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A Boson Representation for SU (N) Lattice Gauge Theories

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Abstract. SU(N) lattice gauge theories are reformulated in terms of fields varying over non-compact spaces $\mathbb{C}^N$, transforming as N dimensional representations of SU(N) and integrated with Gaussian measure. This reformulation is equivalent to a boson operator representation. Strong coupling expansions based on this formalism do not involve SU(N) vector coupling coefficients.

1. Introduction

In pure Euclidean Yang–Mills field theories on a lattice field, variables range over the group manifold itself. This manifold is compact and a non-trivial Riemannian space. The gauge groups we will consider are SU(N), $N=2,3$ but our results can be immediately generalized to any N. In this article we reformulate such theories in an equivalent fashion in terms of fields taken from the flat non-compact space $\mathbb{C}^N$. They transform as N dimensional representations of SU(N). We will therefore call them "bosonic spinorial variables" for the gauge field. The integration is over a Gaussian measure instead of a Haar measure. A straightforward change of notation leads then to a boson operator formulation of Yang–Mills lattice field theories.

Our approach is based on Bargmann's realization of group representations of SU(N) [1], which makes use of Hilbert spaces of entire analytic functions over $\mathbb{C}^N$ or powers of $\mathbb{C}^N$. This formalism is equivalent to the so-called boson operator calculus [2]. For technical reasons and for the sake of mathematical clarity we prefer to use spaces of analytic functions in this article.

The lattice Λ is assumed to be hypercubic, to have dimension D and the boundary conditions are presumed to be periodic. Let ℓ denote the links and p the plaquettes of Λ. We define the partition function by the standard ansatz

$$Z = \int \prod_{\substack{\text{links}\\ \ell \in \Lambda}} (du_\ell) e^{S(\{u_l\})}.$$

$$u_\ell \in \mathrm{SU}(N) \tag{1}$$

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The action S can be represented as a sum of contributions S_p of each plaquette p

$$S(\{u_\ell\}) = \sum_{\substack{\text{plaquettes} \\ p \in \Lambda}} S_p. \tag{2}$$

We assume that each S_p can be expanded into characters

$$e^{S_p} = \sum_{\{R\}} \dim R \chi^R(u_{\partial p}) f_R(\beta). \tag{3}$$

Here $\{R\}$ is the set of irreducible unitary representations of SU(N), χ^R are the corresponding characters. β is the inverse temperature or the inverse coupling constant squared and $f_R(\beta)$ are "dynamical factors" that specify the action. The Wilson action [3] or the generalized Villain action [4] are included as special cases. The argument $u_{\partial p} \in \mathrm{SU}(N)$ is the usual product of group elements u_ℓ along the boundary ∂p of p.

For the purpose of generality we use (3) as a starting point. After the introduction of bosonic spinorial variables all integrations over gauge group variables can be performed exactly. If we are able to sum over R for certain given functions $f_R(\beta)$, we can study the reformulated Yang–Mills theory both for $\beta \to 0$ (the strong coupling limit) as for $\beta \to \infty$ (the weak coupling limit). Without this summation over R the reformulated Yang–Mills theory can only be studied in the strong coupling domain. Since all integrations in the strong coupling expansion are now Gaussian, they are elementary. Vector coupling coefficients of SU(N) do not arise. This is of particular interest, since for $N \geqq 3$ analytic expressions for vector coupling coefficients are not known.

Of course one cannot expect that a complicated invariant contraction of SU(N) vector coupling coefficients can be replaced by an elementary integral, but it is certainly possible to replace it by several or many such integrals. We can only hope that up to a certain order the number of terms generated is small enough to be listed up by a computer. We can say that our result combines the algebra of the group with the combinatorics of the lattice and admits a unified graphical approach to strong coupling expansions.

The method of integrating over the group SU(N) can be extended to groups $U(N)$ as well. It can then be compared with the technique developed in [5] which does not yield vector coupling coefficients of $U(N)$ either. Our method sums up contributions to one irreducible representation which leads to a considerable reduction of the number of terms.

We introduce Bargmann spaces for $SU(2)$ and $SU(3)$ in Sect. 2. Whereas each representation of SU(2) is self-conjugate, representations of SU(3) occur in conjugate pairs in general. Some relevant properties of the conjugation matrix are derived in Sect. 3. Using the delta function kernels for the Bargmann spaces, we integrate tensor products of representation operators T^R_u over the group SU(N) in Sect. 3. As a final tool we derive the projection kernels that allow us to contract the operators $T^R_{u_l}$ to characters $\chi^R(u_{\partial p})$ in Sect. 5. Section 6 is devoted to some miscellaneous remarks on strong coupling expansions and their generating functions.

Notations are the same as in [6], those for links and plaquettes of the lattice

Λ and the cells of the dual lattice Λ^* are again compiled in an Appendix.

2. The Bargmann Spaces

The Bargmann space for SU(2) [1] is a Hilbert space $H_\mu(\mathbb{C}^2)$ of entire analytic functions over $\mathbb{C}^2$ with a Gaussian measure μ

$$z=\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\in\mathbb{C}^2, z_i=x_i+iy_i, i=1,2, \tag{4}$$

$$d\mu_2(z)=\pi^{-2}\prod_{i=1,2}[dx_i dy_i e^{-x_i^2-y_i^2}], \tag{5}$$

and a scalar product

$$(f,g)_\mu=\int d\mu_2(z)\overline{f(z)}g(z). \tag{6}$$

We define a unitary representation of SU(2) in $H_\mu(\mathbb{C}^2)$ by the definition

$$T_u f(z)=f(u^T z), u\in \mathrm{SU}(2). \tag{7}$$

The irreducible unitary representation $R=j, j\in\frac{1}{2}\mathbb{Z}_+$ of SU(2) is carried by a subspace Q_j of $H_\mu(\mathbb{C}^2)$ consisting of analytic homogeneous polynomials of degree $2j$. The standard basis in Q_j consists of the polynomials

$$v_m^j(z)=\frac{z_1^{j+m}z_2^{j-m}}{[(j+m)!(j-m)!]^{1/2}}, \quad -j\leqq m\leqq j. \tag{8}$$

The projection $f\to f|Q_j$ of any vector onto its component in Q_j is established by a kernel

$$f|_{Q_j}(z)=\int d\mu_2(z')Q^j(z,z')f(z') \tag{9}$$

with

$$Q^j(z,z')=\sum_{m=-j}^{+j} v_m^j(z)\overline{v_m^j(z')}$$
$$=\frac{1}{(2j)!}(z\cdot\bar{z}')^{2j}. \tag{10}$$

Here we denote

$$z\cdot z' = \sum_{i=1,2} z_i z_i'. \tag{11}$$

The linear functional δ_z

$$f(z)=(\bar{\delta}_z, f)_\mu \tag{12}$$

is bounded and can consequently be given as a kernel

$$\delta_z(z')=\exp(z\cdot z'). \tag{13}$$

This is the basic property of the Bargmann spaces which will be exploited in this article. We obtain immediately

$$\delta_z(z') = \sum_{j=0}^{\infty} Q^j(z,z') \tag{14}$$

so that $H_\mu(\mathbb{C}^2)$ decomposes into the orthogonal direct sum

$$H_\mu(\mathbb{C}^2) = \sum_{j=0}^{\infty} \oplus Q_j \tag{15}$$

The Bargmann space for SU(3) is a similar space $H_\mu(\mathbb{C}^6)$ of entire analytic functions over $\mathbb{C}^6$ with a Gaussian measure μ

$$(z, z') \in \mathbb{C}^6, \ z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}, \ z' = \begin{pmatrix} z'_1 \\ z'_2 \\ z'_3 \end{pmatrix}, \ z_i = x_i + iy_i, \quad i = 1,2,3 \tag{16}$$

and

$$\mathrm{d}\mu_6(z,z') = \pi^{-6} \prod_{i=1,2,3} [dx_i dy_i e^{-x_i^2 - y_i^2} dx'_i dy'_i e^{-x_i'^2 - y_i'^2}] \tag{17}$$

and a scalar product similar to (6). A unitary representation of SU(3) is defined in $H_\mu(\mathbb{C}^6)$ analogously to (7).

Define

$$w_i(z,z') = \sum_{j,k=1}^{3} \varepsilon_{ijk} z_j z'_k. \tag{18}$$

Then all homogeneous polynomials of the type

$$v^{\lambda,\mu}_{\{\lambda_i,\mu_i\}}(z, w(z,z')) = \prod_{i=1}^{3} [z_i^{\lambda_i} w_i(z,z')^{\mu_i}],$$
$$\sum_{i=1}^{3} \lambda_i = \lambda, \ \sum_{i=1}^{3} \mu_i = \mu, \ \lambda, \mu \in \mathbb{Z}_+ \tag{19}$$

span a subspace $Q_{(\lambda,\mu)}$ of $H_\mu(\mathbb{C}^6)$ that carries the irreducible unitary representation $R = (\lambda, \mu)$ of SU(3) [7].

The set of functions (19) is linearly dependent due to

$$\sum_{i=1}^{3} z_i w_i(z,z') = 0. \tag{20}$$

An orthonormal basis system can be extracted from it by the method of Gelfand and Zetlin [8], [9]. In this work we rely only on the existence of such a basis.

Denote a state of the Gelfand–Zetlin basis for the representation R by

$$v^R_\alpha(z, w(z,z')).$$

We may then introduce the kernel for the projection operator onto the subspace Q_R by

$$Q^R(\zeta, w(\zeta,\zeta'); z, w(z,z')) = \sum_{\alpha} v^R_\alpha(\zeta, w(\zeta,\zeta'))\overline{v^R_\alpha(z, w(z,z'))} \tag{21}$$

However, we are now forced to compute this kernel by other arguments. Homogeneity and invariance suggest the ansatz $[R=(\lambda,\mu)]$

$$Q^R(\zeta, w(\zeta,\zeta'); z, w(z,z')) = N^R(\zeta\cdot\bar{z})^\lambda(w(\zeta,\zeta')\cdot\overline{w(z,z')})^\mu. \tag{22}$$

The normalization constant results from arguments presented below

$$N^R = \frac{\lambda+1}{\mu!(\lambda+\mu+1)!}. \tag{23}$$

If we exchange the rôle of z, z' in (21) we obtain a projection operator on a different subspace (if $\lambda \neq 0$) of $H_\mu(\mathbb{C}^6)$. In fact the space $H_\mu(\mathbb{C}^6)$ contains the representation "3" of SU(3) (i.e., $\lambda=1, \mu=0$) twice and "$\bar{3}$" (i.e., $\lambda=0, \mu=1$) only once. The sum of Q^R over R cannot yield the δ kernel of $H_\mu(\mathbb{C}^6)$, which by direct generalization of (13) is

$$\delta_{\zeta,\zeta'}(z,z') = \exp(\zeta\cdot\bar{z} + \zeta'\cdot\bar{z}'). \tag{24}$$

Instead we consider the "generating function" for the kernels Q^R

$$\exp(\zeta\cdot\bar{z} + w(\zeta,\zeta')\cdot\overline{w(z,z')}) = \sum_{\lambda,\mu=0}^{\infty} \frac{(\lambda+\mu+1)!}{(\lambda+1)!} Q^{(\lambda,\mu)}(\zeta, w(\zeta,\zeta'); z, w(z,z')). \tag{25}$$

We multiply one such function by a conjugate one and integrate

$$\int d\mu_6(\zeta,\zeta')\exp\{\zeta\cdot\bar{z} + w(\zeta,\zeta')\cdot\bar{w}\}\exp\{\bar{\zeta}\cdot z_0 + \overline{w(\zeta,\zeta')}\cdot w_0\} = (1-w_0\cdot w)^{-2}\exp\frac{z_0\cdot\bar{z} - (z_0\cdot w_0)(\bar{z}\cdot\bar{w})}{1-w_0\cdot\bar{w}}. \tag{26}$$

This integral converges whenever $w_0, \bar{w}$ are small enough. Both z, w and z_0, w_0 have been considered as independent variables.

If we set

$$w_0 = w(z_0, z_0'), \quad \text{or} \quad w = w(z,z'),$$

then the second term in the exponent vanishes. Assume that w_0 is expressed in this fashion but that w is independent of z. In this case we expand both sides of (26) in powers and obtain by homogeneity arguments

$$[\lambda!\mu!\lambda'!\mu'!]^{-1}\int d\mu_6(\zeta,\zeta')(\zeta\cdot\bar{z})^\lambda(w(\zeta,\zeta')\cdot\bar{w})^\mu(\bar{\zeta}\cdot z_0)^{\lambda'}(\overline{w(\zeta,\zeta')}\cdot w_0)^{\mu'} = \delta_{\lambda\lambda'}\delta_{\mu\mu'}\frac{(2+\lambda)_\mu}{\lambda!\mu!}(z_0\cdot\bar{z})^\lambda(w_0\cdot\bar{w})^\mu. \tag{27}$$

Any polynomial $f(\zeta,\zeta')\in Q_{(\lambda,\mu)}$ can be obtained from

$$(\zeta\cdot\bar{z})^\lambda(w(\zeta,\zeta')\cdot\bar{w})^\mu$$

by fixing the values for z, w and taking linear combinations (z, w are independent!). Thus we obtain from (27)

$$\frac{\lambda+1}{\mu!(\lambda+\mu+1)!}\int d\mu_6(\zeta,\zeta')f(\zeta,\zeta')(\bar{\zeta}\cdot z)^\lambda(\overline{w(\zeta,\zeta')}\cdot w(z,z'))^\mu = f(z,z'). \tag{28}$$

This proves (22), (23).

It is a typical complication of SU(3) compared with SU(2) that the two functions (24) and (25) are different!

3. Conjugate Representations

If T_u is a representation then

$$T_u^c = T_{u^{-1,T}} \tag{29}$$

is called the "conjugate" representation. For SU(2) we have

$$\begin{aligned} u^{-1,T} &= \varepsilon u \varepsilon^{-1} \\ \varepsilon &= i\sigma_2 \in \mathrm{SU}(2) \end{aligned} \tag{30}$$

so that both representations are equivalent. In the basis (8) we have in fact

$$D^j_{mm'}(u) = (v^j_m, T_u v^j_{m'})_\mu \tag{31}$$

$$D^j_{mm'}(u^{-1,T}) = (-1)^{j-m+j-m'} D^j_{-m,-m'}(u). \tag{32}$$

In the case of SU(3) and relying on the basis (19) we define a substitution C

$$\begin{aligned} z_i &\underset{C}{\rightarrow} w_i(z,z') \\ w_i(z,z') &\underset{C}{\rightarrow} z_i \end{aligned} \tag{33}$$

and introduce the C matrix with respect to the Gelfand–Zetlin basis

$$v^R_\alpha(z, w(z,z')) \rightarrow v^R_\alpha(w(z,z'), z) = \sum_{\alpha'} v^{R^c}_{\alpha'}(z, w(z,z')) C^R_{\alpha'\alpha}. \tag{34}$$

By inspection of Eq. (19) we obtain

$$R^C = (\lambda^C, \mu^C), \lambda^C = \mu, \mu^C = \lambda. \tag{35}$$

By definition of the Gelfand–Zetlin basis [8], [9] we have reality of C^R

$$(C^R)^\dagger = (C^R)^T, \tag{36}$$

and by repeated application we get

$$C^{R^c} = (C^R)^{-1}. \tag{37}$$

If we introduce matrix elements

$$D^R_{\alpha\alpha'}(u) = (v^R_\alpha, T_u v^R_{\alpha'})_\mu, \tag{38}$$

we have in addition (in matrix notation)

$$C^R D^R(u^{-1,T}) = D^{R^c}(u) C^R. \tag{39}$$

Using

$$D^R_{\alpha\alpha'}(u) = D^R_{\alpha'\alpha}(u^T) \tag{40}$$

and applying (39) once for u and once for u^T we can argue by means of Schur's lemma that

$$(C^R)^{-1} = \gamma^R (C^R)^T. \tag{41}$$

The constant γ^R turns out below to be

$$\gamma^R = \frac{(\mu+1)!}{(\lambda+1)!}. \tag{42}$$

Besides the invariant sum over the basis of the space Q_R we have also to deal with invariant sums of the kind

$$\sum_{\alpha,\alpha'} v_{\alpha'}^{R^c}(z, w(z,z'))C_{\alpha'\alpha}^{R} v_{\alpha}^{R}(\zeta, w(\zeta,\zeta')) = \sum_{\alpha} v_{\alpha}^{R}(w(z,z'), z) v_{\alpha}^{R}(\zeta, w(\zeta,\zeta'))$$
$$= N^R(\zeta \cdot w(z,z'))^{\lambda}(w(\zeta,\zeta')\cdot z)^{\mu}. \tag{43}$$

On the other hand we may use (41) and get

$$= \sum_{\alpha,\alpha'} v_{\alpha'}^{R^c}(z, w(z,z')) v_{\alpha}^{R}(\zeta, w(\zeta,\zeta')) C_{\alpha\alpha'}^{R^c}(\gamma^R)^{-1}$$
$$= \sum_{\alpha'} v_{\alpha'}^{R^c}(z, w(z,z')) v_{\alpha'}^{R^c}(w(\zeta,\zeta'),\zeta)(\gamma^R)^{-1}$$
$$= \frac{N^{R^c}}{\gamma^R}(z\cdot w(\zeta,\zeta'))^{\lambda^c}(w(z,z')\cdot\zeta)^{\mu^c}. \tag{44}$$

Comparison of (44) and (43) yields

$$\gamma^R = \frac{N^{R^c}}{N^R}. \tag{45}$$

Inserting (23) gives (42).

4. Integration over the Group

From the introduction we know that for each link $\ell = (x,\rho)$, $\rho = 1, 2, \ldots, D$[1]), we have an integration

$$\int du_{(x,\rho)}$$

over the group. Each plaquette having this link on its boundary contributes a factor $D_{\alpha\alpha'}^{R}(u)$; more precisely, we have the integral

$$\int du_{(x,\rho)} \prod_{\sigma \neq \rho} D_{\alpha_\sigma \alpha'_\sigma}^{R(x,\rho,\sigma)}(u_{(x,\rho)}) \prod_{\tau \neq \rho} D_{\alpha_\tau \alpha'_\tau}^{R(x-\underline{\tau},\rho,\tau)}(u_{(x,\rho)}^{-1}). \tag{46}$$

In the second product we use (39) and (40) to transform the factors into

$$D_{\alpha'_\tau \alpha_\tau}^{R(x-\underline{\tau},\rho,\tau)}(u_{(x,\rho)}^{-1,T}) = (C^{R^c(x-\underline{\tau},\rho,\tau)} D^{R^c(x-\underline{\tau},\rho,\tau)}(u_{(x,\rho)}) C^{R(x-\underline{\tau},\rho,\tau)})_{\alpha'_\tau \alpha_\tau}. \tag{47}$$

We know already from [6] that for SU(2)

$$D_{mm'}^{j}(u) = (v_m^j, T_u v_{m'}^j)_\mu = \int d\mu_2(\zeta) d\mu_2(z) \overline{v_m^j(\zeta)} K_2(u;\zeta,z) v_{m'}^j(z) \tag{48}$$

with

$$K_2(u;\zeta,z) = \exp\{(u^T\zeta)\cdot\bar{z}\}. \qquad \zeta, z \in \mathbb{C}^2 \tag{49}$$

For SU(3) we have similarly

$$D_{\alpha\alpha'}^{R}(u) = (v_\alpha^R, T_u v_{\alpha'}^R)_\mu = \int d\mu_6(\zeta,\zeta') d\mu_6(z,z') \overline{v_\alpha^R(\zeta, w(\zeta,\zeta'))}$$
$$\cdot K_3(u;\zeta,z) K_3(u;\zeta',z') v_{\alpha'}^R(z, w(z,z')) \tag{50}$$

1 For the notations of links and plaquettes, see the Appendix

with

$$K_3(u;\zeta, z) = \exp\{(u^T\zeta)\cdot\bar{z}\},$$
$$\zeta, z \in \mathbb{C}^3. \tag{51}$$

Both kernels in the integrals (48) and (50) are obtained by operating with T_u on the left arguments of the delta functions (13) and (24), respectively.

Thus we are left with integrals of the type

$$L_{K,N}(\{\zeta(i)\}, \overline{\{z(i)\}}) = \int du \exp\left\{\sum_{i=1}^{K} (u^T\zeta(i))\cdot\overline{z(i)}\right\} \tag{52}$$

with $K = 2D - 2$ for SU(2) and $K = 2(2D - 2)$ for SU(3). For SU(2) the result was already reported in [6].

$$L_{K,2}(\{\zeta(i)\}, \overline{\{z(i)\}}) = \sum_{n=0}^{\infty} \frac{\Omega_2((\zeta(i)\}, \overline{\{z(i)\}})^n}{n!(n+1)!} \tag{53}$$

with

$$\Omega_2(\{\zeta(i)\}, \overline{\{z(i)\}}) = \tfrac{1}{2} \sum_{i,j=1}^{K} \begin{vmatrix} \zeta(i)\cdot\overline{z(i)}, & \zeta(i)\cdot\overline{z(j)} \\ \zeta(j)\cdot\overline{z(i)}, & \zeta(j)\cdot\overline{z(j)} \end{vmatrix}. \tag{54}$$

For SU(3) the result is similar

$$L_{K,3}(\{\zeta(i)\}, \overline{\{z(i)\}}) = 2\sum_{n=0}^{\infty} \frac{\Omega_3(\{\zeta(i)\}, \overline{\{z(i)\}})^n}{n!(n+1)!(n+2)!} \tag{55}$$

with

$$\Omega_3(\{\zeta(i)\}, \overline{\{z(i)\}}) = \tfrac{1}{6} \sum_{i,j,k=1}^{K} \begin{matrix} \zeta(i)\cdot z(i), & \zeta(i)\cdot z(j), & \zeta(i)\cdot z(k) \\ \zeta(j)\cdot z(i), & \zeta(j)\cdot z(j), & \zeta(j)\cdot z(k) \\ \zeta(k)\cdot z(i), & \zeta(k)\cdot z(j), & \zeta(k)\cdot z(k) \end{matrix} \tag{56}$$

The expression for general SU(N) can easily be guessed at, the first term in the series for $L_{K,N}$ is always one.

The result for SU(2), (53), (54) has been derived by explicitly performing the integrations [6]. For SU(3) the integration proceeds as follows. Each matrix $u \in$ SU(3) up to a set of Haar measure zero can be decomposed into

$$u = u_\xi v, \tag{57}$$

$$v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{SU}(2) \tag{58}$$

so that u_ξ boosts the vector

$$\eta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{C}^3 \tag{59}$$

into

$$u_\xi \eta = \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \in \mathbb{C}^3, \sum_{i=1}^{3} |\xi_i|^2 = 1, \tag{60}$$

implying $\xi \in \mathcal{S}^5$. It is easy to see that the Haar measure on SU(3) is the product of the Haar measure on SU(2) with the normalized uniform measure on $\mathcal{S}^5$

$$du = d\omega(\xi) dv, \tag{61}$$

where

$$dv = \frac{1}{\pi^2} d^2\alpha d^2\beta \delta(|\alpha|^2 + |\beta|^2 - 1) \tag{62}$$

and

$$d\omega(\xi) = \frac{2}{\pi^3} \mathrm{d}^2\xi_1 \mathrm{d}^2\xi_2 \mathrm{d}^2\xi_3 \delta\left(\sum_{i=1}^{3} |\xi_i|^2 - 1 \right). \tag{63}$$

Denote

$$u_\xi = \begin{pmatrix} \xi_1 b_1 c_1 \\ \xi_2 b_2 c_2 \\ \xi_3 b_3 c_3 \end{pmatrix} \in \mathrm{SU}(3), \tag{64}$$

where b_i, c_i are functions of ξ but not of v due to

$$\bar{\xi}_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} b_j c_k. \tag{65}$$

Now integrate first over v, using the known result for SU(2). Ω_2 (54) is obtained in the form [using (65)]

$$\Omega_2 = A \cdot \bar{\xi}. \tag{66}$$

In addition we are left with exponential factors

$$\exp(B \cdot \xi).$$

Both A and B are certain functions of $\zeta(i), \overline{z(i)}$. Integration over the measure (63) yields

$$2 \sum_{n=0}^{\infty} \frac{(A \cdot B)^n}{n!(n+1)!(n+2)!}. \tag{67}$$

This method of integration can be used for an inductive derivation of the integral $L_{K,N}$ for arbitrary SU(N).

5. Evaluation of Traces

Though we could well use the original lattice we prefer to rely on the dual lattice Λ^* to formulate the evaluation of traces. It simplifies our language. The combinatorics of these contractions has been disentangled in [6]. We follow the procedure developed in that article. In the figure we project a D cell of Λ^* on the $\rho\sigma$ coordinate

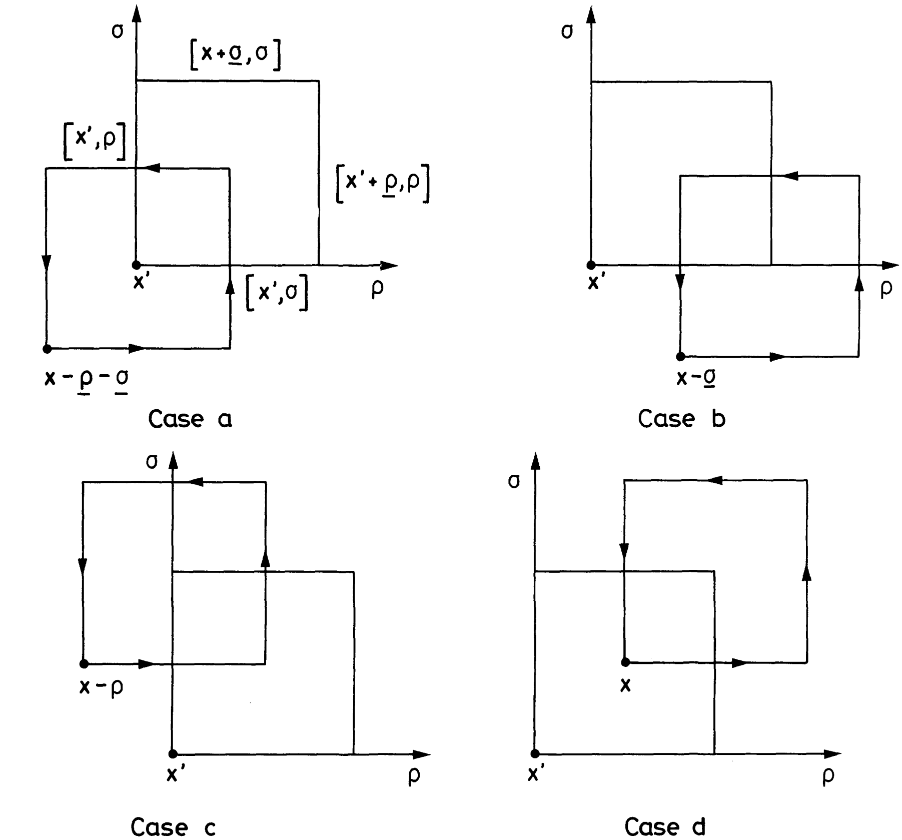

Fig. 1. The $[x']$ D cell projected onto the $\rho\sigma$ plane. $(D-1)$ cells appear as lines, $(D-2)$ cells as points. The plaquettes of Λ encircling the $(D–2)$ cells are drawn

plane and draw the plaquettes in the $\rho\sigma$ plane of Λ along which the traces have to be performed. Each plaquette encircles a $(D-2)$ cell of Λ^* which, in the figure, appears as a point.

Consider a $(D-1)$ cell $[x', \rho]$ of Λ^*. On its boundary it has the $(D-2)$ cells $[x', \rho, \sigma]$, $\sigma \neq \rho$, and $[x'+\underline{\sigma}, \rho, \sigma]$, $\sigma \neq \rho$. To each $(D-1)$ cell belongs one "link kernel" $L_{K,N}$, $K=(N-1)(2D-2)$, (52) [for SU(N)]. For each boundary $(D-2)$ cell of $[x', \rho]$ $L_{K,N}$ depends on an $(N-1)$-plet of pairs of variables ζ, z. We label these variables by

$$(\zeta^{[x',\rho,\sigma]}_{[x',\rho]}, z^{[x',\rho,\sigma]}_{[x',\rho]}), (\zeta'^{[x',\rho,\sigma]}_{[x',\rho]}, z'^{[x',\rho,\sigma]}_{[x',\rho]}), \ldots$$

and

$$(\zeta^{[x'+\underline{\sigma},\rho,\sigma]}_{[x',\rho]}, z^{[x'+\underline{\sigma},\rho,\sigma]}_{[x',\rho]}), \ldots .$$

Thus $L_{K,N}$ depends on $4(N-1)(D-1)$ arguments ranging over $\mathbb{C}^N$. We may graphically represent these variables by pairs of bundles of $N-1$ lines each, one a z bundle and the other a ζ bundle, connecting the centre of the $(D-1)$ cell with the centre of any of its $(D-2)$ cells on the boundary.

A $(D-2)$ cell is common to the boundaries of four $(D-1)$ cells. Thus four pairs of bundles meet on one $(D-2)$ cell. Evaluating the traces connects these bundles in a certain fashion. We study first the case of SU(2) where each bundle consists of a single line. We abbreviate the label of a $(D-2)$ cell by p (as for a plaquette).

case a. $p = [x', \rho, \sigma]$

$$\sum_m D^{j_p}_{.,m}(u_{[x',\sigma]}) D^{j_p}_{m,.}(u^{-1}_{[x',\rho]}) \Rightarrow \sum_m (-1)^{j_p - m} v^{j_p}_m(z(1)) v^{j_p}_{-m}(z(2))$$

$$= Q^{j_p}(z(1), \overline{w(z(2))}), z(1) = z^p_{[x',\sigma]},\ z(2) = z^p_{[x',\rho]} \tag{68}$$

with

$$w(z)_i = \sum_j \varepsilon_{ij} z_j, \varepsilon_{12} = +1. \tag{69}$$

Case b. $p = [x' + \underline{\rho}, \rho, \sigma]$

$$\sum_m D^{j_p}_{.,m}(u^{-1}_{[x'+\underline{\rho},\rho]}) D^{j_p}_{m,.}(u^{-1}_{[x',\sigma]}) \Rightarrow \sum_m \overline{v^{j_p}_m(\zeta(1))} v^{j_p}_m(z(2))$$

$$= Q^{j_p}(z(2), \zeta(1)), z(2) = z^p_{[x',\sigma]},\ \zeta(1) = \zeta^p_{[x'+\underline{\rho},\rho]}. \tag{70}$$

Case c. $p = [x' + \underline{\sigma}, \rho, \sigma]$

$$\sum_m D^{j_p}_{.,m}(u_{[x',\rho]}) D^{j_p}_{m,.}(u_{[x'+\underline{\sigma},\sigma]}) \Rightarrow \sum_m v^{j_p}_m(z(1)) v^{j_p}_m(\zeta(2)) = Q^{j_p}(z(1), \zeta(2)),$$

$$z(1) = z^p_{[x',\rho]},\quad \zeta(2) = \zeta^p_{[x'+\underline{\sigma},\sigma]}. \tag{71}$$

Case d. $p = [x' + \underline{\rho} + \underline{\sigma}, \rho, \sigma]$

$$\sum_m D^{j_p}_{.,m}(u^{-1}_{[x'+\underline{\sigma},\sigma]}) D^{j_p}_{m,.}(u_{[x'+\underline{\rho},\rho]}) \Rightarrow \sum_m (-1)^{j_p - m} \overline{v^{j_p}_{-m}(\zeta(1))}\ \overline{v^{j_p}_m(\zeta(2))}$$

$$= Q^{j_p}(\overline{w(\zeta(1))}, \zeta(2)), \zeta(1) = \zeta^p_{[x'+\underline{\sigma},\sigma]},$$

$$\zeta(2) = \zeta^p_{[x'+\underline{\rho},\rho]}. \tag{72}$$

Thus at one $(D-2)$ cell $p = [x', \rho, \sigma]$ we have contractions with the following "plaquette kernel" $P^{j_p}_N$:

$$P^{j_p}_2(\{\zeta^p_{\ell(p)}\}, \{z^p_{\ell(p)}\}) = Q^{j_p}(z^p_{[x',\sigma]}, \overline{w(z^p_{[x',\rho]})}) Q^{j_p}(z^p_{[x'-\underline{\rho},\sigma]}, \zeta^p_{[x',\rho]})$$
$$\cdot Q^{j_p}(z^p_{[x'-\underline{\sigma},\rho]}, \zeta^p_{[x',\sigma]}) Q^{j_p}(\overline{w(\zeta^p_{[x'-\underline{\rho},\sigma]})}, \zeta^p_{[x'-\underline{\sigma},\rho]}). \tag{73}$$

In the case of SU(3) the situation is very similar and we obtain the plaquette kernel $P^{R_p}_3$:

$$P^{R_p}_3(\{\zeta^p_{\ell(p)}\}, \{z^p_{\ell(p)}\}) = Q^{R_p}(z^p_{[x',\sigma]}, w(z^p_{[x',\sigma]}, z'^p_{[x',\sigma]}); \overline{w(z^p_{[x',\rho]}, z'^p_{[x',\rho]}), z^p_{[x',\rho]}})$$
$$\cdot Q^{R^c_p}(z^p_{[x'-\underline{\rho},\sigma]}, w(z^p_{[x'-\underline{\rho},\sigma]}, z'^p_{[x'-\underline{\rho},\sigma]}); \zeta^p_{[x',\rho]}, w(\zeta^p_{[x',\rho]}, \zeta'^p_{[x',\rho]}))$$
$$\cdot Q^{R_p}(z^p_{[x'-\underline{\sigma},\rho]}, w(z^p_{[x'-\underline{\sigma},\rho]}, z'^p_{[x'-\underline{\sigma},\rho]}); \zeta^p_{[x',\sigma]}, w(\zeta^p_{[x',\sigma]}, \zeta'^p_{[x',\sigma]}))$$

$$\cdot Q^{R^c_p}(\overline{w(\zeta^p_{[x'-\underline{\sigma},\rho]}, \zeta'^p_{[x'-\underline{\sigma},\rho]})}, \overline{\zeta^p_{[x'-\underline{\sigma},\rho]}}; \\ \zeta^p_{[x'-\underline{\rho},\sigma]}, w(\zeta^p_{[x'-\underline{\rho},\sigma]}, \zeta'^p_{[x'-\underline{\rho},\sigma]})) \tag{74}$$

With the reduction obtained so far we can write the partition function as

$$Z = \int \prod_{p,\ell(p)} \{d\mu_6(z^p_{\ell(p)}) d\mu_6(\zeta^p_{\ell(p)})\} \\ \cdot \prod_p \left\{ \sum_{R_p} \dim R_p f_{R_p}(\beta) P_3^{R_p}(\{\zeta^p_{\ell(p)}\}, \{z^p_{\ell(p)}\}) \right\} \\ \cdot \prod_\ell L_{2(2D-2),3}(\{\zeta^{p(\ell)}_\ell\}, \{z^{p(\ell)}_\ell\}) \tag{75}$$

for SU(3), say.

The gauge invariance of the original Yang–Mills theory has been fully preserved. In other words, the number of superfluous field variables has not been reduced. In fact, a gauge transformation

$$u_{(x,\rho)} \to u^{-1}_{(x)} u_{(x,\rho)} u_{(x+\underline{\rho})} \tag{76}$$

entails

$$z^p_{[x',\rho]} \to u_{[x']} z^p_{[x',\rho]} \\ \zeta^p_{[x',\rho]} \to u_{[x'-\underline{\rho}]} \zeta^p_{[x',\rho]} \tag{77}$$

and similarly for z', ζ', etc. Both the plaquette kernels and the link kernels are invariant under the gauge transformation (77).

6. Remarks

In the case of SU(2) we can perform half of the integrations in (75) by means of an auxiliary integration as follows. We make use of

$$Q^j(\zeta, z) = \frac{1}{2\pi i} \int_{0+} \frac{d\tau}{\tau^{2j+1}} \exp(\tau \zeta \cdot \bar{z}). \tag{78}$$

This leads to the replacement of

$$P_2^{j_p}(\{\zeta^p_{\ell(p)}\}, \{z^p_{\ell(p)}\}) = (2\pi i)^{-4} \iiiint_{0+} \prod_{i=1}^4 \left\{ \frac{d\tau^p_i}{(\tau^p_i)^{2j_p+1}} \right\} \cdot \exp\{\tau^p_1 z^p_{[x',\sigma]} \\ \cdot w(z^p_{[x,\rho]}) + \tau^p_2 z^p_{[x'-\rho,\sigma]} \cdot \overline{\zeta^p_{[x',\rho]}} + \tau^p_3 z^p_{[x'-\underline{\sigma},\rho]} \cdot \overline{\zeta^p_{[x',\sigma]}} \\ + \tau^p_4 \overline{w(\zeta^p_{[x'-\underline{\rho},\underline{\sigma}]})} \cdot \overline{\zeta^p_{[x'-\underline{\sigma},\rho]}} \}. \tag{79}$$

The exponentials are delta functions [see Eq. (13)]. The integration over these delta functions introduces ζ into the arguments of the link kernels. It can easily be shown that the integration over appropriately chosen remaining variables ζ, z converges absolutely for any finite lattice whenever

$$|\tau^p_i| \leqq \varepsilon \text{ for some } \varepsilon > 0 \text{ and all } p, i. \tag{80}$$

There results then an analytic function $G(\{\tau\})$ which is holomorphic around zero and whose Taylor expansion yields the strong coupling expansion of Z. This "generating function" $G(\{\tau\})$ was first introduced in [6].

In order to study the weak coupling domain of SU(2) Yang–Mills theories one can sum the character expansion. Denote

$$\tau^p = \prod_{i=1}^{4} \tau_i^p, \tag{81}$$

and sum

$$\theta(\tau, \beta) = \sum_{j=0}^{\infty} (2j+1)\tau^{-2j} f_j(\beta). \tag{82}$$

For standard forms of actions such as Wilson's [3] or the generalized Villian action [4], $\theta(\tau, \beta)$ is entirely analytic in τ^{-1}. We may modify the definition (82) by adding irrelevant holomorphic functions that vanish at $\tau = 0$, e.g.,

$$\tilde{\theta}(\tau, \beta) = \sum_{j=0}^{\infty} (2j+1)(\tau^{-2j} - \tau^{2j+2}) f_j(\beta). \tag{83}$$

For the Wilson action this sum yields

$$\tilde{\theta}_W(\tau, \beta) = (1 - \tau^2) e^{(1/2)\beta(\tau + \tau^{-1})}. \tag{84}$$

In that case we obtain

$$Z = \int \prod_{p \in \Lambda^*} \left\{ \prod_{i=1}^{4} \left[\frac{d\tau_i^p}{2\pi i \tau_i^p} \right] \tilde{\theta}_W(\tau^p, \beta) \right\} G(\{\tau\}) \tag{85}$$

G also determines the dynamics in the weak coupling domain.

For SU(3) the kernel (25)

$$\exp\left[\tau \zeta \cdot \bar{z} + \omega w(\zeta, \zeta') \cdot \overline{w(z, z')}\right] \tag{86}$$

can be used for an auxiliary integration still analogous to (79). However, this does not render the integrations trivial, it increases the polynomial order of Ω_3 by the integration, and analyticity of the generating function $G(\{\tau, \omega\})$ is doubtful. Nevertheless $G(\{\tau, \omega\})$ still exists as a formal power series connected with the strong coupling expansion.

Finally we mention how the transition to the boson operator formalism is achieved from (75). This can be formulated rigorously for the strong coupling expansion for which each order is a polynomial expression in the variables ζ_ℓ^p, z_ℓ^p. In each polynomial expression let all holomorphic factors appear to the right of all antiholomorphic factors. Then replace

$$\zeta_\ell^p, z_\ell^p \quad \text{by} \quad b_\ell^{p+}, a_\ell^{p+}$$
$$\bar{\zeta}_\ell^p, \bar{z}_\ell^p \quad \text{by} \quad b_\ell^p, a_\ell^p,$$

each of which is an N-plet of bosonic creation or annihilation operators. Finally replace the integration by taking the standard Fock space vacuum expectation value.

Appendix

Notations for the lattice Λ and its dual Λ^*. The lattice Λ consists of sites

$$x = n_1\underline{1} + n_2\underline{2} + n_3\underline{3} + \ldots + n_D\underline{D}.$$

A link starting at x and ending at $x + \underline{\rho}$ is denoted by (x, ρ). A plaquette spanned by two links (x, ρ), (x, σ), $\rho \neq \sigma$, is denoted by (x, ρ, σ) or (x, σ, ρ). The dual lattice Λ^* has the same structure (hypercubic) as Λ, denote its sites by x! Its links, plaquettes, etc., may be denoted in the same fashion as for Λ. However, for our purposes another notation is more practical. We denote a D cell spanned by the links (x', ρ), all ρ, by $[x']$; a $(D-1)$ cell spanned by the links (x', ρ), $\rho \neq \sigma$, by $[x, \sigma]$; a $(D-2)$ cell spanned by the links (x', τ), $\tau \neq \rho$, $\tau \neq \sigma$, *by* $[x', \rho, \sigma]$ or $[x', \sigma, \rho]$.

The duality transformation is a one-to-one map of Λ on Λ^* so that

$$x' \leftrightarrow x, \quad x = x' + \frac{1}{2}\sum_{\rho} \underline{\rho}$$

and

$$\begin{aligned} &\text{sites } x \leftrightarrow D \text{ cells } [x'], \\ &\text{links } (x,\rho) \leftrightarrow (D-1) \text{ cells } [x' + \underline{\rho}, \rho], \\ &\text{plaquettes } (x,\rho,\sigma) \leftrightarrow (D\text{-}2) \text{ cells } [x' + \underline{\rho} + \underline{\sigma}, \rho, \sigma]. \end{aligned}$$

These objects of Λ are orthogonal to their images in Λ^* so that the centre of the object coincides with the centre of its image.

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