

## Lax Representation for the Systems of S. Kowalevskaya Type

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**Abstract.** We describe a number of dynamical systems that are generalizations of the S. Kowalevskaya system and admit the Lax representation.

It is well known (see for example [1, 2]), that the equations of motion of a three-dimensional heavy rigid body rotated about a fixed point is a completely integrable dynamical system only in the cases of Euler [3], Lagrange [4], and S. Kowalevskaya [5]. In the present note we describe a number of systems that are of the Kowalevskaya type and admit the Lax representation<sup>1</sup>.

1. Let  $\mathcal{G}$  be the Lie algebra of a group  $G$ ,  $\mathcal{G}^*$  be the space dual to  $\mathcal{G}$ ,  $\{x_\alpha\}$  be the coordinates of a point in the space  $\mathcal{G}^*$ . In the space  $\mathcal{F}(\mathcal{G}^*)$  of smooth functions on  $\mathcal{G}^*$ , let us define the Poisson bracket<sup>2</sup>

$$\{f, g\} = C_{\alpha\beta}^\gamma x_\gamma \partial^\alpha f \partial^\beta g, \quad \partial^\alpha = \partial / \partial x_\alpha. \quad (1)$$

Here  $C_{\alpha\beta}^\gamma$  are the structure constants of the Lie algebra. The space  $\mathcal{F}(\mathcal{G}^*)$  is endowed by the above formula with the structure of the Lie algebra. A dynamical system in  $\mathcal{G}^*$  is determined by a Hamiltonian function  $H(x) \in \mathcal{F}$ , so that the equations of motion have the form

$$\dot{x}_\alpha = \{H, x_\alpha\}. \quad (2)$$

The coadjoint representation of the group  $G$  acts on the space  $\mathcal{G}^*$ . Orbits of this representation are invariant with respect to an arbitrary Hamiltonian  $H$ , and are the phase spaces of the considered systems.

2. Let  $G$  be a compact simple Lie group,  $K$  be its subgroup such that the factor space  $G/K$  is a symmetric space [6]. Then  $\mathcal{G} = \mathcal{K} \oplus \mathcal{S}$ ,  $\mathcal{K}$  is the Lie al-

<sup>1</sup> Note that the Lax representation for equations of motion of a rigid body was discovered first by S. Manakov [8] for some particular cases

<sup>2</sup> In this note we use the tensor notations; in particular, everywhere repeated indices imply summation

gebra of the group  $K$ ,  $\mathcal{S}$  is the orthogonal complement to  $\mathcal{K}$  in  $\mathcal{G}$  relative to the Killing-Cartan form.

It is well known that a certain irreducible representation  $T^0(k)$  of the group  $K$  acts on the space  $\mathcal{S}$ . Let us consider an irreducible representation  $T(g)$  of the group  $G$ , such that under the restriction of this representation on the subgroup  $K$ , the irreducible representation  $T^0(k)$  is contained in  $T(g)$  with the unit multiplicity. Let  $V$  be a vector space in which the representation  $T(g)$  acts. Then it is possible to define the group  $\tilde{G} = G \cdot V$  that is a semidirect product of the group  $G$  and the abelian vector group  $V$ . Let us denote by  $V^0$  the subspace of  $V$  on which the representation  $T^0(k)$  acts, and by  $A$  the projection operator on the subspace  $V^0$ . Let  $V'$  be the orthogonal complement to  $V^0$ :  $V = V^0 \oplus V'$ . Note that the spaces  $V^0$  and  $\mathcal{S}$  are isomorphic, because  $T(g)$  contains  $T^0(k)$  only once. Therefore we have the decomposition

$$\tilde{\mathcal{G}} = \mathcal{K} \oplus \mathcal{S} \oplus \mathcal{V}^0 \oplus \mathcal{V}^1,$$

and the analogous decomposition for the space  $\tilde{\mathcal{G}}^*$ :

$$\tilde{\mathcal{G}}^* = \mathcal{L} \oplus \mathcal{N} \oplus \mathcal{P} \oplus \mathcal{Q}. \tag{3}$$

Here  $\mathcal{L} = \mathcal{K}^*$ ,  $\mathcal{N} = \mathcal{S}^*$ ,  $\mathcal{P} = \mathcal{V}^{(0)*}$ ,  $\mathcal{Q} = \mathcal{V}^{(1)*}$ . In addition,  $\dim \mathcal{N} = \dim \mathcal{P} = n$ , and  $\mathcal{N}$  and  $\mathcal{P}$  are isomorphic relative to the action of the group  $K$ .

Let  $\hat{\ell}$  be the matrix of the representation  $T$  of the Lie algebra  $\mathcal{G}$ ,  $\ell = \{\ell_{jk}\}$  corresponds to the matrix of the representation  $T^0$  of the algebra  $\mathcal{K}$ , acting in the space  $\mathcal{P}$ . Let us introduce the matrix, which is important in the sequel,

$$L = A[-\hat{\ell}^2 + (\gamma \otimes p + p \otimes \gamma)] A, \tag{4}$$

where  $p \in \mathcal{P} \oplus \mathcal{Q}$  and  $\gamma \in \mathcal{P}$  is the constant vector in  $\mathcal{P}$ . We identify  $\mathcal{P}$  and  $\mathcal{P}^*$  by means of the  $K$ -invariant scalar product on  $\mathcal{P}$ .

The system of S. Kowalevskaya type is defined by the Hamiltonian

$$H = \frac{1}{2} \text{tr } L = \alpha I_2(\ell) + \beta \mathcal{J}_2(n) + (\gamma p). \tag{5}$$

Here  $I_2(\ell)$  and  $\mathcal{J}_2(n)$  are the quadratic functions on  $\mathcal{L}$  and  $\mathcal{N}$  respectively that are invariant relative to the coadjoint representation of the group  $K$  and  $p \in \mathcal{P}$ .

3. Let us consider in detail the case:  $G = SO(n+1)$ ,  $K = SO(n)$ ,  $\tilde{G} = G \cdot V = E(n+1)$  is the motion group of the  $(n+1)$ -dimensional Euclidean space  $V = \mathbb{R}^{n+1}$ ,  $V^0 = \mathbb{R}^n$ ,  $V^1 = \mathbb{R}^1$ . Let  $\hat{\ell}_{jk} = -\hat{\ell}_{kj}$  and  $\hat{p}_m$  ( $j, k = 1, \dots, (n+1)$ ) be the standard basis in the space of linear functions on  $\mathcal{G}^*$  and  $\mathcal{V}^*$  respectively with the standard Poisson brackets.

The Hamiltonian (5) takes now the form

$$2H = 2 \sum_{j < k}^n \ell_{jk}^2 + \sum_{j=1}^n n_j^2 + 2 \sum_{j=1}^n \gamma_j p_j, \tag{6}$$

where  $n_j = \hat{\ell}_{j, n+1}$ .

**Theorem.** *The equations of motion (2) of the system with Hamiltonian (6) are equivalent to the Lax equation*

$$\dot{L} = [L, M], \quad (7)$$

where  $L$  is given by the formula (4), and  $M = c\ell = cA\hat{\ell}A$ ,  $c$  is a constant.

The theorem is verified by a direct calculation.

From the Lax representation (7) it follows immediately that the eigenvalues of the matrix  $L$  or the quantities  $I_{2k} = k^{-1} \text{tr}(L^k)$ ,  $k=1, \dots, n$  are integrals of motion for Eqs. (2). Notice  $I_2 = 2H$ .

There are also  $(n-1)(n-2)/2$  linear integrals related to the invariance of  $H$  with respect to the group  $G_0 = SO(n-1)$  that leaves the vector  $\gamma$  invariant, and also  $([n/2] + 1)$  independent polynomials that are invariant with respect to the coadjoint representation of the group  $\tilde{G}$ . Using the theory of reduction of hamiltonian systems with symmetries [2, 7] we may show, that the above system is reduced to a system with a  $2n$ -dimensional phase space. It can be verified that integrals  $I_2, I_4, \dots, I_{2n}$  are functionally independent and in involution. Therefore, the systems under consideration are completely integrable.

Finally, we wish to note the following facts.

1. The system considered by S. Kowalevskaya corresponds to  $n=2$ .
2. Similar results are valid for other symmetric spaces. The simplest ones are the cases of the spaces of rank one:  $SU(n+1)/SU(n) \times U(1)$ ,  $Sp(n+1)/Sp(n) \times Sp(1)$  and  $F_4/SO(9)$ .
3. If  $p \in \mathcal{Q}$  in formula (5), then we deal with the generalization of the Lagrange case. In this case the constants  $\alpha$  and  $\beta$  may be arbitrary.
4. The results of this sections are valid also for similar systems, related to group  $\tilde{G} = SO(n+2)$ .

The detailed presentation of results will be published elsewhere.

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## References

1. Motion of a rigid body around a fixed point. A collection of papers, dedicated to the memory of S. Kowalevskaya. Acad. of Sci. USSR, Moscow (1940) (in Russian)
2. Arnold, V.I.: Mathematical methods of classical mechanics. New York: Springer 1978
3. Euler, L.: Decouverte d'un nouveau principe de Mecanique. Mem. de l'Acad. des Sciences de Berlin 1758
4. Lagrange, J.L.: Mecanique analytique. Paris 1788
5. Kowalevskaya, S.: Acta Math. **12**, H. 2 177-232 (1889)
6. Helgason, S.: Differential geometry and symmetric spaces. New York: Acad. Press 1962
7. Marsden, J., Weinstein, A.: Rep. Math. Phys. **5**, 121-130 (1974)
8. Manakov, S.: Funkt. Anal. Priloz. **10**, 93-94 (1976)

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