Commun. Math. Phys. 81, 495–499 (1981)

Communications in
Mathematical
Physics

A Poisson Random Walk is Bernoulli*

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Abstract. Particles distributed on the integers in Poisson distribution, each independently taking a random walk, form a stationary Markov chain. The canonical shift in this space is Bernoulli.

1. Introduction

Consider particles distributed on $\mathbb{Z}$ with Poisson distribution, parameter 1. (That is for each $n \in \mathbb{Z}$, there is a nonnegative number ω_n of particles on n. The values $\{\omega_n\}_{n\in\mathbb{Z}}$ are independent identically distributed (i.i.d.) random variables, each with Poisson distribution parameter 1.) Let each particle take a random walk, described as follows. (Results can be shown to remain valid for any random walk but we choose this one for simplicity. A discussion of generalizing this result to arbitrary random walk will be made in a closing remark at the end of this paper.) We let each particle stand still, move one step forward, or one step backward, each with probability 1/3. They all move independently. Then the particles will remain Poisson distributed. By continuing this procedure we obtain a sequence of configurations $\{X_i\}_{i\in\mathbb{N}}$, where each X_i is a Poisson-distributed Ω-valued random variable, where $\Omega = \mathbb{N}^{\mathbb{Z}}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$. Since $\{X_i\}_{i\in\mathbb{N}}$ forms a stationary Markov chain, we can extend the process to a stationary Markov chain $\{X_i\}_{i\in\mathbb{Z}}$. This process will henceforth be called Poisson Random Walk, abbreviated P-walk. The shift $\{X_i\} \to \{Y_i\}$, $Y_i = X_{i+1}$ for all i, is a stationary shift. Here it is shown to be a Bernoulli shift.

My result can easily be extended to a process where particles are labeled by their past and future, thereby strengthening a result of Goldstein and Lebowitz (Theorem 5.3, page 10 of [1]).

2. Probabilistic Statement of Theorem to be Proved

Let $\{X_i\}_{i\in\mathbb{Z}}$ be as in the Introduction. The theorem proved by this paper is

Theorem 1. *$\{X_i\}_{i\in\mathbb{Z}}$ is Bernoulli.*

* This research was supported in part by NSF grant MCS 78–07739–A03

0010-3616/81/0081/0495/$01.00

The purpose of this section is to reduce this theorem to a probabilistic statement.

$\Omega = \mathbb{N}^{\mathbb{Z}}$. For any $\omega \in \Omega$ and any finite set $S \subset \mathbb{Z}$; let ω_s be the restriction of ω to the set S. Thus, when we speak of the particles of $(X_i)_S$ we mean the particles of X_i which lie on S. $(X_i)_S$ refers to the function taking each $n \in S$ to the number of particles which lie on the integer n at time i.

From a theorem of Ornstein, Theorem 4, p. 53 of [3], it follows that if the process $\{(X_i)_S\}_{i \in \mathbb{Z}}$ is Bernoulli for each finite $S \subset \mathbb{Z}$, then the entire process $\{X_i\}_{i \in \mathbb{Z}}$ is Bernoulli. By definition, a process is Bernoulli if it is isomorphic to an independent process. However, that condition is difficult to check so instead we use here a condition which is equivalent.

The condition we use is Shields' alteration of the very weak Bernoulli condition of Ornstein [2]. Proving that condition for the $\{X_i\}_s$ process is easily seen to be equivalent to proving the following theorem.

Theorem 2. *Given $\varepsilon > 0$ and a finite set $S \subset \mathbb{Z}$, we can construct a sequence of jointly distributed random variables $\{(X_i, Y_i)\}_{i \in \mathbb{Z}}$ such that*

1) *$\{X_i\}_{i \leqq 0}$ and $\{Y_i\}_{i \in \mathbb{Z}}$ are independent.*
2) *$\{X_i\}$, $\{Y_i\}$ are both P-walks.*
3) *$P((X_n)_S \neq (Y_n)_S) < \varepsilon$ for all sufficiently large n.*

Thus the remainder of this paper is a proof of Theorem 2.

3. Proof of Theorem 2

First construct the $\{Y_i\}$ process and construct $\{X_i\}_{i \leqq 0}$ to be independent of the $\{Y_i\}$ process. We allow the particles of X_0 to take a random walk in order to form configurations $X_1, X_2, \ldots$. As the definition of P-walk implies, the particles of X_0 walk independently of each other. However, we will allow their walk to depend upon the walk of the Y_0 particles as follows.

A one-to-one correspondence between a certain subset of the Y_0 particles and a certain subset of the X_0 particles will be established. The uncorresponded X_0 particles will walk independently of all Y_0 particles and all other X_0 particles. Each corresponded X_0 particle will similarly walk independent of all other particles until it meets its corresponding Y_0 particle (i.e., until the two particles are both on the same integer). This will eventually happen with probability 1 because the difference between the positions of the two particles takes a symmetric, aperiodic random walk with bounded step sizes; all such walks eventually reach zero. Once the corresponded X_0 particle meets its corresponding Y_0 particle it will continue to move exactly as does that Y_0 particle so that the two particles will stay together from that time onward.

In order to complete the definition of $\{X_i\}$ process, all that remains is to choose the subsets of X_0 and Y_0 which we will correspond and to exhibit that correspondence. Choose $L > 0$. We will later specify how L is chosen more precisely. Let

$$I_k = \{i \in \mathbb{Z} : kL + 1 \leqq i \leqq (k+1)L\}.$$

Our correspondence will consist of a correspondence, for each k, between the elements of a subset of the particles of $(X_0)_{I_k}$ and those of a subset of the particles of $(Y_0)_{I_k}$. (Recall that $(X_0)_{I_k}$ are those particles of X_0 that lie in I_k.) Let $N(X, k)$ (respectively, $N(Y, k)$) be the total number of particles of $(X_0)_{I_k}$ (respectively, $(Y_0)_{I_k}$). Let

$$N_k = \min\{N(X, k), N(Y, k)\}.$$

The subset we choose from $(X_0)_{I_k}$ will consist of N_k particles of $(X_0)_{I_k}$ randomly chosen in a uniform manner, i.e., for any given set of N_k elements of X_0, we will choose that set with probability $1\Big/\binom{N(X,k)}{N_k}$ where $\binom{N(X,k)}{N_k}$ is the binomial coefficient

$$\frac{N(X,k)!}{N_k!\,(N(X,k)-N_k)!}.$$

In a similar uniform manner choose N_k particles from $(Y_0)_{I_k}$. Since both sets have N_k members, arbitrarily choose a bijection between them. This set of bijections, for all k, collectively forms one large bijection of a subset of X_0 with a subset of Y_0. This completes our construction of a $\{X_i, Y_i\}_{i \geqq 0}$ process satisfying conditions 1 and 2.

We now check condition 3.$P((X_n)_S \neq (Y_n)_S) \leqq P$ (at time n, S contains uncorresponded X_0 particles) $+ P$ (at time n, S contains uncorresponded Y_0 particles) $+ P$ (at time n, S contains corresponded X_0 particles which have not yet met their corresponding Y_0 particle) $+ P$ (at time n, S contains corresponded Y_0 particles which have not yet met their corresponding X_0 particle).

Let A, A_1, B, B_1 be the four terms of the right-hand side of the above inequality in order. By symmetry, $A = A_1$, $B = B_1$. To complete the proof of Theorem 2 we need only show

1) By properly choosing L, we can insure that both $A < \varepsilon/4$ and $B < \varepsilon/4$ for sufficiently large n.

For $z \in \mathbb{Z}$, let c_z be the number of uncorresponded X_0 particles which are on point z. Note that $E(c_z)$ is independent of z. For $z_1, z_2 \in \mathbb{Z}$, let $P_n(z_1, z_2)$ be the probability that a particle which is on z_1 at time 0 will be at z_2 at time n. Note that $P_n(z_1, z_2)$ depends only on $z_2 - z_1$. Thus for any fixed $z_0 \in \mathbb{Z}$

$$\sum_{z \in \mathbb{Z}} P_n(z, z_0) = \sum_{z \in \mathbb{Z}} P_n(z_0, z) = 1.$$

2) $A \leqq E(\#\{\text{uncorresponded } X_0 \text{ particles which reach } S \text{ at time } n\})$

$$= \sum_{\substack{z_1 \in \mathbb{Z} \\ z_2 \in S}} E(c_{z_1}) P_n(z_1, z_2) = E(c_0) \sum_{\substack{z_1 \in \mathbb{Z} \\ z_2 \in S}} P_n(z_1, z_2)$$

$$= E(c_0) \sum_{z_2 \in S} (1) = \# S E(c_0) = \frac{1}{L} \#(S) E \sum_{i=1}^{L} (c_i)$$

$$= \frac{1}{L} \# S E(N(X, 0)) - \min(\{N(X, 0), N(Y, 0)\}).$$

For any i.i.d. random variables X, Y with variance L,

$$E(X - \min(X, Y)) = E(X - Y)^+ = \tfrac{1}{2}E|X - Y| \leqq \tfrac{1}{2}\sqrt{E(X - Y)^2} = \sqrt{L/2}.$$

Hence the right-hand side of 2 can be made arbitrarily small by choosing L sufficiently large.

Now suppose an X_0 particle at x is corresponded to a Y_0 particle at y. Let $P_n(x, y, z)$ be the probability that the two particles do not meet by time n and that the X_0 particle reaches z at time n. Note that $P_n(x, y, z)$ depends only on $z - x$ and $y - x$. Also recall that $y - x$ lies somewhere between $-L$ and L. Thus

$$P_n(x, y, z) \leqq \sum_{i=-L}^{L} P_n(0, i, z - x).$$

For $-L \leqq i \leqq L$, let $a_{n,i}$ be the probability that an X_0 particle at 0 and a Y_0 particle at i will not meet by time n. Then $\lim_{n\to\infty} a_{n,i} = 0$, and hence

$$\lim_{n\to\infty} \sum_{i=-L}^{L} a_{n,i} = 0.$$

Note also that for any $z \in \mathbb{Z}$,

$$\sum_{x\in\mathbb{Z}} P_n(0, i, z - x) = a_{n,i}.$$

Recall that the expected number of X_0 particles at any site is 1. Since the probability that a corresponded particle at x reaches z at time n without reaching its corresponding Y_0 particle is bounded above by

$$\sum_{i=-L}^{L} P_n(0, i, z - x),$$

we get

$$\begin{aligned} B &\leqq \sum_{\substack{z\in S\\ x\in\mathbb{Z}}} \left(E\,\#\{\text{corresponded } X_0 \text{ particles at } x\} \sum_{i=-L}^{L} P_n(0, i, z - x)\right) \\ &\leqq \sum_{\substack{z\in S\\ x\in\mathbb{Z}}} 1\left(\sum_{i=-L}^{L} P_n(0, i, z - x)\right) \\ &= \sum_{z\in S} \sum_{i=-L}^{L} \left(\sum_{x\in\mathbb{Z}} P_n(0, i, z - x)\right) = \#S \sum_{i=-L}^{L} a_{n,i} \end{aligned}$$

which goes to zero as n approaches ∞. □

Remark. For simplicity I only mentioned the 1/3, 1/3, 1/3 random walk, but the proof goes through word for word for any aperiodic random walk whose step size has finite expectation. It is almost the same if we drop the aperiodic assumption. Here we carry out the same coupling, making sure we do not correspond two particles on non-communicating sites.

If the random walk has infinite expectation we use a coupling due to D. S. Ornstein. Couple corresponding particles so that they take the same step size if that size is larger than one large prechosen number N in absolute value, and take independent steps conditioned on these step sizes not exceeding N in absolute value. Then the difference between the two random walks is bounded and will eventually reach zero if periodicity is right and N is chosen large enough so that the difference is not deterministic.

Acknowledgements. I am grateful to Paul Shields for bringing this problem to my attention.

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Communicated by J. L. Lebowitz

Received November 17, 1980