quad + \langle \partial_{v_{s-1}} f_s(x'_0, x'_1, x_2, \ldots, x_{s-1}, \bar{y}), \partial_{q_{s-1}} U(|q_0 - q'_1|)\rangle = 0, \\ &\qquad x'_0 \in \mathcal{O} \times R^\nu, \ x'_s \in \mathcal{O}^{(s)} \times R^\nu.\end{aligned} \tag{1.4}$$

Dividing by $U'(|q'_0 - q'_1|)\, U'(|q_{s-1} - q'_s|)\, |q'_0 - q'_1|^{-1}\, |q_{s-1} - q'_s|^{-1}$ which is non-zero by (1.3), we arrive at the equality

$$\begin{aligned}&\langle \mathbf{a}^{(1)}(x'_1, x_2, \ldots, x_{s-1}, x'_s, \bar{y}), q_{s-1} - q'_s\rangle \\ &\quad + \langle \mathbf{a}^{(s)}(x'_0, x'_1, x_2, \ldots, x_{s-1}, \bar{y}), q_{s-1} - q'_s\rangle = 0, \\ &\qquad x'_0 \in \mathcal{O} \times R^\nu, \ x'_s \in \mathcal{O}^{(s)} \times R^\nu,\end{aligned} \tag{1.5}$$

where

$$\begin{aligned}&\mathbf{a}^{(1)}(x'_1, x_2, \ldots, x_{s-1}, x'_s, \bar{y}) \\ &\quad = [U'(|q_{s-1} - q'_s|)]^{-1}\, |q_{s-1} - q'_s|\, \partial_{v'_1} f_s(x'_1, x_2, \ldots, x_{s-1}, x'_s, \bar{y}),\end{aligned} \tag{1.6a}$$

$$\begin{aligned}&\mathbf{a}^{(s)}(x'_0, x'_1, x_2, \ldots, x_{s-1}, \bar{y}) \\ &\quad = [U'(|q'_0 - q'_1|)]^{-1}\, |q'_0 - q'_1|\, \partial_{v_{s-1}} f_s(x'_0, x'_1, x_2, \ldots, x_{s-1}, \bar{y}).\end{aligned} \tag{1.6b}$$

By application of $\partial_{q'_0}$ and $\partial_{q'_s}$ to (1.5) we get

$$\begin{aligned}&\partial_{q'_s} \mathbf{a}^{(1)}(x'_1, x_2, \ldots, x_{s-1}, x'_s, \bar{y}) + [\partial_{q'_0} \mathbf{a}^{(s)}(x'_0, x'_1, x_2, \ldots, x_{s-1}, \bar{y})]^* = 0, \\ &\qquad x'_0 \in \mathcal{O} \times R^\nu, \ x'_s \in \mathcal{O}^{(s)} \times R^\nu.\end{aligned} \tag{1.7}$$

It follows that the matrix $\partial_{q'_s} \mathbf{a}^{(1)}(x'_1, x_2, \ldots, x_{s-1}, x'_s, \bar{y})$ is constant on $x'_s \in \mathcal{O}^{(s)} \times R^\nu$. Denoting this matrix by $A^{(1)}(x'_1, x_2, \ldots, x_{s-1}, x'_s, \bar{y})$, we have

$$\begin{aligned}&\mathbf{a}^{(1)}(x'_1, x_2, \ldots, x_{s-1}, x'_s, \bar{y}) = q'_s A^{(1)}(x'_1, x_2, \ldots, x_{s-1}, x'_s, \bar{y}) \\ &\quad + \mathbf{b}^{(1)}(x'_1, x_2, \ldots, x_{s-1}, x'_s, \bar{y}), \ x'_s \in \mathcal{O}^{(s)} \times R^\nu,\end{aligned} \tag{1.8a}$$

where the vector $\mathbf{b}^{(1)}(x'_1, x_2, \ldots, x_{s-1}, x'_s, \bar{y})$ *is constant on* $x'_s \in \mathcal{O}^{(s)} \times R^\nu$. *Due to* (1.7)

$$\begin{aligned}&\mathbf{a}^{(s)}(x'_0, x'_1, x_2, \ldots, x_{s-1}, \bar{y}) = -q'_0 (A^{(1)})^*(x'_1, x_2, \ldots, x_{s-1}, x'_s, \bar{y}) \\ &+ \mathbf{b}^{(s)}(x'_0, x'_1, x_2, \ldots, x_{s-1}, \bar{y}), \ x'_0 \in \mathcal{O} \times R^\nu, \ x'_s \in \mathcal{O}^{(s)} \times R^\nu,\end{aligned} \tag{1.8b}$$

where $\mathbf{b}^{(s)}(x_0', x_1', x_2, \dots, x_{s-1}, \bar{y})$ is constant on $x_0' \in \mathcal{O} \times R^\nu$. Substituting (1.8a, b) into (1.5) and applying $\partial_{q_0'}$ to both sides of this equality we get

$$\mathbf{b}^{(1)}(x_1', x_2, \dots, x_{s-1}, x_s', \bar{y}) = -q_{s-1} A^{(1)}(x_1', x_2, \dots, x_{s-1}, x_s', \bar{y}), \quad x_s' \in \mathcal{O}^{(s)} \times R^\nu.$$

From this and from (1.8a), (1.6a), we obtain (1.2a). □

Proposition 1.2. *Let* $\nu \geqq 1$ *and let* $(x_1, \dots, x_s, \bar{y}) \in \mathcal{I}\mathcal{A}^{(s-1)}$, $x_0 \in B_{q_1}^a \times R^\nu$. *Then*

$$A^{(1)}(x_1, \dots, x_s, \bar{y}) + (A^{(s)})^*(x_0, \dots, x_{s-1}, \bar{y}) = 0. \tag{1.9}$$

Proof. Choose neighbourhoods, $\mathcal{O}_0, \mathcal{O}_s$, of q_0, q_s such that conditions (1.3) hold for $q_0' \in \mathcal{O}_0$, $x_s' \in \mathcal{O}_s \times R^\nu$. Notice that if we write Eq. (0.4) with $x_0' = (q_0', v_0')$ replacing x_0 and $x_s' = (q_s', v_s')$ replacing x_s, we get (1.4).

We then substitute equalities (1.2a, b) with argument $(x_0', x_1, \dots, x_{s-1}, \bar{y})$ into (1.4) [it is easy to see that $(x_0', x_1, \dots, x_{s-1}, \bar{y}) \in \mathcal{I}\mathcal{A}^{(1)}$]. By cancelling the factor $U'(|q_0' - q_1|)\,|q_0' - q_1|^{-1}\, U'(|q_{s-1} - q_s'|)\,|q_{s-1} - q_s'|^{-1}$ and using the fact that the matrices $A^{(1)}$, $A^{(s)}$ are locally constant (see Proposition 1.1), we obtain (1.9). □

Proposition 1.3. *Let* $\nu \geqq 1$ *and let* $(x_1, \dots, x_s, \bar{y}) \in \mathcal{I}\mathcal{A}^{(1)} \cap \mathcal{I}\mathcal{A}^{(s-1)}$. *Then*

$$\begin{aligned}&[\partial^2_{q_1^2} U(|q_1 - q_2|)\, A^{(s)}(x_1, \dots, x_s, \bar{y})]^* \\ &\quad + \partial^2_{q_s^2} U(|q_{s-1} - q_s|)\, A^{(1)}(x_1, \dots, x_s, \bar{y}) = 0.\end{aligned} \tag{1.10}$$

Proof. To Eq. (0.2) we apply ∂_{v_1} and ∂_{v_s}. If $(x_1, \dots, x_s, \bar{y}) \in \mathcal{I}\mathcal{A}^{(1)} \cap \mathcal{I}\mathcal{A}^{(s-1)}$, then using Proposition 1.1 we obtain (1.10). □

Corollary 1.4. *Let* $\nu \geqq 1$. *Assume that for* $s \geqq 3$, $(x_1, \dots, x_s, \bar{y}) \in \mathcal{I}\mathcal{A}^{(s-2)} \cap \mathcal{I}\mathcal{A}^{(s-1)}$ *and for* $s = 2$, $(x_1, x_2, \bar{y}) \in \mathcal{I}\mathcal{A}^{(1)}$. *Assume further that* $x_0 \in B_{q_1}^a \times R^\nu$. *Then,*

$$\begin{aligned}&\partial^2_{q_{s-2}, q_{s-1}} U(|q_{s-2} - q_{s-1}|)\, A^{(1)}(x_0, \dots, x_{s-1}, \bar{y}) \\ &\quad = A^{(1)}(x_1, \dots, x_s, \bar{y})\, \partial^2_{q_0, q_1} U(|q_0 - q_1|).\end{aligned} \tag{1.11}$$

Proof. We apply Proposition 1.3 to $(x_0, \dots, x_{s-1}, \bar{y})$ and then use (1.9). □

Denote by $\mathcal{A}_{\text{sym}}$ the set of configurations $(x_1, \dots, x_s, \bar{y}) \in \mathcal{A}$ for which q_s is an external point in $x_1 \cup \dots \cup x_s \cup \bar{y}$. In other words, $(x_1, \dots, x_s, \bar{y}) \in \mathcal{A}_{\text{sym}}$ if and only if $(x_1, \dots, x_s, \bar{y}), (x_s, \dots, x_1, \bar{y}) \in \mathcal{A}$. Notice that when $\nu = 1$, $\mathcal{A}_{\text{sym}}$ is non-empty only if $n(\bar{y}) = 0$.

Proposition 1.5. *Let* $\nu \geqq 1$ *and let* $(x_1, \dots, x_s, \bar{y}) \in \mathcal{I}\mathcal{A}^{(s-1)} \cap \mathcal{A}_{\text{sym}}$. Then

$$A^{(1)}(x_1, \dots, x_s, \bar{y}) = A^{(s)}(x_s, \dots, x_1, \bar{y}). \tag{1.12}$$

Proof. According to the Basic Lemma, f_s is symmetric. By using (1.2a, b) we see that

$$\begin{aligned}&\partial_{q_s} U(|q_{s-1} - q_s|)\, A^{(1)}(x_1, \dots, x_s, \bar{y}) \\ &\quad - \partial_{q_s} U(|q_{s-1} - q_s|)\, A^{(s)}(x_s, \dots, x_1, \bar{y}) = 0, \ (x_1, \dots, x_s, \bar{y}) \in \mathcal{I}\mathcal{A}^{(s-1)} \cap \mathcal{A}_{\text{sym}}.\end{aligned}$$

By cancelling the factor $U'(|q_{s-1} - q_s|)\,|q_{s-1} - q_s|^{-1}$ we see that

$$(q_{s-1} - q_s)\,[A^{(1)}(x_1, \dots, x_s, \bar{y}) - A^{(s)}(x_s, \dots, x_1, \bar{y})] = 0.$$

Using the fact that the matrices $A^{(1)}$ and $A^{(s)}$ are locally constant we arrive at (1.12). □

Corollary 1.6. *Let $v \geqq 1$. Assume that $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}^{(s-1)}$ and that $x_0 \in B^a_{q_1} \times R^v$ is such that $(x_0, \ldots, x_{s-1}, \bar{y}) \in \mathscr{A}_{\mathrm{sym}}$. Then*

$$(A^{(1)})^* (x_1, \ldots, x_s, \bar{y}) + A^{(1)} (x_{s-1}, \ldots, x_0, \bar{y}) = 0. \tag{1.13}$$

Proof. Notice that the conditions stated in Proposition 1.2 hold for $(x_1, \ldots, x_s, \bar{y})$ and x_0. Hence, (1.9) holds. Moreover, since $(x_{s-1}, \ldots, x_0, \bar{y}) \in \mathscr{I}\mathscr{A}^{(s-1)} \cap \mathscr{A}_{\mathrm{sym}}$, Equality (1.12) holds for $(x_{s-1}, \ldots, x_0, \bar{y})$ by virtue of Proposition 1.5. Combining these two equalities, we obtain (1.13). □

Proposition 1.7. *Let $v \geqq 1$ and let $(x_1, \ldots, x_s \bar{y})$, $(x'_1, \ldots, x'_s, \bar{y}) \in \bigcap_{i=1}^{s-1} \mathscr{I}\mathscr{A}^{(i)}$. Further, assume that $q_1, \ldots, q_s, q'_1, \ldots, q'_s$ lie on a straight line, $L \subset R^v$, satisfying the condition* dist $(L, \mathrm{Conv}\, \bar{q}(\bar{y})) > d_1$ *and assume that q'_1 lies on the same side of q'_2 as q_1 of q_2. Then*

$$A^{(1)} (x_1, \ldots, x_s, \bar{y}) = A^{(1)} (x'_1, \ldots, x'_s, \bar{y}). \tag{1.14}$$

Proof. From conditions $(I''_1 - I''_3)$ on U it follows that in any neighbourhood of d_1 there are points, r, for which $U'(r) \neq 0$. Using this fact we choose $r_1, r'_1 \in (d_0, d_1)$ and natural k_1, $k'_1 > s/2$ such that the following conditions hold

$$U'(r_1)\, U'(r'_1) \neq 0, \tag{1.15a}$$

$$r_1, r'_1 > d_1/2, \qquad r_1 + |q_2 - q_1| > d_1, \qquad r'_1 + |q'_2 - q'_1| > d_1, \tag{1.15b}$$

$$q_1 + 2k_1 r_1 (q_1 - q_2)/|q_1 - q_2| = q'_1 + 2k'_1 r'_1 (q'_1 - q'_2)/|q'_1 - q'_2|. \tag{1.15c}$$

Further, set

$$q_i = q_1 + r_1(-i+1)(q_1 - q_2)/|q_2 - q_1|, \qquad i = 0, -1, -2, \ldots,$$

$$q'_i = \begin{cases} q'_1 + r'_1(-i+1)(q'_1 - q'_2)/|q'_1 - q'_2|, & i = 0, -1, \ldots, -2k'_1 + 1, \\ q'_1 + [r'_1 \cdot 2k'_1 + (-i+1-2k'_1) r_1](q_1 - q_2)/|q_1 - q_2|, & i = -2k'_1, -2k'_1 - 1, \ldots \end{cases}$$

(see Fig. 1). Finally, choose arbitrary $v_i \in R^v$ for $i = 0, -1, \ldots$ and arbitrary $v'_i \in R^v$ for $i = 0, \ldots, -2k'_1 + 2$. Set $v'_i = v_{i+2k'_1-2k_1}$ for $i = -2k'_1 + 1, -2k'_1, \ldots$.

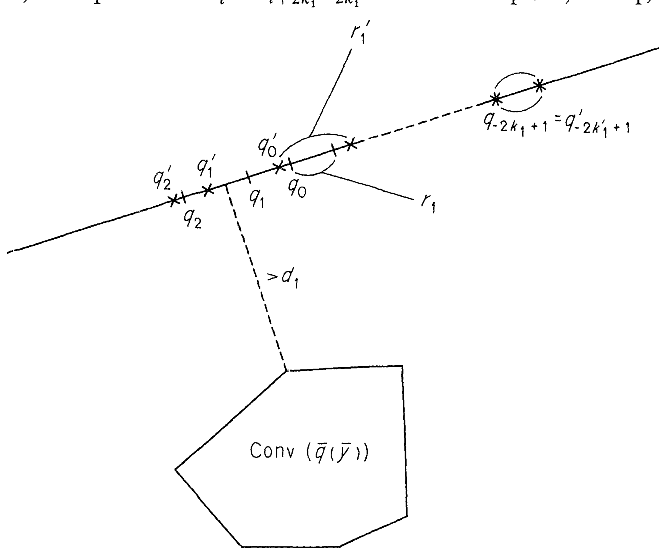

Fig. 1

Consider configurations $(x_i, x_{i+1}, \ldots, x_{i+s-1}, \bar{y})$ and $(x'_i, x'_{i+1}, \ldots, x'_{i+s-1}, \bar{y})$, $i=1, 0, -1, \ldots$, where $x_j=(q_j, v_j)$, $x'_j=(q'_j, v'_j)$, $j=s, s-1, \ldots$. From (1.15a, b) and the conditions of the proposition under consideration it follows that for all $i \leqq 0$

$$(x_{i+1}, \ldots, x_{i+s}, \bar{y}) \in \mathcal{I}\mathcal{A}^{(s-1)}, \quad (x_i, \ldots, x_{i+s-1}, \bar{y}) \in \mathcal{A}_{\mathrm{sym}},$$
$$x_i \in B^a_{q_{i+1}}(x_{i+1} \cup \ldots \cup x_{i+s-1} \cup \bar{y}) \times R^\nu,$$
$$(x'_{i+1}, \ldots, x'_{i+s}, \bar{y}) \in \mathcal{I}\mathcal{A}^{(s-1)}, \quad (x'_i, \ldots, x'_{i+s-1}, \bar{y}) \in \mathcal{A}_{\mathrm{sym}},$$
$$x'_i \in B^a_{q'_{i+1}}(x'_{i+1} \cup \ldots \cup x'_{i+s-1} \cup \bar{y}) \times R^\nu.$$

Using Corollary 1.6, we obtain

$$A^{(1)}(x_1, \ldots, x_s, \bar{y}) = A^{(1)}(x_{2j+1}, \ldots, x_{2j+s}, \bar{y}),$$
$$A^{(1)}(x'_1, \ldots, x'_s, \bar{y}) = A^{(1)}(x'_{2j+1}, \ldots, x'_{2j+s}, \bar{y}),$$
$$j=0, -1, \ldots. \tag{1.16}$$

From the construction and from (1.15c) it follows that if $i \leqq -2k_1+1$ and $i'=i+2k_1-2k'_1$, then $q_i=q'_{i'}$ and $v_i=v'_{i'}$. Hence, for $j \leqq -k_1-(s-1)/2$, $j'=j+k_1-k'_1$ the configurations $(x_{2j+1}, \ldots, x_{2j+s}, \bar{y})$ and $(x'_{2j'+1}, \ldots, x'_{2j'+s}, \bar{y})$ coincide. Together with (1.16) this gives (1.14). □

Notice that if $\nu=1$, the conditions of Proposition 1.7 hold only when $n(\bar{y})=0$.

Proposition 1.8. *Let $\nu \geqq 1$. Assume that $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{i=1}^{s-1} \mathcal{I}\mathcal{A}^{(i)}$ and that the points $q_1, \ldots, q_s$ lie on a straight-line, $L \subset R^\nu$, satisfying the conditions $L \cap B^e_{q_1} \neq \emptyset$,* dist $(L, \mathrm{Conv}(\bar{q}(\bar{y})) > d_1$. *Then*

$$A^{(1)}(x_1, \ldots, x_s, \bar{y}) = 0. \tag{1.17}$$

Proof. We will assume that $s \geqq 3$ (as in the proof of Proposition 1.1, for $s=2$ the formulas below need to be modified in a superficial manner). Choose a sequence of points, $q_1(i)$, $i=0, 1, \ldots$, on L, lying on the same side of q_2 as q_1, with $q_1(0)=q_1$ and such that (a) $U'(|q_1(i)-q_2|) \neq 0$, $|q_1(i)-q_2| > |q_1-q_2|$ for all i, (b) $\lim_{i \to \infty} |q_1(i)-q_2| = d_1$.

By application of Propositions 1.1, 1.7, we can find a neighbourhood, $\mathcal{O}^{(s)}$, of q_s such that

$$A^{(1)}(x_1, \ldots, x_s, \bar{y}) = A^{(1)}(x'_1(i), x_2, \ldots, x_{s-1}, x'_s, \bar{y}),$$
$$x'_s \in \mathcal{O}^{(s)} \times R^\nu, \quad x'_1(i) = (q_1(i), v'_1), \quad v'_1 \in R^\nu, \quad i=0, 1, \ldots. \tag{1.18}$$

By (1.2a) and (1.18),

$$f_s(x'_1(i), x_2, \ldots, x_{s-1}, x'_s, \bar{y}) = \langle v'_1, \partial_{q'_s} U(|q_{s-1}-q'_s|) A^{(1)}(x_1, \ldots, x_s, \bar{y})\rangle$$
$$+ a(x'_1(i), x_2, \ldots, x_{s-1}, x'_s, \bar{y}),\ x'_s \in \mathcal{O}^{(s)} \times R^\nu,\ x'_1(i) = (q_1(i), v'_1),$$
$$v'_1 \in R^\nu,\ i=0, 1, \ldots, \tag{1.19}$$

where $a(x'_1(i), x_2, \ldots, x_{s-1}, x'_s, \bar{y})$ does not depend on $v'_1 \in R^\nu$. As assumed in the Basic Lemma,

$$\lim_{i \to \infty} f_s(x'_1(i), x_2, \ldots, x_{s-1}, x'_s, \bar{y}) = 0. \tag{1.20}$$

By using (1.19) with v'_1 set equal to zero we get

$$\lim_{i\to\infty} a(x'_1(i), x_2, \ldots, x_{s-1}, x'_s, \bar{y}) = 0. \tag{1.21}$$

From (1.19)–(1.21) it is clear that

$$\langle v'_1, \partial_{q'_s} U(|q_{s-1} - q'_s|) A^{(1)}(x_1, \ldots, x_s, \bar{y})\rangle = 0, \quad q'_s \in \mathcal{O}^{(s)}, \quad v'_s \in R^\nu,$$

and therefore,

$$\partial_{q'_s} U(|q_{s-1} - q'_s|) A^{(1)}(x_1, \ldots, x_s, \bar{y}) = 0, \quad q'_s \in \mathcal{O}^{(s)}.$$

By cancelling $U'(|q_{s-1} - q'_s|)$ and using the fact that $\mathcal{O}^{(s)}$ is non-empty and open we obtain (1.17). □

Proposition 1.9. *Let* $\nu \geqq 2$ *and let* $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{i=1}^{s-1} \mathcal{A}^{(i)}$. *Then*

$$A^{(1)}(x_1, \ldots, x_s, \bar{y}) = 0.$$

Remark. This statement also holds when $\nu = 1$ and $n(\bar{y}) = 0$. In this case it is implied by Proposition 1.8.

Proof. We employ a geometric construction which will also be useful below. This construction can be used only when $\nu \geqq 2$. According to the definition of $B^a_{q_1}$ we can find a point, $q' \in B^a_{q_1}(x_1 \cup \ldots \cup x_s \cup \bar{y})$, such that $|q' - q_1| > d_1/\sqrt{2}$ and $U''(|q' - q_1|) \neq 0$. Set $r_1 = |q' - q_1|$ and let

$$q'_i = q_1 + (-i+1)(q' - q_1), \quad x'_i = (q'_i, v'_i), \quad i = 0, -1, \ldots,$$

where $v'_i \in R^\nu$, $i = 0, -1, \ldots$, are arbitrary. It is not difficult to check that for $|i_0| > s$ sufficiently large there is a point q such that:

a) the half-line $L_1 = \{\tilde{q} \in R^\nu : \tilde{q} = q'_{i_0} + t(q - q'_{i_0}), \quad t > 0\}$ lies in $B^e_{q'_{i_0}}(x'_{i_0} \cup \ldots \cup x_{i_0+s-1} \cup \bar{y})$,

b) dist $(L, \text{Conv}(\bar{q}(\bar{y}))) > d_1$, where L is the straight line containing L_1,

c) $|q - q'_{i_0}| = r_1$

(see Fig. 2). For such an i_0 we set

$$x_i = x'_i, \quad i = 0, -1, \ldots, i_0,$$

$$q_i = q_{i_0} + (i_0 - 1)(q - q'_{i_0}), \quad x_i = (q_i, v_i), \quad i = i_0 - 1, \quad i_0 - 2, \ldots,$$

where $v_i \in R^\nu$, $i = i_0 - 1$, $i_0 - 2, \ldots$, is arbitrary (recall how the sequence x'_i, $i = 0, -1, \ldots$, was constructed).

Let's consider the configurations $(x_i, \ldots, x_{i+s-1}, \bar{y})$, $i = 1, 0, -1, \ldots$. By construction, $(x_i, \ldots, x_{i+s-1}, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathcal{I}\mathcal{A}^{(j)}$, $x_{i-1} \in B^a_{q_i}(x_i \cup \ldots \cup x_{i+s-1} \cup \bar{y}) \times R^\nu$ and for $(x_i, \ldots, x_{i+s-1}, \bar{y})$, $i \leqq i_0 - s + 1$, the conditions of Proposition 1.8 hold. By Corollary 1.4 and our choice of q_0 which guarantees the non-degeneracy of $\partial^2_{q_0, q_1} U(|q_0 - q_1|)$, it is enough to prove that

$$A^{(1)}(x_0, \ldots, x_{s-1}, \bar{y}) = 0. \tag{1.22}$$

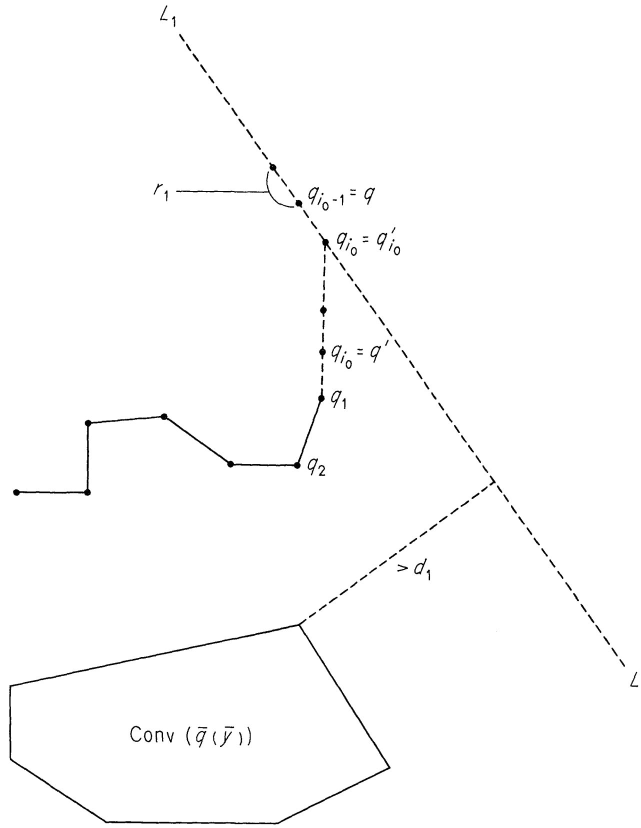

Fig. 2

By applying Corollary 1.4 to $(x_i, \ldots, x_{i+s-1}, \bar{y})$ and x_{i-1}, $i=0, -1, \ldots$, it is clear that we will be able to prove equality (1.22) if we can find an i such that

$$A^{(1)}(x_i, \ldots, x_{i+s-1}, \bar{y})=0.$$

But by Proposition 1.8 the last equality holds for all $i \leqq i_0 - s + 1$. □

Proposition 1.10. *Let* $\nu \geqq 2$. *Assume that* $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathscr{I}\mathscr{A}^{(j)}$ *and that* $U''(|q_1 - q_2|) \neq 0$. *Then*

$$A^{(s)}(x_1, \ldots, x_s, \bar{y})=0. \tag{1.23}$$

Proof. According to Proposition 1.9, $A^{(1)}(x_1, \ldots, x_s, \bar{y})=0$. Using Proposition 1.3 and the fact that $\partial^2_{q_2^2} U(|q_1 - q_2|)$ is non-degenerate we get (1.23). □

Proposition 1.11. *Let* $\nu \geqq 2$ *and let* $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathscr{I}\mathscr{A}^{(j)}$. *Then equality* (1.23) *holds.*

Proof. We choose an arbitrary $q_0 \in B^e_{q_1}$ and consider the half-line $L=\{q \in R^\nu : q=q_1+t(q_0-q_1), t \geqq 0\}$. Obviously, $(x_1(t), x_2, \ldots, x_s, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathscr{I}\mathscr{A}^{(j)}$ for $0 \leqq t < t_0$, where

$$t_0 = \min[t \geqq 0 : U'(|q_1 + t(q_0 - q_1) - q_2|)=0].$$

Since $A^{(s)}$ is locally constant and continuous (see Proposition 1.1),

$$A^{(s)}(x_1(t), x_2, \ldots, x_s, \bar{y}) = A^{(s)}(x_1, \ldots, x_s, \bar{y}), \qquad 0 \leqq t < t_0 .$$

If we can find $\tau \in [0, t_0)$ such that $U''(|q_1(\tau) - q_2|) \neq 0$, then, by Proposition 1.10, $A^{(s)}(x_1(\tau), x_2, \ldots, x_s, \bar{y}) = 0$ and (1.23) follows.

If $U''(|q_1(t) - q_2|) = 0$ for all $t \in [0, t_0)$, then $U'(|q_1(t) - q_2|) = c$, a constant, for $0 \leqq t < t_0$, and, by our choice of t_0 and the continuity of U', $c = 0$. This contradicts the condition $U'(|q_1 - q_2|) \neq 0$. □

Assertion (1) of Theorem 1 follows immediately from Propositions 1.1, 1.9, and 1.11.

2. Analysis of the Middle Function for $\nu \geqq 2$ (An Application of the Middle Equation)

We now pass to the second part of the proof of the Basic Lemma. In this second section we study the properties of f_{s-1} using Eq. (0.2) and Theorem 1 of Sect. 1. We will assume throughout this section that $\nu \geqq 2$.

We will consider the following sets:

$$\bar{\mathscr{A}}^{(i)} = \{(x_1, \ldots, x_s, \bar{y}) \in \mathscr{A}^{(i)} : U''(|q_i - q_{i+1}|) \neq 0\}, \qquad i = 1, \ldots, s-1 .$$

In addition, we will consider the set $\bar{\mathscr{A}}_1$ composed of those configurations $(x_1, \ldots, x_{s-1}, \bar{y}) \in \bigcup_{k_1} \mathscr{B}(n-1, m-1, k_1, s-1)$ for which there exists an open set, $\mathcal{O} \in \mathrm{R}^\nu$, having the property that $(x, x_1, \ldots, x_{s-1}, \bar{y}) \in \bigcap_{i=1}^{s-1} \bar{\mathscr{A}}^{(i)}$ for all $x \in \mathcal{O} \times R^\nu$.

We define

$$G(q, q') = \partial^2_{q, q'} U(|q - q'|), \qquad q, q' \in R^\nu, \qquad |q - q'| > d_0 .$$

It is easy to see that $(x_1, \ldots, x_{s-1}, \bar{y}) \in \bar{\mathscr{A}}_1$, $s \geqq 3$, implies that the matrix $\prod_{i=1}^{s-2} G(q_i, q_{i+1})$ is non-degenerate.

The major result of this section is:

Theorem 2. a) *If* $s \geqq 3$, *then*

$$\begin{aligned} &\partial^2_{v_1, v_{s-1}} f_{s-1}(x_1, \ldots, x_{s-1}, \bar{y}) \\ &\quad = b(x_1, \ldots, x_{s-1}, \bar{y}) \prod_{i=1}^{s-2} G(q_i, q_{i+1}), \qquad (x_1, \ldots, x_{s-1}, \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1 ; \end{aligned} \tag{2.1a}$$

b) if $s = 2$, *then*

$$\partial^2_{v_1^2} f_1(x_1, \bar{y}) = b(x_1, \bar{y}) E, \qquad (x_1, \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1 . \tag{2.1b}$$

Here, b is a function on $\mathscr{I}\bar{\mathscr{A}}_1$ having the property that $(x_1, \ldots, x_{s-1}, \bar{y})$, $(x_1', \ldots, x_{s-1}', \bar{y}') \in \mathscr{I}\bar{\mathscr{A}}_1$ together with $\bar{y} = \bar{y}'$, implies

$$b(x_1, \ldots, x_{s-1}, \bar{y}) = b(x_1', \ldots, x_{s-1}', \bar{y}'), \tag{2.2}$$

E is the identity matrix.

We now begin the proof of Theorem 2.

Proposition 2.1. *Let* $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{i=1}^{s-1} \mathscr{I}\mathscr{A}^{(i)}$. *Then*

$$\partial^2_{v_2, v_s} f_{s-1}(x_2, \ldots, x_s, \bar{y})\, G(q_1, q_2) = G(q_{s-1}, q_s)\, \partial^2_{v_1, v_{s-1}} f_{s-1}\,(x_1, \ldots, x_{s-1}, \bar{y}). \tag{2.3}$$

Proof. We shall only consider the case $s \geqq 3$. Let $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{i=1}^{s-1} \mathscr{I}\mathscr{A}^{(i)}$. Choose neighbourhoods, $\mathscr{O}_1$, $\mathscr{O}_s$, of q_1, q_s, such that $(x_1', x_2, \ldots, x_{s-1}, x_s', \bar{y}) \in \bigcap_{i=1}^{s-1} \mathscr{I}\mathscr{A}^{(i)}$ when $x_1' = (q_1', v_1') \in \mathscr{O}_1 \times R^\nu$ and $x_s' = (q_s', v_s') \in \mathscr{O}_s \times R^\nu$. By Theorem 1 (1),

$$\partial_{v_1'} f_s(x_1', x_2, \ldots, x_{s-1}, x_s', \bar{y}) = \partial_{v_s'} f_s(x_1', x_2, \ldots, x_{s-1}, x_s', \bar{y}) = 0. \tag{2.4}$$

Consider Eq. (0.2) with argument $(x_1', x_2, \ldots, x_{s-1}, x_s', \bar{y})$. By applying $\partial_{v_s'}$ and $\partial_{v_1'}$ to this equation and using (2.4), we obtain

$$\begin{aligned} &\partial_{q_s'} f_s(x_1', x_2, \ldots, x_{s-1}, x_s', \bar{y}) \\ &= \partial^2_{v_2, v_s'} f_{s-1}(x_2, \ldots, x_{s-1}, x_s', \bar{y})\, \partial_{q_2} U(|q_1' - q_2|) \end{aligned} \tag{2.5a}$$

and

$$\begin{aligned} \partial_{q_1'} f_s(x_1', x_2, \ldots, x_{s-1}, x_s', \bar{y}) &= \partial^2_{v_{s-1}, v_1'} f_{s-1}(x_1', x_2, \ldots, x_{s-1}, \bar{y})\, \partial_{q_{s-1}} U(|q_{s-1} - q_s'|), \\ x_1' \in \mathscr{O}_1 \times R^\nu, &\qquad x_s' \in \mathscr{O}_s \times R^\nu. \end{aligned} \tag{2.5b}$$

By applying $\partial_{q_1'}$ to (2.5a) and $\partial_{q_s'}$ to (2.5b) and then comparing the results we arrive at (2.3). □

For $(x_1, \ldots, x_{s-1}, \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1$ we set

$$B(x_1, \ldots, x_{s-1}, \bar{y}) = \begin{cases} \partial^2_{v_{s-1}, v_1} f_{s-1}(x_1, \ldots, x_{s-1}, \bar{y}) \left(\prod_{i=1}^{s-2} G(q_i, q_{i+1}) \right)^{-1}, & s \geqq 3, \\ \partial^2_{v_1^2} f_1(x_1, \bar{y}), \; s = 2. \end{cases} \tag{2.6}$$

Proposition 2.2. *Let* $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{i=1}^{s-1} \mathscr{I}\bar{\mathscr{A}}^{(i)}$. *Then*

$$G(q_1, q_2)\, B(x_2, \ldots, x_s, \bar{y}) = B(x_1, \ldots, x_{s-1}, \bar{y})\, G(q_1, q_2). \tag{2.7}$$

Proof. Notice that if $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{i=1}^{s-1} \mathscr{I}\bar{\mathscr{A}}^{(i)}$, then $(x_1, \ldots, x_{s-1}, \bar{y})$, $(x_2, \ldots, x_s, \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1$. Under these conditions Eq. (2.3) of Proposition 2.1 holds. We substitute (2.6) into this equation. Using the fact that $G(q, q')$ is symmetric we obtain (2.7). □

Proposition 2.3. *Let* $(x_1, \ldots, x_{s-1}, \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1$. *There are neighbourhoods,* $\mathscr{O}_i \subset R^\nu$, *of* q_i, $i = 1, \ldots, s-1$, *such that for* $x_i' \in \mathscr{O}_i \times R^\nu$, $i = 1, \ldots, s-1$,

$$(x_1', \ldots, x_{s-1}', \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1, \qquad B(x_1, \ldots, x_{s-1}, \bar{y}) = B(x_1', \ldots, x_{s-1}', \bar{y}). \tag{2.8}$$

Proof. We first choose $x_0' = (q_0', v_0')$ such that $(x_0', x_1, \ldots, x_{s-1}, \bar{y}) \in \bigcap_{i=1}^{s-1} \mathscr{I}\bar{\mathscr{A}}^{(i)}$. We then choose a sequence of points, $x_{-1}' = (q_{-1}', v_{-1}')$, $x_{-2}' = (q_{-2}', v_{-2}'), \ldots,$ and neighbourhoods, $\mathscr{O}_1, \ldots, \mathscr{O}_{s-1}$, of $q_1, \ldots, q_{s-1}$ such that for every

$x'_i = (q'_i, v'_i) \in \mathcal{O}_i \times R^\nu$, $i = 1, \ldots, s-1$, $(x'_1, \ldots, x'_{s-1}, \bar{y}) \in \mathcal{I}\bar{\mathcal{A}}_1$, and for every $j \leqq 0$

$$(x'_j, \ldots, x'_{j+s-1}, \bar{y}) \in \bigcap_{l=1}^{s-1} \mathcal{I}\bar{\mathcal{A}}^{(l)}.$$

By Proposition 2.2, for $j \leqq 0$ we have

$$B(x'_j, \ldots, x'_{j+s-2}, \bar{y})\, G(q'_j, q'_{j+1}) = G(q'_j, q'_{j+1})\, B(x'_{j+1}, \ldots, x'_{j+s-1}, \bar{y}),$$

where $x'_i = (q'_i, v'_i) \in \mathcal{O}_i \times R^\nu, \quad i = 1, \ldots, s-1$.

By repeatedly applying this equation we obtain

$$\prod_{i=-s+2}^{0} G(q'_i, q'_{i+1})\, B(x'_1, \ldots, x'_{s-1}, \bar{y}) = B(x'_{-s+2}, \ldots, x'_0, \bar{y}) \prod_{i=-s+2}^{0} G(q'_i, q'_{i+1}).$$

The right-hand side of this equality is constant on $x'_i \in \mathcal{O}_i \times R^\nu$, $i = 1, \ldots, s-1$. From this and the fact that $\prod_{i=-s+2}^{0} G(q'_i, q'_{i+1})$ is non-degenerate, we obtain the required result. □

Proposition 2.4. *Let $(x_1, \ldots, x_{s-1}, \bar{y}) \in \mathcal{I}\bar{\mathcal{A}}_1$. Suppose that the points $q_1, \ldots, q_{s-1}$ lie on a straight line, $L \subset R^\nu$, such that* $\operatorname{dist}(L, \operatorname{Conv}(\bar{q}(\bar{y}))) > d_1$. *Assume further that for $s \geqq 3$ these points satisfy the condition $|q_{i+1} - q_i| = r$, $i = 1, \ldots, s-2$, where* $\max(d_0, d_1/\sqrt{2}) < r < d_1$ *and*

$$0 \neq U''(r) \neq U'(r)/r \neq 0. \tag{2.9}$$

Then if $\tilde{q} \in L \setminus q_1$, the vector $q_1 - \tilde{q}$ is an eigenvector of $B(x_1, \ldots, x_{s-1}, \bar{y})$.

Proof. We will consider the cases $\nu = 2$ and $\nu \geqq 3$ separately. For $\nu = 2$ we construct a closed polygonal line, $\gamma \subset R^2$, with the following properties:

1) γ is the boundary of a rectangle, Γ, which is not a square;

2) the length of the edges forming γ are multiples of r;

3) one of these edges belongs to L and contains the points $q_1, \ldots, q_{s-1}$. The distances between q_1, q_{s-1} and the two ends of this edge are both multiples of r;

4) $\operatorname{Conv}(\bar{q}(\bar{y})) \subset \Gamma$, and $\operatorname{dist}(\gamma, \operatorname{Conv}(\bar{q}(\bar{y}))) > d_1$ (see Fig. 3a).

For $\nu \geqq 3$ we will employ the following geometric result: if $C \subset R^\nu$ is a bounded convex set and $R \subset R^\nu$ is a proper subspace, then there is a hyperplane, $H \subset R^\nu$, such that $R \subset H$ and $\operatorname{dist}(C, R) = \operatorname{dist}(C, H)$. We take L for R and $\operatorname{Conv}(\bar{q}(\bar{y}))$ for C and construct a closed polygonal line, γ, in H with the properties 1)–3) mentioned above (see Fig. 3b).

We now subdivide the edges of γ into semi-intervals of length r. By construction, $q_1, \ldots, q_{s-1}$ are among the endpoints of these semi-intervals. We denote the remaining endpoints $q_s, \ldots, q_N$ in such a way that $|q_{i+1} - q_i| = r$ for $i = s-1, \ldots, N-1$ and $|q_N - q_1| = r$. Choosing arbitrary $v_s, \ldots, v_N \in R^\nu$, we set $x_i = (q_i, v_i)$, $i = s, \ldots, N$. Set

$$[j] = \begin{cases} j & \text{for } 1 \leqq j \leqq N, \\ j - N & \text{for } j \geqq N, \end{cases} \qquad j = 1, 2, \ldots.$$

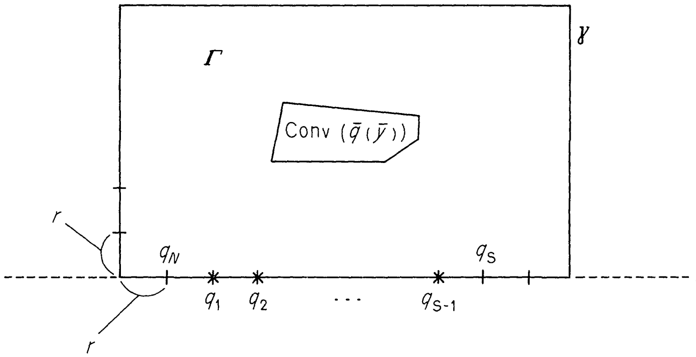

Fig. 3a

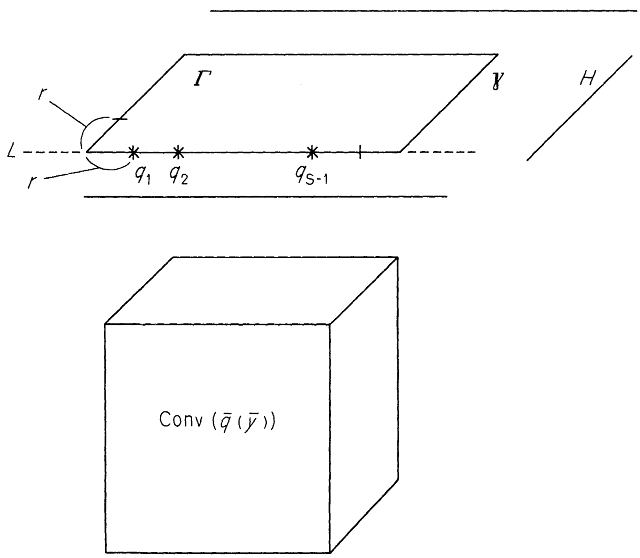

Fig. 3b

It follows from the construction and the conditions of the proposition under consideration that for both cases, $\nu=2$ and $\nu \geqq 3$,

$$(x_j, x_{[j+1]}, \ldots, x_{[j+s-1]}, \bar{y}) \in \bigcap_{i=1}^{s-1} \mathscr{I}\overline{\mathscr{A}}^{(i)}, \quad j=1, \ldots, N.$$

By Proposition 2.2,

$$\begin{aligned} & B(x_j, x_{[j+1]}, \ldots, x_{[j+s-2]}, \bar{y})\, G(q_j, q_{[j+1]}) \\ & \quad = G(q_j, q_{[j+1]})\, B(x_{[j+1]}, \ldots, x_{[j+s-1]}, \bar{y}), \quad j=1, \ldots, N. \end{aligned}$$

Iterating, we obtain

$$B(x_1, \ldots, x_{s-1}, \bar{y})\, G_1^N = G_1^N\, B(x_1, \ldots, x_{s-1}, \bar{y}), \tag{2.10}$$

where $G_1^N = \prod_{i=1}^{N} G(q_i, q_{[i+1]})$.

In order to solve Eq. (2.10), we represent $G(q,q')$ as

$$G(q,q') = -U''(|q-q'|)P_{q-q'} - \frac{U'(|q-q'|)}{|q-q'|}(E-P_{q-q'}), \qquad q,q'\in R^\nu,\ |q-q'|>d_0, \tag{2.11}$$

where $P_{q-q'}$ is the orthogonal projector onto the subspace generated by $q-q'$. From (2.11) we obtain

$$G_1^N = (U''(r))^{2l_1/r}(U'(r)/r)^{2l_2/r}P_1 + (U'(r)/r)^{2l_1/r}(U''(r))^{2l_2/r}P_2 + (U'(r)/r)^N(E-P_1-P_2), \tag{2.12}$$

where l_1 is the length of that edge of γ which contains q_1, q_{s-1}, P_1 is the orthogonal projector onto the subspace generated by a vector collinear to this edge (for example, by the vector $q_1-\tilde{q}$ where $\tilde{q}\in L$, $\tilde{q}\neq q_1$), l_2 and P_2 are the corresponding quantities for the orthogonal edge of γ.

From (2.10) and (2.12) it follows that

$$G_1^N B(x_1,\ldots,x_{s-1},\bar{y})(q_1-\tilde{q}) = (U''(r))^{2l_1/r}(U'(r)/r)^{2l_2/r}B(x_1,\ldots,x_{s-1},\bar{y})(q_1-\tilde{q}),$$

i.e., $B(x_1,\ldots,x_{s-1},\bar{y})(q_1-\tilde{q})$ is an eigenvector of G_1^N with eigenvalue $\lambda=(U''(r))^{2l_1/r}(U'(r)/r)^{2l_2/r}$. According to (2.9) and since $l_1\neq l_2$, the coefficients of P_1, P_2 and E-P_1-P_2 on the right hand side of (2.12) are different. Hence every eigenvector of G_1^N with eigenvalue λ is proportional to $q_1-\tilde{q}$. Therefore, $q_1-\tilde{q}$ is an eigenvector of the matrix $B(x_1,\ldots,x_{s-1},\bar{y})$. □

Proposition 2.5. *Let the conditions of Proposition 2.4 hold. Then $B(x_1,\ldots,x_{s-1},\bar{y})$ is a scalar matrix.*

Proof. Consider the case $s=2$. Let $q^{\sim}\in L$ be chosen arbitrarily from the points distinct from q_1. There is a neighbourhood, $\mathcal{O}\subset R^\nu$, of $q^{\sim}$, not containing q_1, such that for every point $q'\in\mathcal{O}$ the line L' going through the points q_1 and q' satisfies the condition $\operatorname{dist}(L', \operatorname{Conv}(\bar{q}(\bar{y})))>d_1$. By Proposition 2.4, q_1-q' is an eigenvector of $B(x_1,\bar{y})$. Therefore this matrix has an open set of eigenvectors, and hence, is scalar.

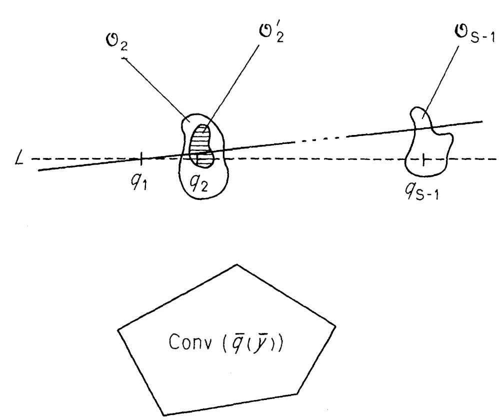

Fig. 4

Now let $s \geqq 3$. Consider the neighbourhoods $\mathcal{O}_2, \ldots, \mathcal{O}_{s-1}$ of $q_2, \ldots, q_{s-1}$ figuring in Proposition 2.3. It is easy to check that one can find a neighbourhood, $\mathcal{O}_2' \subset \mathcal{O}_2$, of q_2 such that for any $x_2' = (q_2', v_2') \in \mathcal{O}_2' \times R^\nu$ there exist points $x_i' = (q_i', v_i') \in \mathcal{O}_i \times R^\nu$, $2 \leqq i \leqq s-1$, $i \neq 2$, for which $(x_1, x_2', \ldots, x_{s-1}', \bar{y})$ satisfies the conditions of Proposition 2.4 (see Fig. 4). It follows from Propositions 2.3, 2.4 that for any $q_2' \in \mathcal{O}_2'$ the vector $q_1 - q_2'$ is an eigenvector of the matrix $B(x_1, \ldots, x_{s-1}, \bar{y})$. As above, we deduce from this fact that $B(x_1, \ldots, x_{s-1}, \bar{y})$ is a scalar matrix. □

Proposition 2.6. *Let* $(x_1, \ldots, x_{s-1}, \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1$. *Then the matrix* $B(x_1, \ldots, x_{s-1}, \bar{y})$ *is a scalar*, i.e.,

$$B(x_1, \ldots, x_{s-1}, \bar{y}) = b(x_1, \ldots, x_{s-1}, \bar{y})\, E. \tag{2.13}$$

Proof. By the definition of $\mathscr{A}_1$, we can choose a point, $q_0 \in R^\nu$, so that $(x_0, x_1, \ldots, x_{s-1}, \bar{y}) \in \bigcap_{i=1}^{s-1} \mathscr{I}\bar{\mathscr{A}}^{(i)}$, where $x_0 = (q_0, v_0)$, $v_0 \in R^\nu$. Using the geometrical construction employed in the proof of Proposition 1.9 we can find a point, $q' \in B^a_{q_0}(x_0, \ldots, x_{s-1}, \bar{y})$, for which

$$|q' - q_0| > d_1/\sqrt{2}, \qquad 0 \neq U''(|q' - q_0|) \neq U'(|q' - q_0|)/|q' - q_0| \neq 0.$$

We set $r_1 = |q' - q_0|$,

$$q_i' = q_0 - i(q' - q_0), \qquad x_i' = (q_i', v_i'), \qquad i = -1, -2, \ldots,$$

where $v_i' \in R^\nu$, $i = -1, -2, \ldots$, are chosen arbitrarily. If $|i_0| > s$ is large enough, there will be a point, q, satisfying conditions a)–c) listed in the proof of Proposition 1.9. Fixing such i_0 and q, we set

$$x_i = x_i', \qquad i = -1, -2, \ldots, i_0,$$

$$q_i = q_{i_0}' - (i - i_0)(q - q_{i_0}'), \qquad x_i = (q_i, v_i), \qquad i = i_0 - 1, i_0 - 2, \ldots,$$

where $v_i \in R^\nu$, $i \leqq i_0 - 1$, are chosen arbitrarily. By construction, $(x_i, \ldots, x_{i+s-1}, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathscr{I}\bar{\mathscr{A}}^{(j)}$ for $i \leqq 0$. Furthermore, if $i \leqq i_0 - s + 2$, then the conditions of Proposition 2.4 hold for $(x_i, \ldots, x_{i+s-2})$ and hence, the matrix $B(x_i, \ldots, x_{i+s-2}, \bar{y})$ is a scalar.

By repeated application of Proposition 2.2 we find that the matrix $B(x_1, \ldots, x_{s-1}, \bar{y})$ is a scalar. □

Proposition 2.7. *The function b appearing in* (2.13) *has the following property*: *if* $(x_1, \ldots, x_{s-1}, \bar{y})$, $(x_1', \ldots, x_{s-1}', \bar{y}') \in \mathscr{I}\bar{\mathscr{A}}_1$ *and* $\bar{y}' = \bar{y}$, *then*

$$b(x_1, \ldots, x_{s-1}, \bar{y}) = b(x_1', \ldots, x_{s-1}', \bar{y}'). \tag{2.14}$$

Proof. Let $(x_1, \ldots, x_{s-1}, \bar{y})$, $(x_1', \ldots, x_{s-1}', \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1$. Choose points q_0, q_0' so that $(x_0, x_1, \ldots, x_{s-1}, \bar{y})$, $(x_0', x_1', \ldots, x_{s-1}', \bar{y}) \in \bigcap_{j=1}^{s-1} \mathscr{I}\bar{\mathscr{A}}^{(j)}$, where $x_0 = (q_0, v_0)$, $x_0' = (q_0', v_0')$, v_0, $v_0' \in R^\nu$. Next, choose points $q \in B^a_{q_0}(x_0, \ldots, x_{s-1}, \bar{y})$, $q' \in B^a_{q_0'}(x_0', \ldots, x_{s-1}', \bar{y})$ so that: (a) the vectors $q - q_0$ and $q' - q_0'$ are not parallel; (b) $|q - q_0|$, $|q' - q_0'| > d_1/\sqrt{2}$, $U''(|q - q_0|)\, U''(|q' - q_0'|) \neq 0$. We now consider sequences, $q_{1,i}$, $v_{1,i}$, $q_{1,i}'$, $v_{1,i}'$, $i = -1, -2, \ldots$, where

$$q_{1,i} = q_0 - i(q - q_0), \qquad q_{1,i}' = q_0' - i(q' - q_0')$$

and $v_{1,i}$, $v'_{1,i}$ are chosen arbitrarily. Let $x_{1,i}=(q_{1,i}, v_{1,i})$, $x'_{1,i}=(q'_{1,i}, v'_{1,i})$, $i=-1, -2, \ldots$. It is easy to see that when $i\to-\infty$, the sets $B^e_{q_{1,i}}(x_{1,i}, \ldots, x_{1,i+s-1}, \bar{y})$, $B^e_{q'_{1,i}}(x'_{1,i}, \ldots, x'_{1,i+s-1}, \bar{y})$ tend (in a natural way) to half-spaces. By condition (a) the intersection of these sets is non-empty if $|i|$ is large enough. Moreover, one can find open half-lines, $L\subset B^e_{q_{1,i}}(x_{1,i}, \ldots, x_{1,i+s-1}, \bar{y})$, $L'\subset B^e_{q'_{1,i}}(x'_{1,i}, \ldots, x'_{1,i+s-1}, \bar{y})$, originated at $q_{1,i}$, $q'_{1,i}$ which intersect each other with an acute angle at some point $\tilde{q}(i)$ (see Fig. 5). By the properties of the potential U, the conditions

$$U'(\varrho)\, U''(\varrho) \neq 0, \qquad \varrho > d_1/2 \tag{2.15}$$

hold on some finite interval of values of ϱ and therefore, one can guarantee that if dist $(q_{1,i}, \tilde{q}(i))$ and dist $(q'_{1,i}, \tilde{q}(i))$ are large enough [according to condition (a) they tend to $+\infty$ as $i\to-\infty$], each of them is divisible by some number from this interval.

Let $i=i_0$ be such that all of the above holds. Choose half-lines, L and L', intersecting each other with an acute angle at a point, $\tilde{q}(i_0)=\tilde{q}$. With such choices

$$\operatorname{dist}(q_{1,i_0}, \tilde{q}) = jr, \qquad \operatorname{dist}(q'_{1,i_0}, \tilde{q}) = j' r',$$

where j, j' are natural numbers. In addition (2.15) will hold for $\varrho = r, r'$.

Let $q_{2,-1}$, $q'_{2,-1}$ be points on L, L', respectively, such that

$$\operatorname{dist}(q_{1,i_0}, q_{2,-1}) = r, \qquad \operatorname{dist}(q'_{1,i_0}, q'_{2,-1}) = r'.$$

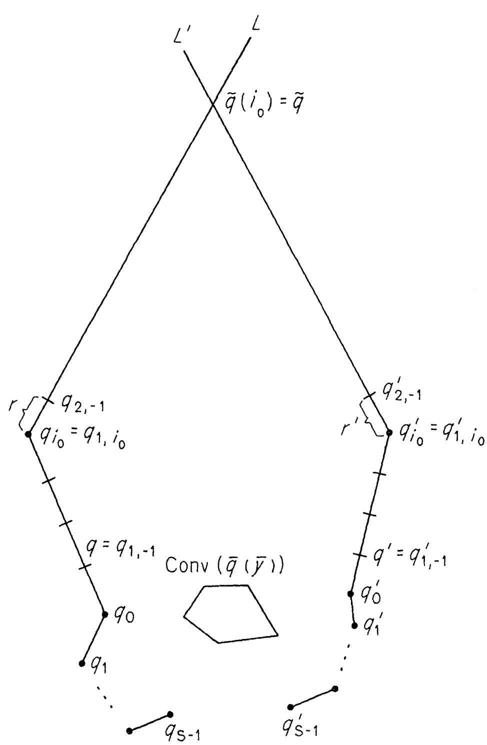

Fig. 5

For $i=-1,-2,\ldots$ we set

$$q_{2,i}=q_{1,i_0}-i(q_{2,-1}-q_{1,i_0}),\qquad q'_{2,i}=q'_{1,i_0}-i(q'_{2,-1}-q'_{1,i_0}),$$

$$q_i=\begin{cases} q_{1,i}, & i_0\leqq i\leqq -1,\\ q_{2,i_0+i}, & i\leqq i_0-1,\end{cases}\qquad q'_i=\begin{cases} q'_{1,i}, & i_0\leqq i\leqq -1,\\ q_{2,i_0+1}, & i_0-j'\leqq i\leqq i_0-1,\\ q_i, & i\leqq i_0-j'-1.\end{cases}$$

Set $v_i=v_{1,i}$, $v'_i=v'_{1,i}$ for $i_0\leqq i\leqq -1$ and choose $v_i, v'_i\in R^\nu$ for $i\leqq i_0-1$ so that $v_i=v'_i$ for $i\leqq i_0-j'-1$. Finally, let $x_i=(q_i,v_i)$, $i=-1,-2,\ldots$.

It follows from the construction that

$$(x_{i_0-j-s+2},\ldots,x_{i_0-j},\bar{y})=(x'_{i_0-j'-s+2},\ldots,x'_{i_0-j'},\bar{y})$$

and for $i\leqq 0$

$$(x_i,\ldots,x_{i+s-1},\bar{y}),\qquad (x'_i,\ldots,x'_{i+s-1},\bar{y})\in\bigcap_{l=1}^{s-1}\bar{\mathscr{A}}^{(l)}.$$

Using Propositions 2.6 and 2.2, we obtain (2.14). □

The result of Theorem 2 is now obtained by applying (2.6) and Propositions 2.6 and 2.7.

3. The One-Dimensional Case

In this section we will prove Theorem 1(1) for the special case where $\nu=1$. In addition we will prove a slightly modified version of Theorem 2 using Propositions 1.1–1.3 and Corollary 1.4 (which we have proved for arbitrary dimension). We use the notation introduced in Sect. 1 noting that vectors and matrices are scalars when $\nu=1$. In this special case $G(q,q')=U''(|q-q'|)$.

3.1. We first prove assertion (1) of Theorem 1. Consider the set

$$\bar{\mathscr{A}}^{(i)}=\{(x_1,\ldots,x_s,\bar{y})\in\mathscr{A}\colon U''(|q_i-q_{i+1}|)\neq 0\},\qquad i=1,\ldots,s-1,$$

and the function

$$A(x_1,\ldots,x_s,\bar{y})=\begin{cases} A^{(1)}(x_1,\ldots,x_s,\bar{y})\left[\prod_{i=1}^{s-2}U''(|q_i-q_{i+1}|)\right]^{-1}, & \\ \qquad (x_1,\ldots,x_s,\bar{y})\in\mathscr{I}\mathscr{A}^{(s-1)}\cap\bigcap_{i=1}^{s-2}\mathscr{I}\bar{\mathscr{A}}^{(i)}, & s\geqq 3,\\ A^{(1)}(x_1,x_2,\bar{y}),\qquad (x_1,x_2,\bar{y})\in\mathscr{I}\mathscr{A}^{(1)}, & s=2.\end{cases}\tag{3.1}$$

Proposition 3.1. *Assume that* $(x_1,\ldots,x_s,\bar{y})\in\mathscr{I}\mathscr{A}^{(s-2)}\cap\mathscr{I}\mathscr{A}^{(s-1)}\cap\bigcap_{i=1}^{s-2}\mathscr{I}\bar{\mathscr{A}}^{(i)}$ *when* $s\geqq 3$, *and that* $(x_1,x_2,\bar{y})\in\mathscr{I}\mathscr{A}^{(1)}$ *when* $s=2$. *Let* $x_0\in B^a_{q_1}\times R^1$, $U''(|q_0-q_1|)\neq 0$. *Then*

$$A(x_0,\ldots,x_{s-1},\bar{y})=A(x_1,\ldots,x_s,\bar{y}).$$

Proof. We apply Corollary 1.4 and use the definition of A. □

Proposition 3.2. *Assume that* $(x_1, \dots, x_s, \bar{y}), (x'_1, \dots, x'_s, \bar{y}') \in \bigcap_{i=1}^{s-1} \mathscr{A}^{(i)} \cap \bigcap_{i=1}^{s-2} \mathscr{I}\bar{\mathscr{A}}^{(i)}$ *when* $s \geqq 3$, *and that* $(x_1, x_2, \bar{y}), (x'_1, x'_2, \bar{y}') \in \mathscr{I}\mathscr{A}^{(1)}$ *when* $s = 2$, *where* $\bar{y} = \bar{y}'$ *and the points* q_i, q'_i, $1 \leqq i \leqq s$, *all lie on the same side of* $\bar{q}(\bar{y})$. *Then*

$$A(x_1, \dots, x_s, \bar{y}) = A(x'_1, \dots, x'_s, \bar{y}'). \tag{3.2}$$

Proof. Using the properties of U choose positive real numbers r, r' and positive integers j, j' such that

$$r > \max(d_0, d_1/2, d_1 - |q_1 - q_2|), \qquad r' > \max(d_0, d_1/2, d_1 - |q'_1 - q'_2|),$$
$$U'(r)\, U''(r) \neq 0 \neq U'(r')\, U''(r'),$$
$$q_1 + jr \operatorname{sign}(q_1 - q_2) = q'_1 + j'r' \operatorname{sign}(q'_1 - q'_2).$$

Let

$$q_i = q_1 + (-i+1)\, r \operatorname{sign}(q_1 - q_2),$$
$$q'_i = \begin{cases} q'_1 + (-i+1) \operatorname{sign}(q'_1 - q'_2), & -j'+1 \leqq i \leqq 0, \\ q_{i+j'-j}, & i \leqq -j', \end{cases}$$
$$x_i = (q_i, v_i), \qquad x'_i = (q'_i, v'_i), \qquad i = 0, -1, \dots,$$

(see Fig. 6), where $v_i, v'_i \in R^1$ are chosen so that $v'_i = v_{i+j'-j}$ for $i \leqq j'$.

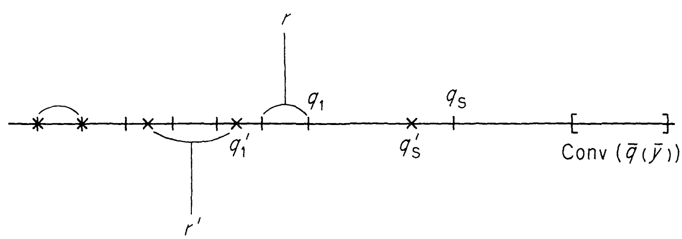

Fig. 6

From the construction it follows that the pairs $(x_i, \dots, x_{i+s-1}, \bar{y}), x_{i-1}$ and $(x'_i, \dots, x'_{i+s-1}, \bar{y}), x'_{i-1}$ both satisfy the conditions of Proposition 3.2. Moreover,

$$(x_{-j-s+2}, \dots, x_{-j+1}, \bar{y}) = (x'_{-j'-s+2}, \dots, x'_{-j'+1}, \bar{y}).$$

We now obtain (3.2) from Proposition 3.1. □

To lighten the notation we shall often write $A(\bar{y})$ for $A(x_1, \dots, x_s, \bar{y})$ when either $s \geqq 3$, $(x_1, \dots, x_s, \bar{y}) \in \bigcap_{i=1}^{s-1} \mathscr{I}\mathscr{A}^{(i)} \cap \bigcap_{i=1}^{s-2} \mathscr{I}\bar{\mathscr{A}}^{(i)}$, or $s = 2$, $(x_1, x_2, \bar{y}) \in \mathscr{I}\mathscr{A}^{(1)}$.

Proposition 3.3. *The function A vanishes on* $\bigcap_{i=1}^{s-1} \mathscr{I}\bar{\mathscr{A}}^{(i)}$ *(see Sect. 2).*

Proof. Let us first assume that $s = 2$. Using Propositions 1.1, 3.2, and formula (3.1) we apply the operator ∂_{v_1} to (0.2) two times. We then get

$$2A(\bar{y})\, U''(|q_1 - q_2|) - \partial_{q_1} U(|q_1 - q_2|)\, \partial^3_{v_1^3} f_1(x_1, \bar{y}) = 0, \qquad (x_1, x_2, \bar{y}) \in \mathscr{I}\mathscr{A}^{(1)}. \tag{3.3}$$

Using Propositions 1.1, 1.3, 3.2, and formula (3.1), we apply ∂_{v_2} to (0.2) two times and get

$$-2A(\bar{y})\, U''(|q_1 - q_2|) - \partial_{q_2} U(|q_1 - q_2|)\, \partial^3_{v_2^3} f_1(x_2, \bar{y}) = 0, \quad (x_1, x_2, \bar{y}) \in \mathscr{I}\mathscr{A}^{(1)}. \tag{3.4}$$

Combining (3.3) and (3.4) we have

$$\partial_{q_1} U(|q_1-q_2|)\,[\partial^3_{v_1^3} f_1(x_1,\bar{y}) - \partial^3_{v_2^3} f_1(x_2,\bar{y})] = 0, \qquad (x_1,x_2,\bar{y}) \in \mathscr{I}\mathscr{A}^{(1)},$$

or

$$\partial^3_{v_1^3} f_1(x_1,\bar{y}) = \partial^3_{v_2^3} f_1(x_2,\bar{y}), \qquad (x_1,x_2,\bar{y}) \in \mathscr{I}\mathscr{A}^{(1)}. \tag{3.5}$$

Using a simpler version of the method of proof used for Proposition 3.2, one can deduce from (3.5) that $\partial^3_{v_1^3} f_1(x_1,\bar{y}) = \partial^3_{(v_1')^3} f_1(x_1',\bar{y}')$ when $\bar{y}'=\bar{y}$ and $(x_1,\bar{y})$, $(x_1',\bar{y}') \in \bigcup_{k_1} \mathscr{I}\mathscr{B}(n-1,m-1,k_1,1)$ (see Sect. 0) and when both q_1, q_1' lie on the same side of $\bar{q}(\bar{y})$. This fact along with (3.3) imply

$$U''(r)/U'(r) = \text{const}$$

on $\{r>d_0 : U'(r) \neq 0\}$, when $A(x_1,x_2,\bar{y}) \neq 0$ for some $(x_1,x_2,\bar{y}) \in \mathscr{I}\mathscr{A}^{(1)}$. But this contradicts conditions (I_1'')–(I_3'') on U, and we must conclude that

$$A(x_1,x_2,\bar{y}) = 0, \qquad (x_1,x_2,\bar{y}) \in \mathscr{I}\mathscr{A}^{(1)}.$$

So the proposition holds when $s=2$.

We now consider $s \geqq 3$. Let $i \in \{1,\dots,s\}$ be fixed. Apply ∂_{v_1}, ∂_{q_s}, ∂_{v_i} to (0.2) successively and subtract from this equality that obtained by applying ∂_{v_s}, ∂_{q_1}, ∂_{v_i} to (0.2). Using (3.1) and Propositions 1.1, 1.3, 3.2 this expression simplifies to

$$2A(\bar{y})\,\partial_{q_i}\left[\prod_{j=1}^{s-1} U''(|q_j-q_{j+1}|)\right] + \partial^3_{v_1,v_i,v_{s-1}} f_{s-1}(x_1,\dots,x_{s-1},\bar{y})\,U''(|q_{s-1}-q_s|)$$
$$-U''(|q_1-q_2|)\,\partial^3_{v_2,v_i,v_s} f_{s-1}(x_2,\dots,x_s,\bar{y}) = 0, \;\; (x_1,\dots,x_s,\bar{y}) \in \bigcap_{j=1}^{s-1} \mathscr{I}\bar{\mathscr{A}}^{(j)} \tag{3.6,i}$$

[if $i=1$ or s, one of the two last terms on the left side of (3.6,i) must be dropped]. By (3.6,s) we have

$$\partial^3_{v_2,v_5^2} f_{s-1}(x_2,\dots,x_s,\bar{y}) = 2A(\bar{y})\,\partial_{q_s}\left[\prod_{i=2}^{s-1} U''(|q_i-q_{i+1}|)\right]. \tag{3.7}$$

For each configuration $(x_1,\dots,x_s,\bar{y}) \in \bigcap_{i=1}^{s-1} \mathscr{I}\bar{\mathscr{A}}^{(i)}$ there is an $x_0 \in R^1 \times R^1$ for which $(x_0,\dots,x_{s-1},\bar{y}) \in \mathscr{I}\bar{\mathscr{A}} \cap \bigcap_{i=1}^{s-1} \mathscr{A}^{(i)}$. Therefore we can replace each x_j in (3.7) by x_{j-1}, $2 \leqq j \leqq s$, and obtain

$$\partial^3_{v_1,v_{s-1}^2} f_{s-1}(x_1,\dots,x_{s-1},\bar{y}) = 2A(\bar{y})\,\partial_{q_{s-1}}\left[\prod_{i=1}^{s-2} U''(|q_i-q_{i+1}|)\right].$$

By substituting this into $(3.6, s-1)$ we get

$$U''(|q_1-q_2|)\,\partial_{v_2,v_{s-1},v_s} f_{s-1}(x_2,\dots,x_s,\bar{y})$$
$$= 2A(\bar{y})\left\{2U''(|q_{s-1}-q_s|)\,\partial_{q_{s-1}}\left[\prod_{i=1}^{s-2} U''(|q_i-q_{i+1}|)\right]\right.$$
$$\left.+\left[\prod_{i=1}^{s-2} U''(|q_i-q_{i+1}|)\right]\partial_{q_{s-1}} U''(|q_{s-1}-q_s|)\right\}. \tag{3.8}$$

When $s \geqq 4$ we can divide (3.8) by $U''(|q_1 - q_2|)$ and obtain

$$\begin{aligned}
&\partial^3_{v_2, v_{s-1}, v_s} f_{s-1}(x_2, \ldots, x_s, \bar{y}) \\
&\quad = 2A(\bar{y}) \left\{ 2U''(|q_{s-1} - q_s|)\, \partial_{q_{s-1}} \left[\prod_{i=2}^{s-2} U''(|q_i - q_{i+1}|) \right] \right. \\
&\qquad \left. + \left[\prod_{i=2}^{s-2} U''(|q_i - q_{i+1}|) \right] \partial_{q_{s-1}} U''(|q_{s-1} - q_s|) \right\}. \qquad (3.9)
\end{aligned}$$

Replacing x_j by x_{j-1}, $2 \leqq j \leqq s$, in (3.9) we obtain a formula for $\partial^3_{v_1, v_{s-2}, v_{s-1}}$. Substituting this into (3.6,s–2) and repeatedly applying this procedure we get

$$\begin{aligned}
&\partial^3_{v_2, v_j, v_s} f_{s-1}(x_2, \ldots, x_s, \bar{y}) \\
&\quad = 2A(\bar{y}) \left\{ (s-j+1)\, U''(|q_j - q_{j+1}|)\, \partial_{q_j} \left[\prod_{\substack{2 \leqq l \leqq s-1, \\ l \neq j}} U''(|q_l - q_{l+1}|) \right] \right. \\
&\qquad \left. + (s-j) \left[\prod_{\substack{2 \leqq l \leqq s-1, \\ l \neq j}} U''(|q_l - q_{l+1}|) \right] \partial_{q_j} U''(|q_j - q_{j+1}|) \right\}, \quad 3 \leqq j \leqq s-1, \qquad (3.10)
\end{aligned}$$

which can be checked by induction going from j to $j-1$, using (3.6,$j-1$) and replacing $x_2, \ldots, x_s$ with $x_1, \ldots x_{s-1}$ [notice that as $j = s-1$, (3.10) coincides with (3.9)]. This same substitution in (3.10) implies that

$$\begin{aligned}
&\partial^3_{v_1, v_2, v_{s-1}} f_{s-1}(x_1, \ldots, x_{s-1}, \bar{y}) \\
&\quad = 2A(\bar{y}) \left\{ (s-2)\, U''(|q_2 - q_3|)\, \partial_{q_2} \left[\prod_{\substack{1 \leqq l \leqq s-2, \\ l \neq 2}} U''(|q_l - q_{l+i}|) \right] \right. \\
&\qquad \left. + (s-3) \left[\prod_{\substack{1 \leqq l \leqq s-2, \\ l \neq 2}} U''(|q_l - q_{l+1}|) \right] \partial_{q_2} U''(|q_2 - q_3|) \right\}, \quad s \geqq 4. \qquad (3.11)
\end{aligned}$$

For $s \geqq 4$ we can substitute (3.11) into (3.6,2) and obtain

$$\begin{aligned}
&U''(|q_1 - q_2|)\, \partial^3_{v_2^2, v_s} f_{s-1}(x_2, \ldots, x_s, \bar{y}) \\
&\quad = 2A(\bar{y}) \left\{ (s-1)\, U''(|q_2 - q_3|)\, \partial_{q_2} \left[\prod_{\substack{1 \leqq l \leqq s-1, \\ l \neq 2}} U''(|q_l - q_{l+1}|) \right] \right. \\
&\qquad \left. + (s-2) \prod_{\substack{1 \leqq l \leqq s-1, \\ l \neq 2}} U''(|q_l - q_{l+1}|)\, \partial_{q_2} U''(|q_2 - q_3|) \right\}. \qquad (3.12)
\end{aligned}$$

Notice that (3.12) also holds for $s = 3$ since in this case (3.12) coincides with (3.8).

We now consider equality (3.6,1). Using the conditions on $(x_1, \ldots, x_s, \bar{y})$ and the fact that $s \geqq 3$, we can divide (3.6,1) by $U''(|q_{s-1} - q_s|)$ and get

$$2A(\bar{y})\, \partial_{q_1} \left[\prod_{j=1}^{s-2} U''(|q_j - q_{j+1}|) \right] + \partial^3_{v_1^3, v_{s-1}} f_{s-1}(x_1, \ldots, x_{s-1}, \bar{y}) = 0. \qquad (3.13)$$

Now suppose that

$$(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathscr{I}\mathscr{A}^{(j)}, \qquad \operatorname{dist}(q_s, \bar{q}(\bar{y})) > 3d_1. \qquad (3.14)$$

[The validity of the last condition follows by shifting q_i, $1 \leqq i \leqq s$, by some vector. This does not alter $A(x_1, \ldots, x_s, \bar{y})$.] Then there will be a point, $x_{s+1} \in R^1 \times R^1$, such that $(x_2, \ldots, x_{s+1}, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathscr{I}\bar{\mathscr{A}}^{(j)}$. We can then replace each x_i in (3.13) by x_{i+1}, $1 \leqq i \leqq s-1$. Hence

$$2A(\bar{y})\, \partial_{q_2} \left[\prod_{j=2}^{s-1} U''(|q_j - q_{j+1}|) \right] + \partial^3_{v_2^2, v_s} f_{s-1}(x_2, \ldots, x_s, \bar{y}) = 0. \tag{3.15}$$

From (3.15), (3.12) it follows that if (3.14) holds, then

$$2A(\bar{y})(s-1) \prod_{\substack{2 \leqq j \leqq s-1, \\ j \neq 2}} U''(|q_j - q_{j+1}|)\, [U''(|q_1 - q_2|)\, \partial_{q_2} U''(|q_2 - q_3|)$$
$$+ U''(|q_2 - q_3|)\, \partial_{q_2} U''(|q_1 - q_2|)] = 0.$$

Dividing by the non-zero factors we get

$$2A(\bar{y})\, [U''(|q_1 - q_2|)\, U'''(|q_2 - q_3|) + U'''(|q_1 - q_2|)\, U''(|q_2 - q_3|)] = 0. \tag{3.16}$$

Suppose $(x_1, \ldots, x_s, \bar{y})$ satisfies (3.14) and is such that $A(\bar{y}) = A(x_1, \ldots, x_s, \bar{y}) \neq 0$. If

$$r > |q_2 - q_1|, \quad U'(r)\, U''(r) \neq 0, \tag{3.17}$$

then the configuration $((q_1', v_1), x_2, \ldots, x_s, \bar{y})$, where $q_1' = q_2 + r \operatorname{sign}(q_1 - q_2)$, also satisfies (3.14) and the pair of configurations $(x_1, \ldots, x_s, \bar{y})$, $((q_1', v_1), x_2, \ldots, x_s, \bar{y})$ satisfy the conditions of Proposition 3.2. Due to this Proposition and the above hypothesis, $A((q_1', v_1), x_2, \ldots, x_s, \bar{y}) \neq 0$. From (3.16) it follows that if conditions (3.17) are satisfied, then

$$U'''(r)\, (U''(r))^{-1} + c = 0,$$

where $c = U'''(|q_3 - q_2|)\, (U''(|q_3 - q_2|))^{-1}$. Thus for $r > |q_2 - q_1|$ we have

$$[U'''(r) + c\, U''(r)]\, U''(r)\, U'(r) = 0, \tag{3.18}$$

which contradicts conditions (I_1'')–(I_3'') on U. To show this let us divide the open set $\mathcal{O} = \{r : r > |q_2 - q_1|,\ U'(r) \neq 0\}$ into disjoint intervals. Let (α, β) be one of these intervals. It then follows from (3.18) that $(U''' + c\, U'')\, U'' = 0$ on (α, β) whose general solution is $U(r) = c_1 \exp(-cr) + c_2 r + c_3$. If $\alpha > |q_2 - q_1|$, then $U'(\alpha) = U'(\beta) = 0$. Since the derivative $U'(r) = -cc_1 \exp(-cr) + c_2$ is monotone on (α, β), $U'(r) \equiv 0$, $r \in (\alpha, \beta)$. But this contradicts the definition of (α, β). Hence $\alpha = |q_2 - q_1|$, i.e. $\mathcal{O}$ consists of a single interval. From (I_1'')–(I_3'') it follows that $\beta = d_1$. Thus $U(r)$ has the above form for each $r \in (|q_2 - q_1|, d_1)$. But this contradicts (I_1'')–(I_3'') as claimed.

We now see that $A(x_1, \ldots, x_s, \bar{y}) = 0$ for each $(x_1, \ldots, x_s, \bar{y})$ satisfying (3.14). By Proposition 3.2 we can extend this equality to all $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathscr{I}\bar{\mathscr{A}}^{(j)}$. □

Corollary 3.4. *Suppose that* $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathscr{I}\bar{\mathscr{A}}^{(j)}$. *Then*

$$A^{(1)}(x_1, \ldots, x_s, \bar{y}) = 0.$$

Proof. Use (3.1) and Proposition 3.3. □

Proposition 3.5. *Suppose that* $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathscr{I}\mathscr{A}^{(j)}$. *Then*

$$A^{(1)}(x_1, \ldots, x_s, \bar{y}) = 0.$$

The proof follows the line of that of Proposition 1.9, but we use Corollary 3.4 rather than Proposition 1.8. We shall not go into details. □

Proposition 3.6. *Suppose that* $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathscr{I}\mathscr{A}^{(j)}$. *Then*

$$A^{(s)}(x_1, \ldots, x_s, \bar{y}) = 0.$$

The proof is obtained in two steps. We first obtain the 1-dimensional version of Proposition 1.10 and then of Proposition 1.11 using arguments identical to those used in the proofs of these propositions. □

For the $v = 1$ case assertion (1) of Theorem 1 is an immediate consequence of Propositions 3.5, 3.6, and 1.1 [see formulas (1.2a, b)].

3.2. We now proceed to Theorem 2. Recall the function b in the statement of this theorem. For the $v = 1$ case b satisfies the following condition: if $(x_1, \ldots, x_{s-1}, \bar{y})$, $(x'_1, \ldots, x'_{s-1}, \bar{y}') \in \mathscr{I}\bar{\mathscr{A}}_1$, $\bar{y}' = \bar{y}$ and the points q_i, q'_i, $i = 1, \ldots, s-1$, all lie on the same side of $\bar{q}(\bar{y})$, then $b(x_1, \ldots, x_{s-1}, \bar{y}) = b(x'_1, \ldots, x'_{s-1}, \bar{y}')$.

In order to prove this modified version of Theorem 2 we use the $v = 1$ version of Theorem 1 (1) and then literally repeat the arguments used in Sect. 2 which lead to the proof of Proposition 2.2. Since, when $v = 1$, $B(x_1, \ldots, x_{s-1}, \bar{y})$ is a scalar, all that remains is to prove the following 1-dimensional version of Proposition 2.7.

Proposition 3.7. *Assume that* $(x_1, \ldots, x_{s-1}, \bar{y}), (x'_1, \ldots, x'_{s-1}, \bar{y}') \in \mathscr{I}\bar{\mathscr{A}}_1, \bar{y} = \bar{y}'$ *and assume that the points* q_i, q'_i, $i = 1, \ldots, s-1$, *all lie on the same side of* $\bar{q}(\bar{y})$. *Then* (2.14) *holds.*

Proof. The construction used below is a simplified version of that used in the proof of Proposition 2.7. Taking $\tilde{q} \in R^1$ sufficiently far from $\bar{q}(\bar{y})$ and lying on the same side of $\bar{q}(\bar{y})$ as q_i, q'_i, $i = 1, \ldots, s-1$, we can choose $r, r' > 0$ such that (2.15) holds when $\varrho = r, r'$ and such that

$$|q_1 - \tilde{q}| = jr, \qquad |q'_1 - \tilde{q}| = j' r',$$

where j, j' are positive integers.

Let

$$q_i = q_1 - (i-1)\, r \operatorname{sign}(q_1 - q_2), \qquad i = 0, -1, \ldots,$$

$$q'_i = \begin{cases} q'_1 - (i-1)\, r' \operatorname{sign}(q'_1 - q'_2), & i = 0, \ldots, -j'+1, \\ \tilde{q} - (i+j'-1)\, r \operatorname{sign}(q_1 - q_2), & i = -j', -j'-1, \ldots, \end{cases}$$

$$x_i = (q_i, v_i), \qquad x'_i = (q'_i, v'_i), \qquad i = 0, -1, \ldots,$$

where $v_i, v'_i \in R^1$ are chosen so that $v_i = v'_{i+j-j'}$ as $i \leqq -j+1$ (see Fig. 7).

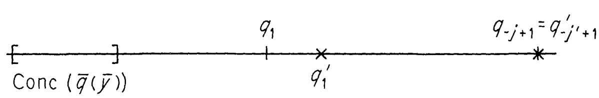

Fig. 7

Due to the construction

$$(x_{-j-s+3}, \ldots, x_{-j+1}, \bar{y}) = (x'_{-j'-s+3}, \ldots, x'_{-j'+1}, \bar{y}),$$

$$(x_i, \ldots, x_{i+s-1}, \bar{y}), \quad (x'_i, \ldots, x'_{i+s-1}, \bar{y}) \in \bigcap_{l=1}^{s-1} \mathscr{I}\bar{\mathscr{A}}^{(l)}, \quad i \leqq 0.$$

Applying Proposition 2.2, we get (2.14). □

4. Further Study of the Middle Function (An Application of the Middle and Lower Equations)

In preceding sections we obtained essential information about the function f_{s-1}. In this section we continue this investigation. Here we deal with arbitrary $v \geqq 1$. We emphasize that Theorem 1 and a slightly modified version of Theorem 2 hold for the case $v = 1$ (see Sect. 3). Our aim is to prove the following theorem.

Theorem 3. *The equality*

$$\partial^2_{v_1, v_{s-1}} f_{s-1}(x_1, \ldots, x_{s-1}, \bar{y}) = 0$$

holds on $\mathscr{I}\bar{\mathscr{A}}_1$ *(see Sect. 2).*

In view of Theorem 2 it suffices to show that

$$b(x_1, \ldots, x_{s-1}, \bar{y}) = 0, \quad (x_1, \ldots, x_{s-1}, \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1. \tag{4.1}$$

We introduce the following notation:

$$C(x_1, \ldots, x_{s-1}, \bar{y}) = b(x_1, \ldots, x_{s-1}, \bar{y}) \prod_{j=1}^{s-2} G(q_j, q_{j+1}),$$

$$(x_1, \ldots, x_{s-1}, \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1, \ s \geqq 3, \tag{4.2a}$$

$$C(x_1, \bar{y}) = b(x_1, \bar{y}) E, \quad (x_1, \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1, \ s = 2. \tag{4.2b}$$

Proposition 4.1. *The following equality holds*

$$f_{s-1}(x_1, \ldots, x_{s-1}, \bar{y}) = \langle C(x_1, \ldots, x_{s-1}, \bar{y}) v_{s-1}, v_1 \rangle \tag{4.3a}$$
$$+ b_1(x_1, \ldots, x_{s-1}, \bar{y}) + b_{s-1}(x_1, \ldots, x_{s-1}, \bar{y}), \quad (x_1, \ldots, x_{s-1}, \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1, \ s \geqq 3,$$

$$f_1(x_1, \bar{y}) = (1/2) C\langle (x_1, \bar{y}) v_1, v_1 \rangle + \langle \mathbf{b}_1(x_1, \bar{y}), v_1 \rangle + b_1(x_1, \bar{y}),$$
$$(x_1, \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1, \ s = 2, \tag{4.3b}$$

where the functions b_1, b_{s-1}, *and* $\mathbf{b}_1$ *are such that*

$$\partial_{v_{s-1}} b_1(x_1, \ldots, x_{s-1}, \bar{y}) = \partial_{v_1} b_{s-1}(x_1, \ldots, x_{s-1}, \bar{y}) = 0, \quad s \geqq 3, \tag{4.4a}$$

$$\partial_{v_1} b_1(x, \bar{y}) = 0, \quad \partial_{v_1} \mathbf{b}_1(x_1, \bar{y}) = 0, \quad s = 2. \tag{4.4b}$$

This Proposition immediately follows from Theorem 2.

Proposition 4.2. *Let* $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathscr{I}\bar{\mathscr{A}}^{(j)}$. *Then*

$$\partial_{q_1} f_s(x_1, \ldots, x_s, \bar{y}) = C(x_1, \ldots, x_{s-1}, \bar{y})\, \partial_{q_{s-1}} U(|q_{s-1} - q_s|), \tag{4.5}$$

$$\partial_{q_s} f_s(x_1, \ldots, x_s, \bar{y}) = C(x_2, \ldots, x_s, \bar{y})^* \,\partial_{q_2} U(|q_1 - q_2|). \tag{4.6}$$

Proof. We first apply ∂_{v_1} to Eq. (0.2). Using Theorems 1 and 2, we then obtain (4.5). The other equality can be proved similarly. □

Proposition 4.3. *Let* $s=2$. *Then* (4.1) *holds.*

Proof. First let $v_1=0$ in (4.3b). Then from condition (4) of the Basic Lemma it follows that

$$\lim_{|q_1|\to\infty} b_1(x_1,\bar{y})=0. \tag{4.7}$$

We now make a double application of ∂_{v_1} to Eq. (0.3) for the case $s=2$. Taking into account (4.2b), (4.3b), (4.4b), and the properties of b [see Theorem 2(b)], we obtain an equality which shows that $\mathbf{b}_1(x_1,\bar{y})$ is linear in q_1, i.e.

$$\mathbf{b}_1(x_1,\bar{y})=B_1(x_1,\bar{y})\,q_1+\mathbf{b}_2(x_1,\bar{y}), \qquad (x_1,\bar{y})\in\mathcal{I}\bar{\mathcal{A}}_1, \tag{4.8}$$

where

$$\partial_{q_1}B(x_1,\bar{y})=\partial_{v_1}B_1(x_1,\bar{y})=0, \qquad \partial_{q_1}\mathbf{b}_2(x_1,\bar{y})=\partial_{v_1}\mathbf{b}_2(x_1,\bar{y})=0. \tag{4.9}$$

With the facts proved above and condition (4) of the Basic Lemma we successively deduce from (4.3b) that $B_1(x_1,\bar{y})=0$, $(1/2)\langle C(x_1,\bar{y})\,v_1,v_1\rangle+\langle\mathbf{b}_2(x_1,\bar{y}),v_1\rangle=0$, and, finally, that $b(x_1,\bar{y})=0$. □

So we have shown that when $s=2$ Theorem 3 is true.

In the remaining part of this section we will assume that $s\geqq 3$. The values of our functions will be 3-tensors. We will denote such tensors by bold Roman capital letters: $\mathbf{A}$, $\mathbf{B}$, etc. The components of such a 3-tensor $\mathbf{A}$ will be written as $\mathbf{A}^{i_1,i_2,i_3}$, $1\leqq i_1,i_2,i_3\leqq\nu$.

Given a tensor $\mathbf{A}$, let $\mathbf{A}^*_{1,2}$, $\mathbf{A}^*_{1,3}$, $\mathbf{A}^*_{2,3}$ denote the tensors $\mathbf{A}^{i_1,i_2,i_3}=(\mathbf{A}^*_{1,2})^{i_2,i_1,i_3}=(\mathbf{A}^*_{1,3})^{i_3,i_2,i_1}=(\mathbf{A}^*_{2,3})^{i_1,i_3,i_2}$. We say that $\mathbf{A}$ is symmetric if $\mathbf{A}^*_{i,j}=\mathbf{A}$, $1\leqq i<j\leqq 3$.

If $\mathbf{a}=(a^1,\ldots,a^\nu)\in R^\nu$ is a vector let $\mathbf{A}^{1\cdot 1}\mathbf{a}$, $\mathbf{A}^{2\cdot 1}\mathbf{a}$, $\mathbf{A}^{3\cdot 1}\mathbf{a}$ denote the matrices

$$(\mathbf{A}^{1\cdot 1}\mathbf{a})^{i_1,i_2}=\sum_{j=1}^{\nu}\mathbf{A}^{j,i_1,i_2}a^j,\ (\mathbf{A}^{2\cdot 1}\mathbf{a})^{i_1,i_2}=\sum_{j=1}^{\nu}\mathbf{A}^{i_1,j,i_2}a^j,\ (\mathbf{A}^{3\cdot 1}\mathbf{a})^{i_1,i_2}=\sum_{j=1}^{\nu}\mathbf{A}^{i_1,i_2,j}a^j.$$

Finally, given a matrix A with entries $A^{i,j}$, $1\leqq i,j\leqq\nu$, let $\mathbf{A}^{1\cdot 1}A$, $\mathbf{A}^{1\cdot 2}A$, $\mathbf{A}^{2\cdot 1}A,\ldots$ denote the tensors

$$(\mathbf{A}^{1\cdot 1}A)^{i_1,i_2,i_3}=\sum_{j=1}^{\nu}\mathbf{A}^{j,i_1,i_2}A^{j,i_3},\ (\mathbf{A}^{1\cdot 2}A)^{i_1,i_2,i_3}$$
$$=\sum_{j=1}^{\nu}\mathbf{A}^{j,i_1,i_2}A^{i_3,j},\ (\mathbf{A}^{2\cdot 1}A)^{i_1,i_2,i_3}=\sum_{j=1}^{\nu}\mathbf{A}^{i_1,j,i_2}A^{j,i_3},\ \ldots.$$

Similarly one defines the tensors $A^{1\cdot 1}\mathbf{A}$, $A^{2\cdot 1}\mathbf{A}$, $A^{1\cdot 2}\mathbf{A},\ldots$.[4]

We will adopt the common notation $\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a}$, $\mathbf{a}\otimes A$ and $A\otimes\mathbf{a}$ for tensor products of vectors and matrices.

Given a matrix-valued C^1-function A of $q_1,\ldots,q_N$, $v_1,\ldots,v_N\in R^\nu$, its derivative $\partial_{q_i}A$ (respectively $\partial_{v_i}A$) will be a 3-tensor with

$$(\partial_{q_i}A)^{i_1,i_2,i_3}=\frac{\partial}{\partial q_i^{i_1}}A^{i_2,i_3}\left[\text{respectively, } (\partial_{v_i}A)^{i_1,i_2,i_3}=\frac{\partial}{\partial v_i^{i_1}}A^{i_2,i_3}\right].$$

4 These binary operations can be described using multiplication and contraction of tensors if we consider vectors and matrices as 1-tensors and 2-tensors respectively

From this it follows that for any scalar C^3-function f of $q_1, \dots, q_N, v_1, \dots, v_N \in R^\nu$ the tensors $\partial^3_{q_\alpha, q_\beta, q_\gamma} f$, $\partial^3_{q_\alpha, q_\beta, v_\gamma} f, \dots$ have the form

$$(\partial^3_{q_\alpha, q_\beta, q_\gamma} f)^{i_1, i_2, i_3} = \frac{\partial^3 f}{\partial q_\alpha^{i_3} \partial q_\beta^{i_2} \partial q_\gamma^{i_1}}, \; (\partial^3_{q_\alpha, q_\beta, v_\gamma} f)^{i_1, i_2, i_3} = \frac{\partial^3 f}{\partial q_\alpha^{i_3} \partial q_\beta^{i_2} \partial v_\gamma^{i_1}}, \dots$$

(one should have in mind the definition of the second derivative).

For $q, q' \in R^\nu$ with $|q - q'| > d_0$ we set

$$\mathbf{F}(q, q') = \partial^3_{q, q, q'} U(|q - q'|).$$

Obviously $\mathbf{F}(q, q')$ is a symmetric tensor and moreover $\mathbf{F}(q', q) = \mathbf{F}(q, q')$.

Proposition 4.4. *Let* $(x_1, \dots, x_s, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathscr{I}\mathscr{A}^{(j)}$. *Then*

$$\begin{aligned}
&\partial^2_{v_1, q_{s-1}} b_1(x_1, \dots, x_{s-1}, \bar{y}) - [G(q_{s-1}, q_s)]^{-1} [\partial^2_{v_2, q_s} b_1(x_2, \dots, x_s, \bar{y})] \, G(q_1, q_2) \\
&\quad + 2 \sum_{(q,v) \in x_1 \cup \dots \cup x_{s-2} \cup \bar{y}} [(\partial_q C(x_1, \dots, x_{s-1}, \bar{y}))^*_{1,3}]^{3 \cdot 1} v \\
&\quad + 3 [\partial_{q_{s-1}} C(x_1, \dots, x_{s-1}, \bar{y})]^{3 \cdot 1} v_{s-1} \\
&\quad + \{[G(q_{s-1}, q_s)]^{-1 \, 2 \cdot 1} [\mathbf{F}(q_{s-1}, q_s)^{2 \cdot 2} C(x_1, \dots, x_{s-1}, \bar{y})]^*_{2,3}\}^{3 \cdot 1} v_{s-1} \\
&\quad - 2 \sum_{(q,v) \in \bar{y}} \left[(\partial_v C(x_1, \dots, x_{s-1}, \bar{y}))^{1 \cdot 1} \sum_{q' : q \neq q' \in \bar{y}} \partial_q U(|q - q'|) \right]^* = 0.
\end{aligned} \tag{4.10}$$

Proof. Substituting (4.3a), (4.5) and (4.6) into (0.2) and using (4.4a), we obtain an equality in which the variables v_1, v_s do not appear. We then successively apply ∂_{q_1} and ∂_{q_s} to this equality. For $(x_1, \dots, x_s, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathscr{I}\mathscr{A}^{(j)}$ we get

$$\begin{aligned}
&\sum_{(q,v) \in x_2 \cup \dots \cup x_{s-1} \cup \bar{y}} [G(q_{s-1}, q_s)^{2 \cdot 3} \partial_q C(x_1, \dots, x_{s-1}, \bar{y})]^{2 \cdot 1} v \\
&\quad + [\mathbf{F}(q_{s-1}, q_s)^{2 \cdot 2} C(x_1, \dots, x_{s-1}, \bar{y})]^*_{2,3} {}^{3 \cdot 1} v_{s-1} \\
&\quad - (\partial^2_{v_2, q_s} b_1(x_2, \dots, x_{s-1}, x_s, \bar{y})) \, G(q_1, q_2) - G(q_{s-1}, q_s) \, \partial^2_{q_1, v_{s-1}} b_{s-1}(x_1, \dots, x_{s-1}, \bar{y}) \\
&\quad - \sum_{(q,v) \in \bar{y}} G(q_{s-1}, q_s) \left[\partial_v C(x_1, \dots, x_{s-1}, \bar{y})^{1 \cdot 1} \sum_{q' : q \neq q' \in \bar{y}} \partial_q U(|q - q'|) \right]^* = 0.
\end{aligned} \tag{4.11}$$

By substituting (4.3a) into (0.3) and successively applying ∂_{v_1} and $\partial_{v_{s-1}}$ we get

$$\begin{aligned}
&(\partial_{q_1} C^*(x_1, \dots, x_{s-1}, \bar{y}))^*_{1,2} {}^{3 \cdot 1} v_1 + \sum_{(q,v) \in x_1 \cup \dots \cup x_{s-1} \cup \bar{y}} (\partial_q C(x_1, \dots, x_{s-1}, \bar{y}))^*_{1,3} {}^{3 \cdot 1} v \\
&\quad + (\partial_{q_{s-1}} C(x_1, \dots, x_{s-1}, \bar{y}))^{3 \cdot 1} v_{s-1} \\
&\quad + \partial^2_{v_1, q_{s-1}} b_1(x_1, \dots, x_{s-1}, \bar{y}) + \partial^2_{q_1, v_{s-1}} b_{s-1}(x_1, \dots, x_{s-1}, \bar{y}) \\
&\quad - \sum_{(q\,v) \in \bar{y}} \left[\partial_v C(x_1, \dots, x_{s-1}, \bar{y})^{1 \cdot 1} \sum_{q' : q \neq q' \in \bar{y}} \partial_q U(|q - q'|) \right]^* = 0.
\end{aligned} \tag{4.12}$$

We multiply (4.11) on the left with $(G(q_{s-1}, q_s))^{-1}$ and add the resulting expression to (4.12). This gives (4.10). □

Assuming that $(x_1, \ldots, x_{s-1}, \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1$ we introduce the following notation:

$$C_1(x_1, \ldots, x_{s-1}, \bar{y}) = \partial^2_{v_1, q_{s-1}} b_1(x_1, \ldots, x_{s-1}, \bar{y}), \tag{4.13}$$

$$\mathbf{T}_{(q,v)}(x_1, \ldots, x_{s-1}, \bar{y}) = 2(\partial_q C(x_1, \ldots, x_{s-1}, \bar{y}))^*_{1,3}, \quad (q,v) \in x_1 \cup \ldots \cup x_{s-2} \cup \bar{y}, \tag{4.14}$$

$$\begin{aligned} \mathbf{T}_{(q_{s-1}, v_{s-1})}(x_1, \ldots, x_s, \bar{y}) &= 3\,\partial_{q_{s-1}} C(x_1, \ldots, x_{s-1}, \bar{y}) \\ &+ [G(q_{s-1}, q_s)]^{-1\; 2\cdot 1} [\mathbf{F}(q_{s-1}, q_s)^{\,2\cdot 2}\, C(x_1, \ldots, x_{s-1}, \bar{y})]^*_{2,3} \end{aligned} \tag{4.15}$$

$$\begin{aligned} &Y(x_1, \ldots, x_{s-1}, \bar{y}) \\ &= -2 \sum_{(q,v)\in\bar{y}} \left[\partial_v C(x_1, \ldots, x_{s-1}, \bar{y})^{\,1\cdot 1} \sum_{q': q \neq q' \in \bar{y}} \partial_q U(|q-q'|)\right]^*. \end{aligned} \tag{4.16}$$

With (4.13)–(4.16) Eq. (4.10) assumes the form

$$\begin{aligned} &C_1(x_1, \ldots, x_{s-1}, \bar{y}) - [G(q_{s-1}, q_s)]^{-1} C_1(x_2, \ldots, x_s, \bar{y})\, G(q_1, q_2) \\ &+ \sum_{(q,v)\in x_1\cup\ldots\cup x_{s-2}\cup\bar{y}} \mathbf{T}_{(q,v)}(x_1, \ldots, x_{s-1}, \bar{y})^{\,3\cdot 1}\, v + \mathbf{T}_{(q_{s-1}, v_{s-1})}(x_1, \ldots, x_s, \bar{y})^{\,3\cdot 1}\, v_{s-1} \\ &+ Y(x_1, \ldots, x_{s-1}, \bar{y}) = 0, \qquad (x_1, \ldots, x_s, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathscr{I}\bar{\mathscr{A}}^{(j)}. \end{aligned} \tag{4.17}$$

Proposition 4.5. *There exist tensor-valued functions, $\mathbf{W}_1, \ldots, \mathbf{W}_{s-2}$, and a matrix-valued function, Z, on $\mathscr{I}\bar{\mathscr{A}}_1$, constant in $v_1, \ldots, v_{s-1}$, such that*

$$\begin{aligned} &C_1(x_1, \ldots, x_{s-1}, \bar{y}) \\ &= \sum_{i=1}^{s-2} [G(q_{s-2}, q_{s-1})^{\,2\cdot 1}\, \mathbf{W}_i(x_1, \ldots, x_{s-1}, \bar{y})]^{\,3\cdot 1}\, v_i + Z(x_1, \ldots, x_{s-1}, \bar{y}). \end{aligned} \tag{4.18}$$

Proof. We shall show that, for $j = 1, \ldots, s-2$, there exist tensor-valued functions, $\mathbf{W}_{i,j}$, $s-j-1 \leqq i \leqq s-2$, and a matrix-valued function, Z_j, on $\mathscr{I}\bar{\mathscr{A}}_1$, constant in v_i, $s-j-1 \leqq i \leqq s-2$, such that

$$\begin{aligned} &C_1(x_1, \ldots, x_{s-1}, \bar{y}) \\ &= \sum_{i=s-j-1}^{s-2} [G(q_{s-2}, q_{s-1})^{\,2\cdot 1}\, \mathbf{W}_{i,j}(x_1, \ldots, x_{s-1}, \bar{y})]^{\,3\cdot 1}\, v_i + Z_j(x_1, \ldots, x_{s-1}, \bar{y}). \end{aligned} \tag{4.19}$$

For $j = s-2$, (4.19) reduces to (4.18) when $\mathbf{W}_i = \mathbf{W}_{i,s-2}$, $\mathbf{Z} = \mathbf{Z}_{s-2}$.

We use induction on j. First consider the case $j = 1$. Let $(x_1, \ldots, x_{s-1}, \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1$. Choose $x_0 \in B^a_{q_1} \times R^\nu$ such that $(x_0, x_1, \ldots, x_{s-1}, \bar{y}) \in \bigcap_{l=1}^{s-1} \mathscr{I}\bar{\mathscr{A}}^{(l)}$. Equation (4.17) for $(x_0, x_1, \ldots, x_{s-1}, \bar{y})$ has the form

$$\begin{aligned} &C_1(x_0, x_1, \ldots, x_{s-2}, \bar{y}) - [G(q_{s-2}, q_{s-1})]^{-1} C_1(x_1, \ldots, x_{s-1}, \bar{y})\, G(q_0, q_1) \\ &+ \sum_{(q,v)\in x_0\cup\ldots\cup x_{s-3}\cup\bar{y}} \mathbf{T}_{(q,v)}(x_0, \ldots, x_{s-2}, \bar{y})^{\,3\cdot 1}\, v + \mathbf{T}_{(q_{s-2}, v_{s-2})}(x_0, \ldots, x_{s-1}, \bar{y})^{\,3\cdot 1}\, v_{s-2} \\ &+ Y(x_0, \ldots, x_{s-2}, \bar{y}) = 0. \end{aligned} \tag{4.20}$$

From (4.2a), (4.4a), (4.13)–(4.15), and the properties of b (see Theorem 2) it follows that a) there are no terms in (4.20) containing v_{s-1}, b) there are no terms except for the second and the fourth, containing v_{s-2}, c) the fourth term is linear in

v_{s-2}. Multiplying (4.20) on the left by $G(q_{s-2}, q_{s-1})$ and on the right by $(G(q_0, q_1))^{-1}$ and using (4.2a), (4.14), (4.15) we obtain (4.19) for $j=1$.

Notice that for $s=3$, (4.19) coincides with (4.18), when $\mathbf{W}_1 = \mathbf{W}_{1,1}$, $Z = Z_1$. So far $s=3$ Proposition 4.5 is true.

We now suppose (4.19) to be valid for some $j_0 \leqq s-3$. For $(x_1, \ldots, x_{s-1}, \bar{y}) \in \mathcal{I}\bar{\mathcal{A}}_1$ choose $x_0 \in B^a_{q_1} \times R^\nu$ such that $(x_0, x_1, \ldots, x_{s-1}, \bar{y}) \in \bigcap_{l=1}^{s-1} \mathcal{I}\bar{\mathcal{A}}^{(l)}$. By setting $j=j_0$ and replacing x_i by x_{i-1}, for $1 \leqq i \leqq s-1$, in (4.19) we obtain $C_1(x_0, \ldots, x_{s-2}, \bar{y})$. By substituting this expression into (4.20) and multiplying the result on the left and right by $G(q_{s-2}, q_{s-1})$ and $(G(q_0, q_1))^{-1}$ respectively we get (4.19) for $j=j_0+1$. □

Assuming that $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathcal{I}\bar{\mathcal{A}}^{(j)}$, we introduce the following abbreviated notation.

$$\mathbf{W}_i = \mathbf{W}_i(x_1, \ldots, x_{s-1}, \bar{y}), \quad \mathbf{W}_i^+ = \mathbf{W}_i(x_2, \ldots, x_s, \bar{y}), \quad 1 \leqq i \leqq s-1, \tag{4.21}$$

$$\mathbf{T}_i = \mathbf{T}_{(q_i, v_i)}(x_1, \ldots, x_{s-1}, \bar{y}), \quad \mathbf{T}_i^+ = \mathbf{T}_{(q_{i+1}, v_{i+1})}(x_2, \ldots, x_s, \bar{y}), \quad 1 \leqq i \leqq s-2, \tag{4.22}$$

$$\mathbf{T}_{s-1} = \mathbf{T}_{(q_{s-1}, v_{s-1})}(x_1, \ldots, x_s, \bar{y}), \tag{4.23}$$

$$G_{i,i+1} = G(q_i, q_{i+1}), \quad 1 \leqq i \leqq s-1. \tag{4.24}$$

Proposition 4.6. *Let* $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathcal{I}\bar{\mathcal{A}}^{(j)}$. *Then*

$$G_{s-2,s-1} \,{}^{2\cdot 1}\, \mathbf{W}_1 + \mathbf{T}_1 = 0, \quad s \geqq 3, \tag{4.25}$$

$$G_{s-2,s-1} \,{}^{2\cdot 1}\, \mathbf{W}_i - (\mathbf{W}_{i-1}^+ \,{}^{2\cdot 1}\, G_{1,2})^*_{2,3} + \mathbf{T}_i = 0, \quad 2 \leqq i \leqq s \leqq 2, \quad s \geqq 4, \tag{4.26}$$

$$-(\mathbf{W}_{s-2}^+ \,{}^{2\cdot 1}\, G_{1,2})^*_{2,3} + \mathbf{T}_{s-1} = 0, \quad s \geqq 3. \tag{4.27}$$

Remark. (4.26) shows that $\mathbf{W}_{i-1}^+$, $2 \leqq i \leqq s-2$, is independent of x_s.

Proof. We first substitute (4.18) into (4.17) and then equate the coefficients of $v_1, \ldots, v_{s-1}$ to zero. □

Henceforth Eqs. (4.25)–(4.27) will play a principal role. By deleting $\mathbf{W}_i$, $\mathbf{W}_i^+$, $1 \leqq i \leqq s-2$, from them we obtain relations among the $\mathbf{T}_1, \ldots, \mathbf{T}_{s-1}, \mathbf{T}_{s-2}^+$ and will show that these relations hold only in the case where $b=0$.

We start with the following auxiliary assertion.

Proposition 4.7. *The following equalities hold*:

$$G_{s-1,s}^{-1} \,{}^{2\cdot 1}\, \mathbf{T}_1^+ = [(\mathbf{T}_1 + \mathbf{T}_2) \,{}^{2\cdot 2}\, (G_{1,2}^{-1})]^*_{2,3}, \qquad s \geqq 4,$$

$$(G_{s-1,s}^{-1}) \,{}^{2\cdot 1}\, \mathbf{T}_i^+ = [\mathbf{T}_{i+1} \,{}^{2\cdot 2}\, (G_{1,2}^{-1})]^*_{2,3}, \qquad s \geqq 5, \quad 2 \leqq i \leqq s-3.$$

The proof can be obtained by a straightforward computation.

Proposition 4.8. *Let* $s \geqq 4$ *and let* $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{l=1}^{s-1} \mathcal{I}\bar{\mathcal{A}}^{(l)}$. *Assume that for some* i, $2 \leqq i \leqq s-2$, *there are* $x_{s+1}, \ldots, x_{s+i-1} \in R^\nu \times R^\nu$ *such that* $(x_j, \ldots, x_{j+s-1}, \bar{y}) \in \bigcap_{l=1}^{s-1} \mathcal{I}\bar{\mathcal{A}}^{(l)}$ *for* $j=2, \ldots, i$. *Then*

$$\mathbf{W}_i = -(G_{s-2,s-1}^{-1}) \,{}^{2\cdot 1} \left[(i-1)\mathbf{T}_i + \sum_{l=1}^{i} \mathbf{T}_l\right]. \tag{4.28}$$

Remark. Equation (4.28) also holds for $i=1$, $s \geqq 3$, $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{l=1}^{s-1} \mathcal{I}\bar{\mathcal{A}}^{(l)}$, as immediately follows from (4.25).

The proof is obtained by induction in i using (4.26) and Proposition 4.7. We leave out details.

Proposition 4.9. *Let* $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{l=1}^{s-1} \mathcal{I}\bar{\mathcal{A}}^{(l)}$ *and assume that there are* $x_{s+1}, \ldots, x_{2s-2} \in R^\nu \times R^\nu$ *such that* $(x_j, \ldots, x_{j+s-1}, \bar{y}) \in \bigcap_{l=1}^{s-1} \mathcal{I}\bar{\mathcal{A}}^{(l)}$, $2 \leqq j \leqq s-1$. *Then*

$$\mathbf{T}_1^{+\;2\cdot 1}\, G_{1,2} + G_{2,3}{}^{2\cdot 1}\, (\mathbf{T}_2)^*_{2,3} = 0, \qquad s=3, \tag{4.29a}$$

$$(s-2)\,\mathbf{T}_{s-2}^{+\;2\cdot 1}\, G_{1,2} + G_{s-1,s}{}^{2\cdot 1} \sum_{i=1}^{s-1} (\mathbf{T}_i)^*_{2,3} = 0, \qquad s \geqq 4. \tag{4.29b}$$

Proof. In view of the above condition on $(x_1, \ldots, x_s, \bar{y})$ we apply (4.28) with $i=s-2$ and obtain $\mathbf{W}_{s-2}$ and then $\mathbf{W}_{s-2}^+$. By substituting this last expression into (4.27) and then using Proposition 4.7 we obtain (4.29a, b). □

Proposition 4.10. *Assume that the assumptions of Proposition* 4.9 *hold. Then*

$$b(-G_{s-1,s}{}^{2\cdot 2}\, \mathbf{F}_{s-2,s-1} + \mathbf{F}_{s-1,s}{}^{2\cdot 2}\, G_{s-2,s-1}) = 0, \tag{4.30}$$

where

$$b = b(x_1, \ldots, x_{s-1}, \bar{y}), \qquad \mathbf{F}_{i,i+1} = \mathbf{F}(q_i, q_{i+1}).$$

Proof. As some computations are cumbersome, we will only outline the proof. Firstly, from (4.2a), (4.14), (4.15), (4.22), and (4.23) we obtain $\mathbf{T}_{s-2}^+$, $\mathbf{T}_i$, $1 \leqq i \leqq s-1$, and then substitute those expressions into (4.29a, b). From the equality thus obtained and by using the fact that b is locally constant (see Theorem 2) and by the symmetry of $G_{i,i+1}$ and $\mathbf{F}_{i,i+1}$, find that

$$(2s-3)\, b(\mathbf{F}_{s-1,s}{}^{2\cdot 2}\, G_1^{s-1} - G_{s-1,s}{}^{2\cdot 2}\, \mathbf{F}_{s-2,s-1}{}^{2\cdot 2}\, G_1^{s-2}) = 0, \tag{4.31}$$

where $G_i^j = \prod_{l=i}^{j-1} G_{l,l+1}$ for $1 \leqq i < j \leqq s-1$, and $G_i^j = E$ for $1 \leqq i = j \leqq s-1$. Denoting the left side of (4.31) by $\mathbf{W}$ we rewrite (4.31) in the form $\mathbf{W}^{\,3\cdot 1}\, (G_1^{s-2})^{-1} = 0$ which can be reduced to (4.30). □

We will need the following explicit formula for the matrix $\partial^2_{q^2} U(|q|)$ and the tensor $\partial^3_{q^3} U(|q|)$ $(\nu \geqq 2)$:

$$\partial^2_{q^2} U(|q|) = \psi_0(|q|)\, E + \psi_2(|q|)\,(q \otimes q), \tag{4.32}$$

$$\begin{aligned}\partial^3_{q^3} U(|q|) = {} & \psi_1(|q|)\,[q \otimes E + E \otimes q + (E \otimes q)^*_{2,3}] \\ & + \psi_3(|q|)\,(q \otimes q \otimes q),\end{aligned} \tag{4.33}$$

where

$$\begin{aligned}&\psi_0(r) = r^{-1} U'(r), \qquad \psi_1(r) = \psi_2(r) = r^{-2} U''(r) - r^{-3} U'(r), \\ &\psi_3(r) = r^{-5}\,[r^2 U'''(r) - 3r\, U''(r) + 3U'(r)].\end{aligned} \tag{4.34}$$

From now on the cases $\nu \geqq 2$ and $\nu = 1$ will be considered separately.

Proposition 4.11. *Let $v \geqq 2$ and the conditions of Proposition 4.9 hold. Then*

$$b(x_1, \ldots, x_{s-1}, \bar{y})\, \psi_3(|q_{s-1} - q_s|) = 0. \tag{4.35}$$

Proof.[5] We first remark that if $(x_1, \ldots, x_s, \bar{y})$ satisfies the conditions of Proposition 4.9, then so does every configuration $(x_1, \ldots, x_{s-1}, (q, v), \bar{y})$ such that $q - q_{s-1}$ is within a sufficiently small neighbourhood $\mathcal{O}$ of $q_s - q_{s-1}$. Fixing $x_1, \ldots, x_{s-1}$, and $\bar{y}$ we consider the left side of (4.30) to be a tensor-valued function on $q \in \mathcal{O}$. We denote this function by $\mathbf{V}$. Due to (4.30), (4.32), and (4.33), an arbitrary component of $\mathbf{V}(q)$ has the form

$$(\mathbf{V}(q))^{i_1, i_2, i_3} = b \sum_{i=0}^{3} \psi_i(|q|)\, P_i(q), \tag{4.36}$$

where P_i, $0 \leqq i \leqq 3$, is a homogeneous polynomial of degree i in the components of the vector $q = (q^1, \ldots, q^v)$. Specifically,

$$P_3(q) = -\sum_{i=1}^{v} q^{i_1} q^{i_2} q^i\, G^{i_3, i}_{s-2, s-1}. \tag{4.37}$$

We now show that P_3 is not constant on the sphere $S = \{q \in R^v : |q| = |q_s - q_{s-1}|\}$. Suppose that $P_3(q) = \text{const} = c$, $q \in S$. Since P_3 is a homogeneous polynomial of an odd degree it follows that $c = 0$ and hence $P_3(q) = 0$, $q \in R^v$. Equality (4.37) implies that the matrix $G_{s-2,s-1}$ maps the open set $\{q \in R^v : q^{i_1} q^{i_2} \neq 0\}$ in the subspace $\{q \in R^v : q^{i_3} = 0\}$. But this contradicts the non-degeneracy of $G_{s-2,s-1}$.

Choose a neighbourhood Γ of $q_s - q_{s-1}$ on S and a neighbourhood Λ of 1 on R^1 so that $\lambda q \in \mathcal{O}$ for any $q \in \Gamma$, $\lambda \in \Lambda$. Then by virtue of (4.30), (4.36) for $q \in \Gamma$, $\lambda \in \Lambda$ we have

$$b[\psi_0(\lambda\varrho) P_0(q) + \lambda \psi_1(\lambda\varrho) P_1(q) + \lambda^2 \psi_2(\lambda\varrho) P_2(q) + \lambda^3 \psi_3(\lambda\varrho) P_3(q)] = 0, \tag{4.38}$$

where $\varrho = |q_s - q_{s-1}|$. For fixed λ the left side of (4.38) is a polynomial in the components of $q \in R^v$. Since this polynomial vanishes on $\Gamma \subset S$ it vanishes on S.

Now suppose that $b\psi_3(\varrho) \neq 0$. Since ψ_3 is continuous [see Condition (I_1'')], there is a neighbourhood, $\Lambda_1 \subset \Lambda$, such that

$$b\psi_3(\lambda\varrho) \neq 0, \qquad \lambda \in \Lambda_1. \tag{4.39}$$

Since P_3 is not constant on S (see above), there is a point $q^* \in S$ and a non-empty open subset Π of S such that $P_3(q^*) \neq P_3(q)$ for $q \in \Pi$. From (4.38) we get

$$\begin{aligned} &b\lambda\psi_1(\lambda\varrho)[P_1(q) - P_1(q^*)] + b\lambda^2\psi_2(\lambda\varrho)[P_2(q) - P_2(q^*)] \\ &\quad + b\lambda^3\psi_3(\lambda\varrho)[P_3(q) - P_3(q^*)] = 0. \end{aligned} \tag{4.40}$$

By (4.39) the last term on the left side of (4.40) differs from zero when $\lambda \in \Lambda_1$ and $q \in \Pi$. Since $\psi_1 \equiv \psi_2$, it follows from (4.40) that $\psi_1(\lambda\varrho) = \psi_2(\lambda\varrho) \neq 0$ when $\lambda \in \Lambda_1$. So the functions Ψ_1 and Ψ_2, where $\Psi_i : \lambda \mapsto \lambda^i \psi_i(\lambda\varrho)$, $1 \leqq i \leqq 3$, $\lambda \in \Lambda_1$, are linearly independent. But (4.40) shows that Ψ_1, Ψ_2, and Ψ_3 are linearly dependent. Thus for $q \in \Pi$ the 3-dimensional vector $(P_1(q) - P_1(q^*), P_2(q) - P_2(q^*), P_3(q) - P_3(q^*))$ is

5 The idea of the proof was suggested to us by Yu. S. Il'iašenko

proportional to a vector (c_1, c_2, c_3) independent of q and such that $(c_1^2 + c_2^2)\, c_3 \neq 0$ [see (4.40) and the definition of Π]. For every $q \in \Pi$ we have

$$c_2\,[P_3(q) - P_3(q^*)] = c_3\,[P_2(q) - P_2(q^*)], \tag{4.41 a}$$

$$c_1\,[P_3(q) - P_3(q^*)] = c_3\,[P_1(q) - P_1(q^*)]. \tag{4.41 b}$$

Since these equalities hold on a non-empty open subset of S, they hold on S. Replace q in (4.41 a) by $-q$ and subtract the resulting equality from (4.41 a). Since P_2 and P_3 are homogeneous, we find that $c_2 P_3(q) = 0$, $q \in S$, and hence that $c_2 = 0$, $c_1 \neq 0$. Similarly, using (4.41 b) we deduce that $P_3(q) = c_1^{-1} c_3 P_1(q)$, $q \in S$, i.e.

$$P_3(q) = c_1^{-1} \varrho^{-2} c_3 |q|^2 P_1(q), \qquad q \in S.$$

Both the left and right hand sides of this equality are homogeneous polynomials of degree 3 in $q^1, \ldots, q^\nu$. Since their restrictions to S coincide, they are identical, that is

$$P_3(q) = \text{const}\, P_1(q) \sum_{i=1}^{\nu} (q^i)^2, \qquad q \in R^\nu. \tag{4.42}$$

By the non-degeneracy of $G_{s-2,s-1}$ (4.42) contradicts (4.37) and hence $b\,\psi_3(\varrho) = 0$ as desired. □

Proposition 4.12. *Let* $\nu \geqq 2$ *and the conditions of Proposition* 4.9 *hold. Then*

$$b(x_1, \ldots, x_{s-1}, \bar{y})\, \psi_1(|q_{s-1} - q_s|) = 0. \tag{4.43}$$

Proof. Let $\mathcal{O}$, $\mathbf{V}$, S, Γ, Λ, ϱ be as in the proof of the previous proposition. By (4.30), (4.32), and (4.33) the polynomial P_1 appearing in (4.36) has the form

$$P_1(q) = -q^{i_1} G_{s-2,s-1}^{i_2,i_3} - q^{i_2} G_{s-2,s-1}^{i_1,i_3} - \delta_{i_1,i_2} \sum_{i=1}^{\nu} q^i G_{s-2,s-1}^{i,i_3}, \tag{4.44}$$

δ_{i_1,i_2} being the Kroneker symbol. If $i_1 = i_2$, P_1 cannot be constant on S. This follows by an argument identical to that used in the case of P_3 (see the proof of Proposition 4.11).

By Proposition 4.11, when $\lambda \in \Lambda$, $q \in \Gamma$, equality (4.38) assumes the form

$$b\,[\psi_0(\lambda\varrho) P_0(q) + \lambda \psi_1(\lambda\varrho) P_1(q) + \lambda^2 \psi_2(\lambda\varrho) P_2(q)] = 0. \tag{4.45}$$

As in the above we can be assured that (4.45) holds for $q \in S$. Assuming (4.43) false, choose a neighbourhood $\Lambda_2 \subset \Lambda$ such that

$$b\,\psi_1(\lambda\varrho) \neq 0, \qquad \lambda \in \Lambda_2. \tag{4.46}$$

Let $i_1 = i_2$. Since in this case $P_1(q) \neq \text{const}$, $q \in S$, there exist q^*, $q^{**} \in S$ such that $P_1(q^*) \neq P_1(q^{**})$. From (4.45) we get

$$b\lambda\psi_1(\lambda\varrho)\,[P_1(q^{**}) - P_1(q^*)] + b\lambda^2 \psi_2(\lambda\varrho)\,[P_2(q^{**}) - P_2(q^*)] = 0, \qquad \lambda \in \Lambda. \tag{4.47}$$

By (4.46) and the identity $\psi_1 \equiv \psi_2$, the functions Ψ_1 and Ψ_2 are linearly independent on Λ_2. Then from (4.47) it follows that $P_1(q^{**}) = P_1(q^*)$ which contradicts the choice of q^* and q^{**}. □

Proposition 4.13. *There is a point* r, *arbitrarily close to* d_1, *such that* $\psi_1(r)\,U'(r)\,U''(r) \neq 0$.

Proof. Assuming the proposition to be false, choose a point $d_2 \in (d_0, d_1)$ such that $\psi_1(r)\,U'(r)\,U''(r) = 0$ for all $r \in (d_2, d_1)$, that is (see (4.34)), $U''(r)\,[(U'(r))^2/r^2]' = 0$. It is easy to see that the identity $U''(r) \equiv 0$, $r \in (d_2, d_1)$, contradicts conditions (I_1'')–(I_3'') on U. So the set $(d_2, d_1) \cap \{r : U''(r) \neq 0\}$ is non-empty and open. Let (α, β) be an interval being a connected component of this set. The equation $[(U'(r))^2/r^2]' = 0$ has the general solution $U(r) = c_0 r^2 + c_1$ on (α, β). By the definition of (α, β), $U''(\beta) = 2c_0 = 0$. Whence $c_0 = 0$ and $U''(r) = 0$ for $r \in (\alpha, \beta)$. But this contradicts the definition of (α, β). □

Proposition 4.14. *Let* $\nu \geqq 1$ *and let* $(x_1, \ldots, x_{s-1}, \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1$. *Then* (4.1) *holds.*

Proof. We first consider the case $\nu \geqq 2$. Proposition 4.12 then implies that if $(x_1', \ldots, x_{s-1}', \bar{y}') \in \mathscr{I}\bar{\mathscr{A}}_1$ and there is a point $x_s' \in R^\nu \times R^\nu$ such that $(x_1', \ldots, x_{s-1}', x_s', \bar{y}')$ satisfies the conditions of Proposition 4.12 and moreover if $\psi_1(|q_{s-1}' - q_s'|) \neq 0$, then $b(x_1', \ldots, x_{s-1}', \bar{y}') = 0$. But given a configuration $(x_1, \ldots, x_{s-1}, \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1$, Proposition 4.13 assures us that there is another configuration $(x_1', \ldots, x_{s-1}', \bar{y}')$ with $\bar{y}' = \bar{y}$ and with the properties just mentioned. Since b is constant when $\bar{y}$ is fixed (see Theorem 2), we have $b(x_1, \ldots, x_{s-1}, \bar{y}) = 0$.

We now deal with $\nu = 1$. In this case equation (4.30) assumes the form

$$b(x_1, \ldots, x_{s-1}, \bar{y})\,\psi(|q_{s-1} - q_s|) = 0, \tag{4.48}$$

where

$$\psi(r) = U'''(r) - [U'''(|q_{s-2} - q_{s-1}|)/U''(|q_{s-2} - q_{s-1}|)]\,U''(r)$$

($x_1, \ldots, x_{s-1}$, and $\bar{y}$ are assumed to be fixed).

From (4.48) it follows that if $(x_1', \ldots, x_{s-1}', \bar{y}') \in \mathscr{I}\bar{\mathscr{A}}_1$, if there is a point $x_s' \in R^1 \times R^1$ such that $(x_1', \ldots, x_{s-1}', x_s', \bar{y}')$ satisfies the conditions of Proposition 4.9 and moreover if $\psi(|q_{s-1}' - q_s'|) \neq 0$, then $b(x_1', \ldots, x_{s-1}', \bar{y}') = 0$.

In the proof of Proposition 3.3 it was shown that given $c \in R^1$ and a neighbourhood $\mathcal{O}$ of d_1, there is a point $r \in \mathcal{O}$ for which

$$[U'''(r) + c\,U''(r)]\,U''(r)\,U'(r) \neq 0.$$

Keeping this in mind, for every $(x_1, \ldots, x_{s-1}, \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1$, it is easy to find a configuration $(x_1', \ldots, x_{s-1}', \bar{y}')$ with $\bar{y}' = \bar{y}$ which satisfies the conditions just mentioned. Since $b(x_1, \ldots, x_{s-1}, \bar{y})$ is independent of $x_1, \ldots, x_{s-1}$, we have $b(x_1, \ldots, x_{s-1}, \bar{y}) = 0$. □

This finishes the proof of Theorem 3.

5. Completion of the Proof of the Basic Lemma

We let $\mathscr{A}_s$ denote configurations $(x_1, \ldots, x_{s-1}, \bar{y}) \in \bigcup_{k_1} \mathscr{B}(n-1, m-1, k_1, s-1)$ satisfying the following condition: there is a point, $x \in R^\nu \times R^\nu$, such that $(x_1, \ldots, x_{s-1}, x, \bar{y}) \in \bigcap_{i=1}^{s-1} \mathscr{A}^{(i)}$ (see Sect. 1).

Proposition 5.1. *Let* $(x_1, \ldots, x_{s-1}, \bar{y}) \in \mathscr{I}\mathscr{A}_s$. *Then*

$$\partial^2_{v_1, v_{s-1}} f_{s-1}(x_1, \ldots, x_{s-1}, \bar{y}) = 0. \tag{5.1}$$

Proof. We shall consider only the case $s \geqq 3$, because when $s=2$, $\mathscr{A}_s \subset \bar{\mathscr{A}}_1$ and hence our proposition follows from Theorem 3. It is easy to find $x_i = (q_i, v_i) \in R^v \times R^v$, $-s+3 \leqq i \leqq 0$, such that $(x_{-s+3}, \ldots, x_1, \bar{y}) \in \mathscr{I}\bar{\mathscr{A}}_1$ and such that $(x_i, \ldots, x_{i+s-1}, \bar{y}) \in \bigcap_{j=1}^{s-1} \mathscr{I}\mathscr{A}^{(j)}$ for $i = 0, \ldots, -s+3$. By Theorem 3

$$\partial^2_{v_{-s+3}, v_1} f_{s-1}(x_{-s+3}, \ldots, x_1, \bar{y}) = 0.$$

By Proposition 2.1 and the non-degeneracy of the matrix $G(q_i, q_{i+1})$, $-s+3 \leqq i \leqq 0$, we obtain (5.1). □

Corollary 5.2. *Let* $(x_1, \ldots, x_{s-1}, \bar{y}) \in \bigcap_{i=1}^{s-1} \mathscr{I}\mathscr{A}^{(i)}$. *Then*

$$\partial_{q_1} f_s(x_1, \ldots, x_s, \bar{y}) = 0. \tag{5.2}$$

One can get (5.2) by first applying ∂_{v_1} to (0.2) and then using Proposition 5.1.

Proposition 5.3. *Let* $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}$ *(see Sect. 0). Then*

(a) *if* $U'(|q_1 - q_2|) = 0$, *then*

$$\partial_{q_s} f_s(x_1, \ldots, x_s, \bar{y}) = 0; \tag{5.3}$$

(b) *if* $U'(|q_{s-1} - q_s|) = 0$, *then* (5.2) *holds.*

Proof. (a) In this case Eq. (0.5) has the form

$$U'(|q_s - q_{s+1}|)\, |q_s - q_{s+1}|^{-1} \langle \partial_{v_s} f_s(x_1, \ldots, x_s, \bar{y}), q_s - q_{s+1} \rangle = 0, \qquad q_{s+1} \in B^a_{q_s}.$$

Dividing by the non-zero factors and using the fact that $B^a_{q_s}$ is a non-empty open set we get

$$\partial_{v_s} f_s(x_1, \ldots, x_s, \bar{y}) = 0.$$

Using this equality, an application of ∂_{v_s} to (0.2) gives (5.3).

Assertion (b) can be proved similarly [use (0.5) instead of (0.4)]. □

Proposition 5.4. *Let* $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}$ *and let* $U'(|q_{s-1} - q_s|) = 0$. *Then*

$$f_s(x_1, \ldots, x_s, \bar{y}) = 0. \tag{5.4}$$

Proof. Set

$$q_1(t) = q_1 + t\mathbf{l}, \qquad t \geqq 0,$$

where a) if $v \geqq 2$, $\mathbf{l}$ is the unit exterior normal vector of an arbitrary supporting hyperplane of Conv $(\bar{q}(x_1 \cup \ldots \cup x_s \cup \bar{y}))$ passing through q_1; b) if $v=1$, $\mathbf{l} = l = \operatorname{sign}(q_1 - q_2)$. Let τ be such that $|q_1(\tau) - q_2| = d_1$. If $s \geqq 3$, then the configuration $((q_1(t), v_1), x_2, \ldots, x_s, \bar{y})$ for $t \in [0, \tau)$ satisfies the assumptions of Proposition 5.3 (b) which in turn implies that

$$\partial_q f_s((q, v_1), x_2, \ldots, x_s, \bar{y})\,|_{q=q_1(t)} = 0, \qquad 0 \leqq t < \tau. \tag{5.5}$$

If $s=2$, then the above mentioned configurations satisfy either the assumptions of Proposition 5.3(b) or those of Proposition 5.2. In both cases (5.5) holds. From (5.5) and the identity $f_s \equiv 0$ on $\mathscr{D}\mathscr{A}(n, m, k, s)$ (see the statement of the Basic Lemma) we obtain (5.4). $\square$

We will treat the cases $\nu \geqq 2$ and $\nu = 1$ separately and will start with the first. Here we need two easy geometric results whose proof we leave to the reader.

Lemma 5.5 *Let $K \subset R^\nu$, $\nu \geqq 2$, be convex and let $p_1 \neq p_2$ lie on its boundary. For $i=1,2$ let L_i be a supporting hyperplane of K passing through p_i and let l_i be the exterior (with respect to K) normal of L_i passing through p_i. Then for $y_1 \in l_1$, $y_2 \in l_2$, and $y \in K$ we have*

(1) $\operatorname{dist}(y_i, y) \geqq \operatorname{dist}(p_i, y), \quad i=1,2;$

(2) $\operatorname{dist}(p_1, p_2) \leqq \operatorname{dist}(y_1, y_2).$

Lemma 5.6. *Let K be as in Lemma 5.5 and let $p_1, p_2 \in R^\nu \setminus K$, $p_1 \neq p_2$. Then at least one of the points p_1, p_2 is an extreme point of* $\operatorname{Conv}(K \cup p_1 \cup p_2)$.

Proposition 5.7. *Let $\nu \geqq 2$ and let $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{i=1}^{s-1} \mathscr{I}\mathscr{A}^{(i)} \cap \mathscr{A}_{\mathrm{sym}}$ (see Sect. 1). Then (5.4) holds.*

Proof. Let $K = \operatorname{Conv}(\bar{q}(x_1 \cup \ldots \cup x_s \cup \bar{y}))$. For $i=1,2$ let q_i be the p_i appearing in Lemma 5.5 and let l_i be the corresponding exterior normal. Let $\mathbf{l}_i$ be the vector corresponding to l_i and set

$$q_1(t) = q_1 + t\mathbf{l}_1, \qquad q_s(t) = q_s + t\mathbf{l}_2, \qquad t \geqq 0.$$

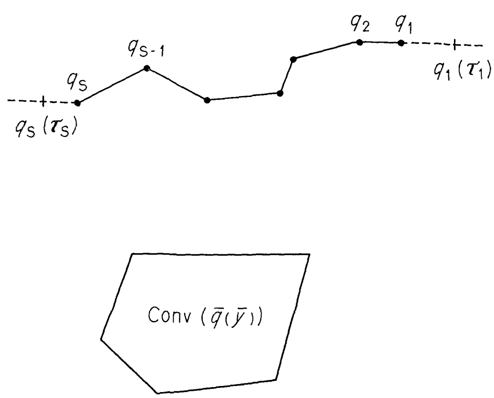

Fig. 8

Let τ_1, τ_s be such that

$$|q_1(\tau_1) - q_2| = |q_s(\tau_s) - q_{s-1}| = d_1$$

(see Fig. 8). If

$$\tau_1' = \min\{t > 0 : U'(|q_1(t) - q_2|) = 0\}, \tag{5.6}$$

then $0 < \tau_1' \leqq \tau_1$, and $((q_1(t), v_1), x_2, \ldots, x_s, \bar{y}) \in \bigcap_{i=1}^{s-1} \mathscr{I}\mathscr{A}^{(i)}$ when $0 \leqq t < \tau_1'$, and for such t and, moreover, for $t = \tau_1'$ (by Corollary 5.2 and the continuity of f_s)

$$f_s(x_1, \ldots, x_s, \bar{y}) = f_s((q_1(t), v_1), x_2, \ldots, x_s, \bar{y}). \tag{5.7}$$

If $\tau_1' = \tau_1$, then $((q_1(\tau_1'), v_1), x_2, \ldots, x_s, \bar{y})$ is a boundary configuration and, by (5.7) and by the assumptions of the Basic Lemma, (5.4) holds. If $\tau_1' < \tau_1$ and $s = 2$, then $((q_1(\tau_1'), v_1), x_2, \bar{y})$ satisfies the assumptions of Proposition 5.4 which together with (5.7) imply that $f_2(x_1, x_2, \bar{y}) = 0$. Thus for $s = 2$ our proposition is true.

We now suppose that $s \geqq 3$ and $\tau_1' < \tau_1$. First consider the case where for every $t \in [0, \tau_s)$, $q_1(\tau_1')$ is an external point in $(q_1(\tau_1'), v_1) \cup x_2 \cup \ldots \cup x_{s-1} \cup (q_s(t), v_s) \cup \bar{y})$. In this case Lemma 5.5 implies that $((q_1(\tau_1'), v_1), x_2, \ldots, x_{s-1}, (q_s(t), v_s), \bar{y})$, $t \in [0, \tau_s)$, satisfies the assumptions of Proposition 5.3(a) which in view of the continuity of f_s implies that

$$\begin{aligned} &f_s((q_1(\tau_1'), v_1), x_2, \ldots, x_s, \bar{y}) \\ &\quad = f_s((q_1(\tau_1'), v_1), x_2, \ldots, x_{s-1}, (q_s(t), v_s), \bar{y}), \qquad 0 \leqq t \leqq \tau_s. \end{aligned} \tag{5.8}$$

If $t = \tau_s$ the right side of (5.8) vanishes. This tegether with (5.7) imply (5.4).

We now consider the case where there is a $t \in [0, \tau_s)$ such that $q_1(\tau_1')$ is not an external point of $(q_1(\tau_1'), v_1) \cup x_2 \cup \ldots \cup (q_s(t), v_s) \cup \bar{y}$. It is easy to see that there exists a minimal t with this property. Denote it by τ_s'. By Proposition 5.3(a) and the continuity of f_s,

$$\begin{aligned} &f_s((q_1(\tau_1'), v_1), x_2, \ldots, x_s, \bar{y}) \\ &\quad = f_s((q_1(\tau_1'), v_1), x_2, \ldots, x_{s-1}, (q_s(t), v_s), \bar{y}), \qquad 0 \leqq t \leqq \tau_s'. \end{aligned} \tag{5.9}$$

The definition of τ_s' and Lemma 5.6 together imply that $q_s(\tau_s')$ is an external point of $(q_1(\tau_1'), v_1) \cup x_2 \cup \ldots \cup x_{s-1} \cup (q_s(\tau_s'), v_s) \cup \bar{y}$. Hence $((q_s(\tau_s'), v_s), x_{s-1}, \ldots, x_2, (q_1(\tau_1'), v_1), \bar{y})$ satisfies the assumptions of Proposition 5.4, from which it follows that

$$f_s((q_s(\tau_s'), v_s), x_{s-1}, \ldots, x_2, (q_1(\tau_1'), v_1), \bar{y}) = 0. \tag{5.10}$$

Using (5.7), (5.9), (5.10), and the symmetry of f_s (see the statement of the Basic Lemma) we get (5.4). □

Proposition 5.8. *Let* $v \geqq 2$, *let* $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}$, *and assume that the following condition holds: there exists a supporting hyperplane, L, of* Conv $(\bar{q}(x_1 \cup \ldots \cup x_s \cup \bar{y}))$ *passing through* q_1 *and such that* q_i *for* $i = 1, \ldots, s$ *is an extreme point of* Conv $(\bar{q}(x_1 \cup \ldots \cup x_s \cup \bar{y}) \cup l))$, *where l is the exterior (with respect to* Conv $(\bar{q}(x_1 \cup \ldots \cup x_s \cup \bar{y}))$ *normal of L, passing through* q_1. *Then* (5.4) *holds.*

Proof. Let $\mathscr{C}_i$, $0 \leqq i \leqq s-1$, denote those configurations $(x_1, \ldots, x_s, \bar{y})$ satisfying the conditions of our proposition and in addition (for $0 \leqq i \leqq s-2$) the following condition: $\prod_{j=1}^{s-i-1} U'(|q_j - q_{j+1}|) \neq 0$. We have to show that $f_s \equiv 0$ on $\mathscr{C}_{s-1}$. For this we shall show that $f_s \equiv 0$ on $\mathscr{C}_i$, $0 \leqq i \leqq s-1$, using induction.

Evidently, $\mathscr{C}_0 \subseteq \bigcap_{j=1}^{s-1} \mathscr{I}\mathscr{A}^{(j)} \cap \mathscr{A}_{\text{sym}}$. Therefore Proposition 5.7 implies that $f_s \equiv 0$ on $\mathscr{C}_0$. Assume $f_s \equiv 0$ on $\mathscr{C}_i$ for some i, $0 \leqq i \leqq s-2$. Since $\mathscr{C}_i$ is open, all derivatives of f_s vanish on $\mathscr{C}_i$. Keeping this in mind we apply ∂_{v_s} to Eq. (0.2) where we assume $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{C}_i$. It follows that

$$U'(|q_1 - q_2|)\,|q_1 - q_2|^{-1}\,[\partial^2_{v_2, v_s} f_{s-1}(x_2, \ldots, x_s, \bar{y})](q_1 - q_2) = 0.$$

By definition of $\mathscr{C}_i$ and since $i \leqq s-2$, the scalar factor on the left side is non-zero. Hence for all $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{C}_i$

$$[\partial^2_{v_2, v_s} f_{s-1}(x_2, \ldots, x_s, \bar{y})](q_1 - q_2) = 0. \tag{5.11}$$

As $\mathscr{C}_i$ is open, (5.11) remains true when we replace q_1 by an arbitrary point sufficiently close to q_1. Therefore for $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{C}_i$

$$\partial^2_{v_2, v_s} f_{s-1}(x_2, \ldots, x_s, \bar{y}) = 0. \tag{5.12}$$

Now let $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{C}_{i+1}$. Using the properties of U choose a point q' on l appearing in the statement of the proposition, such that

$$|q' - q_1| > d_1, \qquad U'(|q' - q_1|) \neq 0, \qquad \min_{\substack{q \in x_1 \cup \ldots \cup x_s \cup \bar{y}, \\ q \neq q_1}} |q' - q| > d_1.$$

By the definition of $B^a_{q_1}$ (see Sect. 0) and $\mathscr{C}_i$, we have

$$q' \in B^a_{q_1}, \qquad (x', x_1, \ldots, x_{s-1}, \bar{y}) \in \mathscr{C}_i, \tag{5.13}$$

where $x' = (q', v')$, $v' \in R^\nu$. As $B^a_{q_1}$ and $\mathscr{C}_i$ are open, there is a neighbourhood, $\mathcal{O}$, of q' such that (5.13) remains true when q' is replaced by an arbitrary point $q_0 \in \mathcal{O}$ and x' is replaced by $x_0 = (q_0, v_0)$, $v_0 \in R^\nu$. Therefore for every $x_0 = (q_0, v_0) \in \mathcal{O} \times R^\nu$ equality (0.4) holds which, by the induction assumption, assumes the form

$$U'(|q_0 - q_1|)\,|q_0 - q_1|^{-1} \langle \partial_{v_1} f_s(x_1, \ldots, x_s, \bar{y}), q_0 - q_1 \rangle = 0.$$

By the properties of $\mathcal{O}$ it follows that

$$\partial_{v_1} f_s(x_1, \ldots, x_s, \bar{y}) = 0. \tag{5.14}$$

The inclusion $(x_0, \ldots, x_{s-1}, \bar{y}) \in \mathscr{C}_i$, $x_0 \in \mathcal{O} \times R^\nu$, allows us to apply (5.12) to $(x_0, \ldots, x_{s-1}, \bar{y})$ which gives

$$\partial^2_{v_1, v_{s-1}} f_{s-1}(x_1, \ldots, x_{s-1}, \bar{y}) = 0. \tag{5.15}$$

In view of (5.14), (5.15) an application of ∂_{v_1} to (0.2), where $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{C}_{i+1}$, yields

$$\partial_{q_1} f_s(x_1, \ldots, x_s, \bar{y}) = 0. \tag{5.16}$$

As in the proof of Proposition 5.7, let $q_1(t) = q_1 + t\mathbf{l}$, where $\mathbf{l}$ is the direction vector of the normal l. Let $\tau_1' = \min\{t \geqq 0 : U'(|q_1(t) - q_2|) = 0\}$. It is easy to see that if $0 \leqq t < \tau_1'$, then $((q_1(t), v_1), x_2, \ldots, x_s, \bar{y}) \in \mathscr{C}_{i+1}$. By (5.16) and the continuity of f_s we have

$$f_s(x_1, \ldots, x_s, \bar{y}) = f_s((q_1(t), v_1), x_2, \ldots, x_s, \bar{y}), \qquad 0 \leqq t \leqq \tau_1', \tag{5.17}$$

(we remark that $\tau_1' > 0$ when $i < s-2$ and $\tau_1' \geqq 0$ when $i = s-2$). If $|q_1(\tau_1') - q_2| = d_1$ then $((q_1(\tau_1'), v_1), x_2, \ldots, x_s, \bar{y})$ is a boundary configuration, and (5.4) follows from (5.17) and condition (1) of the Basic Lemma. If $|q_1(\tau_1') - q_2| < d_1$, then $(x_s, \ldots, x_2, (q_1(\tau_1'), v_1), \bar{y})$ satisfies the assumptions of

Proposition 5.4 which in view of (5.17) and the summetry of f_s also implies (5.4). □

Proposition 5.9. *Let* $\nu \geqq 2$, *let* $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}$, *and assume that the following condition holds: there exists a supporting hyperplane,* L, *of* $\operatorname{Conv}(\bar{q}(x_1 \cup \ldots \cup x_s \cup \bar{y}))$ *passing through* q_1 and such that the straight line passing through q_1 *and orthogonal to* L *has no intersection with* $\operatorname{Conv}(\bar{q}(\bar{y}))$. *Then* (5.4) *holds.*

Proof. Let $\mathscr{D}_i$, $0 \leqq i \leqq s-1$, be the set of configurations $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}$ satisfying the following condition: there exists a supporting hyperplane, L, of $\operatorname{Conv}(\bar{q}(x_1 \cup \ldots \cup x_s \cup \bar{y}))$ passing through q_1 such that q_j for $j=1, \ldots, s-i$ is an extreme point of $\operatorname{Conv}((\bar{q}(x_1 \cup \ldots \cup x_j \cup \bar{y}) \cup l)$, where l is the exterior normal of L passing through q_1. One can easily check that $\mathscr{D}_0 \subset \mathscr{D}_1 \subset \ldots \subset \mathscr{D}_{s-1}$ and that $\mathscr{D}_0$, $\mathscr{D}_{s-1}$ coincide with the sets of configurations satisfying the assumptions of Propositions 5.8 and 5.9 respectively.

We shall now show by induction that $f_s \equiv 0$ on $\mathscr{D}_i$, $0 \leqq i \leqq s-1$. For $i=0$ this follows from Proposition 5.8. Suppose now that $f_s \equiv 0$ on $\mathscr{D}_i$ for some i, $0 \leqq i \leqq s-2$. Just as in the proof of Proposition 5.8 one can obtain the equality

$$U'(|q_1-q_2|)\,|q_1-q_2|^{-1}\,\partial^2_{v_2,v_s} f_{s-1}(x_2, \ldots, x_s, \bar{y})(q_1-q_2)=0,$$
$$(x_1, \ldots, x_s, \bar{y}) \in \mathscr{D}_i, \tag{5.18}$$

which implies (5.11) provided that $U'(|q_1-q_2|) \neq 0$. If, however, $U'(|q_1-q_2|)=0$, one could find a point, $q \in l$, such that $U'(|q-q_2|) \neq 0$. Since $((q, v_1), x_2, \ldots, x_s, \bar{y}) \in \mathscr{D}_i$ and $\mathscr{D}_i$ is an open set, we have $((q', v_1), x_2, \ldots, x_s, \bar{y}) \in \mathscr{D}_i$ for every q' from a sufficiently small neighbourhood of q. Replacing q_1 in (5.18) by q' we get (5.11) and then (5.12). The subsequent arguments do not differ from those involved in the proof of Proposition 5.8.

Proposition 5.10. *Let* $\nu \geqq 2$ *and let* $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}$. *Then* (5.4) *holds.*

Proof. Let $\mathscr{E}_0$ be the set of configurations satisfying the conditions of Proposition 5.9 and let $\mathscr{E}_i$, $i=1,2,\ldots$, be the set of configurations $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}$ satisfying the following condition: there exists a supporting hyperplane, L, of $\operatorname{Conv}(\bar{q}(x_1 \cup \ldots \cup x_s \cup \bar{y}))$ passing through q_1 and such that on the exterior normal l of L passing through q_1 there are points, $q_0, \ldots, q_{-i+1}$, such that (a) $U'(|q_j-q_{j+1}|) \neq 0, j=0, \ldots, -i+1$; (b) when $x'_j=(q_j, v'_j)$, $v'_j \in R^\nu$ for $j \leqq 0$ and when $x'_j=x_j$ for $j>0$, $(x'_{j+1}, \ldots, x'_{j+s}, \bar{y}) \in \mathscr{I}\mathscr{A}$ for $j=-1, \ldots, -i$, and $(x'_{-i+1}, \ldots, x'_{-i+s}, \bar{y})$ satisfies the assumptions of Proposition 5.9.

We show by induction that $f_s \equiv 0$ on $\mathscr{E}_i$ for every i. For $i=0$ this follows from Proposition 5.9. Adopting the same procedure as that used in the proof of Propositions 5.8 and 5.9 we can go from i to $i+1$. Finally we note that $\mathscr{I}\mathscr{A}=\bigcup_i \mathscr{E}_i$. This completes the proof of Proposition 5.10 and also the proof of the Basic Lemma for $\nu \geqq 2$.

In the case $\nu=1$ the following two propositions play the role of Propositions 5.7 and 5.8 above.

Proposition 5.11. *Let* $\nu=1$, *let* $(x_1, \ldots, x_s, \bar{y}) \in \bigcap_{i=1}^{s-1} \mathscr{I}\mathscr{A}^{(i)}$, *and assume that* dist $(q_s, \bar{q}(\bar{y}))>3d_1$. *Then* (5.4) *holds.*

Proposition 5.12. *Let* $v=1$, *let* $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}$, *and assume that* $\operatorname{dist}(q_s, \bar{q}(\bar{y})) > 3d_1$. *Then* (5.4) *holds.*

Proposition 5.11 can be proved as was Proposition 5.7. However the inequality $\operatorname{dist}(q_s, \bar{q}(\bar{y})) > 3d_1$ replaces the condition that q_s is an external point in $x_1, \cup \ldots \cup x_s \cup \bar{y}$. Moreover, the fact that for t, $t' > 0$ the point $q_1(t) = q_1 + t \operatorname{sign}(q_1 - q_2)$ is external in $(q_1(t), v_1) \cup x_2 \cup \ldots \cup x_{s-1} \cup (q_s(t'), v_s)$, where $q_s(t') = q_s + t' \operatorname{sign}(q_s - q_{s-1})$, allows the arguments to be simplified.

The proof of Proposition 5.12 is similar to that of Proposition 5.8.

The following proposition completes the proof of the Basic Lemma for $v=1$.

Proposition 5.13. *Let* $v=1$ *and* $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}$. *Then* (5.4) *holds.*

Proof. Let $\mathscr{F}_0$ be the configurations satisfying the assumptions of Proposition 5.12 and let $\mathscr{F}_i$, $i=1, 2, \ldots$, be the configurations $(x_1, \ldots, x_s, \bar{y}) \in \mathscr{I}\mathscr{A}$ for which there exist $q_0, \ldots, q_{-i+1} \in R^1$ such that (a) $U'(|q_j - q_{j+1}|) \neq 0$, $-i+1 \leqq j \leqq 0$; (b) when $x'_j = (q_j, v'_j)$, $v'_j \in R^1$, for $j \leqq 0$ and when $x'_j = x_j$ for $j > 0$, $(x'_{j+1}, \ldots, x'_{j+s}, \bar{y}) \in \mathscr{I}\mathscr{A}$, $-i \leqq j \leqq -1$, and the $(x'_{-i+1}, \ldots, x'_{-i+s}, \bar{y})$ satisfies the assumptions of Proposition 5.12. By the latter proposition $f_s \equiv 0$ on $\mathscr{F}_0$. We go from $\mathscr{F}_i$ to $\mathscr{F}_{i+1}$ using the method adopted in the proof of Proposition 5.10. It remains to note that $\mathscr{I}\mathscr{A} = \bigcup_{i=1}^{4} \mathscr{F}_i$. □

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