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A Refinement of Simon's Correlation Inequality★

Elliott H. Lieb

Departments of Mathematics and Physics, Princeton University, Princeton, NJ 08544, USA

Abstract. A general formulation is given of Simon's Ising model inequality: $\langle\sigma_\alpha\sigma_\gamma\rangle \leqq \sum_{b\in B}\langle\sigma_\alpha\sigma_b\rangle\langle\sigma_b\sigma_\gamma\rangle$ where B is any set of spins separating α from γ. We show that $\langle\sigma_\alpha\sigma_b\rangle$ can be replaced by $\langle\sigma_\alpha\sigma_b\rangle_A$ where A is the spin system "inside" B containing α. An advantage of this is that a finite algorithm can be given to compute the transition temperature to any desired accuracy. The analogous inequality for plane rotors is shown to hold if a certain conjecture can be proved. This conjecture is indeed verified in the simplest case, and leads to an upper bound on the critical temperature. (The conjecture has been proved in general by Rivasseau. See notes added in proof.)

In an accompanying paper [1] in this volume Simon proves a correlation inequality with important consequences. For a finite range pairwise interacting (generalized) Ising ferromagnet (the spins take on values $2M, 2M-2, \ldots, -2M$), Simon shows that

$$\langle\sigma_\alpha\sigma_\gamma\rangle \leqq \sum_{b\in B}\langle\sigma_\alpha\sigma_b\rangle\langle\sigma_b\sigma_\gamma\rangle, \tag{1}$$

where B is any set of spins separating α from γ (i.e. any path from α to γ must run through B). Aizenman and Simon [2] have proved a related inequality for N-component spins. In this paper we shall generalize (1) in the following way: $\langle\sigma_\alpha\sigma_b\rangle$ can be replaced by $\langle\sigma_\alpha\sigma_b\rangle_A$, where A is the connected component of the lattice containing α and B and $\langle\cdot\rangle_A$ denotes expectation values in the A system alone. The possibility of extending this inequality to plane rotors is also discussed, but the proof is carried to completion only in a special case. (See notes added in proof.)

In [1] Simon discusses the consequences of (1) and our generalization. We shall not repeat them, except to note that the most interesting consequence of the extension is that for the first time one has an algorithm for computing the transition temperature, T_c (in the sense that above, but not below T_c there is

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exponential decay of the two point function $\langle\sigma_0\sigma_x\rangle$), to arbitrary accuracy. Take $\alpha=0$ and let B be the spins on the boundary of a square of side L centered at 0. By boundary we mean all points within a distance R of the geometric boundary, where R is not less than the range of the interaction. The A system is the inside of the square alone. $\langle\sigma_0\sigma_b\rangle_A$ can be computed explicitly, and if

$$\sum_{b\in B}\langle\sigma_0\sigma_b\rangle<1 \tag{2}$$

for some T, then there is exponential decay for that T. This sets an upper bound to T_c. It is easy to see [1], however, that as $L\to\infty$, T_L [the T for which equality holds in (2)] approaches T_c. While the convergence of T_L to T_c is expected to be extremely slow, the mere existence of the algorithm is an interesting matter of principle. It is not known if T_L is necessarily monotone decreasing in L; this is an open question.

A consequence of our generalization is the continuity of the mass gap as function of the interaction, for nearest neighbor ferromagnetic interactions, proven in [1]. A more general stability of the mass gap, m, under perturbations was pointed out by Aizenman (private communication). It is expressed by the *lower semicontinuity of m, as function of the interaction, in the cone of pairwise ferromagnetic interactions of any fixed finite range.* This is proven in the following way. Suppose the (finite range) Hamiltonian H is given and T is such that for the infinite system

$$\langle\sigma_0\sigma_x\rangle<C_\varepsilon\exp[(-m+\varepsilon)|x|]$$

for all x, all $\varepsilon>0$, but not $\varepsilon<0$. m is then the mass gap and it will be assumed that $m>0$. Given $\varepsilon>0$, it is easy to see that for any R there must be a finite box such that

$$\sum_{b\in B}\langle\sigma_0\sigma_b\rangle\exp[\mu|b|]<1 \tag{3}$$

for $\mu=m-\varepsilon$. Conversely, our generalization of (1) shows that if (3) holds with some μ for some box, then the mass gap is not less than μ.

Since condition (3) (with $\mu=m-\varepsilon$) refers to a finite system, by continuity it continues to hold (with $\mu=m-2\varepsilon$) when the Hamiltonian is changed from H to $H+K$ and $\|K\|<\delta_0$, for some $\delta_0>0$ and independent of K. If we also require that $H+K$ is pairwise ferromagnetic and has range $\leqq R$, then (3) (with $\mu=m-2\varepsilon$) and our generalization of (1) imply that the new mass gap is not less than $m-2\varepsilon$.

Simon's proof of (1) uses a graphical expansion. The analysis presented here will not use this explicitly, but instead will use certain "gaussian correlation inequalities" of Newman [3]. While it is true that Newman's inequalities are themselves proved by graphical means, it is hoped that the decomposition of the problem into the two steps given here will be useful.

Let us begin with some definitions. The system under consideration is viewed as the union of two subsystems of spins A and C.

$$A\cap C=B$$

is the set of spins common to both. To say that the B spins separate A from C means that

$$H_{A+C}=H_A+H_C, \tag{4}$$

where the H's are Hamiltonians. The symbols Z denote partition functions, $\langle\cdot\rangle$ denote expectation values and $(\cdot)=Z\langle\cdot\rangle$ denote unnormalized expectation values – all at reciprocal temperature β. Thus, for example,

$$\begin{aligned}(\sigma_A)_A&=\operatorname{Tr}\sigma_A\exp(-\beta H_A)\\ Z_A&=(1)_A=\operatorname{Tr}\exp(-\beta H_A)\\ \langle\sigma_A\rangle_A&=(\sigma_A)_A/Z_A.\end{aligned} \tag{5}$$

Here σ_A is some observable in the A system. It may, of course, depend on the B spins since they are in A.

The spins that are mostly relevant to our analysis are the B spins. The word "spin" is to some extent a misnomer, for the only hypothesis is that at each point $b\in B$ there is an independent a-priori probability measure $d\mu_b$ on some measure space Ω_b. For simplicity we take these to be independent of b. Let $\{\phi^n\}$ be a *complete* orthonormal family of functions in $L^2(d\mu)$. The choice of the $\{\phi^n\}$ is important because the hypotheses made later can be expected to hold, if at all, only for special choices. With $\mathbf{n}=(n_1,n_2,\ldots)$ a multi-index on $B=(b_1,b_2,\ldots)$, we denote the following orthonormal functions on $\prod_{b\in B}\Omega_b$:

$$\phi_B^{\mathbf{n}}=\phi_{b_1}^{n_1}\phi_{b_2}^{n_2}\ldots. \tag{6}$$

Example 2 (Spin $\frac{1}{2}$ Ising Model). Here $\Omega=\{-1,1\}$, μ gives weight $\frac{1}{2}$ to each point and $\{\phi^n\}=\{\phi^0,\phi^1\}$ with $\phi^0(\sigma)=1$, $\phi^1(\sigma)=\sigma$.

Example 2 (Plane Rotor). Ω is the unit circle $0\leqq\theta<2\pi$, $d\mu(\theta)=d\theta/2\pi$ is the uniform measure, and $\phi^n(\theta)=\exp(in\theta)$ with $n=0,\pm1,\pm2,\ldots$.

The constitution of the remainder of the A and C systems is irrelevant to the general formalism we present. It can be composed of quarks, for example. σ_A (resp. σ_C)will denote observables in the A (resp. C) systems and they can both depend on the B spins. Note that the functions $\phi_B^{\mathbf{n}}$ can be regarded either as A or as C observables.

A formula connecting A, C and $A+C$ expectations is required. In other words, we have to "glue" the A and C systems together to form the $A+C$ system.

Lemma 1.

$$(\sigma_A\sigma_C)_{A+C}=\sum_{\mathbf{n}}(\sigma_A\phi_B^{\mathbf{n}})_A(\sigma_C\bar{\phi}_B^{\mathbf{n}})_C. \tag{7}$$

In particular,

$$Z_{A+C}=\sum_{\mathbf{n}}(\phi_B^{\mathbf{n}})_A(\bar{\phi}_B^{\mathbf{n}})_C. \tag{8}$$

Proof. In a schematic notation, let x, y, and z respectively stand for the B variables, the A variables other than B, and the C variables other than B. The

Boltzmann factor is $M(x,y)N(x,z)$ where $M(x,y)=\exp[-\beta H_A(x,y)]$ and $N(x,z)=\exp[-\beta H_C(x,z)]$. Let the a-priori measure be $d\mu_\beta(x)\,d\mu_\alpha(y)\,d\mu_\gamma(z)$ and let $F(x)=\int d\mu_\alpha(y)\,\sigma_A(x,y)\,N(x,y)$, $G(x)=\int d\mu_\gamma(z)\sigma_C(x,z)M(x,z)$. Then, by Parseval's theorem,

$$(\sigma_A\sigma_C)_{A+C}=\int d\mu_\beta(x)F(x)G(x)=\sum_{\mathbf{n}} D_{\mathbf{n}}E_{\mathbf{n}}$$

with $D_{\mathbf{n}}=\int d\mu_\beta(x)\phi_B^{\mathbf{n}}(x)F(x)$ and $E_{\mathbf{n}}=\int d\mu_\beta(x)\bar{\phi}_B^{\mathbf{n}}(x)G(x)$. But this sum on $\mathbf{n}$ is precisely the right side of (7). □

Henceforth we fix the observables σ_A and σ_C, the Hamiltonians H_A and H_C, and make the following hypotheses (with respect to σ_A and σ_C) about the A and C systems.

H.C1 (Positivity). $\langle\sigma_C\bar{\phi}_B^{\mathbf{n}}\rangle_C\geqq 0$ for all $\mathbf{n}$. (9)

H.A1 (The Gaussian-Type Inequality [3]*).* There exists a function $F(\mathbf{n})$, not necessarily nonnegative, of the multi-index such that

$$\langle\sigma_A\phi_B^{\mathbf{m}}\rangle_A\leqq\sum_{\mathbf{n}}F(\mathbf{n})\langle\sigma_A\phi_B^{\mathbf{n}}\rangle_A\langle\bar{\phi}_B^{\mathbf{n}}\phi_B^{\mathbf{m}}\rangle_A \tag{10}$$

for all $\mathbf{m}$ such that $\langle\sigma_C\bar{\phi}_B^{\mathbf{m}}\rangle_C>0$.

The meaning of H.A1 will become clear later when we consider the Ising and plane rotor models as examples. For now we note that comparatively little is required of system C. The main theorem is the following:

Theorem 1. *Under hypotheses H.A1 and H.C1*

$$\langle\sigma_A\sigma_C\rangle_{A+C}\leqq\sum_{\mathbf{n}}F(\mathbf{n})\langle\sigma_A\phi_B^{\mathbf{n}}\rangle_A\langle\bar{\phi}_B^{\mathbf{n}}\sigma_C\rangle_{A+C}\,. \tag{11}$$

Proof. Multiply (11) by Z_AZ_{A+C} and use Lemma 1. We require that

$$\begin{aligned}&Z_A\sum_{\mathbf{m}}(\sigma_A\phi_B^{\mathbf{m}})_A(\bar{\phi}_B^{\mathbf{m}}\sigma_C)_C\\&\quad\leqq\sum_{\mathbf{m}}\left\{\sum_{\mathbf{n}}F(\mathbf{n})(\sigma_A\phi_B^{\mathbf{n}})_A(\bar{\phi}_B^{\mathbf{n}}\phi_B^{\mathbf{m}})_A\right\}(\bar{\phi}_B^{\mathbf{m}}\sigma_C)_C\,.\end{aligned} \tag{12}$$

Here, $\phi_B^{\mathbf{n}}$ has been regarded as an A observable. In view of H.C1 it suffices to prove (12) for each $\mathbf{m}$ but, if we divide by Z_A^2, this is seen to be H.A1. □

The analogue of Simon's inequality [1] would have $\langle\ \rangle_{A+C}$ instead of $\langle\ \rangle_A$ on the right side of (11). There are then two natural questions: When does the Simon type of inequality hold and when is it weaker than Theorem 1, as it is for the Ising model? The following hypotheses help to answer this.

H.C2. $\langle\bar{\phi}_B^{\mathbf{n}}\rangle_C\geqq 0$, all $\mathbf{n}$. (13)

H.A2 (inequality of the second Griffiths type).

$$\langle\sigma_A\phi_B^{\mathbf{n}}\rangle_A\langle\phi_B^{\mathbf{m}}\rangle_A\leqq\langle\sigma_A\phi_B^{\mathbf{n}}\phi_B^{\mathbf{m}}\rangle_A,\quad\text{all }\mathbf{n} \tag{14}$$

whenever $\langle\bar{\phi}_B^{\mathbf{m}}\rangle_C>0$.

Theorem 2. *Suppose H.A2 and H.C2 hold. Then*

$$\langle \sigma_A \phi_B^{\mathbf{n}} \rangle_A \leqq \langle \sigma_A \phi_B^{\mathbf{n}} \rangle_{A+C}, \quad \text{all } \mathbf{n}. \tag{15}$$

Proof. (15) is equivalent to

$$\sum_{\mathbf{mn}} (\phi_B^{\mathbf{m}})_A (\bar{\phi}_B^{\mathbf{m}})_C (\sigma_A \phi_B^{\mathbf{n}})_A \leqq Z_A \sum_{\mathbf{m},\mathbf{n}} (\sigma_A \varphi_B^{\mathbf{n}} \phi_B^{\mathbf{m}})_A (\bar{\phi}_B^{\mathbf{m}})_C,$$

but this is implied by (13), (14). □

Corollary 1. *Suppose* (11) *and* (15) *hold and* $F(\mathbf{n}) \geqq 0$. *Then*

$$\langle \sigma_A \sigma_C \rangle_{A+C} \leqq \sum_{\mathbf{n}} F(\mathbf{n}) \langle \sigma_A \phi_B^{\mathbf{n}} \rangle_{A+C} \langle \bar{\phi}_B^{\mathbf{n}} \sigma_C \rangle_{A+C}. \tag{16}$$

Moreover, the right side of (16) is not less than the right side of (11).

If $F(\mathbf{n})$ is not nonnegative, (16) can still be proved under a further hypothesis:

H.A3.

$$\langle \sigma_A \phi_B^{\mathbf{m}} \rangle_A \langle \phi_B^{\mathbf{k}} \rangle_A \leqq \sum_{\mathbf{n}} F(\mathbf{n}) \langle \sigma_A \phi_B^{\mathbf{n}} \phi_B^{\mathbf{k}} \rangle_A \langle \bar{\phi}_B^{\mathbf{n}} \phi_B^{\mathbf{m}} \rangle_A. \tag{17}$$

whenever both $\langle \sigma_C \bar{\phi}_B^{\mathbf{m}} \rangle_C > 0$ and $\langle \bar{\phi}_B^{\mathbf{k}} \rangle_C > 0$.

Theorem 3. (16) *holds under hypothese H.C1, H.C2, and HA3.*

The proof of Theorem 3 is an imitation of the proofs of Theorems 1 and 2. Note that under these hypothese one cannot say that (16) is weaker than (11).

The following is a trivial consequence of the definitions

Lemma 2. *If* $F(\mathbf{n}) \geqq 0$ *then H.A1 and H.A2 imply H.A3.*

The Ising Model as an Example

Spin 1/2 Ising Models

The ϕ^n are given in Example 1. We take σ_A and σ_C each to be products of an odd number of spins. H.C1, H.C2, and H.A2 are Griffiths' inequalities. Newman's inequality [3] states, in particular, that if F is a family of partitions of $K = \{1, \ldots, k\}$ into two disjoint subsets then (with $\sigma_D = \sigma_a \sigma_b \ldots \sigma_d$ when $D = \{a, b, \ldots, d\}$)

$$\langle \sigma_K \rangle \leqq \sum_{f \in F} \langle \sigma_{f_1} \rangle \langle \sigma_{f_2} \rangle, \tag{18}$$

whenever $|K| = 2L$ is even and *every partition of* K into L pairs is a refinement of some $f \in F$. Sylvester [14] also gives a proof of (18).

Let the spins in B be labeled $\sigma_1, \ldots, \sigma_M$. In (10), $\mathbf{m}$ can be thought of as a subset of $\{1, \ldots, M\}$. Clearly, $\langle \sigma_A \phi_B^{\mathbf{m}} \rangle_A > 0$ implies that $|\mathbf{m}|$ is odd.

Assume that σ_A is just one spin, σ_α, and, without loss, that $\alpha \notin B$. Taking $K = \{\alpha\} \cup \mathbf{m}$, and all f_1 of the form $\{\alpha, i\}$ with $i \in \mathbf{m}$, (18) implies (10) with

$$\begin{aligned} F(\mathbf{n}) &= 1 \quad \text{if} \quad |\mathbf{n}| = 1 \\ &= 0 \quad \text{otherwise}. \end{aligned} \tag{19}$$

[Note: There are more terms on the right side of (10) than the right side of (18). The excess terms are nonnegative by Griffiths first inequality.] In this case we conclude that

$$\langle\sigma_\alpha\sigma_C\rangle_{A+C}\leqq\sum_{b\in B}\langle\sigma_\alpha\sigma_b\rangle_A\langle\sigma_b\sigma_C\rangle_{A+C} \tag{20}$$

as stated in the introduction. It was not assumed that $|C|=1$.

If σ_A is a product of N(odd) spins then (18) implies (10) with

$$\begin{aligned} F(\mathbf{n}) &= 1 \quad \text{if} \quad |\mathbf{n}|=1,3,\ldots,N \\ &= 0 \quad \text{otherwise}. \end{aligned} \tag{21}$$

Then (20) changes to

$$\langle\sigma_A\sigma_C\rangle_{A+C}\leqq\sum_{\substack{b\subset B\\|b|\leqq|A|}}\langle\sigma_A\sigma_b\rangle_A\langle\sigma_b\sigma_C\rangle_{A+C}\,. \tag{22}$$

Other Ising Models

One generalization is to spin $M>\frac{1}{2}$ with $\sigma=2M,\ldots,-2M$. A way to proceed would be to use an appropriate orthonormal basis $\{\phi^n\}$ of dimensions $2M+1$. We have not pursed this possibility. A second method is to use Griffiths' trick [5] of writing a spin M as M ferromagnetically coupled spin $\frac{1}{2}$ spins. H.C1, H.C2, and H.A2 follow from this, as does (20) and (21) by summing over the "component" spins. Much is lost this way, however.

Another generalization, which we shall not explicate, is to allow multi-spin interactions.

The Plane Rotor Model

We consider pairwise ferromagnetic interactions; the interaction between two spins $\vec{\sigma}_a$ and $\vec{\sigma}_b$ is $-J_{ab}\vec{\sigma}_a\cdot\vec{\sigma}_b=-J_{ab}\cos(\theta_a-\theta_b)$, with $J_{ab}\geqq0$. The basis $\{\phi^n\}$ is given in Example 2.

There is some reason to believe that the analogue of (20) holds in the following sense:

$$\langle\vec{\sigma}_a\cdot\vec{\sigma}_c\rangle_{A+C}\leqq\sum_{b\in B}\langle\vec{\sigma}_a\cdot\vec{\sigma}_b\rangle_A\langle\vec{\sigma}_b\cdot\vec{\sigma}_c\rangle_{A+C}\,. \tag{23}$$

when $\vec{\sigma}_a$ and $\vec{\sigma}_c$ are single spins. In terms of the ϕ^n we have

$$2\vec{\sigma}_a\cdot\vec{\sigma}_c=\phi_a^1\phi_c^{-1}+\phi_a^{-1}\phi_c^1\,.$$

Therefore we require that (9) and (10) hold when σ_C (resp. σ_A) is ϕ_c^1 or ϕ_c^{-1} (resp. ϕ_a^1 or ϕ_a^{-1}) and F is given by (19). With these choices for σ_A and σ_C, (9), and also (14) hold [6]. The difficulty lies with (10).

We do not have a proof of (10), but believe it to be true. A possibility would be to try to imitate the graphical proof [3,4] that is successful in the Ising case. It would then be necessary to deal with directed graphs (digraphs). The following, if it were true would immediately yield a proof of (10):

Conjecture. Let G be a finite direted graph (possibly with several edges between two vertices) and let the valence at vertex i (the number of arrows in minus the number out of i) be M_i. (Clearly, $\Sigma M_i = 0$.) Suppose $M_1 = +1$, and $M_2, M_3, \ldots, M_k < 0$, and $M_i \geqq 0$ otherwise. Let N be the number of subgraphs (subsets of edges) of G, including the empty graph, having valence 0 at each vertex. Let K be the number of subgraphs of G with the following property: vertex 1 has valence $+1$, some vertex j, with $2 \leqq j \leqq k$, has valence -1, and all other valences are 0. Then $N \leqq K$.

There is one special but important case in which the conjecture, and hence (23) holds. Suppose $\vec{\sigma}_a$ is connected to n nearest neighbors, which we take to be B. It is immaterial whether the B spins are connected together, for any such interaction can be regarded as part of H_C. In a v dimensional cubic lattice $n = 2v$.

The graph G in the conjecture then has the following structure: it has $n+1$ vertices and is star-like with edges only between the central vertex V_1 (which is really the original vertex marked a) and the other n. Suppose $M_2 < 0$. Then it is easy to see that the conjecture is verified if the following is true: Let $\tilde{G}$ be the subgraph of G consisting of vertices 1 and 2 and all the edges between them. Let $\tilde{N}$ be the number of valence 0 subgraphs of $\tilde{G}$, and let $\tilde{K}$ be the number of subgraphs of $\tilde{G}$ with $M_1 = +1, M_2 = -1$. Then $\tilde{N} \leqq \tilde{K}$. The easy proof of this is left to the reader.

This simple case can also be conveniently viewed in terms of (10) directly. We require that

$$\langle \phi_a^1 \phi_B^{\mathbf{m}} \rangle \leqq \sum_{b=1}^{n} \langle \phi_a^1 \phi_b^{-1} \rangle \langle \phi_b^1 \phi_B^{\mathbf{m}} \rangle . \tag{24}$$

Both sides of (24) vanish unless $\sum_1^n m_i = -1$. Let $P(m)$ be the Fourier transform of $\exp[\beta \cos\theta]$, namely

$$P(m) = I_m(\beta) > 0, m \in \mathbb{Z}, \tag{25}$$

where I_m is the modified Bessel function. Then (24) reads

$$\{P(0)^n\} \left\{ \prod_{i=1}^{n} P(m_i) \right\} \leqq \sum_{i=1}^{n} \{P(0)^{n-1} P(1)\} \left\{ P(m_i + 1) \prod_{j \neq i} P(m_j) \right\} . \tag{26}$$

Suppose $m_1 < 0$, say. It is sufficient to have

$$P(0)P(m_1) \leqq P(1)P(m_1 + 1) . \tag{27}$$

If both sides of (27) are expanded in a power series in β, (27) is true term by term. This is just the graphical exercise mentioned above. However, the following stronger result, which implies (27), holds.

Lemma 2. *Fix $\beta \geqq 0$. The function $m \to I_m(\beta)$ is log concave on the integers, i.e.*

$$I_m(\beta)^2 \geqq L_{m+1}(\beta) I_{m-1}(\beta) , \qquad m \in \mathbb{Z} . \tag{28}$$

Proof. $I_m(\beta)=I_{-m}(\beta)$. (28) is trivial for $m=0$, so it is sufficient to consider $m\geqq 1$. If both sides of (28) are expanded in a power series in β, we claim (28) holds termwise. Use

$$I_m(\beta)=\sum_{j=0}^{\infty}(\beta/2)^{2j+m}[j!(j+m)!]^{-1}.$$

Thus for the coefficient of β^{2m+2t}, $t\geqq 0$, we require

$$\sum_{j=0}^{t} g_0(j-t/2)g_m(j-t/2)\geqq \sum_{j=0}^{t} g_0(j-t/2)g_m(j+1-t/2), \tag{29}$$

where $g_p(x)=[\Gamma(t/2+1+p+x)\Gamma(t/2+1+p-x)]^{-1}\theta(x)$ and $\theta(x)=1$ if $|x|\leqq t/2+p$ and $=0$ otherwise. Now $\Gamma(x)$ is log convex for $x>0$ and hence $g_p(x)$ is even and log concave. Thus, for j an integer in $[0,t]$, $g_0(j-t/2)$ is a positive sum of functions of the form $\mu_a(j)=1$ if $a\leqq j\leqq t-a$ and $=0$ otherwise, for $a=0,1,\ldots[t/2]$. Hence, it suffices to have $\sum_{j=a}^{t-a} g_m(j-t/2)-g_m(j+1-t/2)\geqq 0$. But this is true because $g_m(j-t/2)$ is also a positive sum of the $\mu_b(j)$ functions (with $-m\leqq b\leqq[t/2]$) and $\sum_{j=a}^{t-a}\mu_b(j+r)$ is decreasing for $r\geqq 0$, $r\in\mathbb{Z}$. $\square$

Since (23) holds when $\vec{\sigma}_a$ is a single spin and B are its neighbors, we can conclude using (2) that

Theorem 4. *For the plane rotor model on a v-dimensional hypercubic lattice there is exponential fall-off of the two-point function if* $\langle\vec{\sigma}_a\cdot\vec{\sigma}_b\rangle<1/2v$ *for the two-spin system consisting of a and b alone. This is equivalent to* $I_1(\beta)/I_0(\beta)<1/2v$ *if* $J_{ab}=1$ *is assumed for the nearest neighbor coupling constant. In particular*

$$\beta_c\geqq 0.52 \quad (v=2); \qquad \beta_c\geqq 0.34 \quad (v=3).$$

For $v=2$, Fröhlich [7] has shown that $\beta_c\geqq 0.64$, and Aizenman and Simon [8] have shown that $\beta_c\geqq 0.88$.

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Notes added in proof. (1) Simon's inequality (1) for the Ising model is a special case of a class of inequalities and identities discussed by Boel and Kasteleyn [9, 10]. They found necessary and sufficient conditions for such inequalities to hold; therefore (1) can be proved by their methods.

(2) The conjecture in this paper has been proved by Rivasseau [11]. Thus inequality (23) for rotors holds for all subsystems A, not merely for the case of the star graph proved here.

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