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Some Relations Between Dimension and Lyapounov Exponents

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Abstract. We consider differentiable maps and compact invariant sets. We introduce dimensional quantities related to the ergodic invariant measures, and prove some simple relations.

We consider differentiable maps and compact invariant sets. An estimate from above for the Hausdorff dimension of such a set has been given by A. Douady and J. Oesterlé [DO] and by Mañe [M_1]. In this paper we discuss some other relations of this kind. We first show how to deduce an estimate involving Lyapunov exponents of the system. We also introduce the fractal dimension $f(m)$ of a measure m on a compact space, which weights an "essential" dimension of (X, m).

The results are the following: for any ergodic invariant probability measure, we consider the spectrum of the linear tangent map (the so-called Lyapunov exponents) and the "dilating dimension" of this spectrum dim dil Sp m; the dimension of a compact invariant set is bounded from above by the supremum of dim dil Sp m over all invariant probability measures; individually, for any ergodic invariant probability measure, we have

$$f(m) \leqq \operatorname{dim}\operatorname{dil}\operatorname{Sp} m.$$

This inequality is generally a strict inequality, as is shown by considering maps of the interval, where $f(m)$ is related to the entropy $h(m)$ and the positive Lyapunov coefficient λ by $f(m) = \frac{h(m)}{\lambda}$.

This notion of dimension of a measure is closer to what is actually measured in experiments like those performed by P. Frederikson, J. Kaplan and J. Yorke [FKY]. It leads us to reformulate these conjectures there and to discuss some other questions.

I. Notations and Results

Let L be a linear operator from an euclidean space E of dimension d in an euclidean space F. Define s-numbers of L, denoted $[L]$, as the decreasing sequence

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$\lambda_1 \geqq \lambda_2 \dots \geqq \lambda_d$ of logarithms of the eigenvalues of the positive operator $(L^*L)^{1/2}$ (multiple eigenvalues are repeated according to their multiplicity).

For any such sequence $s = \lambda_1, \dots, \lambda_d$, and any real α, $0 \leqq \alpha \leqq d$, we define $c^\alpha(s)$ by $c^\alpha(s) = \sum_{i=1}^{[\alpha]} \lambda_i + (\alpha - [\alpha])\lambda_{[\alpha]+1}$, where $[\alpha]$ denotes the integer part of the number α, and dim dil s by

$$\dim \operatorname{dil} s = 0 \text{ if } \lambda_1 < 0,$$
$$\dim \operatorname{dil} s = \sup\{\alpha, 0 \leqq \alpha \leqq d, c^\alpha(s) \geqq 0\} \text{ otherwise.}$$
$$(\text{We can put } c^\alpha(s) = -\infty \text{ for } \alpha > d).$$

Let K be a compact set in a metric space. For any $\beta > 0$, any $\varepsilon > 0$, and any cover u of K by sets A_i of diameter r_i, $i \in I$, $r_i \leqq \varepsilon$, we compute $N_\beta(u, \varepsilon) = \sum_{i \in I} r_i^\beta$ and

$$M_\beta(K) = \lim_{\varepsilon \downarrow 0} \inf_u N_\beta(u, \varepsilon).$$

The Hausdorff dimension of K dim K is defined by

$$\dim K = \inf\{\beta \beta \geqq 0 | M_\beta(K) = 0\}.$$

If f is a differentiable map of Riemannian manifold X, and K a compact invariant set, we have by Douady and Oesterlé's formula

$$\dim K \leqq \sup_x \{\dim \operatorname{dil} [D_x f]\}.$$

For any measure m on a compact metric space (K, d), for any $\varepsilon > 0$ and $\delta > 0$, we call $N_d(\varepsilon, \delta, m)$ the smallest number of balls of radius ε we need to cover the space up to measure δ. We then define the dimension of m by

$$f(m) = \lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \frac{\log N_d(\varepsilon, \delta, m)}{\log 1/\varepsilon}.$$

For an invariant ergodic measure m by a map f, we shall use a definition of the entropy very close to the preceding one (cf. Katok [K]): let d_n be the metric on K defined by

$$d_n(x, y) = \max\{d(f^i x, f^i y), 0 \leqq i \leqq n\}.$$

We define the entropy $h(m)$ by

$$h(m) = \lim_{\varepsilon \searrow 0} \lim_{\delta \searrow 0} \limsup_{n \to \infty} \frac{1}{n} \log N_{d_n}(\varepsilon, \delta, m).$$

Let $P = \{p_1, p_2, \dots, p_n, \dots\}$ be a partition of X into measurable sets and define $r(x)$ by $x \in p_r(x)$. If $H(P) = -\int \log m(P_{r(x)}) m(dx) < +\infty$, we have, by Shannon–Mac Millan's theorem (cf [B])

$$\overline{\lim_n} -\frac{1}{n} \log m(p_r(x) \cap f^{-1} p_{r(fx)} \cap \dots \cap f^{-n+1} p_{r(f^{n-1}x)}) \leqq h(m). \tag{1.1}$$

If we consider a differentiable map f of a compact Riemannian manifold X, we denote $\mathscr{E}(x,f)$ the set of invariant ergodic probability measures. For any m in $\mathscr{E}(x,f)$, there exists a sequence Sp m of s numbers such that

$$\frac{1}{n}[D_x f^n] \to \text{Sp } m \text{ a.e.and in } L^1. \tag{1.2}$$

Furthermore by Oseledets' theorem (cf. [O], [$\bar{\text{R}}$], [$\bar{\text{R}}$a]), there exists almost everywhere a decreasing family of subspaces of $T_x X$,

$$T_x X = E^x_{s_1} \supset E^x_{s_2} \supset \dots \supset E^x_{s_r} \supset E^x_{s_{r+1}} = \{0\}$$

such that the map $x \to E^x_{s_i}$ is measurable and for almost all x, the sequence $\frac{1}{n}\operatorname{Log} \| D_x f^n v \|$ converges uniformly on v in $E^x_{s_j} \backslash E^x_{s_{j+1}}$ towards some element λ_{s_j} of Sp m.

Our results are the following:

Proposition 1. *If f is a differentiable map of a compact Reimannian manifold X, and m an invariant ergodic probability measure, $f(m) \geqq \frac{h(m)}{\lambda_1}$, where λ_1 is the first element of* Sp m. (Proposition 1 is empty when $\lambda_1 = 0$).

Proposition 2. *If f is a differentiable map of a Reimannian manifold X and K a invariant subset,*

$$\inf_n \sup_{x \in K} \dim \operatorname{dil} [D_x f^n] = \sup_{m \in \mathscr{E}(K,f)} \dim \operatorname{dil} \text{Sp } m.$$

Corollary. *With the same conditions, we have*

$$\dim K \leqq \sup_{m \in \mathscr{E}(K,f)} \dim \operatorname{dil} \text{Sp } m.$$

Proposition 3. *If f is a differentiable map of a compact Riemannian manifold X* such that *$D_x f$ is Hölder continuous on X and if $m \in \mathscr{E}(X,f)$, we have*

$$f(m) \leqq \dim \operatorname{dil} \text{Sp } m.$$

Let us make some comments on these results.

Proposition 1 gives a rough estimate, but in certain cases it may be the best one; for instance, in dimension 1, we have the following generalization of known results (cf. [B], [Bo], [C], [E], [F]).

Proposition 4. *Let f be a piecewise differentiable map from an interval I into itself, such that f' is piecewise monotone—Let Q be the partition of I defined by the critical points of f and of f'—. Let m be an invariant ergodic probability measure such that $H(Q) < +\infty$* and $\int \log |f'| dm > 0$. *We have*

$$f(m) = \frac{h(m)}{\int \log |f'| dm}.$$

There are examples where the estimate in the corollary is an equality, for instance when an ergodic smooth measure exists, but there are also examples where this is not true (cf. the discussion in [FKY] and below). The result of Proposition 3 is also an equality for an ergodic smooth measure and in a forthcoming paper, it will be shown that we still have $f(m) = \dim \operatorname{dil} \operatorname{Sp} m$ if the following conditions are satisfied: there is only one nonstrictly positive exponent and the measure m is absolutely continuous with respect to the unstable foliation. (L. S. Young has also obtained a related result.)

Let us also remark that all these results can be extended to a differentiable map of a Hilbert space satisfying suitable compactness conditions as long as ergodic theorems (cf. [R_2]) and [D.O] are still valid. In the case of a differentiable map of a Banach space, Spectrum and Lyapunov are defined by an extension of the Oseledets theorem [M_2], and Mañe ([M_1]) gave a formula for the capacity of a compact invariant set very close to the one we get by corollary here. The capacity of a compact set K is given by $c(K) = \lim_\varepsilon \sup \dfrac{\log N_d(\varepsilon, o, m)}{\log 1/\varepsilon}$, where m is any measure with support the whole set K. Although it is not explicitly stated that way, Mañe's estimation could actually give with the help of Proposition 2,

$$c(K) \leqq \sup_{m \in \mathscr{E}(K,f)} \dim \operatorname{dil} \operatorname{Sp} m.$$

II. Some Proofs

We first prove Proposition 1.

Let f be a differentiable map of a compact Riemannian manifold X, and $r > 0$. We put

$$\| D_x f \|_r = \max\left(\| D_x f \|, \frac{1}{r} \right).$$

For any $\chi > 0$, there exists $\varepsilon > 0$ such that $d(x, y) < \varepsilon$ implies $\| D_y f \| \leqq \| D_x f \|_r (1 + \chi)$. Let m be an invariant ergodic measure, $\delta > 0$ and let us denote $v_r(f) = \int \log \| D_x f \|_r m(dx)$.

By the ergodic theorem, there exists n_0 such that if $n \geqq n_0$ $m(A_n) \geqq 1 - \delta$, where

$$A_n = \left\{ x ;\ \sup_{0 \leqq j \leqq n} \prod_{k=0}^{j-1} \| D_{f^k_x} f \|_r \leqq e^{n v_r(f)(1+\chi)} \right\}$$

Let $\varepsilon' < \varepsilon$, if $x \in A_n$ and $d(y, x) \leqq \varepsilon' e^{-n v_r(f)(1+\chi)} (1 + \chi)^{-n}$, we have clearly by induction on $j \leqq n$,

$$d(f^j y, f^i x) < \varepsilon' e^{-n v_r(f)(1+\chi)} \prod_{k=0}^{j-1} \| D_{f^k_x} f \|_r (1 + \chi)^{j-n}.$$

In the other words the d ball of radius $\varepsilon' e^{-n v_r(f)(1+\chi)} (1 + \chi)^{-n}$ and center x is contained in the d_n ball of radius ε'. Furthermore, in a cover up to measure δ by a family of sets, the subcover made of the sets which meet A_n by a nonempty intersection covers X up to a measure 2δ.

And putting this together we have just shown

$$N_{d_n}(\varepsilon', 2\delta, m) \leqq N_d\left(\frac{\varepsilon' e^{-n v_r(f)(1+\chi)} (1 + \chi)^{-n}}{2}, \delta, m \right).$$

By taking $\lim_{\varepsilon' \to 0} \lim_{\delta \to 0} \lim_n \sup \frac{1}{n} \log$, we have, if $v_r(f) > 0$

$$h(m) \leqq f(m). \left[v_r(f) (1 + \chi) + \log(1 + \chi) \right].$$

But χ and r can be arbitrarily small. Therefore we get $h(m) \leqq f(m)$. $(\int \log \| D_x f \| m(dx))^+$ and by applying this result to $f^n n \geqq 0$, we have finally $h(m) \leqq f(m) \cdot \frac{1}{n} (\int \log \| D_x f^n \| m(dx))^+$ which by (1.2) proves Proposition 1.

Before proving Proposition 2, we have the following Lemma.

Lemma 2.1. *Let L and L' be linear operators from respectively euclidean spaces E to E' and E' to E''. We have for any real α*

$$c^\alpha([L'. L]) \leqq c^\alpha([L]) + c^\alpha([L']). \tag{2.1}$$

Proof. It is clear that if one of $c^\alpha([L])$ or $c^\alpha([L'])$ is $-\infty$, that means that either rank of L or L' is smaller than α, so that the rank of $L'.L$ is also smaller than α and (2.1) is valid.

Remark also that $c^1([L])$ is the logarithm of the norm of the operator L and that proves (2.1) for $\alpha = 1$. The proof of (2.1) for an integer α follows by considering the wedge product $\overset{\alpha}{\wedge} L$ and applying $c^\alpha([L]) = c^1([\overset{\alpha}{\wedge} L])$.

Then (2.1) follows for a rational $\alpha = \frac{p}{q}$ by considering the direct sum of q copies of the spaces E, E', E'' and the maps L, L' and by applying $c^{p/q}([L]) = \frac{1}{q} c^p([\bigoplus_q L])$.

Finally (2.1) extends to all real α by continuity.

We now consider a differentiable map f of a Riemannian manifold X and an invariant subset K and let us call $\alpha_0 = \inf_n \sup_x \dim \operatorname{dil} [D_x f^n]$.

We have clearly $\alpha_0 \geqq \sup_x \inf_n \dim \operatorname{dil} [D_x f^n] \geqq \dim \operatorname{dil} \operatorname{Sp} m$ and proving Proposition 2 is proving the converse inequality.

For all n we choose x_n such that $\dim \operatorname{dil} [D_{x_n} f^n] \geqq \alpha_0$, i.e. $c^\alpha([D_{x_n} f^n]) \geqq 0$ for all $\alpha < \alpha_0$.

Let us fix $\alpha < \alpha_0$, N, $n > N$ and $0 \leqq j < N$. We have, by Lemma 2.1:

$$\sum_{k=0}^{[n/N]-1} c^\alpha([D_{f^{Nk+j} x_n} f^N]) + 2\alpha N \sup_x \log \| D_x f \| \geqq 0.$$

By summing over j, $0 \leqq j < N$, we get:

$$\sum_{k=0}^{n-1} c^\alpha([D_{f^k x_n} f^N]) \geqq -3\alpha N^2 \sup_x \log \| D_x f \|, \text{ or}$$

$$\int c^\alpha([D_x f^N]) \left(\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x_n} \right)(dx) \geqq - \frac{3\alpha N^2 \sup_x \log \| D_x f \|}{n}, \tag{2.2}$$

where δ_z denotes the Dirac measure at the point z. Let M be the set of vague limit points of the sequence of probability measures $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k x_n}$. All measures in M are invariant and (2.2) implies $\inf_{m\in M}\int c^\alpha([D_x f^N])m(dx) \geqq 0$. So for all m in M and all N we get $\int c^\alpha\left(\frac{1}{N}[D_x f^N]\right)m(dx) \geqq 0$ and this implies, if $m = \int m_x\, m(dx)$ is the ergodic decomposition of m, by (1.2): $\int c^\alpha(\mathrm{Sp}\; m_x)m(dx) \geqq 0$, which means that there exists an ergodic measure m_0 with $c^\alpha(\mathrm{Sp}\; m_0) \geqq 0$, i.e. dim dil Sp $m_0 \geqq \alpha$. This last relation being true for any $\alpha < \alpha_0$ proves Proposition 2.

We also now prove Proposition 4: we consider a piecewise differentiable map of the interval and an invariant ergodic measure m, such that if Q is the partition in critical points, we have $H(Q) < \infty$ and $\lambda = \int \log|f'|dm > 0$. For any δ, $\frac{1}{2} > \delta > 0$, we call P_n^δ the set $P_n^\delta = \{e^{-n\delta} < |f'| \leqq e^{-(n-1)\delta}\}$ for n in $\mathbb{Z}$. Remark that P_n^δ is empty when n is mall enough.

The function $\log^+\frac{1}{|f'|}$ being integrable, we have

$$\sum_{n>0} nm(P_n^\delta) \leqq \sum_{n>0}\int_{P_n^\delta}\log\frac{1}{|f'|} < +\infty,$$

and therefore, if we call $P \vee Q$ the partition defined by the critical points and the sets P_n^δ, we have $H(P\vee Q) \leqq H(Q) - \sum m(P_n^\delta)\log m(P_n^\delta) < +\infty$. We fix now $\varepsilon > 0$ and $\chi > 0$. Call $m_n(x)$ the measure of the atom of $\bigvee_{i=0}^{n-1} f^{-i}(P\vee Q)$ which contains x. By the ergodic theorem and (1.1) there exists n_0 so that if $n \geqq n_0$,

$$m(A_n) \geqq 1 - \varepsilon/2 \text{ and } m(B_n) \geqq 1 - \varepsilon/2,$$

where

$$A_n = \left\{\frac{1}{n}\log|(f^n)'| \geqq \lambda(1-\chi)\right\}$$

$$B_n = \left\{-\frac{1}{n}\log m_n \leqq h(m) + \chi\right\}.$$

We call c_n the set of atoms of $\bigvee_{i=0}^{n-1} f^{-i}(P\vee Q)$ which intersects $A_n \cap B_n$. We have the following properties:

$m(c_n) \geqq 1 - \varepsilon$

c_n is made of less than $e^{n(h(m)+\chi)}$ atoms

c_n is an interval where f^n is monotone, so that the length of an atom a in c_n is smaller than

$\inf_{y\in a}|(f^n)'(y)|)^{-1}$.

For any two points y and z in the same atom of $\bigvee_{i=0}^{n-1} f^{-i}(P\vee Q)$, we have

$\left[\frac{(f^n)'(y)}{(f^n)'(z)}\right] \leqq e^{n\delta}$. The two last properties imply that the length of an atom in c_n is smaller than $\frac{e^{n\delta}}{e^{n\lambda(1-\chi)}}$, so that we have proved that $N_d(e^{n\delta}/e^{n\lambda(1-\chi)}, \varepsilon, m) \leqq e^{n(h(m)+\chi)}$. Therefore

$$f(m) = \lim_{\varepsilon} \overline{\lim_{n}} \frac{\log N_d(e^{n\delta}/e^{n\lambda(1-\chi)}, \varepsilon, m)}{n(\lambda(1-\chi)-\delta)} \leqq \frac{h(m)+\chi}{\lambda(1-\chi)-\delta}.$$

We first let χ be arbitrarily small and then δ to prove $f(m) \leqq \frac{h(m)}{\lambda}$. The converse inequality comes from Proposition 1 if f is differentiable, or is proved in the same way if f is only piecewise differentiable and m such that $H(Q) < +\infty$.

III. Proof of Proposition 3

The proof of proposition 3 consists in making rigorous the heuristic argument of [FKY]. We consider a differentiable map f of a compact Riemannian manifold X, such that $D_x f$ satisfies a Hölder condition of order ε. For x and y close enough, let us call τ_y^x the isometry from $T_y X$ to $T_x X$ defined by parallel transport along the geodesic. The Hölder condition means that there exists C_0 and $\varepsilon > 0$ such that

$$\| D_x f - \tau_{fy}^{fx} D_y f \tau_x^y \| \leqq C_0 (d(x, y))^{\varepsilon}.$$

From this it follows that there exists C_1 such that, if $f^i x$ and $f^i y$ stay close enough for $0 \leqq i < n$,

$$\| D_x f^n - \tau_{f^n y}^{f^n x} D_y f^n \tau_x^y \| \leqq C_1^n (d(x, y))^{\varepsilon}. \tag{3.1}$$

Let m be an invariant ergodic probability measure on X and consider the spectrum $\operatorname{Sp} m = \{\lambda_1 \geqq \ldots . \geqq \lambda_d\}$ of $D_x f$ for m. If $\lambda_1 < 0$, by [R] Corollary 6.2, m is carried by a finite set of points and $f(m) = 0$. If $\lambda_d \geqq 0$, dim dil Sp $m = d$ and Proposition 3 is also true. So for proving Proposition 3, we may choose j with $\lambda_j < 0$ and dim dil Sp $m = j - 1 - \frac{\sum_{i=1}^{j-1} \lambda_i}{i}$. Then by Oseledets' theorem, for m almost every x in X, there exists a decreasing sequence of subspaces $E_{s_1}^x = T_x X \supset E_{s_2}^x \supset \ldots . \supset E_{s_r}^x \supset E_{s_{r+1}}^x = \{0\}$, such that the map $x \to E_{s_i}^x$ is measurable and $\frac{1}{n} \log \| D_x f^n v \| \to \lambda_{s_i}$ as $n \to \infty$, uniformly on v in $E_{s_i}^x \backslash E_{s_{i+1}}^x$, $\| v \| = 1$, $i = 1, \ldots, r$. We now fix $\chi > 0$, $\delta > 0$. There exists $n_0(\delta, \chi)$ such that for any $n \geqq n_0$, $m(A_n) \geqq 1 - \delta$ where

$$A_n = \{x \in X, \| D_x f^n u \| \leqq (1+\chi)^n e^{n\lambda_{s_k}} \text{ for all } u \text{ in } E_{s_k}, \| u \| = 1 \text{ and all } k\}.$$

We choose now $\delta' > 0$, $n \geqq n_0$ and a set of balls $\{B_i, i \in I\}$ of radius r_n such that:

$$\frac{1}{2^{\varepsilon}(1+\chi)^n}\left\langle\frac{1}{2}, m\left(\bigcup_I B_i\right)\right\rangle 1-\delta',$$

$$r_n=\frac{1}{2}\left(\frac{e^{\lambda_j}}{(1+\chi)C_1}\right)^{n/\varepsilon}, |I| \leqq 2N_d(r_n, \delta', m).$$

We consider $I' \subset I$ such that if $i \in I'$, B_i intersects A_n in x_i and we consider for $i \in I'$ the ball B_i' of radius $2r_n$ and center x_i. We have clearly $|I'| \leqq I$ and $m\left(\bigcup_{I'} B_i'\right) > 1-\delta'-\delta$; the main step of the proof is proving Lemma 3.1:

Lemma 3.1. *For i in I', $f^n B_i'$ can be covered by less than K balls of radius $r_n e^{n\lambda_j}$,*

$$K=(1+\chi)^{nd}(2d)^d \prod_{i=1}^{j} e^{n(\lambda_i-\lambda_j)}.$$

We show now how Lemma 3.1 implies Proposition 3. The set $\bigcup_{I'} f^n B_i'$ can be covered by less than $K|I'|$ balls of radius $r_n e^{n\lambda_j}$ and has a measure bigger that $1-\delta'-\delta$. This means

$$\begin{aligned} N_d(r_n e^{n\lambda_j}, \delta+\delta', m) &\leqq K|I'| \\ &\leqq 2(1+\chi)^{nd}(2d)^d \prod_{i=1}^{j-1} e^{n(\lambda_i-\lambda_j)} N_d(r_n, \delta', m). \end{aligned}$$

Remark also that $r_n=\frac{1}{2}(B(\chi))^n$, where $B(\chi)=\left(\frac{e^{\lambda_j}}{(1+\chi)C_1}\right)^{1/\varepsilon}$. We have, by taking $\lim_{\delta\downarrow 0}\lim_{\delta'\downarrow 0}\overline{\lim_{n\to\infty}} \frac{\log}{n(\lambda_j-\log B(\chi))}$,

$$f(m) \leqq \frac{\log(1+\chi)^d+\sum_{i=1}^{j-1}(\lambda_i-\lambda_j)}{-(\lambda_j+\log B(\chi))}+\frac{-\log B(\chi)}{-(\lambda_j+\log B(\chi))} f(m).$$

Hence: $-\lambda_j f(m) \leqq \log(1+\chi)^d+\sum_{i=1}^{j-1}(\lambda_i-\lambda_j)$, and by taking χ arbitrarily small, we get $f(m) \leqq j-1-\frac{\sum_{i=1}^{j-1}\lambda_i}{\lambda_j}=\dim \operatorname{dil} \operatorname{Sp} m$ by our choice of λ_j.

Proof of Lemma 3.1: For any point y in B_i' we consider the points $f^n y$ and z, where z is such that $\exp_{f^n x_i}^{-1} z = D_{x_i} f^n(\exp_{x_i}^{-1} y)$, where $\exp_v$ denotes the exponential map from a neighbourhood of 0 in $T_v X$ into X.

By (3.1) we have

$$\begin{aligned} d(f^n y, z) &= \|\exp_{f^n x_i}^{-1} f^n y - \exp_{f^n x_i}^{-1} z\| \\ &\leqq C_1^n (d(x,y))^{1+\varepsilon} \\ &\leqq C_1^n r_n r_n^{\varepsilon} = \frac{C_1^n r_n e^{n\lambda_j}}{2(1+\chi)^n C_1^n} = \frac{r_n e^{n\lambda_j}}{2}. \end{aligned}$$

If we consider now $C_i = \exp_{f^n x_i}[D_{x_i} f^n(\exp_{x_i}^{-1} B_i')]$, the former computation shows that for any point $f^n y$ in $f^n B_i'$, there exists a point z in C_i with $d(f^n y, z) \leqq \frac{1}{2} r_n e^{n\lambda_j}$.

From a cover of C_i by K balls of radius $\frac{1}{2} r_n e^{n\lambda_j}$, we can therefore deduce a cover of B_i' by K balls of radius $r_n e^{n\lambda_j}$. We compute now such a K: we choose in $T_{x_i}X$ an orthonormal basis $u_1, u_2, \ldots u_d$, by first choosing a basis in $E_{s_r}^{x_i}$, then completing it into a basis in $E_{s_{r-1}}^{x_i}$, and so on, so that the asymptotic behaviour of $\| Df^n u_k \|$ is given by λ_{d-k}.

We have $\exp_{x_i}^{-1} B_i' \subset \{v \in T_{x_i}X \mid v = \sum v_k u_k, |v_k| \leqq 2r_n\}$ and therefore

$$\exp_{f^n x_i}^{-1} C_i \subset \{v \in T_{f^n x_i}X \mid v = \textstyle\sum v_k (D_{x_i} f^n) u_k, |v_k| \leqq 2r_n\},$$

where $(D_{x_i} f^n) u_k$ are independent vectors of $T_{f^n x_i}X$, satisfying

$$\| (D_{x_i} f^n) u_k \| \leqq e^{n\lambda_{d-k}} (1+\chi)^n.$$

Now cover $\exp_{f^n x_i}^{-1} C_i$ by parallelepipeds in $T_{f^n x_i}X$ of the following form,

$$\left\{ z \,\middle|\, z - x_0 = \sum t_k \frac{(D_{x_i} f^n) u_k}{\| (D_{x_i} f^n) u_k \|}, |t_k| \leqq \frac{r_n e^{n\lambda_j}}{2d} \right\},$$

with less than K such parallelepipeds, where

$$K \leqq (1+\chi)^{nd} (2d)^d \prod_{i=1}^{j-1} e^{n(\lambda_i - \lambda_j)}.$$

By using the exponential map, any such parallelepiped becomes contained in some ball of radius $\frac{r_n e^{n\lambda_j}}{2}$ and this proves the lemma.

IV. Conclusions

In this paper we introduced a number associated to a dynamical system and an invariant measure which expresses some geometrical property of the system.

For a compact metric K and a map f, one can also define

$$f(K) = \sup_{m \in \mathscr{E}(K,f)} f(m).$$

Under which conditions is the relation $\dim K \geqq f(K)$ true? Is there an abstract definition for $f(K)$, by cleverly mixing topological entropy and capacity (cf. Bowen [Bo] for instance)?

Let us come now to the comparison with numerical experiment. In [FKY], they consider some map f of the square, and the computer gives a pseudo-orbit $\{\alpha_n 1 \leqq n \leqq N\}$. What is computed is the number L of squares of radius 1/8 which cover the pseudo-orbit (except some of the very first terms) and the coefficient $s = \log_8 L$.

If one admits that the statistic of the pseudo-orbit is given by some invariant measure m, what is computed is actually $\log_8 N_d(1/8, \delta(N), m)$ with $\delta(N) \to 0$ as $N \to \infty$.

If one admits also that the system has very strong self similarity properties, then the convergence of $\frac{\log N_d(\varepsilon, \delta, m)}{\log 1/\varepsilon}$ is much faster when $\varepsilon \to 0$ than when $\delta \to 0$.

So it is sensible to believe that the number s is closer to $f(m)$ than to $\overline{\lim}_{\varepsilon \to 0} \lim_{\delta \to 0} \frac{\log N_d(\varepsilon, \delta, m)}{\log 1/\varepsilon} = c(m)$ which is the capacity of the support of the measure m.

It turns out that for an Axiom A attractor such that the stable direction is of dimension 1, and Ruelle's measure m, both numbers $f(m)$ and $c(m)$ can be computed and that we have generically

$$f(m) = \dim \operatorname{dil} \operatorname{Sp} m < c(m).$$

This remark is due to A. Manning and L. S. Young

A sensible reformulation fo [FKY]'s conjectures is therefore, under some conditions, a strange attractor generically admits an invariant measure satisfying $f(m) = \dim \operatorname{dil} \operatorname{Sp} m$.

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