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Outer Automorphisms and Reduced Crossed Products of Simple C^*-Algebras

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Abstract. Every outer automorphism of a separable simple C^*-Algebra is shown to have a pure state which is mapped into an inequivalent state under this automorphism. The reduced crossed product of a simple C^*-algebra by a discrete group of outer automorphisms is shown to be simple.

0. Introduction

We consider two problems both of which depend on one technical lemma.

One of them is the problem, stated by Lance in [4], whether or not any universally weakly inner automorphism of a simple C^*-algebra is inner. In Sect. 2 we shall answer this affirmatively in case the C^*-algebra is separable. The idea of the proof is based on [3], the corresponding result in case of one-parameter automorphism groups.

The other is the problem whether or not the reduced crossed product of a simple C^*-algebra by a discrete group of outer automorphisms is simple. In Sect. 3 we shall answer this affirmatively. The proof is essentially the same as that of the result which has been obtained by Elliott in [1] in case the C^*-algebra is AF (i.e., approximately finite-dimensional).

In Sect. 1 we shall give a main lemma on outer automorphisms; a similar result has also been obtained by Elliott in the AF case.

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1. Outer Automorphisms

Let A be a C^*-algebra and α an automorphism of A. In [2] we have defined the strong Connes spectrum $\tilde{\mathbb{T}}(\alpha)$ of α. In this case $\tilde{\mathbb{T}}(\alpha)$ is the set of $\lambda \in \mathbb{T} = \hat{\mathbb{Z}}$ such that $\hat{\alpha}_\lambda(I) = I$ for any primitive ideal I of the crossed product $A \times_\alpha \mathbb{Z}$ of A by α, where $\hat{\alpha}$ is the dual action. $\tilde{\mathbb{T}}(\alpha)$ depends on α up to inner automorphisms, i.e., $\tilde{\mathbb{T}}(\mathrm{Ad}\,u \circ \alpha) = \tilde{\mathbb{T}}(\alpha)$, where u is a unitary multiplier of A. If A is α-simple, i.e., A does not have

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any non-trivial α-invariant closed two-sided ideals, then $\tilde{\mathbb{T}}(\alpha)$ equals the Connes spectrum of α. By Olesen's result in [5], if A is simple and $\tilde{\mathbb{T}}(\alpha) = \{1\}$, then α is inner.

1.1. Lemma. *Let A be a C^*-algebra and α an automorphism of A. If $\tilde{\mathbb{T}}(\alpha) \neq \{1\}$, then for any non-zero hereditary C^*-subalgebra B of A and for any $a \in A$ (admitting $a = 1$),*

$$\inf\{\|xa\alpha(x)\|; \quad 0 \leqq x \in B, \quad \|x\| = 1\} = 0.$$

Proof. (See [1, 2.3] for a similar result.) Suppose that there is a non-zero hereditary C^*-subalgebra B of A and $a \in A$ such that the infimum in the lemma is positive, say δ.

Let x be a positive element of B with $\|x\| = 1$. Then $x^n \geqq 0$ and $\|x^n\| = 1$, and so $\|x^n a\alpha(x^n)\| \geqq \delta$ for any $n \in \mathbb{N}$. It follows that for any m, $n \in \mathbb{N}$, $\|x^m a\alpha(x^n)\| \geqq \delta$.

Since $\alpha(x^n)a^* x^{2m} a\alpha(x^n)$ is positive and has norm not less than δ^2, there is a state f_m of A such that

$$f_m(\alpha(x^n)a^* x^{2m} a\alpha(x^n)) \geqq \delta^2.$$

Then for $k \leqq m$,

$$f_m(\alpha(x^n)a^* x^{2k} a\alpha(x^n)) \geqq \delta^2, \tag{*}$$

since $x^{2k} \geqq x^{2m}$. Let f be a weak* limit point of (f_m). Then (*) holds with f in place of f_m, for any $k \in \mathbb{N}$. Since (x^{2k}) is a decreasing sequence of positive elements, it has a strong limit p in A^{**}, which is a projection. Thus

$$f(\alpha(x^n)a^* p a\alpha(x^n)) \geqq \delta^2.$$

Since $\|f\| \leqq 1$, it follows that $\|pa\alpha(x^n)\|^2 = \|\alpha(x^n)a^* pa\alpha(x^n)\| \geqq \delta^2$.

Since $pa\alpha(x^{2n})a^* p$ is positive and has norm not less than δ^2, there is a state g_n of A such that $g_n(p) = 1$ and

$$g_n(pa\alpha(x^{2n})a^* p) \geqq \delta^2 - n^{-1}.$$

Hence g_n satisfies that $g_n(x^{2m}) = 1$ for any $m \in \mathbb{N}$, and

$$g_n(a\alpha(x^{2k})a^*) \geqq \delta^2 - n^{-1}$$

for any $k \leqq n$. Let g be a weak* limit point of (g_n). Then

$$g(x^{2k}) = 1 \quad \text{and} \quad g(a\alpha(x^{2k})a^*) \geqq \delta^2$$

for any $k \in \mathbb{N}$. Thus $g(p) = 1$ and $g(a\bar{\alpha}(p)a^*) \geqq \delta^2$, where $\bar{\alpha} = \alpha^{**}$. In particular, since g is a state, it follows that $\|pa\bar{\alpha}(p)\| \geqq \delta$.

Let φ be a pure state of B and let

$$K_\varphi = \{x \in B; \quad 0 \leqq x \leqq 1, \quad \varphi(x) = 1\}.$$

Then

$$\bigcap_{x \in K_\varphi} \{f; \quad \text{state of } A \text{ such that } f(x) = 1\} = \{\varphi\}.$$

(To prove this it is enough to show that for any pure state f of B other than φ, there is a positive $x \in B$ such that $\varphi(x) = \|x\| = 1$ and $f(x) < 1$. This can be easily shown by using Kadison's transitivity theorem.) For each $x \in K_\varphi$ let $p(x)$ be the strong limit of

x^k as $k \to \infty$. If $x, y \in K_\varphi$, then

$$z = (x+y)/2 \in K_\varphi$$

and $p(z) \leqq p(x), p(y)$. Thus $\{p(x);\ x \in K_\varphi\}$ forms a downward directed set of non-zero projections of A^{**} so that it has a strong limit p in A^{**}. Since $\varphi(p) = 1$ and $f(p) < 1$ for any state f other than φ, p is a minimal projection in A^{**}. From the preceding paragraph, there is a state g_x of A for each $x \in K_\varphi$ such that

$$g_x(x^{2k}) = 1 \quad \text{and} \quad g_x(a\alpha(x^{2k})a^*) \geqq \delta^2$$

for any $k \in \mathbb{N}$. If $p(x) \leqq p(y)$ with $x, y \in K_\varphi$, then

$$g_x(y^{2k}) = 1 \quad \text{and} \quad g_x(a\alpha(y^{2k})a^*) \geqq \delta^2.$$

Let g be a weak* limit point of the net (g_x). Then the above equations hold with g in place of g_x for all $k \in \mathbb{N}$ and $y \in K_\varphi$. Hence $g = \varphi$ and

$$\varphi(a\bar{\alpha}(p)a^*) \geqq \delta^2.$$

Let $\tilde{B} = B + \mathbb{C}e$ where e is the identity of B^{**} (in A^{**}). For a unitary u in $\tilde{B}$, by using uxu^* in place of x with $x \in K_\varphi$ we obtain

$$\varphi(u^* a\bar{\alpha}(upu^*)a^* u) \geqq \delta^2.$$

In particular

$$\|pu^* a\bar{\alpha}(up)\| \geqq \delta. \tag{**}$$

Let $\bar{\pi}_\varphi$ be the unique extension of π_φ to a representation of A^{**}. $\bar{\pi}_\varphi(p)$ is the one-dimensional projection onto $\mathbb{C}\Omega_\varphi$ and $\bar{\pi}_\varphi(\bar{\alpha}(p))$ is non-zero and so a one-dimensional projection. Let Ψ be a unit vector of H_φ such that

$$\bar{\pi}_\varphi(\bar{\alpha}(p))\Psi = \Psi.$$

Now we define an operator V on H_φ such that

$$V\pi_\varphi(x)\Omega_\varphi = \pi_\varphi \circ \alpha(x)\Psi, \qquad x \in A.$$

Since

$$\begin{aligned} \|\pi_\varphi \circ \alpha(x)\Psi\|^2 &= (\bar{\pi}_\varphi \circ \bar{\alpha}(px^*xp)\Psi, \Psi) \\ &= \varphi(x^*x) \\ &= \|\pi_\varphi(x)\Omega_\varphi\|^2 \end{aligned}$$

by $px^*xp = \varphi(x^*x)p$, and since π_φ is irreducible, V has an extension to a unitary on H_φ, which is denoted by V again. Then V satisfies that

$$V\pi_\varphi(x)V^* = \pi_\varphi \circ \alpha(x), \qquad x \in A.$$

Let $E = \bar{\pi}_\varphi(e)$ be the projection onto $[\pi_\varphi(B)\Omega_\varphi] = [\pi_\varphi(B)H_\varphi]$. Then for any unit vector $\Phi \in EH_\varphi$, there is a unitary u in $\tilde{B}$ such that $\Phi = \bar{\pi}_\varphi(u)\Omega_\varphi$, by the transitivity theorem. Then it follows from (**) that

$$|(\pi_\varphi(a)V\Phi, \Phi)| = |(\bar{\pi}_\varphi(pu^* a\bar{\alpha}(up))\Psi, \Omega_\varphi)| = \|\bar{\pi}_\varphi(pu^* a\bar{\alpha}(up))\| \geqq \delta.$$

Since a numerical range is a convex subset of $\mathbb{C}$, by adjusting a phase factor of V we may assume that

$$\operatorname{Re}(E\pi_\varphi(a)VE\Phi,\Phi)\geqq\delta$$

for any $\Phi\in EH_\varphi$ with $\|\Phi\|=1$. Thus

$$E(\pi_\varphi(a)\mathrm{V}+\mathrm{V}^*\pi_\varphi(a^*))E\geqq 2\delta E.$$

Hence for any $b\in B$,

$$\pi_\varphi(b^*)(\pi_\varphi(a)V+V^*\pi_\varphi(a^*))\pi_\varphi(b)\geqq 2\delta\cdot\pi_\varphi(b^*b).$$

By the remark given before the lemma, $\hat{\alpha}_\lambda$ with $\lambda\in\tilde{\mathbb{T}}(\alpha)$ induces an automorphism β_λ of the C^*-algebra generated by $\pi_\varphi(A)$ and V in the way that

$$\beta_\lambda(\pi_\varphi(x))=\pi_\varphi(x),\qquad x\in A,\qquad \beta_\lambda(V)=\lambda V.$$

Hence for $\lambda\in\tilde{\mathbb{T}}(\alpha)$,

$$\pi_\varphi(b^*)(\lambda\pi_\varphi(a)V+\bar{\lambda}V^*\pi_\varphi(a^*))\pi_\varphi(b)\geqq 2\delta\pi_\varphi(b^*b).$$

Since $\bar{\lambda}\in\tilde{\mathbb{T}}(\alpha)$ if $\lambda\in\tilde{\mathbb{T}}(\alpha)$, this implies that

$$\operatorname{Re}\lambda\cdot\pi_\varphi(b^*)(\pi_\varphi(a)V+V^*\pi_\varphi(a^*))\pi_\varphi(b)\geqq 2\delta\pi_\varphi(b^*b)$$

for $b\in B$. Hence $\operatorname{Re}\lambda>0$ for $\lambda\in\tilde{\mathbb{T}}(\alpha)$, i.e., $\tilde{\mathbb{T}}(\alpha)=\{1\}$ since $\tilde{\mathbb{T}}(\alpha)$ is a group. q.e.d.

We conclude this section by giving simple remarks on the assumption in the lemma.

The condition that $\tilde{\mathbb{T}}(\alpha)\neq\{1\}$ in the above lemma can be replaced by the following weaker one: The set of α-invariant closed two-sided ideals I of A with $\tilde{\mathbb{T}}(\alpha|I)\neq\{1\}$ generates an essential ideal of A. Because then for any non-zero hereditary C^*-subalgebra B of A there is an α-invariant closed two-sided ideal I of A such that $B\cap I\neq\{0\}$ and $\tilde{\mathbb{T}}(\alpha|I)\neq\{1\}$, and hence we can proceede as in the above proof with $B\cap I$ (respectively I) in place of B (respectively A) since a (in the lemma) is allowed to be a multiplier.

This weaker condition on α still implies that α is properly outer [1] (while we do not know about the converse).

To show this suppose that α is not properly outer. Then there is a non-zero α-invariant closed two-sided ideal I of A, a unitary multiplier u of I, and a $*$-derivation δ of I such that

$$\alpha|I=(\operatorname{Ad}u)(\exp\delta)$$

(see [1, 2.2]). Since that $\tilde{\mathbb{T}}(\exp\delta)=\{1\}$ follows easily from the definition of $\tilde{\mathbb{T}}$,

$$\tilde{\mathbb{T}}(\alpha|I)=\tilde{\mathbb{T}}(\operatorname{Ad}u^*\circ\alpha|I)=\{1\}.$$

For any non-zero α-invariant closed two-sided ideal J of A with $J\subset I$, the restriction of α to J has an expression of the same type as $\alpha|I$ since u multiplies J and δ leaves J invariant. Hence $\tilde{\mathbb{T}}(\alpha|J)=\{1\}$. This implies that $\tilde{\mathbb{T}}(\alpha|J)=\{1\}$ for any α-invariant closed two-sided ideal J of A with $J\cap I\neq\{0\}$. This completes the proof.

2. States and Automorphisms

2.1. Theorem. *Let A be a separable C^*-algebra and α an automorphism of A. If $\tilde{\mathbb{T}}(\alpha) \neq \{1\}$, then there exists a pure state φ of A such that $\varphi \circ \alpha$ is disjoint from φ.*

Proof. Let (u_n) be a dense sequence of unitaries in $\tilde{A} = A + \mathbb{C}1$, and write $\mathrm{Ad}\, u_n = \sigma_n$.

By Lemma 1.1 (with $a = 1$ and $\alpha = \sigma_1 \circ \alpha$) there is a positive e_1 in A such that $\|e_1\| = 1$ and $\|e_1 \sigma_1 \circ \alpha(e_1)\| < 1/2$. By changing e_1 slightly if necessary, we may further suppose that there is a positive a_1 in A such that $\|a_1\| = 1$ and $e_1 a_1 = a_1$.

By applying 1.1 to $\overline{a_1 A a_1}$ with $\sigma_2 \circ \alpha$, we have positive e_2 and a_2 in $\overline{a_1 A a_1}$ such that $\|e_2\| = \|a_2\| = 1$, $e_2 a_2 = a_2$, and $\|e_2 \sigma_2 \circ \alpha(e_2)\| < 1/2$. Since $e_1 a_1 = a_1$, it follows that $e_1 e_2 = e_2$.

By induction we construct a_n, e_n successively, i.e., e_n and a_n are chosen from $\overline{a_{n-1} A a_{n-1}}$ so that

$$e_n \geqq 0, \quad a_n \geqq 0, \quad \|e_n\| = \|a_n\| = 1, \quad e_n a_n = a_n, \quad \|e_n \sigma_n \circ \alpha(e_n)\| < 1/2.$$

Then it follows that $e_{n-1} e_n = e_n$ for all n.

Let S be the set of states φ of A such that $\varphi(e_n) = 1$ for all n. Then since $e_{n-1} e_n = e_n$ for all n, S is a non-empty closed face. Let φ be an extreme point of S, which is a pure state.

Since $\varphi(\sigma_n \circ \alpha(e_n)) = \varphi(e_n \sigma_n \circ \alpha(e_n))$, it follows that

$$\varphi(\sigma_n \circ \alpha(e_n)) \leqq \|e_n \sigma_n \circ \alpha(e_n)\| < 1/2.$$

Hence

$$\varphi(e_n - \sigma_n \circ \alpha(e_n)) > 1/2.$$

This implies that for any n,

$$\|\varphi - \varphi \circ \sigma_n \circ \alpha\| > 1/2. \tag{*}$$

If $\varphi \circ \alpha$ is not disjoint from φ, then $\varphi \circ \alpha$ is equivalent to φ and so α is implemented by a unitary in the representation π_φ. By applying Kadison's transitivity theorem, we know that there is a unitary u in $\tilde{A}$ such that $\varphi = \varphi \circ \mathrm{Ad}\, u \circ \alpha$, which contradicts (*) since (u_n) is dense in the unitary group of $\tilde{A}$. Thus $\varphi \circ \alpha$ is disjoint from φ.

2.2. Remark. Under the situation of the above theorem the set of pure states φ of A with the property that $\varphi \circ \alpha$ is disjoint from φ is dense in the set of all pure states (in the weak* topology). Because, in the proof of 2.1, e_1 can be chosen from an arbitrarily specified non-zero hereditary C^*-subalgebra of A, so each of those subalgebras has at least one pure state with the above property (as a state of A).

If A is a simple C^*-algebra, then the condition $\tilde{\mathbb{T}}(\alpha) \neq \{1\}$ is equivalent to α being outer. Hence we obtain

2.3. Corollary. *Let A be a separable simple C^*-algebra and let α be an automorphis of A. If $\varphi \circ \alpha$ is equivalent to φ for any pure state φ of A* (i.e., *α is extendible in every irreducible representation, as in* [4]), *then α is inner.*

When A is UHF, this was obtained by Lance [4].

The following extends Theorem 8 in [3]:

2.4. Theorem. *Let A be a simple C^*-algebra (without identity) and α a one-parameter automorphism group of A. If α^* is strongly continuous on A^* (i.e., $\|\varphi\circ\alpha_t-\varphi\|\to 0$ as $t\to 0$ for any $\varphi\in A^*$), then α is inner (i.e., there exists a one-parameter group u of unitary multipliers such that $\alpha_t=\mathrm{Ad}\,u_t$ and $t\to u_t$ is continuous in the strict topology).*

Proof. If there were a sequence (t_n) in $\mathbb{R}$ such that t_n converges to zero and each α_{t_n} is outer, then by applying the proof of 2.1 with α_{t_n} in place of $\sigma_n\circ\alpha$ to this situation we would obtain a contradiction: $\|\varphi\circ\alpha_{t_n}-\varphi\|>1/2$ for some pure state φ of A. Hence each α_t is inner for small t and so for all t. In case A is separable, this would be enough to obtain the conclusion. In general we adopt the argument in the proof of Theorem 7 in [3], where the continuity property of the implementing group of unitaries is shown under the present assumption. q.e.d.

In the above proof we have used only the norm-continuity of $\varphi\circ\alpha_t$ in t with all pure states φ. This is in fact equivalent to α^* being strongly continuous, even if A is not simple [3, Proposition 9].

3. Reduced Crossed Products

Let G be a discrete group and let α be a representation of G by automorphisms of a C^*-algebra A. We denote by $A\times_{\alpha r}G$ the reduced crossed product of A by α. Along the same lines as Elliott's proof in [1,3.2] we show

3.1. Theorem. *Let (A, G, α) be as above. Suppose that A is α-simple and that $\tilde{\Gamma}(\alpha_g)\neq\{1\}$ for all $g\in G\setminus\{1\}$. (In particular suppose that A is simple and each α_g is outer for $g\in G\setminus\{1\}$.) Then the reduced crossed product $A\times_{\alpha r}G$ is simple.*

3.2. Lemma. *Let a be a positive element of the C^*-algebra A, $\{a_i; i=1,2,\dots,n\}$ elements of A, $\{\alpha_i; i=1,2,\dots,n\}$ automorphisms of A with $\tilde{\Gamma}(\alpha_i)\neq\{1\}$, and $\varepsilon>0$. Then there exists a positive $x\in A$ with $\|x\|=1$ such that*

$$\|xax\|\geqq\|a\|-\varepsilon,\qquad \|xa_i\alpha_i(x)\|\leqq\varepsilon,\qquad i=1,\dots,n.$$

Proof. Define a continuous function f on $\mathbb{R}$ by

$$f(t)=\begin{cases}1, & t\geqq\|a\|\\ \varepsilon^{-1}(t-\|a\|+\varepsilon), & \|a\|-\varepsilon\leqq t<\|a\|\\ 0, & t<\|a\|-\varepsilon.\end{cases}$$

Let B be the hereditary C^*-subalgebra of A generated by $f(a)$. Then for any positive $y\in B$ with $\|y\|=1$, $\|ay-\|a\|y\|\leqq\varepsilon$, hence $\|yay\|\geqq a\|-\varepsilon$.

By 1.1 there is a positive $e_1\in B$ with $\|e_1\|=1$ such that

$$\|e_1a_1\alpha_1(e_1)\|\leqq\varepsilon.$$

We may further suppose that there is a positive $x_1\in B$ such that $\|x_1\|=1$ and $e_1x_1=x_1$. Let $B_1=\overline{x_1Ax_1}\subset B$. Again 1.1 shows that there are positive e_2 and x_2 in B_1

such that $\|e_2\| = \|x_2\| = 1$, $e_2 x_2 = x_2$, and $\|e_2 a_2 \alpha_2(e_2)\| \leqq \varepsilon$. Then x_2 satisfies

$$\|x_2 a_1 \alpha_1(x_2)\| \leqq \|e_1 a_1 \alpha_1(e_1)\| \leqq \varepsilon,$$

$$\|x_2 a_2 \alpha_2(x_2)\| \leqq \|e_2 a_2 \alpha_2(e_2)\| \leqq \varepsilon,$$

$$\|x_2 a x_2\| \geqq \|a\| - \varepsilon.$$

Continuing in this way, we obtain $x_n = x$ satisfying all the conditions in the lemma. q.e.d.

We will not complete the proof of 3.1 because it is the same as [1, 3.2] if the above lemma is used for [1, 3.3]. The extra assumption we have made is that a (in the lemma) is positive. But this is not essential because $\|a\| \leqq \left\|a + \sum_{g \in T} a_g u_g\right\|$ in [1, p. 308] follows from

$$\|a^* a\| \leqq \left\|a^* a + \sum_{g \in T} a^* a_g u_g\right\|.$$

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