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On the Symmetry of the Gibbs States in Two Dimensional Lattice Systems

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Abstract. Under fairly general conditions if a two dimensional classical lattice system has an internal symmetry group G, which is a compact connected Lie group, then all Gibbs states are G-invariant.

1. Introduction

For a large class of classical lattice systems with an internal symmetry described by a continuous group G all Gibbs states are also G-invariant if the space dimension is two [1, 2]. One says that spontaneous symmetry breakdown is impossible. This phenomenon occurs in various other situations. We refer to [3] for examples and rigorous results in the field of statistical mechanics. Results of this kind are established here for classical lattice systems on $\mathbb{Z}^2$ with a compact connected Lie group G. A lattice system is given by a measure space Ω_x, which is the space of configurations of the system at the lattice point x, a measure dw_x on each Ω_x and a potential U describing the interactions in the system. For example $\Omega_x = S^1$, the unit circle, dw_x is the uniform measure on S^1 and U is given by two-body interactions $-J(x-y)\cos(w_x - w_y)$ which are G-invariant with $G = S^1$ in an obvious way. Here $w_x \in \Omega_x$ and $w_y \in \Omega_y$. If $J(x-y) = |x-y|^{-\alpha}$, then the system is ferromagnetic. Theorem 1 below proves that for $\alpha \geqq 4$ all Gibbs states are G-invariant and there is no spontaneous magnetization. On the other hand if $2 < \alpha < 4$ there is spontaneous magnetization at low temperature and therefore there are Gibbs states which are not G-invariant [4]. This remains true with $\Omega_x = S^n$, the n-sphere in $\mathbb{R}^{n+1}$, and $(w_x | w_y)$ instead of $\cos(w_x - w_y)$, where $(-|-)$ is the Euclidean scalar product in $\mathbb{R}^{n+1}$. G is S^n and the results follow from [5] when $2 < \alpha < 4$.

The results of this paper extend previous results obtained by Dobrushin and Shlosman [1] and [2]. First of all the theorem below covers the case of power-law decaying interactions and not only exponentially decaying interactions (see also Remark 2 at the end of Sect. 2). This extension gives complete results for the class of ferromagnetic systems introduced above (see also Remark 1 at the end of

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Sect. 2). Finally the proof is quite different. It is based on a very simple physical argument dealing with the energy of configurations. This leads to a proof free of technical difficulties, which is not the case in [2]. Since the general case can be obtained from the case with $G = S^1$ (see [1]), the next section gives the results in this special situation and for two-body interactions only. Many-body interactions and the general case with a compact connected Lie group G are treated in the last section.

2. Main Results for $G = S^1$

In this section $G = S^1$, the simplest compact connected Lie group, and the potential U is given by two-body interactions $U_{x,y}$ only, x and $y \in \mathbb{Z}^2$. For the sake of simplicity all Ω_x are the same and $U_{x,y}(w_x, w_y) = U_{x,y}(w_y, w_x)$ is translation invariant. The group G acts on Ω_x and the action of $g \in G$ on $w_x \in \Omega$ is denoted by $g \cdot w_x \in \Omega_x$. The main assumptions are the following one.

a) *G-Invariance.* The measure dw_x is G-invariant and the potential U is G-invariant

$$U_{x,y}(g \cdot w_x, g \cdot w_y) = U(w_x, w_y).$$

b) *Smoothness.* For any two-element subset of $\mathbb{Z}^2$, any fixed w_x and w_y, the real-valued function

$$g \to U_{x,y}(w_x, g \cdot w_y)$$

defined on S^1 is twice differentiable. Since $S^1 \cong \mathbb{R}/\mathbb{Z}$ this function may be considered as a periodic function on the real line $\mathbb{R}$. The first and second derivatives are denoted by $U'_{x,y}$ and $U''_{x,y}$.

To express the decay property, which is the next condition, the following notations are used. If $x = (x^1, x^2) \in \mathbb{Z}^2$ then $|x| = \max(|x^1|, |x^2|)$. For each positive integer k let

$$f_k(\lambda) = \max(1, \ln_k \lambda)$$

where $\ln_k \lambda = \ln \ln_{k-1} \lambda$ and let $f_k(\lambda) = 1$ whenever $\ln_k \lambda$ is not defined. For $1 < \beta < 2$ $f_k(\lambda^\beta) \leqq \beta f_k(\lambda)$.

c) *Decay Property.* Let

$$J(|x-y|) = \| U''_{x,y} \|_\infty = \sup_{w_x, w_y} |U''_{x,y}(w_x, g \cdot w_y)|.$$

There exist a positive constant C and an integer p so that

$$\sum_{|y| \leqq L} J(|y|)|y|^2 \leqq C \prod_{k=1}^{p} f_k(L) \equiv CF_p(L).$$

This means that the divergence of the above expression is at most like $\ln L \ln_2 L \ldots \ln_p L$ for large L.

Remark. It is also supposed of course that the Gibbs measures for finite systems are well-defined and so on. In particular

$$\sum_{y} \| U_{x,y} \|_{\infty} < \infty$$

in order that the thermodynamic limit makes sense.

Theorem 1. *If a classical lattice system satisfies conditions A, B and C, then all Gibbs states are G-invariant.*

Proof. The proof is based on a physical intuitive argument of Herring an Kittel [6] showing, in the case of the ferromagnetic models described in the introduction, that there is no state with spontaneous magnetization.

Let $g \in G$ be fixed and let Λ_l be the subset $\{x : |x| \leqq l\}$ of $\mathbb{Z}^2$. The main step in the proof is to show that for any configuration $w = (w_x, x \in \mathbb{Z}^2)$ of the infinite system there exists another configuration $\hat{w} = (\hat{w}_x, x \in \mathbb{Z}^2)$ with the properties
a) $\hat{w}_x = g \cdot w_x$, if $|x| \leqq l$
b) $\hat{w}_x = w_x$, if $|x| \geqq l + L$ for some L
c) $E(\hat{w}) - E(w) \leqq K$, K independent of g and l
where $E(\hat{w}) - E(w)$ is the energy difference between the two configurations. This quantity is well-defined since w and $\hat{w}$ are different only over a finite region. Using the isomorphism between G and $\mathbb{R}/\mathbb{Z}$ the identity element of G is represented by 0 and the element g by $\varphi \in [0, 1)$ or by $\psi \in [-1, 0)$ such that $\psi + 1 = \varphi$. Let

$$0 < \varphi_L < \varphi_{L-1} < \ldots < \varphi_1 \leqq \varphi \text{ and } 0 > \psi_L > \psi_{L-1} > \ldots > \psi_1 \geqq \psi.$$

Each ψ_i or φ_i represents a well-defined element of G denoted by the same symbol. The argument of Herring and Kittel suggests to define $\hat{w}$ as w^1 or w^2 where

$$\begin{array}{lll} w_x^1 = w_x & w_x^2 = w_x, & x \notin \Lambda_{L+l} \\ w_x^1 = \varphi_n \cdot w_x & w_x^2 = \psi_n \cdot w_x, & |x| = n + l,\ 1 \leqq n \leqq L \\ w_x^1 = \varphi \cdot w_x & w_x^2 = \psi \cdot w_x, & x \in \Lambda_l. \end{array}$$

In particular $\varphi \cdot w_x = \psi \cdot w_x = g \cdot w_x$. Let

$$Q(L) = \sum_{1 \leqq k \leqq L} \frac{1}{kF_p(k)}.$$

For large L $Q(L)$ diverges like $\ln_{p+1} L$.

The choice of φ_n and ψ_n is

$$\varphi_n = \frac{\varphi}{Q(L)} \sum_{n \leqq k \leqq L} \frac{1}{kF_p(k)}$$

and

$$\psi_n = \frac{\psi}{Q(L)} \sum_{n \leqq k \leqq L} \frac{1}{kF_p(q)}.$$

Therefore $\varphi_1 = \varphi$ and $\psi_1 = \varphi$. Let φ_x (respectively ψ_x) be the rotation applied at x.

For all x and y $|\varphi_x - \varphi_y| \leqq 1$. For $l+1 \leqq |x| < |y| \leqq L+l$,

$$\begin{aligned}\varphi_x - \varphi_y &= \frac{\varphi}{Q(L)} \sum_{|x|-l \leqq k < |y|-l} \frac{1}{kF_p(k)} \\ &\leqq \frac{\varphi}{Q(L)} \frac{|x-y|}{(|x|-l)F_p(|x|-l)}.\end{aligned} \tag{2.1}$$

Finally for $x \in \Lambda_l$ and $|y| > l$

$$\varphi_x - \varphi_y = \frac{\varphi}{Q(L)} \sum_{1 \leqq k < |y|-l} \frac{1}{kF_p(k)} \leqq \frac{\varphi}{Q(L)} Q(|x-y|). \tag{2.2}$$

Similar estimates hold for ψ_x.
By hypothesis A

$$U(w_x^1, w_y^1) = U(\varphi_x . w_x, \varphi_y \cdot w_y) = U(w_x, (\varphi_y - \varphi_x) \cdot w_y)$$

By hypothesis B and with $\alpha \in (0, 1)$

$$U(w_x, \alpha \cdot w_y) = U(w_x, w_y) + U'(w_x, w_y)\alpha + \tfrac{1}{2} U''(w_x, \theta \cdot w_y)\alpha^2$$

for some θ depending on w_x and w_y, $0 < \theta < \alpha$.
By hypothesis C

$$\sum_y J(|y|)|y|^{2-\varepsilon} \leqq C' < \infty$$

for fixed $\varepsilon > 0$. Therefore

$$\sum_y J(|y|) Q^2(|y|) \leqq C_1 < \infty \tag{2.3}$$

and there exists β, $1 < \beta < 2$, such that

$$\sum_{|y| \geqq L^\beta} J(|y|) Q^2(|y|) \leqq C_2 L^{-3} \tag{2.4}$$

and

$$\sum_{|y| \leqq L^\beta} J(|y|)|y|^2 \leqq C_2 F_p(L) \tag{2.5}$$

For a given w the configuration $\hat{w}$ will be w^1 or w^2 according to the value of $E(w^i) - E(w)$.

$$\begin{aligned}E(w^1) - E(w) &= \sum_{x \in \Lambda_{L+l}} \sum_{y:|y|>|x|} \{U(w_x^1, w_y^1) - U(w_x, w_y)\} \\ &= \sum_{x \in \Lambda_{L+l}} \sum_{\substack{y:|y|>|x| \\ |y|>l}} U'(w_x, w_y)(\varphi_y - \varphi_x) \\ &\quad + \sum_{x \in \Lambda_{L+l}} \sum_{\substack{y:|y|>|x| \\ |y|>l}} \tfrac{1}{2} U''(w_x, \theta \cdot w_y)(\varphi_y - \varphi_x)^2.\end{aligned}$$

The last line is smaller in absolute value than (see (2.1) and (2.2))

$$\frac{1}{Q^2(L)}\sum_{x\in\Lambda_l}\sum_y J(|y|)Q^2(|y|)+\frac{1}{Q^2(L)}\sum_{x\in\Lambda_{L+l}\setminus\Lambda_l}\sum_{|y|\leqq(|x|-l)^\beta} J(|y|)\frac{|y|^2}{(|x|-l)^2F_p^2(|x|-l)}$$

$$+\frac{1}{Q^2(L)}\sum_{x\in\Lambda_{L+l}\setminus\Lambda_l}\sum_{|y|\geqq(|x|-l)^\beta} J(|y|)Q^2(|y|).$$

By (2.3), (2.4) and (2.5) this is smaller than

$$\frac{1}{Q^2(L)}(2l+1)^2K_1+\frac{1}{Q^2(L)}C_2\sum_{n=1}^{L}8(n+l)\frac{F_p(n)}{n^2F_p^2(n)}$$

$$+\frac{1}{Q^2(L)}C_2\sum_{n=1}^{L}8(n+l)\frac{1}{n^3}=\frac{1}{Q^2(L)}(K_1(2l+1)^2+K_2Q(L)+K_3l+K_4).$$

Therefore, for any fixed finite l, the last expression is smaller than some K, independent of l and g when L is large enough. When w^1 is replaced by w^2 a similar estimate holds and the term containing the first derivatives of the potential is in this case

$$\sum_{x\in\Lambda_{L+l}}\sum_{\substack{y:|y|>|x|\\|y|>l}} U'(w_x,w_y)(\psi_y-\psi_x).$$

Since $(\psi_y-\psi_x)=\psi\varphi^{-1}(\varphi_y-\varphi_x)$ and $\psi\varphi^{-1}<0$ the above expression and the corresponding one for w^1 have different signs. Therefore for any configuration w and any finite box Λ_l and any rotation $g\in G=S^1$, there exists a configuration $\hat{w}$ such that

$$E(\hat{w})-E(w)\leqq K.$$

This is the key estimate. The rest of the proof is an easy adaptation of the main result of [7].

Let $g\in G$ be given. Let f be a positive local observable depending only on w_x for $x\in\Lambda_l$. Let $\Lambda=\Lambda_{L+l}$ with L large enough. The Gibbs state for the finite region Λ and a fixed configuration w_{Λ^c} outside Λ is given as usual by the measure on $\prod_{x\in\Lambda}\Omega_x=\Omega^\Lambda$

$$\nu_\Lambda(d\eta_\Lambda|w_{\Lambda^c})=\frac{\exp(-E(\eta_\Lambda|w_{\Lambda^c}))d\eta_\Lambda}{Z_\Lambda(w_{\Lambda^c})}.$$

Let μ be any Gibbs state of the infinite system and μ_g the Gibbs state obtained from μ by a rotation g. By definition of Gibbs state the expectation value of f in the state μ_g is

$$\langle f\rangle_{\mu_g}=\int\mu(dw)f(g\cdot w)=\int\mu(dw)\int\nu_\Lambda(d\eta_\Lambda|w_{\Lambda^c})f(g\cdot\eta_\Lambda). \tag{2.6}$$

Let w_{Λ^c} be fixed. On Ω^Λ the transformations T_1 and T_2 are one-to-one

$$T_1 : w_\Lambda \mapsto w_\Lambda^1$$
$$T_2 : w_\Lambda \mapsto w_\Lambda^2$$

where w_Λ^1 and w_Λ^2 are the configurations studied before. They leave the measure $d\eta_\Lambda$ invariant by hypothesis A. Furthermore they coincide with the rotation g on $\prod_{x \in \Lambda_l} \Omega_x$.

Finally there exists a partition of Ω^Λ in two subsets Ω_1 and Ω_2 such that

$$E(T_i w_\Lambda) - E(w_\Lambda) \leqq K, \forall w_\Lambda \in \Omega_i.$$

Consequently if χ_i is the characteristic function of Ω_i

$$\begin{aligned} Z_\Lambda(w_{\Lambda^c}) \int \nu_\Lambda(d\eta_\Lambda | w_{\Lambda^c}) f(g \cdot \eta_\Lambda) &= \sum_{i=1,2} \int d\eta_\Lambda \exp(-E(\eta_\Lambda | w_{\Lambda^c}))\chi_i(\eta_\Lambda) f(T_i \cdot \eta_\Lambda) \\ &= \sum_{i=1,2} \int d\eta_\Lambda \exp(-E(T_i \cdot \eta_\Lambda | w_{\Lambda^c}))\chi_i(\eta_\Lambda) f(T_i \eta_\Lambda) \\ &\quad \cdot \exp(-E(\eta_\Lambda | w_{\Lambda^c}) + E(T_i \cdot \eta_\Lambda | w_{\Lambda^c})) \\ &\leqq e^K \sum_{i=1,2} \int d\eta_\Lambda \exp(-E(T_i \cdot \eta_\Lambda | w_{\Lambda^c}))\chi_i(\eta_\Lambda) f(T_i \cdot \eta_\Lambda) \\ &\leqq 2 \cdot e^K \int d\eta_\Lambda \exp(-E(T_i \cdot \eta_\Lambda | w_{\Lambda^c})) f(T_i \cdot \eta_\Lambda) \\ &= 2 \cdot e^K \int d\eta_\Lambda \exp(-E(\eta_\Lambda | w_{\Lambda^c})) f(\eta_\Lambda). \end{aligned}$$

Using this inequality in (2–6) and integrating with respect to μ gives

$$\langle f \rangle_{\mu_g} \leqq \tilde{K} \langle f \rangle_\mu$$

where $\tilde{K}$ is independent of f, μ and g. Therefore there exists $0 < \tilde{K} < \infty$ independent of f, μ and g such that

$$\tilde{K}^{-1} \langle f \rangle_{\mu_g} \leqq \langle f \rangle_\mu \leqq \tilde{K} \langle f \rangle_{\mu_g}.$$

Since these last inequalities are true for the characteristic functions of cylindrical subsets, they remain true, by a limiting procedure, for any characteristic functions of subsets of the tail field. Let μ be an extremal Gibbs state. These inequalities show that μ and μ_g coincide on the tail field (since μ_g is extremal) and therefore $\mu = \mu_g$ i.e. μ is g invariant for all $g \in G$. This finishes the proof.

Remark 1. The example of ferromagnetic models described in the introduction shows that the theorem is valid if $\alpha \geqq 4$. It is not valid if $\alpha < 4$. More precisely the theorem is valid for the coupling constants $J(|x-y|)$ behaving for large $|x-y|$ like

$$(|x-y|)^{-4} \ln_2 |x-y| \ldots \ln_p |x-y|.$$

On the other hand the proof fails if the behavior of $J(|x-y|)$ for large $|x-y|$ is like

$$(|x-y|)^{-4} (\ln |x-y|)^\varepsilon$$

or even

$$(|x-y|)^{-4} \ln_2 |x-y| \ldots \ln_{p-1}(|x-y|)(\ln_p(|x-y|))^{1+\varepsilon}$$

with $\varepsilon > 0$. In fact the theorem is not true in these cases. Indeed using the results of [5] it is sufficient to find a reflection positive potential with such a behavior for large $|x-y|$ in order to have a counter-example to the theorem. Potentials with such a behavior can be constructed [9].

Remark 2. If the interactions $U_{x,y}$ have an exponential decay for large $|x-y|$, then the condition C can be weakened and the growth condition $f_1(L)\dots f_p(L)$ replaced by L. One uses the exponential decay as follows.

Let $\|U_{x,y}\|_\infty \leqq c\, e^{-\kappa|x-y|}$. Then there exists $\beta > 0$ such that

$$\sum_{\substack{y:\\ |y-x|\geqq \ln L^\beta}} \|U_{x,y}\|_\infty \leqq \frac{1}{L^3}.$$

Therefore it is sufficient to be able to bound the sum with $U''_{x,y}$ only for $|x-y| \leqq \ln L^\beta$. This is possible if $\varphi_n = \frac{\varphi}{Q(L)} \sum_{k\geqq n}^{L} \frac{1}{k}$ and $Q(L) = \sum_{k=1}^{L} \frac{1}{k}$. Therefore the results of Shlosman [2] are covered.

Remark 3. The results are still valid for systems extended in three dimensions provided the thickness is finite.

Remark 4. No particular property of the space Ω_x of the configurations at x is used in the proof. However since the condition C is expressed through the sup-norm $\|\cdot\|_\infty$ genuine models of unbounded spin systems in statistical mechanics do not satisfy the hypothesis of the theorem.

Remark 5. Examples of systems with a continuous symmetry group and with several phases in two dimensions can be found in the work of Shlosman [8].

Remark 6. It is sufficient to have that $U(w_x, \alpha\cdot w_y) = U(w_x, w_y) + U'(w_x, w_y)\alpha + \theta_{x,y}(w_x, w_y, \alpha)\cdot\alpha^2$ with $|\theta_{x,y}(w_x, w_y, \alpha)| \leqq J(|x-y|)$, where $\theta_{x,y}(w_x, w_y, \alpha)$ is some real valued function. Therefore if $U'_{x,y}$ exists and satisfies a kind of Lipschitz condition the theorem is also valid.

Remark 7. Concerning the idea, at the end of the proof, which was taken in [7], one should mention earlier works of Sakai in [10] and of Rost, reported in [11].

3. Generalizations

3.1. Many Body Interactions

The restriction to two body interactions can be removed. In the general case the potential U is a family of functions U_A indexed by the finite subsets A of $\mathbb{Z}^2$. Let A be the subset $\{x_1, \dots, x_n\} \subset \mathbb{Z}^2$. The function U_A is defined on $\Omega^A = \prod_{x\in A} \Omega_x$ and an element of Ω^A is $w_A = (w_x : x\in A)$. Let U_A be a symmetric function of its arguments $w_{x_1}, \dots, w_{x_n}$. Let φ be a rotation. The function U_A must be G-invariant:

$$U_A(\varphi\cdot w_{x_1}, \dots, \varphi\cdot w_{x_n}) = U_A(w_{x_1}, \dots, w_{x_n}).$$

For fixed w_A and for $\alpha_2, \dots, \alpha_n \in G$, $U_A(w_{x_1}, \alpha_2 \cdot w_{x_2}, \dots, \alpha_n \cdot w_{x_n})$ defines a function on $G \times \dots \times G(n-1$ factors). The function must be twice differentiable in the variables $\alpha_2, \dots, \alpha_n$.

Let

$$J(A) = \sum_{j=2}^{n} \sum_{k=2}^{n} \| U_{A,\alpha_j,\alpha_k} \|_\infty |x_1 - x_j| |x_1 - x_k|$$

where $U_{A,\alpha_j\alpha_k}$ is the derivative of U_A with respect to α_j and α_k. Then the decay condition becomes

$$\sum_{\substack{A \ni x_1 \\ \Lambda_L(x_1) \cap (A \backslash \{x_1\}) \neq \emptyset}} J(A) \leqq C \prod_{i=1}^{p} f_i(L)$$

with $\Lambda_L(x_1) = \{x : |x - x_1| \leqq L\}$. Under these conditions theorem 1 is still valid.

3.2. Compact Connected Lie Groups

An argument used by Dobrushin and Shlosman [1] shows that the case where G is a compact connected Lie group follows from the previous situation. Smoothness and decay conditions are as before. The reduction of the general case to the case $G = S^1$ is done as follows. For any element $g \in G$ there is a one parameter subgroup of G containing g. If this subgroup is closed then it is isomorphic to S^1. Otherwise the closure of this subgroup is isomorphic to a torus. This shows that there exists a dense subset G_0 of G such that any element of G_0 is contained in a subgroup isomorphic to S^1. From the proof of the theorem it is clear that in the general case this is sufficient in order to prove Theorem 1 under the appropriate smoothness and decay conditions.

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