Commun. Math. Phys. 78, 65–74 (1980)

Communications in Mathematical Physics

Application of Dobrushin's Uniqueness Theorem to N-Vector Models

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Abstract. We apply Dobrushin's uniqueness theorem to N-Vector models to derive an upper bound of the critical temperature for unique equilibrium. In the case of isotropic ferromagnetic pair interactions this upper bound is the mean field critical temperature multiplied by a numerical factor.

1. Introduction

Recently, Driessler, Landau and Perez [1], with subsequent improvement by Simon [2], have established that N-Vector models with isotropic ferromagnetic pair interactions do not exhibit spontaneous magnetization for temperatures greater than the mean field critical temperature. In this paper we consider a related problem: to establish an upper bound for the critical temperature above which N-Vector models, with general interactions, exhibit a unique equilibrium state. Applying Dobrushin's uniqueness theorem [3] we derive an upper bound which, for isotropic ferromagnetic pair interactions, is the mean field critical temperature multiplied by a numerical factor of $\sqrt{5}$. For $N > 5$ this result improves a previous estimate of Simon [4] by a factor of $\sqrt{5/N}$.

2. Statement of Main Result

We consider the lattice model on $\mathbb{Z}^\nu$ with single spin space S^{N-1}, $N \geqq 2$. The configuration space of the lattice is the space of all functions $\sigma : \mathbb{Z}^\nu \to S^{N-1}$, denoted by $(S^{N-1})^{\mathbb{Z}^\nu}$. $(S^{N-1})^{\mathbb{Z}^\nu}$ is a topological space with product topology inherited from S^{N-1}. For $\sigma \in (S^{N-1})^{\mathbb{Z}^\nu}$, $\boldsymbol{\sigma}_a$ will denote the value of σ at lattice site a, and σ_a^i the i^{th} component of $\boldsymbol{\sigma}_a$ (with respect to the natural basis $\{\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \dots, \hat{\mathbf{n}}_N\}$ of $\mathbb{R}^N$). The a priori measure μ_0 is the invariant probability measure on S^{N-1}.

To simplify the notation we presently consider only two-body and one-body

* Research supported by U.S. National Science Foundation under grants MCS-78-01885 (B.S.) and PHY-78-25390-A01 (E.H.L.)

0010-3616/80/0078/0065/$02.00

interactions, of the form

$$\sum_{i,k=1,2,\cdots,N} J^{i,k}_{\{a,b\}}\sigma^i_a\sigma^k_b \quad \text{and} \quad \sum_{i=1}^{N} h^i_a\sigma^i_a, \text{ respectively,}$$

where $J^{i,k}_{\{a,b\}}$ and h^i_a are real numbers. The general case is noted later (Remark 3.6).

Form the $N \times N$ matrix $(J_{\{a,b\}})$ with ik^{th} entry

$$(J_{\{a,b\}})_{ik} \equiv J^{i,k}_{\{a,b\}}. \tag{2.1}$$

Viewing $(J_{\{a,b\}})$ as an operator on $\mathbb{R}^N$ (with Euclidean norm), denote its operator norm by $\|(J_{\{a,b\}})\|_{\mathrm{op}}$. Let

$$T^N_C \equiv \left(\sup_{a\in\mathbb{Z}^\nu} \sum_{b\in\mathbb{Z}^\nu\setminus\{a\}} \|(J_{\{a,b\}})\|_{\mathrm{op}}\right)\frac{\sqrt{5}}{N}. \tag{2.2}$$

The main result of this paper is contained in the following theorem.

Theorem 1. *For temperatures $T > T^N_C$, the N-Vector models defined above exhibit at most one equilibrium state*[1].

Remark 2.1. For $N > 5$ this result improves a previous estimate of Simon [4] by a factor of $\sqrt{5/N}$.

Remark 2.2. In the isotropic ferromagnetic case, i.e.

$$(J_{\{a,b\}}) \equiv J_{\{a,b\}}\cdot\mathbb{1} \tag{2.3}$$

with $J_{\{a,b\}}$ a non-negative real number, T^N_C is the mean field critical temperature multiplied by $\sqrt{5}$.

Remark 2.3. It has been known for some time, from techniques different than those employed in this paper, that the mean field temperature is an upper bound of the critical temperature for unique equilibrium for the spin-1/2 Ising model (1-Vector model). The most general formulation of this result appears in [5].

3. Proof of Theorem 1

Let $\sigma\in(S^{N-1})^{\mathbb{Z}^\nu}$ and $a\in\mathbb{Z}^\nu$. Define $\mathbf{h}^{a,\sigma}_{eff}\in\mathbb{R}^N$ by

$$(h^{a,\sigma}_{eff})^i \equiv \sum_{b\in\mathbb{Z}^\nu\setminus\{a\}} \sum_{k=1}^{N} J^{i,k}_{\{a,b\}}\sigma^k_b + h^i_a. \tag{3.1}$$

We denote the conditional Gibbs measure on S^{N-1} at site a, with external boundary condition $\sigma|_{\mathbb{Z}^\nu\setminus\{a\}}$, by $\mu^a(\cdot|\sigma)$. Observe we may write it as

$$d\mu^a(\mathbf{x}|\sigma) = \frac{1}{Z_{\mathbf{h}^{a,\sigma}_{eff}}}\exp(\mathbf{h}^{a,\sigma}_{eff}\cdot\mathbf{x})d\mu_0(\mathbf{x}) \quad (\mathbf{x}\in S^{N-1}), \tag{3.2}$$

$$\text{where } Z_{\mathbf{h}^{a,\sigma}_{eff}} \equiv \int_{S^{N-1}} \exp(\mathbf{h}^{a,\sigma}_{eff}\cdot\mathbf{x})d\mu_0(\mathbf{x}).$$

1 We assume T^N_C is finite.

Definition. For two Borel probability measures μ, ν on S^{N-1} define the Vaserstein distance $V(\mu, \nu)$ between μ and ν by

$$V(\mu, \nu) \equiv \sup_{\|f\|_L = 1} |\mu(f) - \nu(f)|, \tag{3.3}$$

the supremum being taken over all functions f, defined on S^{N-1}, which have Lipschitz norm 1, i.e.

$$\sup_{\substack{\mathbf{x},\mathbf{y} \in S^{N-1} \\ \mathbf{x} \neq \mathbf{y}}} \frac{|f(\mathbf{x}) - f(\mathbf{y})|}{\|\mathbf{x} - \mathbf{y}\|_2} = 1, \tag{3.4}$$

where $\|\cdot\|_2$ denotes Euclidean norm on $\mathbb{R}^N$.

For an ordered pair of lattice sites (a, b) form the quantity

$$D_{ab} \equiv \sup \left\{ \frac{V(\mu^a(\cdot|\sigma), \mu^a(\cdot|\sigma'))}{\|\boldsymbol{\sigma}_b - \boldsymbol{\sigma}'_b\|_2} : \sigma = \sigma' \text{ off } \{b\} \text{ and } \boldsymbol{\sigma}_b \neq \boldsymbol{\sigma}'_b \right\}. \tag{3.5}$$

The proof of Theorem 1 is an application of Dobrushin's uniqueness theorem for N-Vector models.

Theorem 2. (Dobrushin's uniqueness theorem for N-Vector models). *If* $\sup_{a \in \mathbb{Z}^\nu} \sum_{b \in \mathbb{Z}^\nu \setminus \{a\}} D_{ab} < 1$, *the N-Vector models defined above exhibit at most one equilibrium state.*

Remark 3.1. Theorem 2 is a special case of a uniqueness theorem by Dobrushin [3]. We also remark the equivalence of the Vaserstein distance referred in [3] to (3.3) is noted, for example, by Vershik [6] and also Rubinstein [7]. A more general discussion is given in [13].

A direct proof of Theorem 2 is provided by slightly modifying an argument of Gross, used to prove another special form of Dobrushin's uniqueness theorem (see proof of Theorem 1 in [8]). Referring directly to his paper, one makes the following redefinitions

$$\rho_{ab} \equiv D_{ba}, \quad \text{for} \quad a, b \in \mathbb{Z}^\nu \tag{3.6}$$

and for $f \in C[(S^{N-1})^{\mathbb{Z}^\nu}]$,

$$\delta_a(f) \equiv \sup \left\{ \frac{|f(\sigma) - f(\sigma')|}{\|\boldsymbol{\sigma}_a - \boldsymbol{\sigma}'_a\|_2} : \sigma = \sigma' \text{ off } \{a\} \text{ and } \boldsymbol{\sigma}_a \neq \boldsymbol{\sigma}'_a \right\}. \tag{3.7}$$

With these modified definitions, his argument can be applied to prove Theorem 2.

To apply Theorem 2, I use the following estimate.

Theorem 3. *For* $\mathbf{h}, \mathbf{h}' \in \mathbb{R}^N$, *let* $\mu_{\mathbf{h}}, \mu_{\mathbf{h}'}$ *be probability measures on* S^{N-1} *given by* $d\mu_{\mathbf{h}}(\mathbf{x}) = \frac{1}{Z_{\mathbf{h}}} e^{\mathbf{h}\cdot\mathbf{x}} d\mu_0(\mathbf{x})$ *and* $d\mu_{\mathbf{h}'}(\mathbf{x}) = \frac{1}{Z_{\mathbf{h}'}} e^{\mathbf{h}'\cdot\mathbf{x}} d\mu_0(\mathbf{x})$, *where* $Z_{\mathbf{h}} \equiv \int_{S^{N-1}} e^{\mathbf{h}\cdot\mathbf{x}} d\mu_0(\mathbf{x})$ *and* $Z_{\mathbf{h}'} \equiv \int_{S^{N-1}} e^{\mathbf{h}'\cdot\mathbf{x}} d\mu_0(\mathbf{x})$. *Then,*

$$V(\mu_{\mathbf{h}}, \mu_{\mathbf{h}'}) \leqq \frac{\sqrt{5}}{N} \|\mathbf{h}' - \mathbf{h}\|_2. \tag{3.8}$$

We first give the proof of Theorem 1, followed by a proof of Theorem 3.

Proof of Theorem 1. For the Vaserstein distance appearing in (3.5) we obtain from Theorem 3, (3.1) and (3.2)

$$V(\mu^a(\cdot|\sigma), \mu^a(\cdot|\sigma')) \leqq \frac{\sqrt{5}}{N} \| \mathbf{h}^{a,\sigma}_{eff} - \mathbf{h}^{a,\sigma'}_{eff} \|_2 = \frac{\sqrt{5}}{N} \| (J_{\{a,b\}})(\boldsymbol{\sigma}_b - \boldsymbol{\sigma}'_b) \|_2$$

$$\leqq \frac{\sqrt{5}}{N} \| (J_{\{a,b\}}) \|_{\mathrm{op}} \| \boldsymbol{\sigma}_b - \boldsymbol{\sigma}'_b \|_2 . \tag{3.9}$$

Combining (3.9) with (3.5) gives

$$D_{ab} \leqq \frac{\sqrt{5}}{N} \| (J_{\{a,b\}}) \|_{\mathrm{op}} . \tag{3.10}$$

Theorem 1 now follows from (3.10) and Theorem 2.

To prove Theorem 3, we essentially write the Vaserstein distance as the variation of an expectation along an appropriate path in h-space. We use two lemmas to control separately variation parallel to the "magnetic field", and perpendicular to the "magnetic field", respectively, and then combine these two cases via geometric considerations to control a general variation.

Lemma 3.2. *For* $\mathbf{h} \in \mathbb{R}^N$, *let* $\mu_{\mathbf{h}}$ *be a probability measure on* S^{N-1} *given by*

$$d\mu_{\mathbf{h}}(\mathbf{x}) \equiv \frac{1}{Z_{\mathbf{h}}} e^{\mathbf{h}\cdot\mathbf{x}} d\mu_0(\mathbf{x}), \tag{3.11}$$

where $Z_{\mathbf{h}} \equiv \int_{S^{N-1}} e^{\mathbf{h}\cdot\mathbf{x}} d\mu_0(\mathbf{x})$.

Also, let $f \in C(S^{N-1}) : \| f \|_L = 1$, *and let* $\hat{\mathbf{e}} \in S^{N-1} : \hat{\mathbf{e}} \perp \mathbf{h}$. *Then,*

$$|\mu_{\mathbf{h}}[(\mathbf{x}\cdot\hat{\mathbf{e}})f]| \leqq \frac{1}{N}. \tag{3.12}$$

Proof. There exists a rotation transforming $\{\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \dots, \hat{\mathbf{n}}_N\}$ into an O.N. basis[2] $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \dots, \hat{\mathbf{e}}_N\}$ with $\hat{\mathbf{e}}_1 \equiv \hat{\mathbf{e}}$. Denote $\mathbf{x}\cdot\hat{\mathbf{e}}_i$ by x_i. Observing that $\mu_{\mathbf{h}}$ is invariant under the operation

$$(x_1, x_2, \dots, x_N) \to (-x_1, x_2, \dots, x_N),$$

we have

$$\mu_{\mathbf{h}}[(\mathbf{x}\cdot\hat{\mathbf{e}})f] \equiv \mu_{\mathbf{h}}(x_1 f)$$

$$= \tfrac{1}{2}\mu_{\mathbf{h}}[x_1(f(x_1, x_2, \dots, x_N) - f(-x_1, x_2, \dots, x_N))] \tag{3.13}$$

Thus, from (3.13),

$$|\mu_{\mathbf{h}}[(\mathbf{x}\cdot\hat{\mathbf{e}})f]| \leqq \tfrac{1}{2}\mu_{\mathbf{h}}[|x_1| \, |f(x_1, x_2, \dots, x_N) - f(-x_1, x_2, \dots, x_N)|]$$

$$\leqq \mu_{\mathbf{h}}(x_1^2). \tag{3.14}$$

2 In this paper, the symbols "$\perp$", "$\|$" and "O.N." will denote, respectively, "perpendicular to", "parallel to" and "orthonormal".

The last inequality uses the fact that $\| f \|_L = 1$ and Euclidean distance is invariant under rotations.

To conclude the proof we use (3.14) and note that the inequality

$$\mu_{\mathbf{h}}(x_1^2) \leqq \frac{1}{N} \tag{3.15}$$

has been established by Dyson, Lieb and Simon (see [9], Theorem D.2).

Lemma 3.3. *Let the measure* $\mu_{\mathbf{h}}$ *and the function* f *be given as in Lemma 3.2. Let* $\hat{\mathbf{e}} \in S^{N-1} : \hat{\mathbf{e}} \| \mathbf{h}$ *and* $\hat{\mathbf{e}} \cdot \mathbf{h} > 0$.
Then, with $\langle \mathbf{x} \cdot \hat{\mathbf{e}} \rangle_{\mathbf{h}} \equiv \mu_{\mathbf{h}}(\mathbf{x} \cdot \hat{\mathbf{e}})$,

$$|\mu_{\mathbf{h}}[(\mathbf{x} \cdot \hat{\mathbf{e}} - \langle \mathbf{x} \cdot \hat{\mathbf{e}} \rangle_{\mathbf{h}})f]| \leqq \frac{2}{N}. \tag{3.16}$$

Proof. Again we note there exists a rotation transforming $\{\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \ldots, \hat{\mathbf{n}}_N\}$ into O.N. basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_N\}$ with $\hat{\mathbf{e}}_1 \equiv \hat{\mathbf{e}}$. Denote $\mathbf{x} \cdot \mathbf{e}_i$ by x_i. Let $(\theta_1, \theta_2, \ldots, \theta_{N-1})$ be the spherical coordinates for $(x_1, x_2, \ldots, x_N) \in S^{N-1}$ with $x_1 = \cos\theta_1$, etc. Write

$$d\mu_0(\theta_1, \theta_2, \ldots, \theta_{N-1}) = (\text{const.}) \sin^{N-2}\theta_1 d\theta_1 d\omega(\theta_2, \theta_3, \ldots, \theta_{N-1}). \tag{3.17}$$

Now

$$Z_{\mathbf{h}} = \int_{S^{N-1}} e^{\mathbf{h}\cdot\mathbf{x}} d\mu_0(\mathbf{x}) = \text{const} \int d\omega(\theta_2, \theta_3, \ldots, \theta_{N-1}) \int_0^\pi \exp(\|\mathbf{h}\|_2 \cos\theta_1) \sin^{N-2}\theta_1 d\theta_1$$

$$= \omega Z_{\mathbf{h}}^{(1)}, \text{ where } \begin{cases} \omega \equiv \int d\omega(\theta_2, \theta_3, \ldots, \theta_{N-1}) \\ Z_{\mathbf{h}}^{(1)} \equiv (\text{const}) \int_0^\pi \exp(\|\mathbf{h}\|_2 \cos\theta_1) \sin^{N-2}\theta_1 d\theta_1 . \end{cases} \tag{3.18}$$

$$\mu_{\mathbf{h}}[(\mathbf{x} \cdot \hat{\mathbf{e}} - \langle \mathbf{x} \cdot \hat{\mathbf{e}} \rangle_{\mathbf{h}})f]$$

$$= \frac{1}{\omega Z_{\mathbf{h}}^{(1)}} \int_{S^{N-1}} (x_1 - \langle x_1 \rangle_{\mathbf{h}}) f(\mathbf{x}) \exp(\|\mathbf{h}\|_2 x_1) d\mu_0(\mathbf{x}), \text{ from (3.18)}$$

$$= \frac{1}{Z_{\mathbf{h}}^{(1)}} \int_0^\pi \sin^{N-2}\theta_1 d\theta_1 (\cos\theta_1 - \langle x_1 \rangle_{\mathbf{h}}) \exp(\|\mathbf{h}\|_2 \cos\theta_1) \hat{f}(\cos\theta_1),$$

where

$$\hat{f}(\cos\theta_1) \equiv \frac{1}{\omega} \int f(\theta_1, \theta_2, \ldots, \theta_{N-1}) d\omega(\theta_2, \theta_3, \ldots, \theta_{N-1}).$$

Thus,

$$\mu_{\mathbf{h}}[(\mathbf{x} \cdot \hat{\mathbf{e}} - \langle \mathbf{x} \cdot \hat{\mathbf{e}} \rangle_{\mathbf{h}})f] = \mu_{\mathbf{h}}[(x_1 - \langle x_1 \rangle_{\mathbf{h}}) \hat{f}(x_1)]. \tag{3.19}$$

For $x_1, x_1' \in [-1, 1]$ and $x_1' \geqq 0$ we have

$$|\hat{f}(x_1) - \hat{f}(x_1')| \leqq \frac{1}{\omega} \int |f(\theta_1', \theta_2, \ldots, \theta_{N-1}) - f(\theta_1, \theta_2, \ldots, \theta_{N-1})| d\omega. \tag{3.20}$$

Invoking the invariance of Euclidean distance under rotation, and simple geometric considerations we obtain

$$|f(\theta_1,\theta_2,\ldots,\theta_{N-1})-f(\theta'_1,\theta_2,\ldots,\theta_{N-1})| \leqq \sqrt{\frac{2}{1-x'_1}}|x_1-x'_1|. \quad (3.21)$$

Thus, from (3.20) and (3.21),

$$|\hat{f}(x_1)-\hat{f}(x'_1)| \leqq \sqrt{\frac{2}{1-x'_1}}|x_1-x'_1|. \quad (3.22)$$

Now

$$\mu_{\mathbf{h}}[(x_1-\langle x_1\rangle_{\mathbf{h}})\hat{f}(x_1)] = \mu_{\mathbf{h}}[(x_1-\langle x_1\rangle_{\mathbf{h}})(\hat{f}(x_1)-\hat{f}(\langle x_1\rangle_{\mathbf{h}}))] \quad (3.23)$$

and thus, from (3.19), (3.22) and (3.23) [note $\langle x_1\rangle_{\mathbf{h}} \geqq 0$]

$$|\mu_{\mathbf{h}}[(\mathbf{x}\cdot\hat{\mathbf{e}}-\langle\mathbf{x}\cdot\hat{\mathbf{e}}\rangle_{\mathbf{h}})f]| \leqq \mu_{\mathbf{h}}[|x_1-\langle x_1\rangle_{\mathbf{h}}|\,|\hat{f}(x_1)-\hat{f}(\langle x_1\rangle_{\mathbf{h}})|]$$
$$\leqq \sqrt{\frac{2}{1-\langle x_1\rangle_{\mathbf{h}}}}\,\mu_{\mathbf{h}}[(x_1-\langle x_1\rangle_{\mathbf{h}})^2]. \quad (3.24)$$

To conclude the proof we establish the following proposition.

Proposition 3.4.

$$\frac{\mu_{\mathbf{h}}[(x_1-\langle x_1\rangle_{\mathbf{h}})^2]}{\sqrt{1-\langle x_1\rangle_{\mathbf{h}}}} \leqq \frac{\sqrt{2}}{N}. \quad (3.25)$$

To verify the proposition we use the elementary inequality

$$\frac{1}{\sqrt{1-\langle x_1\rangle_{\mathbf{h}}}} \leqq \frac{\sqrt{2}}{1-(\langle x_1\rangle_{\mathbf{h}})^2} \quad (3.26)$$

to obtain

$$\frac{\mu_{\mathbf{h}}[(x_1-\langle x_1\rangle_{\mathbf{h}})^2]}{\sqrt{1-\langle x_1\rangle_{\mathbf{h}}}} \leqq \sqrt{2}\,\frac{\mu_{\mathbf{h}}[(x_1-\langle x_1\rangle_{\mathbf{h}})^2]}{1-(\langle x_1\rangle_{\mathbf{h}})^2}. \quad (3.27)$$

We verify the inequality

$$\frac{\mu_{\mathbf{h}}[(x_1-\langle x_1\rangle_{\mathbf{h}})^2]}{1-(\langle x_1\rangle_{\mathbf{h}})^2} \leqq \frac{1}{N} \quad (3.28)$$

as follows:

Dyson, Lieb and Simon have proven the identity (see [9], eqn. (D4))

$$\mu_{\mathbf{h}}[(x_1-\langle x_1\rangle_{\mathbf{h}})^2] = 1-(\langle x_1\rangle_{\mathbf{h}})^2-\frac{(N-1)}{\|\mathbf{h}\|_2}\langle x_1\rangle_{\mathbf{h}}. \quad (3.29)$$

Explicit calculation shows

$$\langle x_1\rangle_{\mathbf{h}} = I_{N/2}(\|\mathbf{h}\|_2)/I_{N/2-1}(\|\mathbf{h}\|_2), \quad (3.30)$$

where $I_{N/2}, I_{N/2-1}$ are modified Bessel functions of order $\frac{N}{2}, \frac{N}{2}-1$, respectively.

Combining (3.29) and (3.30) with some algebra shows (3.28) is equivalent to the inequality

$$F(\|\mathbf{h}\|_2) \leqq 0, \tag{3.31}$$

where

$$F(\|\mathbf{h}\|_2) \equiv \left(1 - \frac{1}{N}\right)[(I_{N/2-1}(\|\mathbf{h}\|_2))^2 - (I_{N/2}(\|\mathbf{h}\|_2))^2] - \frac{(N-1)}{\|\mathbf{h}\|_2} I_{N/2-1}(\|\mathbf{h}\|_2) I_{N/2}(\|\mathbf{h}\|_2)$$

Using the power series for the product of two Bessel functions [10] we write $F(\|\mathbf{h}\|_2)$ as a power series in $\|\mathbf{h}\|_2$:

$$F(\|\mathbf{h}\|_2) = \sum_{r=0}^{\infty} a_r(N)(\|\mathbf{h}\|_2)^{2r+N-2} \tag{3.32}$$

where

$$a_r(N) \equiv \frac{2^{-(2r+N-2)}\Gamma(2r+N-1)}{r!\Gamma(r+N-1)\left[\Gamma\left(r+\frac{N}{2}\right)\right]^2} \left\{\left(1-\frac{1}{N}\right)\left(1-\frac{r}{r+N-1}\right) - \frac{(N-1)(2r+N-1)}{(r+N-1)(2r+N)}\right\}.$$

It is simple to verify

$$a_r(N) \leqq 0; \quad r = 0, 1, 2, \ldots \tag{3.33}$$

Relations (3.32) and (3.33) establish (3.31), which, combined with (3.27) proves the proposition and hence, by (3.24), the lemma.

Proof of Theorem 3. We have

$$V(\mu_{\mathbf{h}}, \mu_{\mathbf{h}'}) \equiv \sup_{\|f\|_L = 1} |\mu_{\mathbf{h}}(f) - \mu_{\mathbf{h}'}(f)|. \tag{3.34}$$

For $t \in [0, 1]$ write

$$\mathbf{h}(t) \equiv \mathbf{h} + (\mathbf{h}' - \mathbf{h})t. \tag{3.35}$$

Then, from (3.35)

$$\mu_{\mathbf{h}'}(f) - \mu_{\mathbf{h}}(f) = \int_0^1 \frac{d}{dt}[\mu_{\mathbf{h}(t)}(f)]dt. \tag{3.36}$$

It is a straightforward calculation to verify

$$\frac{d}{dt}[\mu_{\mathbf{h}(t)}(f)] = \|\mathbf{h}' - \mathbf{h}\|_2 \mu_{\mathbf{h}(t)}[(\mathbf{x}\cdot\hat{\mathbf{e}} - \langle \mathbf{x}\cdot\hat{\mathbf{e}}\rangle_{\mathbf{h}(t)})f]; \quad \hat{\mathbf{e}} \equiv \frac{\mathbf{h}' - \mathbf{h}}{\|\mathbf{h}' - \mathbf{h}\|_2}. \tag{3.37}$$

Combining (3.36) with (3.37) implies

$$|\mu_{\mathbf{h}'}(f) - \mu_{\mathbf{h}}(f)| \leqq \|\mathbf{h}' - \mathbf{h}\|_2 \sup\{|\mu_{\mathbf{g}}[(\mathbf{x}\cdot\hat{\mathbf{c}} - \langle \mathbf{x}\cdot\hat{\mathbf{c}}\rangle_{\mathbf{g}})f]| : \mathbf{g} \in \mathbb{R}^N; \hat{\mathbf{c}} \in S^{N-1}\}. \tag{3.38}$$

We decompose $\hat{\mathbf{c}}$ as

$$\hat{\mathbf{c}} = \mathbf{c}_{\parallel} + \mathbf{c}_{\perp}, \tag{3.39}$$

where $\mathbf{c}_{\parallel} \parallel \mathbf{g}$ and $\mathbf{c}_{\perp} \perp \mathbf{g}$. Using (3.39) and the fact

$$\langle \mathbf{x} \cdot \mathbf{c}_{\perp} \rangle_{\mathbf{g}} = 0 \tag{3.40}$$

we obtain

$$\begin{aligned}\mu_{\mathbf{g}}[(\mathbf{x}\cdot\hat{\mathbf{c}} - \langle \mathbf{x}\cdot\hat{\mathbf{c}}\rangle_{\mathbf{g}})f] = {} & \|\mathbf{c}_{\parallel}\|_2 \mu_{\mathbf{g}}[(\mathbf{x}\cdot\hat{\mathbf{c}}_{\parallel} - \langle \mathbf{x}\cdot\hat{\mathbf{c}}_{\parallel}\rangle_{\mathbf{g}})f] \\ & + \|\mathbf{c}_{\perp}\|_2 \mu_{\mathbf{g}}[(\mathbf{x}\cdot\hat{\mathbf{c}}_{\perp})f],\end{aligned} \tag{3.41}$$

where

$$\hat{\mathbf{c}}_{\parallel} \equiv \frac{\mathbf{c}_{\parallel}}{\|\mathbf{c}_{\parallel}\|_2}, \quad \hat{\mathbf{c}}_{\perp} \equiv \frac{\mathbf{c}_{\perp}}{\|\mathbf{c}_{\perp}\|_2}.$$

Thus, from (3.41) we have

$$\begin{aligned}|\mu_{\mathbf{g}}[(\mathbf{x}\cdot\hat{\mathbf{c}} - \langle \mathbf{x}\cdot\hat{\mathbf{c}}\rangle_{\mathbf{g}})f]| \leqq {} & \|\mathbf{c}_{\parallel}\|_2 |\mu_{\mathbf{g}}[(\mathbf{x}\cdot\hat{\mathbf{c}}_{\parallel} - \langle \mathbf{x}\cdot\hat{\mathbf{c}}_{\parallel}\rangle_{\mathbf{g}})f]| \\ & + \|\mathbf{c}_{\perp}\|_2 |\mu_{\mathbf{g}}[(\mathbf{x}\cdot\hat{\mathbf{c}}_{\perp})f]|\end{aligned} \tag{3.42}$$

Lemmas 3.2, 3.3 and (3.42) imply

$$|\mu_{\mathbf{g}}[(\mathbf{x}\cdot\hat{\mathbf{c}} - \langle \mathbf{x}\cdot\hat{\mathbf{c}}\rangle_{\mathbf{g}})f]| \leqq \frac{2}{N}\|\mathbf{c}_{\parallel}\|_2 + \frac{1}{N}\|\mathbf{c}_{\perp}\|_2 \quad (\|f\|_L = 1). \tag{3.43}$$

Combining the identity

$$(\|\mathbf{c}_{\parallel}\|_2)^2 + (\|\mathbf{c}_{\perp}\|_2)^2 = 1 \tag{3.44}$$

with (3.43) gives

$$|\mu_{\mathbf{g}}[(\mathbf{x}\cdot\hat{\mathbf{c}} - \langle \mathbf{x}\cdot\hat{\mathbf{c}}\rangle_{\mathbf{g}})f]| \leqq \frac{\sqrt{5}}{N} \quad (\|f\|_L = 1). \tag{3.45}$$

Combining (3.34), (3.38) and (3.45) gives (3.8), concluding the proof of Theorem 3.

Remark 3.5. Let $d(\cdot,\cdot)$ be a semi-metric on $\mathbb{Z}^\nu$.
Define

$$\tilde{T}_C^N \equiv \left(\sup_{a\in\mathbb{Z}^\nu} \sum_{b\in\mathbb{Z}^\nu\setminus\{a\}} \|(J_{\{a,b\}})\|_{\mathrm{op}} e^{d(a,b)}\right)\frac{\sqrt{5}}{N}. \tag{3.46}$$

Parallel to the discussion of Gross (see proof of Theorem 1 in [8]), one can prove for temperatures $T > \tilde{T}_C^N$, the truncated spin–spin correlation functions for N-Vector models decay exponentially. More explicitly, if μ is the equilibrium state (unique by Theorem 1 since $\tilde{T}_C^N > T_C^N$) then

$$|\mu(\sigma_a^i \sigma_b^k) - \mu(\sigma_a^i)\mu(\sigma_b^k)| \leqq \eta e^{-d(a,b)}, \tag{3.47}$$

where η is a number independent of a, b.

Remark 3.6. Theorem 1 may be generalized to include many-body interactions of the form

$$\sum_{\{i_a = 1,2,\cdots,N;\, a\in X\}} J_X^{\{i_a\}} \prod_{a\in X} \sigma_a^{i_a}, \tag{3.48}$$

where X is a finite subset of $\mathbb{Z}^\nu$, and $J_X^{\{i_a\}}$ are real numbers. One again applies Theorems 2 and 3. The only relevant quantity which depends on the explicit form of the interactions is $\| \mathbf{h}_{eff}^{a,\sigma} - \mathbf{h}_{eff}^{a,\sigma'} \|_2$. Leaving the details to the reader one proves for temperatures

$$T > T_C^N \equiv \left(\sup_{a\in\mathbb{Z}^\nu} \sum_{X\ni a} (|X| - 1)\mathscr{J}_X \right) \frac{\sqrt{5}}{N}, \tag{3.49}$$

the N-Vector models exhibit at most one equilibrium state, where

$$\mathscr{J}_X \equiv \sup_{\{\sigma_a \in S^{N-1};\, a\in X\}} \left(\sum_{\{i_a = 1,2,\cdots,N;\, a\in X\}} J_X^{\{i_a\}} \prod_{a\in X} \sigma_a^{i_a} \right)$$

and $|X| \equiv$ number of elements of X.

Remark 3.7. The result of Driessler, et. al. [1, 2] mentioned in the introduction suggests the following conjecture: For N-Vector models with general interactions, the "mean field" critical temperature is an upper bound of the critical temperature for unique equilibrium, i.e. the numerical factor of $\sqrt{5}$ appearing in Theorem 1 can be replaced by 1. In particular, it follows from the proof of Theorem 1 that the conjecture is true if one shows

$$\sup_{\|f\|_L = 1} |\mu_{\mathbf{g}}[(\mathbf{x}\cdot\hat{\mathbf{c}} - \langle \mathbf{x}\cdot\hat{\mathbf{c}}\rangle_{\mathbf{g}})f]| \quad (\mathbf{g}\in\mathbb{R}^N, \hat{\mathbf{c}}\in S^{N-1})$$

attains its maximum value at $\mathbf{g} = 0$.

The idea of computing, for certain classical lattice systems, an appropriate Vaserstein distance in Dobrushin's uniqueness theorem to derive the "mean field" temperature as an upper bound of the critical temperature for unique equilibrium, originates from the work of Cassandro, et al. [11]. Estimating the Vaserstein distance between certain probability measures on the real line (with absolute value norm[3]), they proved the "mean field" temperature is an upper bound of the critical temperature for lattice models with general one-component spins interacting pairwise.

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3 Using the formula of Vallender [12] for the Vaserstein distance between two Borel probability measures on $\mathbb{R}^1$, they derive the basic estimate

$$V(\mu_h, \mu_{h'}) \leqq |h - h'| \sup_{g\in\mathbb{R}^1} \mu_g[(x - \langle x\rangle_g)^2],$$

where

$$h, h' \in \mathbb{R}^1;\ \mu_h \equiv e^{hx}d\mu_0(x) \Big/ \int_{\mathbb{R}^1} e^{hx}d\mu_0(x), \quad \mu_{h'} \equiv e^{h'x}d\mu_0(x) \Big/ \int_{\mathbb{R}^1} e^{h'x}d\mu_0(x)$$

and μ_0 is an a priori probability measure on $\mathbb{R}^1$ with appropriate fall off at $\pm\infty$. We note that both their estimate and the Vallender formula they use to derive it can be easily proven by exploiting the dual "Lipschitz" definition of the Vaserstein distance and the method of the present paper.

Acknowledgements. The author is grateful to Barry Simon for advice and encouragement. The author also wishes to thank Michael Aizenman and Alan Sokal for useful discussions.

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Communicated by E. Lieb

Received March 3, 1980