

Riemannian Structure on Manifolds of Quantum States

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Abstract. A metric tensor is defined from the underlying Hilbert space structure for any submanifold of quantum states. The case where the manifold is generated by the action of a Lie group on a fixed state vector (generalized coherent states manifold hereafter noted G.C.S.M.) is studied in details; the geometrical properties of some wellknown G.C.S.M. are reviewed and an explicit expression for the scalar Riemannian curvature is given in the general case. The physical meaning of such Riemannian structures (which have been recently introduced to describe collective manifolds in nuclear physics) is discussed. It is shown on examples that the distance between nearby states is related to quantum fluctuations; in the particular case of the harmonic oscillator group the condition of zero curvature appears to be identical to that of non dispersion of wave packets.

1. Introduction

In recent years there has been renewed interest in the usefulness of geometrical ideas in quantum mechanics. The geometrical structure which has been most studied is the symplectic one. The reason is that the symplectic forms take an important part in the Hamiltonian formulation of the classical mechanics [1] and that they can also be defined on the Hilbert space of quantum states. The key role of this structure is particularly evident in the geometrical quantization program of Kostant and Souriau [2]. It has also been claimed that this geometrical structure *remains present in the nonlinear generalizations of the quantum mechanics* [3].

Another useful geometrical concept is that of a Riemannian structure. In the framework of quantum mechanics this notion has not been much investigated. An explanation may be that, although the scalar product on the Hilbert space induces naturally a distance between the quantum states, one is not interested in the local properties of the manifold of states. Indeed the physically relevant quantities are

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the transition probability amplitudes which are defined for any two states whatever be their relative distance.

However, such a Riemannian structure has been recently considered by theoreticians of nuclear physics interested in the description of the collective behaviours of the nucleons (such as nucleus deformation etc. ...). In order to extract the meaningful collective subdynamics from the manybody dynamics, they have introduced the concept of "collective submanifold" in the many particle Hilbert space [4]. This manifold, which in general is not a vector space, must be chosen in some optimal way. To this end it has been proposed to consider the Riemannian curvature as a test of the collectivity [5].

Our aim in this paper is twofold.

First, on a mathematical level, we want to describe in some detail how the hermitian product on the projective Hilbert space induces a meaningful metric tensor on any manifold of quantum states. An interesting situation often encountered is the case where the manifolds are generated by the action of a Lie group on a fixed quantum state; following Perelomov [6] we call them generalized coherent states manifolds (G.C.S.M.). Geometrical properties of several explicit examples of G.C.S.M. are considered and a detailed study of the curvature tensor is presented in the general case.

Secondly, we want to initiate a discussion on the physical signification of such a metric from a general point of view (not in the particular framework of nuclear physics where this structure seems to have been first introduced). We show on G.C.S.M. examples that the components of the metric tensor are related to the dispersion of the quantum operators acting on the underlying Hilbert space; roughly speaking, the metric structure on the manifold is fixed by the quantum fluctuations. This has to be compared with the results obtained in a recent paper [7] where it has been shown that, in thermodynamics, a meaningful Riemannian structure can also be defined on the manifold of equilibrium states of a system. In this later case the metric is related to the thermal fluctuations of the system i.e. the Riemannian structure originates from the underlying theory of statistical mechanics. A strict analogy would suggest that a Riemannian manifold of quantum states may be considered in some sense as "classical" and more generally that one should pay attention to metric structures when considering the connection between classical and quantum mechanics.

The paper is organized as follows. In Sect. 2 we introduce the different geometrical structures which can be defined on any manifold of quantum states. We emphasize the gauge invariance of these structures. Several examples are worked out. In Sect. 3 we study G.C.S.M. in some detail. Finally Sect. 4 is devoted to physical comments relative to the metric and the curvature tensors.

2. Geometrical Structures on a Manifold of Quantum States

2.1. Definition of a Meaningful Metric Tensor

As a first step towards obtaining of a metric structure let us consider a family $\{\psi(s)\}$ of normalized vectors of some Hilbert space which smoothly depend on an n -dimensional parameter $s=(s_1 \dots s_n) \in \mathbb{R}^n$. Let $\| \cdot \|$ and (\cdot , \cdot) denote the norm and

the scalar product on the Hilbert space. The distance

$$\|\psi(s_2) - \psi(s_1)\| \tag{2.1}$$

between two close vectors in the family induces a metric in the following way. Writing $s_1 = s$ and $s_2 = s + ds$ we develop the quantity

$$\|\psi(s + ds) - \psi(s)\|^2 = (\psi(s + ds) - \psi(s), \psi(s + ds) - \psi(s)) \tag{2.2}$$

up to second order:

$$\|\psi(s + ds) - \psi(s)\|^2 = (\partial_i \psi, \partial_j \psi) ds_i ds_j \left(\partial_i = \frac{\partial}{\partial s_i} \right). \tag{2.3}$$

Separating the real and the imaginary parts of the hermitian product

$$(\partial_i \psi, \partial_j \psi) = \gamma_{ij} + i\sigma_{ij} \tag{2.4}$$

one observes that

$$\gamma_{ij}(s) = \gamma_{ji}(s) \quad \text{and} \quad \sigma_{ij}(s) = -\sigma_{ji}(s). \tag{2.5}$$

Thus(2.3) reads:

$$\|\psi(s + ds) - \psi(s)\|^2 = \gamma_{ij}(s) ds_i ds_j. \tag{2.6}$$

The quantities $\gamma_{ij}(s)$ so defined have been proposed as the components of the metric tensor on the “manifold of collective states” [5]. [From (2.4) it is clear that the γ_{ij} ’s possess the correct transformation property under a change of the coordinates $s \rightarrow s'(s)$.]

However we emphasize that this tensor is meaningless as a metric tensor on a manifold of physical states in ordinary quantum mechanics. Indeed, as long as the phase of a vector state is not observable [8], the physical states are represented by rays of the Hilbert space and the two vectors $\psi(s)$ and $\psi'(s)$

$$\psi'(s) = e^{i\alpha(s)} \psi(s) \tag{2.7}$$

define the same point on the manifold (of rays). Consequently the metric tensors associated with the families $\{\psi'(s)\}$ for $\alpha(s)$ sufficiently smooth and $\{\psi(s)\}$ should be identical and this is not true; from (2.4) the tensor γ' with components

$$\gamma'_{ij} = \text{Re}(\partial_i \psi', \partial_j \psi') \tag{2.8}$$

is different from γ . More precisely we have:

$$\gamma'_{ij} = \gamma_{ij} + \beta_i(\partial_j \alpha) + \beta_j(\partial_i \alpha) + (\partial_i \alpha)(\partial_j \alpha) \tag{2.9}$$

with

$$\beta_j(s) = -i(\psi(s), \partial_j \psi(s)) \tag{2.10}$$

(the normalization condition on ψ implies that β_j is real).

In order to get a metric tensor whose components are independent of the gauge transformation (2.7). We remark that, with respect to this transformation, the β_j ’s change according to:

$$\beta_j \rightarrow \beta'_j = \beta_j + \partial_j \alpha. \tag{2.11}$$

Then, the comparison of the formulas (2.9) and (2.11) leads to the following definition of a meaningful metric tensor:

$$g_{ij}(s) = \gamma_{ij}(s) - \beta_i(s)\beta_j(s). \tag{2.12}$$

These $g_{ij}(s)$ clearly transform like the components of a tensor under a change of coordinates. They are invariant under the transformation (2.7) by construction. Finally the metric is positive definite. (This last property is verified by writing the distance element $d\ell^2$ between two nearby points on the manifold in terms of the vector states associated with these points:

$$d\ell^2 = g_{ij} ds_i ds_j = (\delta\psi, \delta\psi) - |(\psi, \delta\psi)|^2 \tag{2.13}$$

$$(\delta\psi(s) = \psi(s + ds) - \psi(s)).$$

The positive definiteness occurs since, according to Schwartz's inequality, $d\ell^2$ is different from zero unless the vectors ψ and $\delta\psi$ are proportional, i.e. unless the vectors ψ and $\psi + \delta\psi$ define the same point on the manifold of rays.

Remark. A straight way to recover the metric (2.12) consists in defining a distance on the projective Hilbert space of rays rather than on the Hilbert space of state vectors. The square of the distance $D(\tilde{\psi}_1, \tilde{\psi}_2)$ between any two rays $\tilde{\psi}_1$ and $\tilde{\psi}_2$ with associated normalized vectors $\psi_1 e^{i\alpha_1}$ and $\psi_2 e^{i\alpha_2}$ is defined by

$$D^2(\tilde{\psi}_1, \tilde{\psi}_2) = \text{Inf}_{\alpha_1, \alpha_2} \|\psi_1 e^{i\alpha_1} - \psi_2 e^{i\alpha_2}\|^2 = 2 - 2|(\psi_1, \psi_2)|. \tag{2.14}$$

This distance naturally induces the metric (2.12) on the manifold $\{\tilde{\psi}(s)\}$ of interest; up to second order one has:

$$D^2(\tilde{\psi}(s + ds), \tilde{\psi}(s)) = g_{ij}(s) ds_i ds_j. \tag{2.15}$$

(See Appendix 1 for details.)

2.2. Definition of a Symplectic Structure

Before we consider some explicit examples of manifolds of quantum states we recall, for the sake of completeness, that a symplectic structure can be obtained from the imaginary part of the scalar product in the Hilbert space [3]. Indeed, the antisymmetric tensor $\sigma_{ij}(s)$ allows one to define a 2-form σ :

$$\sigma(s) = \sigma_{ij}(s) ds_i \wedge ds_j \tag{2.16}$$

which has the following nice properties

i) $d\sigma = 0$ (d : exterior differentiation). (2.17)

ii) σ is invariant under the transformation (2.7). These properties are straightforward consequences of the locally true equality:

$$\sigma = d\beta \quad (\beta = \beta_i(s) ds_i) \tag{2.18}$$

and of the transformation law (2.11) written under the form:

$$\beta \rightarrow \beta + d\alpha. \tag{2.19}$$

2.3. *Some Examples*

We now review the geometrical properties of some manifolds without entering into the details of the calculations. Let us only remark that the quantities β_j and $\gamma_{ij} + i\sigma_{ij}$ are easily obtained from the scalar product between any two members of the family $\{\psi(s)\}$:

$$\begin{aligned} \beta_j(s) &= -i \frac{\partial}{\partial s'_j} (\psi(s), \psi(s'))|_{s'=s} \\ \gamma_{ij}(s) + i\sigma_{ij}(s) &= \frac{\partial}{\partial s_i} \frac{\partial}{\partial s'_j} (\psi(s), \psi(s'))|_{s'=s}. \end{aligned} \tag{2.21}$$

Of course, the list of examples considered below is not exhaustive. Our choice is motivated by the fact that these examples will support the general study of Sect. 3 and the physical discussion of Sect. 4. It contains some wellknown families of quantum states and therefore we do not specify our notations which are the standard ones.

2.3a. *Family of Translated States.* Let ψ_0 be any state vector of some quantum system and P_i ($i=1,2,3$) be the generators of the group of translations \mathbb{R}^3 in ordinary space. We consider the family $\{\psi(x)\}$ of translated states

$$\psi(x) = e^{-iP \cdot x} \psi_0 \quad (P \cdot x = P_i x_i) \tag{2.22}$$

with scalar product

$$(\psi(x), \psi(x')) = (\psi_0, e^{-iP(x'-x)} \psi_0). \tag{2.23}$$

If $\langle \rangle_x$ denotes the mean value of an operator in the state $\psi(x)$ one gets:

$$\begin{aligned} d\ell^2 = g_{ij}(x) dx_i dx_j &= \langle P_i P_j \rangle_x - \langle P_i \rangle_x \langle P_j \rangle_x dx_i dx_j \\ &= \langle P_i P_j \rangle_0 - \langle P_i \rangle_0 \langle P_j \rangle_0 dx_i dx_j \end{aligned} \tag{2.24}$$

and

$$\sigma = \sigma_{ij}(x) dx_i \wedge dx_j = 0.$$

The metric structure on the manifold $\{\tilde{\psi}(x)\}$ of translated states is that of an Euclidean space. (This structure was not initially present on the group \mathbb{R}^3 .)

2.3b. *Glauber Coherent States (Harmonic Oscillator).* These states can be defined by:

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{1/2}} |n\rangle \quad (\alpha = \alpha_1 + i\alpha_2 = \varrho e^{i\varphi} \in \mathbb{C}). \tag{2.25}$$

From their scalar product

$$\langle \alpha | \alpha' \rangle = \exp\{\bar{\alpha}\alpha' - \frac{1}{2}|\alpha|^2 - \frac{1}{2}|\alpha'|^2\} \tag{2.26}$$

one gets

$$\begin{aligned} d\ell^2 &= d\alpha_1^2 + d\alpha_2^2 = d\varrho^2 + \varrho^2 d\varphi^2 \\ \sigma &= 2d\alpha_1 \wedge d\alpha_2 = 2\varrho d\varrho \wedge d\varphi. \end{aligned} \tag{2.27}$$

The metric and the symplectic structures are those of a 2-dimensional plane. Therefore the Riemannian curvature of the manifold $\{\tilde{|\alpha\rangle}\}$ of Glauber coherent states is zero.

2.3c. *Atomic Coherent States* [9]. These states which have a component of the angular momentum equal to $-j$ in the (θ, φ) direction read :

$$|\theta, \varphi\rangle = \sum_{m=-j}^j \binom{2j}{m+j}^{1/2} \left(\sin \frac{\theta}{2}\right)^{j+m} \left(\cos \frac{\theta}{2}\right)^{j-m} \exp(-i(j+m)\varphi) |j, m\rangle. \tag{2.28}$$

From their scalar product

$$\langle \theta, \varphi | \theta', \varphi' \rangle = \left(\sin \frac{\theta}{2} \sin \frac{\theta'}{2} e^{-i(\varphi - \varphi')} + \cos \frac{\theta}{2} \cos \frac{\theta'}{2} \right)^{2j} \tag{2.29}$$

one gets :

$$\begin{aligned} d\ell^2 &= \frac{1}{2} j (\sin^2 \theta d\varphi^2 + d\theta^2) \\ \sigma &= j \sin \theta d\theta \wedge d\varphi. \end{aligned} \tag{2.30}$$

The metric and the symplectic structures on the manifold $\{\tilde{|\theta, \varphi\rangle}\}$ are those of the sphere S_2 . Therefore, the Riemannian curvature of this manifold is positive and constant. It is equal to $2j^{-1}$. (This value is the inverse of the square of the radius of curvature.)

2.3d. $|z\rangle$ States (*Harmonic Oscillator*) [10, 6]. These states are defined by

$$|z\rangle = (1 - |z|^2)^{1/2} \sum_{n=0}^{\infty} z^n |n\rangle \quad (z = \varrho e^{i\varphi} \in \mathbb{C}). \tag{2.31}$$

From their scalar product

$$\langle z | z' \rangle = (1 - \bar{z}z')^{-1} (1 - |z|^2)^{1/2} (1 - |z'|^2)^{1/2} \tag{2.32}$$

one gets $\left(\varrho = \text{th} \frac{\theta}{2}\right)$:

$$\begin{aligned} d\ell^2 &= \frac{1}{(1 - \varrho^2)^2} (d\varrho^2 + \varrho^2 d\varphi^2) = \frac{1}{4} (\text{sh}^2 \theta d\varphi^2 + d\theta^2) \\ \sigma &= \frac{2}{(1 - \varrho^2)^2} \varrho d\varrho \wedge d\varphi = \frac{1}{2} \text{sh} \theta d\theta \wedge d\varphi. \end{aligned} \tag{2.33}$$

The metric and the symplectic structures of the manifold $\{\tilde{|z\rangle}\}$ are those of the manifold $\{\tilde{|\theta, \varphi\rangle}\}$ for $j = \frac{1}{2}$ provided one replaces the trigonometric functions by hyperbolic ones. They correspond to the Lobatchevski plane. Therefore the curvature of this manifold is constant and negative.

3. Curvature of G.C.S.M.

3.1. Definition

In all the previous examples the manifold of physical states is obtained by the action of a Lie group G on a fixed state. This is clear on the Example 2.3a for which

$G = \mathbb{R}^3$. In the case of the Example 2.3b, it is wellknown that the Glauber coherent states can also be written

$$|\alpha\rangle = e^{(\alpha a^+ - \bar{\alpha} a)}|0\rangle \quad ([a, a^+] = I) \tag{3.1}$$

and the manifold of rays $\{\widetilde{|\alpha\rangle}\}$ is identical to the manifold $\{\widetilde{\exp(i\lambda I + \alpha a^+ - \bar{\alpha} a)}|0\rangle\}$ ($\lambda \in \mathbb{R}$). This manifold is obtained by the action on the vacuum state $|0\rangle$ of the Heisenberg-Weyl group whose Lie algebra is generated by $I, a,$ and a^+ . Since the rays $\widetilde{|\alpha\rangle}$ are invariant under the action of the subgroup $K = \{e^{i\lambda I}\}$ the manifold $\{\widetilde{|\alpha\rangle}\}$ may be considered as an homogeneous space parametrized by the coset space G/K . The Examples 2.3c and 2.3d can be similarly analyzed [6].

We are therefore naturally lead to consider the case of an arbitrary G.C.S.M. Such a manifold is by definition [6] a set of rays of the type $\widetilde{U(g)\psi_0}$ ($g \in G$) where U is an irreducible unitary representation of the Lie group G on a Hilbert space \mathcal{H} and ψ_0 is a fixed vector in \mathcal{H} . In order to parametrize the G.C.S.M. and to recover the notation $\{\widetilde{\psi(s)}\}$ let us introduce the subgroup K of G which leaves $\widetilde{\psi_0}$ invariant. K is the set of elements $g \in G$ such that

$$\widetilde{U(g)\psi_0} = \widetilde{\psi_0} \quad (U(g)\psi_0 = e^{i\alpha(g)}\psi_0). \tag{3.2}$$

Let \mathcal{G} and \mathcal{K} be the Lie algebras of the groups G and K and consider a decomposition of \mathcal{G} under the form:

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{X}. \tag{3.3}$$

(\mathcal{X} is a vector space in \mathcal{G} but not, in general, a Lie algebra.) Then, in a neighbourhood of $\widetilde{\psi_0}$ the manifold $\{\widetilde{U(g)\psi_0}\}$ can be identified with the manifold $\{\widetilde{\psi(s)}\}$ obtained from the vectors

$$\psi(s) = e^{is_j X_j} \psi_0 \quad (s = (s_1 \dots s_n) \in \mathbb{R}^n), \tag{3.4}$$

where the n hermitian operators X_j form a basis of \mathcal{X} in the representation U . The left-action of an element $g \in G$ on the manifold changes the parameter s into gs such that:

$$\widetilde{U(g)\psi(s)} = \widetilde{\psi(gs)}. \tag{3.5}$$

3.2. Properties of the Metric Tensor

The metric on the G.C.S.M. is that introduced in Sect. 2 [formula (2.12) or (2.15)]. The group structure which enters into the definition of a G.C.S.M. implies that the metric is left-invariant under the action of the group:

$$g_{ij}(s) ds_i ds_j = g_{ij}(gs) d(gs)_i d(gs)_j. \tag{3.6}$$

This invariance follows from (2.15) and the fact that $U(g)$ defines an isometry on the projective Hilbert space equipped with the distance D [formula (2.14)]:

$$D^2(U(g)\psi_1, U(g)\psi_2) = D^2(\psi_1, \psi_2) \tag{3.7}$$

$$(\psi_1 = \psi(s_1); \psi_2 = \psi(s_2)).$$

As a consequence it is sufficient to know the metric at one point of the manifold since it can be carried to the remaining points by a left-translation. We

shall choose the point $\tilde{\psi}_0(s=0)$. At this point the components of the metric tensor $g_{ij}(0)$ are calculated from (2.21) as in the Example 2.3a; the result is:

$$g_{ij}(0) = \text{Re} \langle X_i X_j \rangle_0 - \langle X_i \rangle_0 \langle X_j \rangle_0. \tag{3.8}$$

The tangent space to the manifold at the point $\tilde{\psi}_0$ can be identified with the subspace \mathcal{X} of the Lie algebra \mathcal{G} . Therefore, in this way, the metric structure on the manifold induces a pseudo-scalar product on \mathcal{G} . Denoting the elements of \mathcal{G} by small letters and their hermitian representatives in the representation U by capital letters, this product \langle , \rangle is:

$$\langle a, b \rangle = \text{Re} \langle AB \rangle_0 - \langle A \rangle_0 \langle B \rangle_0. \tag{3.9}$$

It is real and symmetric since A and B are hermitian operators; moreover it is positive definite on the subspace \mathcal{X} since, according to formula (3.8):

$$\langle x_i, x_j \rangle = g_{ij}(0) \quad (x_{i,j} \in \mathcal{X}). \tag{3.10}$$

If however one of the elements belongs to the subalgebra \mathcal{K} , say $a = k_\mu \in \mathcal{K}$, then from (3.2) the vector ψ_0 is an eigenvector of K_μ and the product $\langle k_\mu, b \rangle$ is equal to zero. One can understand that the vectors $k_\mu \in \mathcal{K}$ are isotropic in \mathcal{G} in the following way. If the group K is not trivial the manifold is an homogeneous space of G ; roughly speaking it can nevertheless be identified with the whole group G in so far as the distance between two points which are deduced from each other by the right-action of K is arbitrary small.

3.3. Curvature of a G.C.S.M.

In the examples of Sect. 2.3 the manifolds are 2-dimensional ones. In that case the curvature is well described by a scalar. In more than 2-dimensions one associates a scalar curvature to each 2-plane belonging to the tangent space of the manifold at a point s . These ‘‘Riemannian bisectonal curvatures’’ [11] are the curvatures of the 2-dimensional smooth surfaces generated by the geodesics whose tangent vectors at s lie in the 2-planes. They have the following properties:

i) in the case where the manifold is 2-dimensional the bisectonal curvature reduces to the scalar curvature considered above;

ii) if the manifold is n -dimensional ($n > 2$) let $\{\hat{t}_i\}$ ($i = 1, \dots, n$) be an orthonormal basis in the tangent space at the point s . There are $\frac{n(n+1)}{2}$ bisectonal curvatures $C_{i\hat{t}_i}$, corresponding to all pairs $\{\hat{t}_i, \hat{t}_j\}$ of basis vectors ($i < j$). Their sum

$\sum_{i < j} C_{i\hat{t}_i}$ is invariant under an orthogonal change of basis and is equal to the scalar Riemannian curvature.

The group structure which enters into the definition of a G.C.S.M. and the fact that the metric (2.12) is left-invariant enable us to use an extension of a result by Arnold to calculate the bisectonal curvatures at the point $\tilde{\psi}_0$ ($s=0$). Indeed, Arnold has given an explicit formula for the bisectonal curvatures at the neutral element when the manifold is a Lie group equipped with a left-invariant metric [12]. We show in Appendix 2 how this formula can be extended to the case of an

homogeneous space but, for further discussions, it is sufficient to set here the notations and the general form of the result.

Let $\{\hat{x}_i\}$ be an orthonormal basis in \mathcal{X} with respect to the scalar product (3.10):

$$\langle \hat{x}_i, \hat{x}_j \rangle = \delta_{ij}. \tag{3.11}$$

The commutator of two basis vectors \hat{x}_i and \hat{x}_j splits in two parts belonging respectively to \mathcal{X} and \mathcal{K} :

$$[\hat{x}_i, \hat{x}_j] = i \left(\sum_{\ell} f_{ij}^{\ell} x_{\ell} + k_{ij} \right); \quad k_{ij} \in \mathcal{K}. \tag{3.12}$$

(The quantities f are structural constants of G .)

The bisectional curvature $C_{\hat{x}_i, \hat{x}_j}$ in the 2-dimensional direction $\{\hat{x}_i, \hat{x}_j\}$ is given by:

$$C_{\hat{x}_i, \hat{x}_j} = Q_{ij}(f) - \frac{1}{2} (\langle i[\hat{x}_j, k_{ij}], \hat{x}_i \rangle - \langle i[\hat{x}_i, k_{ij}], \hat{x}_j \rangle), \tag{3.13}$$

where $Q_{ij}(f)$ is the following quadratic function of the structural constants:

$$Q_{ij}(f) = \sum_{\ell} \left(\frac{1}{4} (f_{j\ell}^i + f_{i\ell}^j)^2 + \frac{1}{2} f_{ij}^{\ell} (f_{j\ell}^i - f_{i\ell}^j) - \frac{3}{4} (f_{ij}^{\ell})^2 - f_{i\ell}^j f_{j\ell}^i \right). \tag{3.14}$$

When the manifold is the whole group ($\mathcal{K} \equiv 0$; $\mathcal{X} = \mathcal{G}$) $Q_{ij}(f)$ is equal to $C_{\hat{x}_i, \hat{x}_j}$ and one recovers Arnold's result.

3.4. Some Examples

In order to see how the formula (3.13) works we reconsider and generalize some examples of the Sect. 2.3.

In the case of the Heisenberg-Weyl group \mathcal{K} is generated by $\{I\}$ and a basis of \mathcal{X} is $x_1 = x$ (position operator) and $x_2 = p$ (momentum operator). Therefore the structural constants f_{12}^{ℓ} are zero and k_{12} is in the center of \mathcal{G} . Each term in (3.13) is zero and the curvature is null. This result is true whatever ψ_0 may be. (ψ_0 is the vacuum state for Glauber coherent states.)

In the case of atomic coherent states ψ_0 is the state $|j, -j\rangle$ and the group G is the rotation group. \mathcal{K} and \mathcal{X} are respectively generated by $\{j_3\}$ and $\{j_1, j_2\}$ ($[j_i, j_k] = i\varepsilon_{ik\ell} j_{\ell}$). From formula (3.8) and standard results such as $\langle j_1^2 \rangle_0 = \frac{j}{2}$ one sees that

the elements $\hat{x}_i = \left(\frac{2}{j}\right)^{1/2} j_i$ ($i=1, 2$) are orthonormal vectors in \mathcal{X} . Then, the constants of structure $f_{12}^{\ell} = \varepsilon_{12\ell}$ ($\ell=1, 2$) are zero and the element k_{12} of \mathcal{K} is $\frac{2}{j} j_3$. Finally from (3.13) the curvature is:

$$C_{\hat{x}_1, \hat{x}_2} = -\frac{1}{2} \left(\frac{2}{j}\right)^2 (\langle i[j_2, j_3], j_1 \rangle - \langle i[j_1, j_3], j_2 \rangle) = 2j^{-1}. \tag{3.15}$$

An example of a manifold whose dimension is greater than two is the one obtained by the action on some state $\tilde{\psi}_0$ of the group of the Harmonic oscillator whose Lie algebra is generated by the operators I (identity), x (Galilee transformation), p (space translation), and $H = \frac{1}{2}(x^2 + p^2)$ (time translation). This manifold

in general is 3-dimensional $\{x, p, H\}$ being a basis of the subspace $\mathcal{X} \subset \mathcal{G}$. Let $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ be an orthonormal basis of \mathcal{X} :

$$\hat{x}_1 = a_1 x; \quad \hat{x}_2 = a_2(p + \alpha_1 x); \quad \hat{x}_3 = a_3(H + \alpha_2 x + \alpha_3 p). \quad (3.16)$$

(In these expressions the normalization constants a_i , which are strictly positive, and the α_i depend on $\tilde{\psi}_0$.) The scalar Riemannian curvature

$$C = C_{\hat{x}_1 \hat{x}_2} + C_{\hat{x}_2 \hat{x}_3} + C_{\hat{x}_1 \hat{x}_3} = Q_{12}(f) + Q_{23}(f) + Q_{13}(f) \quad (3.17)$$

is found to be:

$$C = -a_3^2 \left(\alpha_1^2 + \frac{1}{4} \left(\frac{a_2}{a_1} (1 + \alpha_1^2) - \frac{a_1}{a_2} \right)^2 \right). \quad (3.18)$$

It is generally negative but can be zero iff $\alpha_1 = 0$ and $\frac{a_1}{a_2} = 1$, that is iff the state $\tilde{\psi}_0$ is such that:

$$\text{Re} \langle xp \rangle_0 - \langle x \rangle_0 \langle p \rangle_0 = 0; \quad \langle x^2 \rangle_0 - \langle x \rangle_0^2 = \langle p^2 \rangle_0 - \langle p \rangle_0^2. \quad (3.19)$$

Moreover, in that case each term in (3.17) is zero.

Using the notations of (3.16), a similar calculation can be achieved in the case of the group of free motion $\left(H = \frac{p^2}{2}\right)$; the corresponding result is

$$C = -\frac{1}{4} \frac{a_2^2 a_3^2}{a_1^2}. \quad (3.20)$$

Contrary to (3.18) this curvature, as well as the bisectonal curvatures in the planes $\{x, p\}$, $\{x, H\}$ or $\{p, H\}$, can never be set equal to zero. These results (3.18) and (3.20) will be shortly discussed in Sect. 4.

4. Physical Comments

Although the notion of distance between quantum states has been invoked in the context of measurement theory [13], its physical signification has not been much discussed. The same comment may be applied to its infinitesimal version, that of a metric, with which we are concerned in this paper. Therefore we now want to initiate a physical discussion on this subject.

A first remark is that the concept of a metric on a manifold of quantum states is the only structure which remains of the underlying Hilbert space. Therefore the question naturally arises whether there exist quantum physical situations which do not require the full Hilbert structure. We think that this may be the case for macroscopic systems which exhibit collective behaviour. Indeed, for such systems, the possibility of going from one state to another one is not described by a direct transition amplitude (scalar product in Hilbert space) but rather through a succession of infinitesimal steps on the manifold of collective states. The relevant distance between distinct states is then the distance measured along geodesics on the manifold.

A second remark is that the infinitesimal distance $d\ell$ between two nearby states differing only by the parameter s_i is related to the quantum fluctuations of the corresponding generator X_i . From (3.8), $d\ell$ reads:

$$d\ell = (\Delta X_i)^{-1} ds_i \quad (\Delta X_i = (\langle X_i^2 \rangle - \langle X_i \rangle^2)^{1/2}).$$

The distance element $d\ell$ being dimensionless, $(\Delta X_i)^{-1}$ appears to be a natural unit for the parameter s_i on the manifold. Heuristical arguments from Fourier analysis of the type $\Delta X_i \Delta S_i \sim 1$ (S_i operator conjugated to X_i) show that the unit for the parameter s_i is proportional to the fluctuation ΔS_i .

The last comment deals with the results (3.18) and (3.20). It is interesting to note that in the case of the “harmonic oscillator manifold” the condition of zero curvature (3.19) is identical with the condition of non dispersion of the wave packet [14]; similarly, in the case of the free motion where it is known that any wave packet disperses, formula (3.20) shows that the curvature can never be set equal to zero. These examples and the fact that the curvature is constant for any G.C.S.M. suggest that the curvature may be used to define a “collective” dispersion for a non commutative family of quantum operators.

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Appendix 1

If one inserts the Taylor expansion

$$\psi(s + ds) = \psi(s) + (\partial_i \psi)(s) ds_i + \frac{1}{2} (\partial_i \partial_j \psi)(s) ds_i ds_j + \dots$$

in the scalar product $(\psi(s), \psi(s + ds))$ one gets, up to second order:

$$(\psi(s), \psi(s + ds)) = 1 + \beta_i(s) ds_i + \frac{1}{2} (\psi, \partial_i \partial_j \psi)(s) ds_i ds_j.$$

The modulus of this product is:

$$|(\psi(s), \psi(s + ds))| = 1 + \frac{1}{2} [\text{Re}(\psi, \partial_i \partial_j \psi)(s) + \beta_i(s) \beta_j(s)] ds_i ds_j.$$

But:

$$\text{Re}(\psi, \partial_i \partial_j \psi) = -\text{Re}(\partial_i \psi, \partial_j \psi) = -\gamma_{ij}$$

since $\partial_i(\psi, \partial_j \psi) = i \partial_i \beta_j$ is imaginary.

Therefore, with use of (2.14):

$$\begin{aligned} D^2(\widetilde{\psi(s)}, \widetilde{\psi(s + ds)}) &= (\gamma_{ij}(s) - \beta_i(s) \beta_j(s)) ds_i ds_j \\ &= g_{ij}(s) ds_i ds_j. \end{aligned}$$

Appendix 2

Let G be a Lie group equipped with a left-invariant metric, \mathcal{G} its Lie algebra, $\langle \cdot, \cdot \rangle$ the scalar product in \mathcal{G} which determines the metric and $[\cdot, \cdot]$ the Lie bracket in \mathcal{G} . According to Arnold [12] the sectional curvature of G at the point g_0 (g_0 identity

of G) in the two dimensional plane defined by the orthonormal vectors \hat{m} and \hat{n} of \mathcal{G} is:

$$C_{\hat{m}, \hat{n}} = \langle d, d \rangle + 2\langle a, b \rangle - 3\langle a, a \rangle - 4\langle B_{\hat{m}}, B_{\hat{n}} \rangle. \quad (\text{A.2.1})$$

In this expression $a, b, c, B_{\hat{m}}$, and $B_{\hat{n}}$ belong to \mathcal{G} and are defined by:

$$\begin{aligned} 2d &= B(\hat{m}, \hat{n}) + B(\hat{n}, \hat{m}) & 2b &= B(\hat{m}, \hat{n}) - B(\hat{n}, \hat{m}) \\ 2a &= [\hat{m}, \hat{n}] & 2B_{\hat{m}} &= B(\hat{m}, \hat{m}), \end{aligned} \quad (\text{A.2.2})$$

where the operation $B: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ is such that:

$$\langle B(\hat{p}, \hat{q}), \hat{r} \rangle = -\langle i[\hat{q}, \hat{r}], \hat{p} \rangle. \quad (\text{A.2.3})$$

In the case of an homogeneous space G/K , we introduce the decomposition (3.3) $\mathcal{G} = \mathcal{H} \oplus \mathcal{X}$ and recall that the metric is zero on \mathcal{H} . This implies that the operation B is no longer defined. However, one can generalize in a simple way the results by Arnold by considering this metric as the limit of a positive definite metric on \mathcal{G} whose restriction to \mathcal{X} is maintained fixed while its restriction to \mathcal{H} goes to zero.

To this end, let $\{\hat{x}_i\}$ be a fixed orthonormal basis in \mathcal{X} and let $\{k_\mu\}$ be a basis in \mathcal{H} such that:

$$\begin{aligned} \langle k_\mu, k_\nu \rangle &= \|k_\mu\|^2 \delta_{\mu\nu} \\ \langle \hat{x}_i, k_\mu \rangle &= 0. \end{aligned} \quad (\text{A.2.4})$$

We consider the limit where the norms $\|k_\mu\|$ go to zero (isotropic vectors) and show that the quantities (A.2.2) have a definite limit, although the operation B is not defined in the limit.

Writing $[\hat{x}_i, \hat{x}_j]$ under the form

$$[\hat{x}_i, \hat{x}_j] = i \sum_{\ell} f_{ij}^{\ell} \hat{x}_{\ell} + \sum_{\mu} f_{ij}^{\mu} k_{\mu} \quad (\text{A.2.5})$$

the B operation on \mathcal{X} reads

$$B(\hat{x}_i, \hat{x}_j) = \sum_{\ell} f_{j\ell}^i \hat{x}_{\ell} - i \sum_{\mu} \|k_{\mu}\|^{-2} \langle [\hat{x}_j, k_{\mu}], \hat{x}_i \rangle k_{\mu}. \quad (\text{A.2.6})$$

In order to calculate the quantities (A.2.2), one uses the crucial identity

$$\langle i[\hat{x}_i, k_{\mu}], \hat{x}_j \rangle = -\langle i[\hat{x}_j, k_{\mu}], \hat{x}_i \rangle \quad (\text{A.2.7})$$

which occurs since the state $\tilde{\psi}_0$ is invariant under the group K . One easily finds

$$\begin{aligned} \langle a, b \rangle &= \frac{1}{4} \left(\sum_{\ell} f_{ij}^{\ell} (f_{j\ell}^i - f_{i\ell}^j) - i \sum_{\mu} f_{ij}^{\mu} (\langle [\hat{x}_j, k_{\mu}], \hat{x}_i \rangle - \langle [\hat{x}_i, k_{\mu}], \hat{x}_j \rangle) \right) \\ \langle a, a \rangle &= \frac{1}{4} \left(\sum_{\ell} (f_{ij}^{\ell})^2 + \sum_{\mu} (f_{ij}^{\mu})^2 \|k_{\mu}\|^2 \right) \\ \langle d, d \rangle &= \frac{1}{4} \sum_{\ell} (f_{j\ell}^i + f_{i\ell}^j)^2 \end{aligned} \quad (\text{A.2.8})$$

$$\langle B_{\hat{x}_i}, B_{\hat{x}_j} \rangle = \frac{1}{4} \sum_{\ell} f_{i\ell}^i f_{j\ell}^j.$$

All the terms in (A.2.8) clearly have a definite limit when $\|k_\mu\|^2$ goes to zero. From (A.2.1) one recovers immediately the formula (3.13) for the bisectonal curvature $C_{\hat{x}_i, \hat{x}_j}$.

References

1. See, for example Arnold, V.I.: *Mathematical methods of classical mechanics*, Chap. 8. Berlin, Heidelberg, New York: Springer 1978
2. Souriau, J.M.: *Structure des systèmes dynamiques*. Paris: Dunod 1970
- Kostant, B.: *Lecture Notes in Mathematics*, Vol. 170. Berlin, Heidelberg, New York: Springer 1970
3. Kibble, T.W.B.: *Commun. Math. Phys.* **65**, 189–201 (1979)
4. Reinhard, P.G., Goeke, K.: *Phys. Rev. C* **20**, 1546–1559 (1979), and references therein
5. Giraud, B.G., Rowe, D.J.: *Journal de Physique, Lettre* **8**, 177–180 (1979) and Preprint C.E.A. DPh. T. 9/79 (1979)
6. Perelomov, A.M.: *Commun. Math. Phys.* **26**, 222–236 (1972)
7. Ruppeiner, G.: *Phys. Rev. A* **20**, 1608–1613 (1979)
8. Provost, J.P., Rocca, F., Vallee, G.: *Ann. Phys. (New York)* **94**, 307–319 (1975)
9. Arecchi, F.T., Courtens, E., Gilmore, R., Thomas, H.: *Phys. Rev. A* **6**, 2211–2237 (1972)
10. Ifantis, E.K.: *J. Math. Phys.* **12**, 1021–1026 (1971)
11. See Ref. [1], Appendix 1
12. See Ref. [1], Appendix 2, Theorem 10
13. Gudder, S., Marchand, J.P., Wyss, W.: *J. Math. Phys.* **20**, 1963–1966 (1979)
14. Messiah, A.: *Mécanique quantique*, Tome 1, Chap. XII. Paris: Dunod 1969

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