Commun. Math. Phys. 74, 71–95 (1980)

Communications in Mathematical Physics
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The Boltzmann Equation with a Soft Potential

I. Linear, Spatially-Homogeneous

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Abstract. The initial value problem for the linearized spatially-homogeneous Boltzmann equation has the form $\frac{\partial f}{\partial t} + Lf = 0$ with $f(\boldsymbol{\xi}, t=0)$ given. The linear operator L operates only on the ξ variable and is non-negative, but, for the soft potentials considered here, its continuous spectrum extends to the origin. Thus one cannot expect exponential decay for f, but in this paper it is shown that f decays like $e^{-\lambda t^\beta}$ with $\beta < 1$. This result will be used in Part II to show existence of solutions of the initial value problem for the full nonlinear, spatially dependent problem for initial data that is close to equilibrium.

1. Introduction

The initial value problem for the Boltzmann equation of kinetic theory is

$$\frac{\partial F}{\partial t} + \boldsymbol{\xi} \cdot \frac{\partial F}{\partial \mathbf{x}} + Q(F, F) = 0, \qquad F(t=0) = F_0 \tag{1.1}$$

in which

$$F = F(\boldsymbol{\xi}, t, \mathbf{x}), \tag{1.2}$$

$$t \in \mathbb{R}^+, \quad \boldsymbol{\xi} \in \mathbb{R}^3, \quad \mathbf{x} \in \mathbb{R}^3. \tag{1.3}$$

Throughout this paper a boldface letter will represent a vector in $\mathbb{R}^3$, while the non-boldface letter signifies its magnitude. The quadratically nonlinear operator Q vanishes if F is a Maxwellian:

$$F_M = \frac{\varrho}{(2\pi T)^{3/2}} e^{-|\boldsymbol{\xi} - \mathbf{u}|^2/2T}, \tag{1.4}$$

where $\varrho, \mathbf{u}, T$ can be any functions of x and t. If they are constants, F_M is an equilibrium solution of (1.1). We will study solutions of (1.1) which are close to such an equilibrium and which are *independent of space.*

* Supported by the National Science Foundation under Grant MCS78-09525 and MCS76-07039 and by the United States Army under Contract DAAG29-75-C-0024

0010-3616/80/0074/0071/$05.00

Write F in the form

$$F(\boldsymbol{\xi},t)=\omega(\xi)+\sqrt{\omega(\xi)}\,f(\boldsymbol{\xi},t), \tag{1.5}$$

in which

$$\omega(\xi)=\frac{1}{(2\pi)^{3/2}}e^{-1/2\,\xi^2}. \tag{1.6}$$

Note that we have removed the $\varrho, \mathbf{u}$, and T by scaling and translating. The equation for f is

$$\frac{\partial f}{\partial t}+Lf+\nu\Gamma(f,f)=0, \tag{1.7}$$

with

$$Lf=2\omega^{-1/2}Q(\omega,\omega^{1/2}f), \tag{1.8}$$

$$\nu\Gamma(f,f)=\omega^{-1/2}Q(\omega^{1/2}f,\omega^{1/2}f). \tag{1.9}$$

In this paper we consider only the linearized equation with given initial data, i.e.

$$\frac{\partial f}{\partial t}+Lf=0, \tag{1.10}$$

$$f(\boldsymbol{\xi},0)=f_0(\boldsymbol{\xi})\in N(L)^{\perp}, \tag{1.11}$$

where $N(L)^{\perp}$ is the orthogonal complement of the null space of L. This last condition (1.11) on f_0 means that we have chosen the right Maxwellian to perturb around; i.e. all the mass, momentum and energy is in the Maxwellian distribution ω.

The linear operator L was analyzed extensively by Grad [5], and we take our notation as well as the general outline of our procedure from there. Grad showed that

$$(Lf)(\boldsymbol{\xi})=\nu(\xi)f(\boldsymbol{\xi})+(Kf)(\boldsymbol{\xi}), \tag{1.12}$$

where K is a compact integral operator and $\nu(\xi)$ is a function which is essentially of the form

$$\nu(\xi)=(1+\xi)^{\gamma}. \tag{1.13}$$

The operator L is self-adjoint and non-negative, i.e.

$$(Lf,f)\geqq 0, \tag{1.14}$$

and has 0 as an eigenvalue of multiplicity 5. Since a compact perturbation does not disturb the continuous spectrum of a self-adjoint operator [12], the decomposition (1.12) shows that

$$\sigma_{\mathrm{cont}}(L)=\{\lambda:\lambda=\nu(\xi)\text{ for some }\xi\}. \tag{1.15}$$

If the force law between two particles is a power of their distance apart, i.e.

$$F(r)=r^{-s}, \tag{1.16}$$

then the exponent γ is found as

$$\gamma=\frac{s-5}{s-1}. \tag{1.17}$$

The mathematical theory is sensitive to the sign of γ. A *hard potential* is a collision law for which $\gamma \geqq 0$ or $s \geqq 5$. The values of ν go from 1 to ∞ and so $\sigma_{\text{cont}}(L)$ does likewise. All that is left in $\sigma(L)$ is discrete eigenvalues, and there is a lowest non-zero eigenvalue λ_0, which is positive. This shows that the part of f in the range of L decays like $e^{-\lambda_0 t}$. Using this decay various authors [6, 11, 13] have shown existence for all time for the linear and nonlinear problems with spatial homogeneity or inhomogeneity, if the initial data is close to Maxwellian, i.e. if $f_0(\boldsymbol{\xi}, x)$ is small. For the nonlinear spatially homogeneous problem, Arkeryd [1] has shown the global existence for a broad class of initial data.

On the other hand for a *soft potential*, with $3 < s < 5$, the function ν has the expression

$$\nu(\xi) = (1+\xi)^{-\gamma}, \tag{1.18}$$

with

$$\gamma = -\frac{s-5}{s-1} > 0. \tag{1.19}$$

(We have switched the sign of γ to emphasis the negativity of the exponent.) Now the values of ν range from 0 to 1, and so the spectrum of L goes all the way down to 0. Thus we cannot expect exponential decay in (1.10), and none of the existence results mentioned above are applicable. Nonetheless we show in this paper that the part of f in the range of L does decay at the rate $e^{-\lambda t^{\beta}}$, with $\beta = \dfrac{2}{2+\gamma}$ and $\lambda > 0$. This is our main result and is stated precisely in Sect. 3. The reason for this decay is that the small values of λ in $\sigma(L)$ correspond to small values of $\nu(\xi)$ and to large velocities ξ. But we will assume that f_0 looks approximately like $e^{-\alpha \xi^2}$, i.e. that it is comparable to a Maxwellian, so that these large velocities are relatively unimportant.

The exact form of L and a modification to remove its null space are presented in Sect. 2. After the main result is stated in Sect. 3, an outline of the proof is given in Sect. 4. Sections 5 and 6 are devoted to estimates on the compact operator K. Then the spectrum of L restricted to $\mathscr{L}^2(\boldsymbol{\xi} : \xi < w)$ is analyzed in Sect. 7. In Sect. 8 we pick the constants λ_0, β, w, μ which appear in previous sections. Finally in Sect. 9 the iteration equation is solved and in Sects. 10 and 11 it is shown that the iteration procedure converges for all time and that the decay is maintained for the $\mathscr{L}^2$ norm. In Sect. 12 we find that the α-norm is preserved and the sup norm decays.

In Part II we will show the global existence of solutions of the spatially periodic initial value problem for the linear spatially dependent equation and for the full nonlinear Eq. (1.1) with small initial data.

Inverse power repulsive forces are often used as first approximations to more realistic but complicated forces [7]. The power s is usually chosen to give agreement with the coefficient of viscosity or heat flow or some other measurable quantity of the gas. For most gases hard forces, with s between 9 and 15, are most realistic while for a few gases soft forces, with s below but close to 5, are relevant [2]. Many authors [10, 2, 4] have also used the Maxwellian force $s=5$ because of its computational simplicity. Of course there is interest in the very soft Coulomb force with $s=2$, which our treatment of $3 < s < 5$ does not include.

Short-time existence theorems for the full nonlinear, spatially dependent problem (1.1) were proved by Glikson [3], Kaniel and Shinbrot [8], and Lanford [9]. Their work included hard as well as soft potentials. Glikson solved the equation by direct iteration. Kaniel and Shinbrot used decreasing and increasing sequences of functions which squeezed down on the solution. Both allow a very general class of initial data. Our results are more restrictive since we consider only small perturbations from equilibrium, but are stronger since we obtain existence and decay for all time.

Throughout the paper there are estimates with constant coefficients. It is not necessary to keep careful account of these constants, and so *we will use c as a generic constant* replacing any other constant (such as c^2) by c.

I am very grateful to Harold Grad, who suggested this problem and found the improved estimates for soft potentials which are basic to its solution. He also pointed out the decay of the eigenfunctions which is crucial in the analysis of $\sigma(L)$ in Sect. 7. In addition I had a number of helpful and stimulating discussions with Percy Deift and George Papanicolaou. This work was performed at the Courant Institute and at the Mathematics Research Center; I am happy to acknowledge their support.

2. The Linearized Collision Operator

The Boltzmann collision operator has the form

$$Q(F,F)(\boldsymbol{\xi}) = \int (F'F'_1 - FF_1) B(\theta, \mathbf{V})\, d\theta\, d\varepsilon\, d\boldsymbol{\xi}_1 , \tag{2.1}$$

where

$$\mathbf{V} = \boldsymbol{\xi}_1 - \boldsymbol{\xi} , \tag{2.2}$$

$$F' = F(\boldsymbol{\xi}') \quad F'_1 = F(\boldsymbol{\xi}'_1) \quad F_1 = F(\boldsymbol{\xi}_1) , \tag{2.3}$$

$$\begin{aligned} \boldsymbol{\xi}' &= \boldsymbol{\xi} + \boldsymbol{\alpha}(\boldsymbol{\alpha}\cdot\mathbf{V}) , \\ \boldsymbol{\xi}'_1 &= \boldsymbol{\xi}_1 - \boldsymbol{\alpha}(\boldsymbol{\alpha}\cdot\mathbf{V}) , \end{aligned} \tag{2.4}$$

and $\boldsymbol{\alpha}$ is the unit vector in the direction of the apse line. The angle θ range from 0 to $\frac{\pi}{2}$ with $\pi - 2\theta$ being the angle of deflection in center of mass coordinates, and ε is the angular coordinate in the impact parameter plane.

Grad [5] has found exact and convenient forms for the function v and the compact operator K in (1.12). These are

$$v(\xi) = 2\pi \int B(\theta, \mathbf{v})\, \omega(\eta)\, d\theta\, d\boldsymbol{\eta} , \tag{2.5}$$

$$Kf(\boldsymbol{\xi}) = \int k(\boldsymbol{\xi}, \boldsymbol{\eta}) f(\boldsymbol{\eta})\, d\boldsymbol{\eta} , \tag{2.6}$$

$$k = -k_1 + k_2 , \tag{2.7}$$

$$k_1(\boldsymbol{\xi}, \boldsymbol{\eta}) = 2\pi \omega^{1/2}(\xi)\, \omega^{1/2}(\eta) \int B(\theta, \mathbf{v})\, d\theta , \tag{2.8}$$

$$\begin{aligned} k_2(\boldsymbol{\xi}, \boldsymbol{\eta}) = {} & \frac{2}{(2\pi)^{3/2}} \frac{1}{v^2} \exp\left[-\tfrac{1}{8}v^2 - \tfrac{1}{2}\zeta_1^2\right] \\ & \cdot \int \exp\left[-\tfrac{1}{2}|\mathbf{w} + \boldsymbol{\zeta}_2|^2\right] Q(v, w)\, d\mathbf{w} , \end{aligned} \tag{2.9}$$

in which

$$\mathbf{v}\equiv\boldsymbol{\eta}-\boldsymbol{\xi}=\boldsymbol{\alpha}(\boldsymbol{\alpha}\cdot\mathbf{V}) \qquad v=V\cos\theta, \tag{2.10}$$

$$\mathbf{w}=\mathbf{V}-\boldsymbol{\alpha}(\boldsymbol{\alpha}\cdot\mathbf{V}) \qquad w=V\sin\theta, \tag{2.11}$$

$$\boldsymbol{\zeta}_1+\boldsymbol{\zeta}_2=\boldsymbol{\zeta}\equiv\tfrac{1}{2}(\boldsymbol{\xi}+\boldsymbol{\eta}), \tag{2.12}$$

with $\boldsymbol{\zeta}_1$ parallel to $\mathbf{v}$ and $\boldsymbol{\zeta}_2$ perpendicular to $\mathbf{v}$. Note that $\mathbf{w}$ is perpendicular to $\mathbf{v}$ and the integral in (2.9) is over that 2-dimensional plane with $\mathbf{v}$ held constant. We define

$$Q(\mathbf{v},\mathbf{w})=\frac{1}{2|\sin\theta|}\left[B(\theta,\mathbf{V})+B(\tfrac{\pi}{2}-\theta,\mathbf{V})\right] \tag{2.13}$$

and ω is defined in (1.6).

We modify L to eliminate its null space, which is spanned by the five functions $\psi_0,\psi_1,\ldots,\psi_4$ defined by

$$\begin{aligned}&\psi_0(\xi)=\omega^{1/2}(\xi)\\ &\psi_i(\boldsymbol{\xi})=\xi_i\omega^{1/2}(\xi) \qquad i=1,2,3\\ &\psi_4(\xi)=\xi^2\omega^{1/2}(\xi).\end{aligned} \tag{2.14}$$

Replace L by $\bar{L}$ with

$$\bar{L}f=Lf+\sum_{i=0}^{4}\psi_i(\psi_i,f). \tag{2.15}$$

This amounts to replacing k_1 by $\bar{k}_1$ where

$$\bar{k}_1=k_1-\sum_{i=0}^{4}\psi_i(\xi)\psi_i(\eta). \tag{2.16}$$

With this modification, $\bar{L}$ is positive, i.e.

$$(\bar{L}f,f)>0. \tag{2.17}$$

Furthermore the problem

$$\frac{\partial f}{\partial t}+\bar{L}f=0, \tag{2.18}$$

$$f(t=0)=f_0\in N(L)^{\perp} \tag{2.19}$$

is equivalent to the problem (1.10) and (1.11). From now on we will drop the bar and L and k_1 will mean the modification in (2.15) and (2.16). The reason for the modification is that, although it does not change the problem, it does affect the proof. We will be performing a velocity cutoff, multiplying L by χ_w defined in (4.8) and applying $\chi_w L$ to functions $\chi_w f$. But $N(L)^{\perp}$ is not invariant under this multiplication. To get rid of this nuisance we have removed the null space by modifying L.

We study only *soft potentials*, i.e. v must satisfy

$$c_0(1+\xi)^{-\gamma}\leqq v(\xi)\leqq c_1(1+\xi)^{-\gamma}, \tag{2.20}$$

where c_0, c_1 and $0<\gamma<1$ are positive constants.

In addition we assume an angular cutoff to the collision process, which means that

$$B(\theta, \mathbf{V}) \leqq c V^{-\gamma} |\sin\theta \cos\theta| . \tag{2.21}$$

In other words B must approach zero linearly at $\theta = 0$ and $\theta = \frac{\pi}{2}$, and it, as well as the total collisional cross section ν, must decay algebraically for large V and have restricted growth for small V. The angular cutoff assumption was first suggested by Grad [5] and used in many subsequent works (e.g. [1, 3, 6, 8, 11, 13]).

The formulas above are more explicit if the intermolecular force is an inverse power, $\mathscr{F} = \mathscr{K}/r^s$, with $3 < s < 5$. Then

$$B(\theta, V) = V^{-\gamma} \beta(\theta), \tag{2.22}$$

$$\gamma = \frac{5-s}{s-1}. \tag{2.23}$$

Furthermore

$$\nu(\xi) = \beta_0 \int |\eta - \xi|^{-\gamma} \exp(-\tfrac{1}{2}\eta^2) d\boldsymbol{\eta}, \tag{2.24}$$

$$\beta_0 = (2\pi)^{-1/2} \int_0^{\pi/2} \beta(\theta) d\theta, \tag{2.25}$$

which satisfies (2.20). The angular cut-off assumption (2.21) is a restriction on $\beta(\theta)$.

3. Main Result

Before stating the main theorem we first define a few useful norms. The notation is not confusing, although it is not entirely consistent.

Definition.

$$\|f\| \equiv \int_{\mathbb{R}^3} f(\boldsymbol{\xi})^2 d\boldsymbol{\xi}.$$

$$\|f\|_{\alpha,r} \equiv \sup_{\xi \in \mathbb{R}^3} (1+\xi)^r e^{\alpha \xi^2} |f(\boldsymbol{\xi})|.$$

$$\|f\|_\alpha \equiv \|f\|_{\alpha,0}.$$

$$\|f\|_\infty \equiv \|f\|_{0,0}.$$

The subscript α will always signify exponential decay and r algebraic decay. If γ ever appears in a subscript it is in the algebraic part. The algebraic decay is used in the proofs but not in the results.

The following theorem establishes existence, uniqueness, and decay for the spatially-homogeneous linearized Boltzman equation with a soft cut-off potential.

Theorem 3.1. *Let* $0 < \alpha < \frac{1}{4}$. *Let* $f_0 \in N(L)^\perp$ *with* $\|f_0\| < \infty$. *Then there is a unique solution of* (1.10) *and* (1.11). *Its decay in time is given by*

$$\|f(t)\| \leqq c \|f_0\|_\alpha e^{-\lambda t^\beta}, \tag{3.1}$$

$$\|f(t)\|_\infty \leqq c \|f_0\|_\alpha e^{-\lambda t^\beta}, \tag{3.2}$$

$$\|f(t)\|_\alpha \leqq c \|f_0\|_\alpha, \tag{3.3}$$

in which

$$\beta=\frac{2}{2+\gamma}, \tag{3.4}$$

$$\lambda=(1-2\varepsilon)\alpha^{1-\beta}\left(\frac{c_0}{\beta}\right)^{\beta}, \tag{3.5}$$

for any $\varepsilon>0$. *The constants* c *depend on* ε.

Remarks. 1) The constant β comes from the following simpler problem which can be solved exactly. Let

$$\frac{\partial}{\partial t}f(t,\xi)+\xi^{-\gamma}f(t,\xi)=0, \quad \text{for} \quad \xi>1, \tag{3.6}$$

$$f(0,\xi)=e^{-\alpha\xi^2}. \tag{3.7}$$

Then

$$f(t,\xi)=e^{-\alpha\xi^2-t\xi^{-\gamma}}, \tag{3.8}$$

$$\|f(t)\|_{\infty}=ce^{-\varrho t^{\beta}}, \tag{3.9}$$

in which β is given by (3.4).

2) Notice that in both (3.1) and (3.2), the norm on the right is different from that on the left. This will cause complications later (in Part II) when we solve the nonlinear problem for small initial data, but it seems to be necessary.

3) There is a simple existence and uniqueness theorem which does not guarantee decay:

Theorem 3.2. *The Eqs.* (1.10) *and* (1.11) *with* $f_0\in\mathscr{L}^2(\boldsymbol{\xi})$ *has a unique solution* $f(t,\boldsymbol{\xi})$ *in* $\mathscr{L}^2(\boldsymbol{\xi})$, *and it satisfies*

$$\|f(t)\|\leqq e^{\kappa t}\|f_0\|, \tag{3.10}$$

where κ *is a bound on* L, *i.e.*

$$\|L\|\leqq\kappa. \tag{3.11}$$

This simple result proves the uniqueness and existence in Theorem 3.1. The real problem is to obtain the decay, which will be needed for subsequent work on the nonlinear problem.

4. Outline of the Proof of Theorem 3.1

First we give a very rough indication of the proof. Split the velocity space into two parts A and $\bar{A}$ with

$$\begin{aligned} A&=\{\boldsymbol{\xi},\xi<w\}\\ \bar{A}&=\{\boldsymbol{\xi},\xi>w\}. \end{aligned} \tag{4.1}$$

In $\bar{A}$ the solution f of the Boltzmann equation is of size $e^{-\alpha w^2}$. Choose w so that $\alpha w^2=\lambda t^{\beta}$, i.e.

$$w=\sqrt{\frac{\lambda}{\alpha}}\,t^{\beta/2}. \tag{4.2}$$

In A we consider the operator $L_w = \chi_A L = v + \chi_A K$ defined on $\mathscr{L}^2(A)$, where χ_A is the characteristic function of A. Since $\min_{\xi \in A} v(\xi) = v(w) \approx c_0 w^{-\gamma}$, the continuous spectrum of L_w has the lower bound $v(w)$. The crucial fact, stated in Theorem 7.1, is that also there are no eigenvalues below $\mu v(w)$ for any $1 > \mu > 0$. Thus (we omit the μ in this rough statement)

$$\|e^{L(t)}\| \leqq e^{-c_0 w^{-\gamma}} = \exp\left\{-c_0 \left(\frac{\lambda}{\alpha}\right)^{-\gamma/2} t^{-\gamma\beta/2}\right\} \tag{4.3}$$

and

$$\begin{aligned} \left\|e^{\int^t L(S)\,dS}\right\| &\leqq \exp\left\{-c_0 \left(\frac{\lambda}{\alpha}\right)^{-\gamma/2} \int^t S^{-\gamma\beta/2}\,dS\right\} \\ &= \exp\left\{-c_0 \left(\frac{\lambda}{\alpha}\right)^{-\gamma/2} \frac{1}{1-\gamma\beta/2} t^{1-\gamma\beta/2}\right\}. \end{aligned} \tag{4.4}$$

Now to get decay like $e^{-\lambda t^\beta}$ inside A, we ask that

$$\lambda t^\beta = c_0 \left(\frac{\lambda}{\alpha}\right)^{-\gamma/2} \frac{1}{1-\gamma\beta/2} t^{1-\gamma\beta/2}, \tag{4.5}$$

and are led to

$$\beta = \frac{2}{2+\gamma}, \tag{4.6}$$

$$\lambda = \alpha^{\beta-1} \left(\frac{c_0}{\beta}\right)^\beta, \tag{4.7}$$

which is approximately the choice of constants in Theorem 3.1.

The actual proof requires a little more care. We make the splitting velocity w constant in the interval $[T, T+1]$. Define the characteristic functions

$$\begin{aligned} \chi_w(\xi) &= \begin{cases} 1 & \xi \in A \\ 0 & \xi \in \bar{A} \end{cases} \\ \bar{\chi}_w &= 1 - \chi_w. \end{aligned} \tag{4.8}$$

The Boltzmann equation (1.10) can be rewritten as

$$(\chi f)_t + \chi L \chi f = -\chi K \bar{\chi} f, \tag{4.9}$$

$$(\bar{\chi} f)_t + v \bar{\chi} f = -\bar{\chi} K (\chi f + \bar{\chi} f). \tag{4.10}$$

Solve these equations in the time period $[T, T+1]$ using the following iterative scheme

$$(\chi f_{n+1})_t + \chi L \chi f_{n+1} = -\chi K \bar{\chi} f_n, \tag{4.11}$$

$$(\bar{\chi} f_{n+1})_t + v \bar{\chi} f_{n+1} = -\bar{\chi} K (\chi f_n + \bar{\chi} f_n). \tag{4.12}$$

We show in Sect. 9 that f_{n+1} decays if f_n is decaying, and in Sect. 10 that $f_n \to f$, which solves (1.10) and (1.11) and has the same decay rate. But in each interval we pick up a factor of $(1 + cT^{-1/3})$. This results in a small loss in the coefficient in the exponential decay, as shown in Sect. 11.

The above argument provides the decay for $\|f\|$. We show the decay of $\|f\|_\infty$ and the preservation of $\|f\|_\alpha$ in Sect. 12.

5. Estimates on the Integral Kernels

The integral operator K is better behaved for a soft potential than for a hard potential. Grad [5] briefly pointed this out, by noting that his estimate (60) could be improved if the potential was soft. The following estimates on the kernel k are the main results of this section.

Proposition 5.1. *For any* $0<\varepsilon<1$, *and any* $\boldsymbol{\xi}\in\mathbb{R}^3$ *and* $\boldsymbol{\eta}\in\mathbb{R}^3$,

$$|k(\boldsymbol{\xi},\boldsymbol{\eta})|\leqq c\frac{1}{v}(1+\xi+\eta)^{-(\gamma+1)}\exp\{-1-\varepsilon)(\tfrac{1}{8}v^2+\tfrac{1}{2}\zeta_1^2)\}, \tag{5.1}$$

$$\int_{\mathbb{R}^3} k(\boldsymbol{\xi},\boldsymbol{\eta})\,d\boldsymbol{\eta}\leqq c(1+\xi)^{-(\gamma+2)}, \tag{5.2}$$

$$\int_{\mathbb{R}^3} k(\boldsymbol{\xi},\boldsymbol{\eta})^2\,d\boldsymbol{\eta}\leqq c(1+\xi)^{-(2\gamma+3)}. \tag{5.3}$$

For a soft potential the kernel k is Hilbert-Schmidt, since the right hand side of (5.3) *is integrable in* $\boldsymbol{\xi}$.

Note. In (5.1) the constant c may depend on ε. But this does not matter since we only use several choices of ε. The vectors $\mathbf{v}$ and $\boldsymbol{\zeta}$ are defined by (2.10) and (2.12). These estimates are valid for $-1<\gamma<1$, i.e. for hard as well as soft potentials.

These estimates will be proved using the next two propositions.

Propositions 5.2. *For any* $\mathbf{v}\in R^2$, $\boldsymbol{\zeta}_2\in\mathbb{R}^3$ *and* $\mathbf{w}\in\mathbb{R}^3$, *we have*

$$Q(\mathbf{v},\mathbf{w})\leqq cv(v^2+w^2)^{-\frac{\gamma+1}{2}}, \tag{5.4}$$

$$\frac{1}{v}\int_{\Omega}\exp\{-\tfrac{1}{2}(\mathbf{w}+\boldsymbol{\zeta}_2)^2\}\,Q(\mathbf{v},\mathbf{w})\,d\mathbf{w}\leqq c(1+\zeta_2+v)^{-(\gamma+1)}, \tag{5.5}$$

in which $\Omega=\{\mathbf{w}\in\mathbb{R}^3:\mathbf{w}\perp\mathbf{v}\}$.

The inequality (5.5) is an improved version of Grad's estimate (60) in [5].

Proposition 5.3. *For any* $\varrho>-3$ *and any* $a>0$, $b>0$, *there is a constant* c *(depending on* ϱ, a, b*) so that*

$$\int_{\mathbb{R}^3} v^{\varrho}\exp\{-av^2-b\zeta_1^2\}\,d\boldsymbol{\eta}\leqq c(1+\xi)^{-1} \tag{5.6}$$

for any $\boldsymbol{\xi}$. *The vectors* $\boldsymbol{\zeta}_1$ *and* $\mathbf{v}$ *are defined as in* (2.10) *and* (2.12).

These propositions are proved in reverse order.

Proof of Proposition 5.3. Denote the integral by I and change its variable of integration to $\mathbf{v}=\boldsymbol{\eta}-\boldsymbol{\xi}$. Write $\boldsymbol{\xi}\cdot\mathbf{v}=x\xi v$ and change to polar coordinates around $\boldsymbol{\xi}$, so that $d\mathbf{v}=v^2\,dv\,dx\,d\varphi$. We can rewrite ζ_1 as

$$\zeta_1^2=\tfrac{1}{4}\frac{(2\xi\cdot v+v^2)^2}{v^2}=\tfrac{1}{4}(2x\xi+v)^2. \tag{5.7}$$

Since the integrand is independent of φ, the integral in (5.6) is

$$I=2\pi\int_0^{\infty} v^{\varrho+2}e^{-av^2}\int_{-1}^{1} e^{-b1/4(2x\xi+v)^2}\,dx\,dv. \tag{5.8}$$

The inner integral is estimated by

$$\int_{-1}^{1} e^{-b1/4(2x\xi+v)^2}dx = \frac{1}{2\xi}\int_{-2\xi+v}^{2\xi+v} e^{-b1/4y^2}dy \leqq \frac{c}{1+\xi}. \tag{5.9}$$

Therefore

$$\begin{aligned} I &\leqq 2\pi \int_0^\infty v^{\varrho+2}e^{-av^2}\frac{c}{1+\xi}dv \\ &\leqq \frac{c}{1+\xi}. \end{aligned} \tag{5.10}$$

Proof of Proposition 5.2. a) According to the angular cutoff hypothesis (2.21) and the definition (2.13), (2.10), and (2.11),

$$\begin{aligned} Q(v,w) &\leqq c|\cos\theta|\, V^{-\gamma} \\ &\leqq c(1+\tau^2)^{-1/2}(v^2+w^2)^{-\gamma/2}, \end{aligned} \tag{5.11}$$

where $\tau = \tan\theta = w/v$. Therefore

$$\frac{1}{v}Q \leqq c(v^2+w^2)^{-(\gamma+1)/2}. \tag{5.12}$$

as in (5.4).

b) Using the bound (5.12), we estimate

$$\begin{aligned} \frac{1}{v}\int_\Omega \exp\{-\tfrac{1}{2}(\mathbf{w}+\zeta_2)^2 Q(\mathbf{v},\mathbf{w})d\mathbf{w} &\leqq \int_\Omega \exp\{-\tfrac{1}{2}(\mathbf{w}+\zeta_2)^2\}(v^2+w^2)^{-\frac{\gamma+1}{2}}d\mathbf{w} \\ &\leqq \int_\Omega e^{-1/2w^2}(v^2+(\mathbf{w}-\zeta_2))^{-\frac{\gamma+1}{2}}d\mathbf{w}. \end{aligned} \tag{5.13}$$

Denote this integral by I, and split it into two parts: I_1, with $w > \frac{1}{2}\zeta_2$, and I_2, with $w < \frac{1}{2}\zeta_2$. Estimate these two separately. First

$$\begin{aligned} I_1 &= \int_{w>1/2\zeta_2} e^{-1/2w^2}(v^2+(\mathbf{w}-\zeta_2)^2)^{-\frac{\gamma+1}{2}}d\mathbf{w} \\ &\leqq c(v+\zeta_2)^{-(\gamma+1)}. \end{aligned} \tag{5.14}$$

In the domain $\{w < \frac{1}{2}\zeta_2\}$ we have

$$v^2+(\mathbf{w}-\zeta_2)^2 > v^2+\tfrac{1}{4}\zeta_2^2. \tag{5.15}$$

So the integral I_2 is bounded by

$$\begin{aligned} I_2 &= \int_{w<1/2\zeta_2} e^{-1/2w^2}(v^2+(\mathbf{w}-\zeta_2)^2)^{-\frac{\gamma+1}{2}}d\mathbf{w} \\ &\leqq c(v^2+\tfrac{1}{4}\zeta_2^2)^{-\frac{\gamma+1}{2}}. \end{aligned} \tag{5.16}$$

Furthermore since $\gamma < 1$, the integrand in I is integrable even for $\mathbf{v} = \zeta_2 = 0$. Combining this with (5.14) and (5.16), it follows that

$$I \leqq c(1+v+\zeta_2)^{-(\gamma+1)}. \tag{5.17}$$

Proof of Proposition 5.1. First we prove (5.1) for k_1 and k_2 separately [recall that k_1 has been modified as in (2.16)].

a) According to (2.16), (2.8), and (2.21), we know that

$$k_1(\boldsymbol{\xi},\boldsymbol{\eta})=e^{-1/4\xi^2}e^{-1/4\eta^2}\cdot\{2\pi\int B(\theta,v)\,d\theta+1+\boldsymbol{\xi}\cdot\boldsymbol{\eta}+\xi^2\eta^2\}, \tag{5.18}$$

in which

$$B(\theta,v)\leqq v^{-\gamma}|\cos\theta\sin\theta|. \tag{5.19}$$

Therefore (making very crude estimates)

$$\begin{aligned}k_1(\xi,\eta)&\leqq cv^{-\gamma}e^{-1/4(1-\varepsilon/2)(\xi^2+\eta^2)}\\&\leqq cv^{-1}(1+\xi+\eta)^{-(\gamma+1)}e^{-1/4(1-\varepsilon)(\xi^2+\eta^2)}\\&\leqq cv^{-1}(1+\xi+\eta)^{-(\gamma+1)}\exp\{-(1-\varepsilon)(\tfrac{1}{8}v^2+\tfrac{1}{2}\zeta_1^2)\},\end{aligned} \tag{5.20}$$

since $\frac{1}{4}(\xi^2+\eta^2)\geqq(\frac{1}{8}v^2+\frac{1}{2}\zeta_1^2)$.

b) According to (2.9), we know that

$$k_2(\boldsymbol{\xi},\boldsymbol{\eta})=\frac{2}{(2\pi)^{3/2}}\,\frac{1}{v^2}\exp\{-\tfrac{1}{8}v^2-\tfrac{1}{2}\zeta_1^2\}\cdot\int\exp\{-\tfrac{1}{2}|w+\zeta_2|^2\}Q(\mathbf{v},\mathbf{w})\,d\mathbf{w}. \tag{5.21}$$

Proposition 5.2 provides an estimate for the integral on the right, so that

$$\begin{aligned}k_2(\boldsymbol{\xi},\boldsymbol{\eta})&\leqq c\frac{1}{v}\exp\{-\tfrac{1}{8}v^2-\tfrac{1}{2}\zeta_1^2\}(1+v+\zeta_2)^{-(\gamma+1)}\\&\leqq c\frac{1}{v}(1+v+\zeta_1+\zeta_2)^{-(\gamma+1)}\exp\{-(1-\varepsilon)(\tfrac{1}{8}v^2-\tfrac{1}{2}\zeta_1^2)\}.\end{aligned} \tag{5.22}$$

Recall that $v=|\boldsymbol{\eta}-\boldsymbol{\xi}|$ and $\zeta=\frac{1}{2}|\boldsymbol{\xi}+\boldsymbol{\eta}|$, and thus

$$1+v+\zeta_1+\zeta_2\geqq c(1+\xi+\eta). \tag{5.23}$$

Finally

$$k_2(\boldsymbol{\xi},\boldsymbol{\eta})\leqq c\frac{1}{v}(1+\xi+\eta)^{-(\gamma+1)}\exp\{-(1-\varepsilon)(\tfrac{1}{8}v^2-\tfrac{1}{2}\zeta_1^2)\}. \tag{5.24}$$

c) Now that (5.1) has been established the remaining estimates are easy. We will prove (5.2); the proof of (5.3) is similar. We set $\varepsilon=\frac{1}{2}$ and integrate (5.1) with the result that

$$\begin{aligned}\int_{\mathbb{R}^3}k(\boldsymbol{\xi},\boldsymbol{\eta})\,d\boldsymbol{\eta}&\leqq C\int_{\mathbb{R}^3}v^{-1}(1+\xi+\eta)^{-(\gamma+1)}\cdot\exp\{-\tfrac{1}{16}v^2-\tfrac{1}{4}\zeta_2^2\}\,d\boldsymbol{\eta}\\&\leqq c(1+\xi)^{-(\gamma+1)}\int_{\mathbb{R}^3}v^{-1}\exp\{-\tfrac{1}{16}v^2-\tfrac{1}{4}\zeta_2^2\}\,d\boldsymbol{\eta}\\&\leqq c(1+\xi)^{-(\gamma+2)},\end{aligned} \tag{5.25}$$

using Proposition 5.3.

6. Estimates on K

In this section we present a number of estimates on the compact integral operator K. These show that the application of K to f results in extra algebraic decay in ξ. These estimates are valid for hard, as well as soft, potentials.

Proposition 6.1. *For any* $0 \leqq \alpha < \frac{1}{4}$ *and* $r \geqq 0$,

$$\|Kf\|_{0,\gamma+3/2} \leqq c\|f\|, \tag{6.1}$$

$$\|Kf\|_{\alpha,r+\gamma+2} \leqq c\|f\|_{\alpha,r}, \tag{6.2}$$

$$\|Kf\| \leqq c\|f\|_{\infty}. \tag{6.3}$$

In the sequel we also need estimates on K with a cutoff. Define the characteristic functions χ_w and $\bar{\chi}_w$ as in (4.8). The product $\bar{\chi}_w K$ has a simple estimate.

Proposition 6.2. *For any* $0 \leqq \alpha < \frac{1}{4}$ *and any* $w > 0$,

$$\|\bar{\chi}_w K \chi_w f\|_{\alpha,\gamma+3/2} \leqq c e^{\alpha w^2}\|f\|. \tag{6.4}$$

Before proving these we state an elementary lemma.

Lemma 6.3.

$$v^2 + 4\zeta_1^2 - 2\xi^2 + 2\eta^2 > 0, \tag{6.5}$$

for all $\boldsymbol{\xi}$ *and* $\boldsymbol{\eta}$ *with* v *and* ζ_1, *as in* (2.10) *and* (2.12). *For any* $w > 0$,

$$v^2 + 4\zeta_1^2 + 2w^2 - 2\eta^2 > 0 \tag{6.6}$$

if $\xi > w > \eta$.

Proof of Proposition 6.1. a) First we prove (6.1). Using the Schwartz inequality and (5.3), we find that

$$\begin{aligned} |Kf(\boldsymbol{\xi})| &\leqq \|f\| \int k(\boldsymbol{\xi},\boldsymbol{\eta})^2 d\boldsymbol{\eta}^{1/2} \\ &\leqq c\|f\|(1+\xi)^{-1/2(2\gamma+3)}. \end{aligned} \tag{6.7}$$

Then

$$\|Kf\|_{0,\gamma+3/2} \leqq c\|f\|. \tag{6.8}$$

b) Next we prove (6.2). From the estimate (5.1), we get

$$\begin{aligned} |Kf(\xi)| &\leqq c \int_{\mathbb{R}^3} \frac{1}{v}(1+\xi+\eta)^{-(\gamma+1)} \exp\{-(1-\varepsilon)(\tfrac{1}{8}v^2 + \tfrac{1}{2}\zeta_1^2)\}\,|f(\boldsymbol{\eta})|\,d\boldsymbol{\eta} \\ &\leqq c e^{-\alpha\xi^2}\|f\|_{\alpha,r}(1+\xi)^{-(\gamma+1)} \\ &\quad \cdot \int_{\mathbb{R}^3} \frac{1}{v} \exp\{-(1-\varepsilon)(\tfrac{1}{8}v^2 + \tfrac{1}{2}\zeta_1^2) + \alpha\xi^2 - \alpha\eta^2\} \cdot (1+\eta)^{-r} d\boldsymbol{\eta}, \\ &\leqq c e^{-\alpha\xi^2}(1+\varepsilon)^{-(\gamma+1)}\|f\|_{\alpha,r} \\ &\quad \cdot \int_{\mathbb{R}^3} \frac{1}{v}(1+\eta)^{-r} \exp\{-\theta(v^2 + 4\zeta_1^2)\}\,d\boldsymbol{\eta}, \end{aligned} \tag{6.9}$$

after picking $\varepsilon=\dfrac{1-4\alpha}{2}$ and $\theta=\dfrac{1-4\alpha}{16}$ and applying (6.5) in Lemma 6.3. Denote the integral on the right by I and split it into two pieces: I_1, with $\eta<\frac{1}{2}\xi$, and I_2, with $\eta>\frac{1}{2}\xi$. Now I_1 is easy to estimate since $v^2>\frac{1}{4}\xi^2$ in that domain, and

$$I_1\leqq ce^{-\theta 1/4\xi^2}. \tag{6.10}$$

In the domain integration for I_2, we have $(1+\eta)>\frac{1}{2}(1+\xi)$, so that

$$\begin{aligned} I_2&\leqq c(1+\xi)^{-r}\int_{\eta>1/2\xi}\frac{1}{v}\exp\{-\theta(v^2+4\zeta_1^2)\}\,d\boldsymbol{\eta}\\ &\leqq c(1+\xi)^{-r-1}, \end{aligned} \tag{6.11}$$

using Proposition 5.3. Combining (6.10) and (6.11) we see that

$$I=I_1+I_2\leqq c(1+\xi)^{r-1}, \tag{6.12}$$

and thus

$$|Kf(\boldsymbol{\xi})|\leqq c(1+\xi)^{-(\gamma+r+2)}e^{-\alpha\xi^2}\|f\|_{\alpha,r}, \tag{6.13}$$

from which (6.2) follows immediately.

c) We prove (6.3) for K by writing

$$\begin{aligned} \|Kf\|&=\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}k(\boldsymbol{\xi},\boldsymbol{\eta})f(\boldsymbol{\eta})\,d\boldsymbol{\eta}\right)^2 d\boldsymbol{\xi}^{1/2}\\ &\leqq \sup_{\eta}|f(\boldsymbol{\eta})|\int\left(\int k(\boldsymbol{\xi},\boldsymbol{\eta})\,d\boldsymbol{\eta}\right)^2 d\boldsymbol{\xi}^{1/2}\\ &\leqq c\|f\|_\infty\int(1+\xi)^{-2(\gamma+2)}\,d\boldsymbol{\xi}^{1/2}\\ &\leqq c\|f\|_\infty, \end{aligned} \tag{6.14}$$

using the estimate (5.2). This concludes the proof of Proposition 6.1.

Proof of Proposition 6.2. Write

$$\begin{aligned} \bar{\chi}_w K\chi_w f&\leqq\|f\|\left(\int_{\eta<w}k(\boldsymbol{\xi},\boldsymbol{\eta})^2\,d\boldsymbol{\eta}\right)^{1/2}\\ &\leqq c\|f\|\left(\int_{\eta<w}\frac{1}{v^2}(1+\xi+\eta)^{-2(\gamma+1)}\exp\{-(1-\varepsilon)(\tfrac{1}{4}v^2+\zeta_1^2)\}\,d\boldsymbol{\eta}\right)^{1/2}\\ &\leqq c\|f\|e^{\alpha w^2-\alpha\xi^2}(1+\xi)^{-(\gamma+1)}\\ &\quad\cdot\left(\int_{\eta<w}\frac{1}{v^2}\exp\{-(1-\varepsilon)(\tfrac{1}{4}v^2+\zeta_1^2)-2\alpha w^2+2\alpha\xi^2\}\,d\boldsymbol{\eta}\right)^{1/2}. \end{aligned} \tag{6.15}$$

In the last step we used (6.6) in Lemma 6.3 after choosing $\varepsilon=\dfrac{1-4\alpha}{2}$, $\theta=\dfrac{1-4\alpha}{8}$ and recognizing that $\xi>w>\eta$. Therefore

$$\begin{aligned} \bar{\chi}_w K\chi_w f&\leqq c\|f\|e^{\alpha w^2-\alpha\xi^2}(1+\xi)^{-(\gamma+1)}\\ &\quad\cdot\int_{\eta<w}\frac{1}{v^2}\exp\{-\theta(v^2+4\zeta_1^2)\}\,d\eta^{1/2}\\ &\leqq c\|f\|e^{\alpha w^2-\alpha\xi^2}(1+\xi)^{-(\gamma+1)}(1+\xi)^{-1/2}, \end{aligned}$$

from which (6.6) follows immediately.

7. Spectrum of the Cutoff Operator

Consider the linearized collision operator (with modification as in Sect. 2)

$$L = \nu(\xi) + K. \tag{7.1}$$

This operator has a positive numerical range, i.e.

$$(Lf, f) > 0. \tag{7.2}$$

Since K is compact, the continuous spectrum of L comes from the values of ν, which range from 0 to $\nu_0 = \max \nu$. There may be discrete eigenvalues as well. In order to get decay in the solution of the problem (1.10) we need $\sigma(L)$ to be bounded away from the origin. That is not true for L, but it is for L with a velocity cutoff. Define

$$L_w = \chi_w L \tag{7.3}$$

as an operator from $\mathscr{L}^2(\xi < w)$ into itself with χ_w defined as in (4.8).

Theorem 7.1. *Let* $0 < \mu < 1$. *For* w *large enough, the operator* L_w *has spectrum bounded from below by* $\mu\nu(w)$, *i.e.*

$$\sigma(L_w) \subset \{\lambda > \mu\nu(w)\}. \tag{7.4}$$

The μ which we use will be a constant chosen in Sect. 8. Theorem 7.1 is proved using the following proposition about the decay of eigenfunctions of L_w.

Proposition 7.2. *Let* f *be an eigenfunction of* L_w *with eigenvalue* λ, *i.e.*

$$\begin{aligned} &f \in \mathscr{L}^2(\xi < w) \\ &L_w f = \lambda f, \end{aligned} \tag{7.5}$$

and suppose that

$$0 < \lambda < \mu\nu(w). \tag{7.6}$$

Then f *is rapidly decreasing at* ∞, *i.e. for each* m *there is a constant* c_m *such that*

$$\|f(\xi)\|_{0,m} \leqq c_m \|f\|. \tag{7.7}$$

Furthermore c_m *is independent of* f, λ, *and* w, *but depends on* μ.

Proof of Proposition 7.2. Since ν is a decreasing function of ξ, it follows from (7.6) that

$$\nu(\xi) - \lambda > (1-\mu)\nu(\xi). \tag{7.8}$$

The eigen-equation (6.11) can be written as

$$\chi_w K f = -(\nu - \lambda) f. \tag{7.9}$$

As a result of (7.8) and (2.20)

$$\begin{aligned} |Kf| &\geqq (1-\mu)\nu|f| \\ &\geqq c_0(1-\mu)(1+\xi)^{-\gamma}|f|, \end{aligned} \tag{7.10}$$

from which it follows that

$$\|Kf\|_{0,r+\gamma} > c\|f\|_{0,r}. \tag{7.11}$$

Use this in the estimates of Proposition 6.1. From (7.11) and (6.1) we estimate

$$\begin{aligned}\|f\|_{0,3/2} &\leqq c\|Kf\|_{0,3/2+\gamma} \\ &\leqq c\|f\|. \end{aligned} \tag{7.12}$$

Continue by iteration using (7.11) and (6.2) with $\alpha=0$, to find

$$\begin{aligned}\|f\|_{0,7/2} &\leqq c\|Kf\|_{0,3/2+\gamma+2} \\ &\leqq c\|f\|_{0,3/2} \\ &\leqq c\|f\|, \end{aligned} \tag{7.13}$$

$$\|f\|_{0,2n+3/2} \leqq c_n\|f\|, \tag{7.14}$$

from which (7.7) follows.

Proof of Theorem 7.1. Suppose that the theorem is not true. Then there is a sequence $w_n \to \infty$ so that each operator L_{w_n} has a point λ_n in its spectrum with $\lambda_n < \mu\nu(w_n)$. Since the continuous spectrum of L_{w_n} is bounded below by $\nu(w_n)$, in fact each λ_n is a discrete eigenvalue with eigenfunction f_n, i.e.

$$L_{w_n} f_n = \lambda_n f_n, \tag{7.15}$$

with $\|f_n\|=1$. The above Eq. (7.15) is in $\mathscr{L}^2(\xi < w_n)$, but we also want to think of f_n as a member of $\mathscr{L}^2(\mathbb{R}^3)$, by just extending it to be zero on $\{\xi > w_n\}$. We shall show that $f_n \to f$, in which f is a null eigenfunction of the full operator L. This is a contradiction, since L is a positive operator.

The eigen-equation (7.15) can be rewritten as

$$\nu(\xi) f_n + \chi_{w_n} K f_n = \lambda_n f_n. \tag{7.16}$$

Since K is compact, then after restricting to a subsequence

$$Kf_n \to g \quad \text{in} \quad \mathscr{L}^2(\mathbb{R}^3). \tag{7.17}$$

Since $w_n \to \infty$,

$$\chi_{w_n} K f_n \to g \quad \text{in} \quad \mathscr{L}^2(\mathbb{R}^3). \tag{7.18}$$

Also since $\nu(w_n) \to 0$, then $\lambda_n \to 0$, and

$$\lambda_n f_n \to 0 \quad \text{in} \quad \mathscr{L}^2(\mathbb{R}^3). \tag{7.19}$$

So we can take the limit in (7.16) to get

$$\lim_{n\to\infty} \nu(\xi) f_n = -g \quad \text{in} \quad \mathscr{L}^2(\mathbb{R}^3). \tag{7.20}$$

Unfortunately division by $\nu(\xi) \sim (1+\xi)^{-\gamma}$ is not a continuous operator in $\mathscr{L}^2$, but by first restricting to a subsequence we can change (7.20) into convergence almost everywhere. Then it is possible to divide by ν

$$f_n \to -\frac{1}{\nu} g, \qquad \text{a.e.} \tag{7.21}$$

By Proposition 7.2, the f_n's are uniformly bounded by the $\mathscr{L}^2$ function $c(1+|\xi|)^{-2}$. Since they converge pointwise, in fact they converge in $\mathscr{L}^2$, i.e.

$$f_n \to f \quad \text{in} \quad \mathscr{L}^2(\mathbb{R}^3). \tag{7.22}$$

Take the limit again in (7.15) to find

$$Lf=0. \tag{7.23}$$

But since $\|f_n\|=1$, f is not 0. This is the intended contradiction which concludes the proof of the theorem.

As a result of Theorem 7.1, we find

Corollary 7.3.

$$\|e^{-t\chi_w L}\| \leqq e^{-t\mu\nu(w)}. \tag{7.24}$$

8. Choice of Constants

We are now ready to pick the coefficient λ_0, cutoff speed w, the exponent β, and the constant μ. This will be done for an arbitrary time interval $[T, T+1]$, and w will depend on T, while λ_0, β, and μ will be constant. Choose

$$\beta=\frac{2}{2+\gamma}, \tag{8.1}$$

$$\lambda_0=(1-\varepsilon)\alpha^{1-\beta}\left(\frac{c_0}{\beta}\right)^{\beta}, \tag{8.2}$$

$$w=\sqrt{\frac{1}{\alpha}(\lambda_0 T^{\beta}+\tfrac{5}{12}\log T)}, \tag{8.3}$$

$$\mu=(1-\varepsilon^2), \tag{8.4}$$

in which $\varepsilon>0$ is fixed but arbitrarily small. The necessary properties of the parameters are listed in the next proposition.

Proposition 8.1. *If ε is sufficiently small and T is sufficiently large,*

$$\exp\{-\mu(t-\sigma)\nu(w)+\lambda_0(t^{\beta}-\sigma^{\beta})\}\leqq 1, \quad \textit{for} \quad T\leqq\sigma\leqq t\leqq T+1, \tag{8.5}$$

$$\exp\{\alpha w^2-\lambda_0 T^{\beta}\}=T^{5/12}, \tag{8.6}$$

$$\exp\{-\alpha w^2+\lambda_0(T+1)^{\beta}\}\leqq cT^{-5/12}. \tag{8.7}$$

Proof. a) By a simple calculation

$$\sup_{T\leqq\sigma<t}\frac{t^{\beta}-\sigma^{\beta}}{t-\sigma}=\beta T^{\beta-1}. \tag{8.8}$$

To show (8.5) it suffices to have $\mu\nu(w)\geqq\lambda_0\beta T^{\beta-1}$. Now

$$\begin{aligned}\nu(w)&\geqq c_0(1+w)^{-\gamma}\\ &=c_0(\lambda_0/\alpha)^{-\gamma/2}T^{-\gamma\beta/2}+O(T^{-\gamma\beta/2}),\end{aligned} \tag{8.9}$$

so that we need only show that

$$c_0\left(\frac{\lambda_0}{\alpha}\right)^{-\gamma/2}T^{-\gamma\beta/2}+O(T^{-\gamma\beta/2})\geqq\lambda_0\beta T^{\beta-1}. \tag{8.10}$$

Notice that $\gamma\beta/2=1-\beta$, and thus (8.10) becomes

$$c_0\left(\frac{\lambda_0}{\alpha}\right)^{-\gamma/2}+O(1)\geqq\lambda_0\beta\,. \tag{8.11}$$

Finally we can take $\lambda_0=(1-\varepsilon)\lambda_1$, in which

$$c_0\left(\frac{\lambda_0}{\alpha}\right)^{-\gamma/2}=\lambda_1\beta \tag{8.12}$$

i.e.

$$\lambda_1=\alpha^{1-\beta}(c_0/\beta)^{\beta}\,. \tag{8.13}$$

b) The equality (8.6) comes directly from the definition (8.4). The next inequality (8.7) is proved by

$$\begin{aligned}\exp\{-\alpha w^2+\lambda_0(T+1)^{\beta}\}&=\frac{\exp\{\lambda_0(T+1)^{\beta}-T^{\beta})}{\exp\{\alpha w^2-\lambda_0 T^{\beta}\}}\\&\leqq cT^{-5/12}\,,\end{aligned} \tag{8.14}$$

since $\beta<1$ and

$$(T+1)^{\beta}-T^{\beta}\leqq c\,. \tag{8.15}$$

9. The Iteration Equation

In the time interval $T\leqq t\leqq T+1$, we choose w according to (8.3) and denote $\chi=\chi_w$, $\bar{\chi}=\bar{\chi}_w$ as in (4.8). Let

$$\operatorname{supp} g_0\subset A\quad \operatorname{supp} h_0\subset A\,, \tag{9.1}$$

$$\operatorname{supp} g_1\subset \bar{A}\quad \operatorname{supp} h_1\subset \bar{A}\,, \tag{9.2}$$

as defined in (4.1). We solve the following inhomogeneous version of the iteration Eqs. (4.11) and (4.12):

$$g_t+\chi Lg=-\chi Kh_1\,, \tag{9.3}$$

$$h_t+vh=-\bar{\chi}K(g_1+h_1)\,, \tag{9.4}$$

for $T\leqq t\leqq T+1$,

$$g(t=T)=g_0\,,\qquad h(t=T)=h_0\,. \tag{9.5}$$

Suppose that T is large enough for Proposition 8.1 to be applicable and that the inhomogeneities satisfy

$$\|g_0+h_0\|\leqq b_0 e^{-\lambda_0 T^{\beta}}\,, \tag{9.6}$$

$$\|h_0\|_{\alpha}\leqq b_0\,. \tag{9.7}$$

$$\|g_1(t)+h_1(t)\|\leqq b_1 e^{-\lambda_0 t^{\beta}} \tag{9.8}$$

$$\|h_1(t)\|_{\alpha}\leqq b_1\,. \tag{9.9}$$

The main result of this section is

Proposition 9.1. *Let g and h solve Eqs.* (9.3)–(9.5) *for* $T \leqq t \leqq T+1$, *and suppose that* g_0, h_0, g_1, h_1 *have the bounds given by* (9.6)–(9.9). *Then if T is sufficiently large,*

$$\|g(t)+h(t)\| \leqq b_2 e^{-\lambda_0 t^\beta}, \tag{9.10}$$

$$\|h(t)\|_\alpha \leqq b_2, \tag{9.11}$$

with

$$b_2 = (1 + cT^{-1/3})b_0 + cT^{-1/3} b_1. \tag{9.12}$$

We can take $b_2 = b_1$, *if*

$$b_1 \geqq (1 + 3cT^{-1/3})b_0. \tag{9.13}$$

Proof. a) The Eqs. (9.3) and (9.4) are decoupled. First estimate $\|g\|$. Solve (9.3) to get

$$g(t) = e^{-(t-T^3\chi L} g_0 - \int_T^t e^{-(t-\sigma)\chi L} \chi K h_1(\sigma) d\sigma. \tag{9.14}$$

According to (7.24)

$$\|g(t)\| \leqq e^{-\mu(t-T)v(w)} \|g_0\| + \int_T^t e^{-\mu(t-\sigma)v(w)} \|\chi K h_1(\sigma)\| d\sigma. \tag{9.15}$$

We estimate the two terms on the right separately. Using Proposition 6.1 and (9.2) we can bound

$$\begin{aligned} \|\chi K h_1(\sigma)\| &\leqq c\|h_1\|_\infty \\ &\leqq c e^{-\alpha w^2} \|h_1(\sigma)\|_\alpha \\ &\leqq c b_1 e^{-\alpha w 2}. \end{aligned} \tag{9.16}$$

Next use (8.5) and (8.7) from Proposition (8.1) to find that

$$\begin{aligned} e^{-\mu(t-\sigma)v(w)} \|\chi K h_1(\sigma)\| &\leqq c b_1 \exp\{-\mu(t-\sigma)v(w) - \alpha w^2\} \\ &\leqq c b_1 \exp\{-\mu(t-\sigma)v(w) + \lambda_0(t^\beta - \sigma^\beta)\} \\ &\quad \cdot \exp\{\lambda_0 \sigma^\beta - \alpha w^2\} \cdot e^{-\lambda_0 t^\beta} \\ &\leqq c b_1 T^{-5/12} e^{-\lambda_0 t^\beta} \end{aligned} \tag{9.17}$$

So the second term on the right side of (9.17) is estimated by

$$\begin{aligned} \int_T^t e^{-\mu(t-\sigma)v} \|\chi K h_1\| d\sigma &\leqq \int_T^t b_1 c T^{-5/12} e^{-\lambda_0 t^\beta} \\ &\leqq c b_1 T^{-1/3} e^{-\lambda_0 t^\beta}, \end{aligned} \tag{9.18}$$

since $t - T \leqq 1$. The first term on the right of (9.15) is estimated in a similar way as

$$\begin{aligned} e^{-\mu(t-T)v(w)} \|g_0\| &\leqq b_0 \exp\{-\mu(t-T)v(w) - \lambda_0 T^\beta\} \\ &\leqq b_0 e^{-\lambda_0 t^\beta} \end{aligned} \tag{9.19}$$

by Proposition 8.1. Therefore, after using (9.18) and (9.19) in (9.15),

$$\|g(t)\| \leqq (b_0 + c b_1 T^{-1/3}) e^{-\lambda_0 t^\beta}. \tag{9.20}$$

b) Next we estimate $\|h(t)\|_\alpha$. Solve (9.4) by

$$h(t)=e^{-(t-T)\nu}h_0-\int_T^t e^{-(t-\sigma)\nu}\bar{\chi}K(g_1+h_1)(\sigma)d\sigma. \tag{9.21}$$

We just drop the $e^{-\nu}$ terms in our estimate and end up with

$$\|h(t)\|_\alpha \leqq \|h_0\|_\alpha+\int_T^t \|\bar{\chi}K(g_1+h_1)(\sigma)\|_\alpha d\sigma. \tag{9.22}$$

Look at the terms inside the integral. The first one is

$$\begin{aligned}\|\bar{\chi}Kg_1\|_\alpha=\|\bar{\chi}K\chi g_1\|_\alpha &\leqq (1+w)^{-(\gamma+3/2)}\|\bar{\chi}K\chi g_1\|_{\alpha,\gamma+3/2}\\ &\leqq (1+w)^{-(\gamma+3/2)}ce^{\alpha w^2}\|g_1(\sigma)\|, \quad \text{by Proposition 6.2}\\ &\leqq cb_1(1+w)^{-(\gamma+3/2)}e^{\alpha w^2-\lambda\sigma^\beta}, \quad \text{by (9.8)}\\ &\leqq cb_1(1+w)^{-(\gamma+3/2)}T^{5/12}, \quad \text{by Proposition 8.1}\\ &\leqq cb_1T^{-(\gamma+3/2)\beta/2+5/12}, \quad \text{using (8.3)}\\ &\leqq cb_1T^{-1/3},\end{aligned} \tag{9.23}$$

since $\beta=\dfrac{2}{2+\gamma}<1$ and $(\gamma+3/2)\beta/2=1-\beta/4>3/4$.

The second term in the integral in (9.22) is

$$\begin{aligned}\|\bar{\chi}Kh_1(\sigma)\|_\alpha &\leqq (1+w)^{-(\gamma+2)}\|\bar{\chi}Kh_1(\sigma)\|_{\alpha,\gamma+2}\\ &\leqq c(1+w)^{-(\gamma+2)}\|h_1\|_\alpha, \quad \text{using Proposition 6.1.}\\ &\leqq cb_1T^{-(\gamma+2)\beta/2}, \quad \text{using (9.8) and (8.3)}\\ &=cb_1T^{-1/3}.\end{aligned} \tag{9.24}$$

Employing (9.23), (9.24), and (9.7) in (9.22), we find an inequality for h as

$$\|h(t)\|_\alpha \leqq b_0+cb_1T^{-1/3}. \tag{9.25}$$

Therefore (9.11) will be true for b_2 given by (9.12).

c) We next calculate $\|h(t)\|$ and $\|h(t)+g(t)\|$

$$\begin{aligned}\|h\|^2 &\leqq \|h\|_\alpha^2\int_{w\leqq\xi} e^{-2\alpha\xi^2}d\xi\\ &\leqq c\|h\|_\alpha^2 we^{-2\alpha w^2}\\ &\leqq c(b_0+cT^{-1/3}b_1)^2T^{\beta/2}T^{-5/6}e^{-2\lambda_0t^\beta},\end{aligned} \tag{9.26}$$

using (9.25), the definition (8.4), and (8.7). Since $\beta<1$, $5/6-\beta/2>1/3$ and

$$\|h\|^2 \leqq c(b_0+cT^{-1/3}b_1)T^{-1/3}. \tag{9.27}$$

Combining this with (9.20), we find

$$\begin{aligned}\|g+h\|^2 &=\|g\|^2+\|h\|^2\\ &\leqq (b_0+cT^{-1/3}b_1)^2(1+cT^{-1/3})e^{-2\lambda_0t^\beta},\\ \|g+h\| &\leqq b_0+cT^{-1/3}(b_0+b_1)\end{aligned} \tag{9.28}$$

Therefore (9.10) will be true for b_2 as in (9.11).

10. Convergence of the Iteration Scheme

At last we are ready to show the convergence of the iteration scheme in the time interval $[T, T+1]$. We suppose that the Boltzmann equation has been solved up to time T and that

$$N_0 \equiv \max\{e^{\lambda_0 T^\beta}\|f(T)\|, \|\bar{\chi}_w f(T)\|_\alpha\} \tag{10.1}$$

is finite. Now start the iteration procedure by defining

$$f_1(t) = e^{\lambda_0(T^\beta - t^\beta)} f(T) \tag{10.2}$$

and define f_{n+1}, for $n \geqq 1$, by (4.11) and (4.12) with starting values

$$f_{n+1}(T) = f(T). \tag{10.3}$$

Also define

$$l_{n+1} = f_{n+1} - f_n \tag{10.4}$$

and

$$N_n = \max_{T \leqq t \leqq T+1} \{e^{\lambda_0 t^\beta}\|f_n\|, \|\bar{\chi}_w f_n\|_\alpha\}, \tag{10.5}$$

$$M_n = \max_{T \leqq t \leqq T+1} \{e^{\lambda_0 t^\beta}\|l_n\|, \|\chi_w l_n\|_\alpha\}. \tag{10.6}$$

First we find uniform bounds on N_n. For N_1, we know that

$$N_1 = N_0 \leqq (1 + 3cT^{-1/3}) N_0. \tag{10.7}$$

Then we proceed by induction using Proposition 9.1, in which

$$\begin{aligned} g_0 &= \chi_w f(T), \\ h_0 &= \bar{\chi}_w f(T), \\ g_1 &= \chi_w f_n, \\ h_1 &= \bar{\chi}_w f_n, \\ g &= \chi_w f_{n+1} \\ h &= \bar{\chi}_w f_{n+1} \\ b_0 &= N_0 \\ b_1 &= N_n \leqq (1 + 3cT^{-1/3}) N_0, \end{aligned} \tag{10.8}$$

with the result that

$$\begin{aligned} N_{n+1} &\leqq (1 + cT^{-1/3}) N_0 + cT^{-1/3} N_n \\ &\leqq (1 + 3cT^{-1/3}) N_0. \end{aligned} \tag{10.9}$$

This is true for all n.

Next we find bounds on M_n. The first one is

$$M_2 \leqq N_2 + N_1 \leqq (2 + 6cT^{-1/3}) N_0. \tag{10.10}$$

The others are found by applying Proposition 9.1 to the equation for $l_{n+1}=f_{n+1}-f_n$, which has

$$\begin{aligned} g_0&=h_0=0,\\ g_1&=\chi_w l_n,\\ h_1&=\bar{\chi}_w l_n,\\ g&=\chi_w l_{n+1}\\ h&=\bar{\chi}_w l_{n+1}\\ b_0&=0,\\ b_1&=M_n, \end{aligned} \tag{10.11}$$

to find that

$$\begin{aligned} M_{n+1} &\leqq cT^{-1/3}M_n\\ &\leqq (cT^{-1/3})^{n-1}(2+6cT^{-1/3})N_0\,. \end{aligned} \tag{10.12}$$

For T large enough, $cT^{-1/3}<1$, and then the series with term M_n is summable, i.e.

$$\sum_{n=2}^{\infty} M_n \leqq \frac{1}{1-cT^{-1/3}}(2+6cT^{-1/3})N_0\,, \tag{10.13}$$

for T large enough. It follows that $\sum_{n=1}^{\infty}(f_{n+1}-f_n)$ converges and therefore

$$f_n \to f, \tag{10.14}$$

in the sense that

$$\begin{aligned} &e^{-\lambda_0 t^\beta}\|f-f_n\|\to 0\\ &\|\bar{\chi}_w(f-f_n)\|_\alpha \to 0\,. \end{aligned} \tag{10.15}$$

The limit f is a solution of the problem (1.10) and (1.11)

Define

$$N=\max_{T\leqq t\leqq T+1}\{e^{\lambda_0 t^\beta}\|f\|,\|\bar{\chi}_w f\|\}\,. \tag{10.16}$$

Because of (10.9),

$$N \leqq (1+3cT^{-1/3})N_0\,. \tag{10.17}$$

This shows the decay of the solution in any time interval $[T, T+1]$. But we are not finished yet, since we need to examine what changes in going from one interval to the next. That is done in the next section.

11. Propagation of the Estimate

So far we have found that f is exponentially decaying in the time interval $[T, T+1]$, according to (10.17), since N is finite. But we picked up a factor $(1+3cT^{-1/3})$, and in order to see the global time decay we must consider its effect.

Denote $w(t)$ to be the continuously varying time-dependent cut-off velocity as defined in (8.3) with t instead of T. Define

$$N(t)=\max\{e^{\lambda_0 t^\beta}\|f(t)\|, \|\bar{\chi}_{w,t}f\|_\alpha\}. \tag{11.1}$$

In the last two sections w was fixed at $w=w(T)$. Statement (10.17) can be translated as

$$\max\{e^{\lambda_0 t^\beta}\|f(t)\|, \|\bar{\chi}_{w(T)}f(t)\|_\alpha\} \leqq (1+3cT^{-1/3})N(T), \tag{11.2}$$

for $T \leqq t \leqq T+1$. But since w is increasing

$$\|\bar{\chi}_{w(T)}f(t)\|_\alpha \geqq \|\bar{\chi}_{w(t)}f(t)\|_\alpha, \tag{11.3}$$

and

$$N(t) \leqq (1+3cT^{-1/3})N(T), \tag{11.4}$$

for $T \leqq t \leqq T+1$. It follows that, for $T+N-1 \leqq t \leqq T+N$,

$$\begin{aligned} N(t) &\leqq \prod_{K=1}^{N}(1+3c(T+K)^{-1/3})N(T) \\ &\leqq c e^{ct^{2/3}} N(T). \end{aligned} \tag{11.5}$$

Since $\beta > 2/3$ we obtain the global decay for $\|f(t)\|$ by just making the coefficient in the exponential a little smaller, i.e. by changing from λ_0 to λ.

Proposition 11.1. *Fix T large anough, then*

$$\|f(t)\| \leqq c e^{-\lambda t^\beta}\|f(T)\|_\alpha \tag{11.6}$$

for $t > T$.

Proof. According to (11.5)

$$\begin{aligned} e^{\lambda_0 t^\beta}\|f(t)\| &\leqq c e^{ct^{2/3}} N(T) \\ &\leqq c e^{ct^{2/3}}\|f(T)\|_\alpha. \end{aligned} \tag{11.7}$$

Since $\beta > \frac{2}{3}$ and $\lambda < \lambda_0$,

$$\|f(t)\| \leqq c e^{-\lambda t^\beta}\|f(T)\|_\alpha. \tag{11.8}$$

Finally we are ready to see the global decay of $\|f(t)\|$. In all the above theory there has been the premise that T be large. But clearly the estimate in Proposition 11.1 is preserved by a shift in time. Define

$$\bar{f}(t)=f(t-T) \quad \text{for} \quad t > T. \tag{11.9}$$

The argument of $\bar{f}$ is large enough to apply the results, and so

$$\|\bar{f}(t)\| \leqq c e^{-\lambda t^\beta}\|\bar{f}(T)\|_\alpha, \tag{11.10}$$

from which it follows that

$$\|f(t)\| \leqq c e^{-\lambda t^\beta}\|f(0)\|_\alpha. \tag{11.11}$$

This is the result (3.1).

12. The Sup Norms

a) First we show the preservation of the α norm. Rewrite the Boltzmann equation (1.10), (1.11) as

$$f(t)=e^{-t\nu}f_0-\int_0^t e^{-(t-s)\nu}Kf(s)ds. \tag{12.1}$$

Since $e^{-t\nu}\leqq 1$, we have

$$\|f(t)\|_\alpha\leqq\|f_0\|_\alpha+\int_0^t\|e^{-(t-s)\nu}Kf(s)\|_\alpha ds. \tag{12.2}$$

Now

$$\begin{aligned}\|e^{-(t-s)\nu}Kf(s)\|_\alpha\leqq&\sup_\xi e^{-(t-s)\nu}(1+\xi)^{-\gamma-1/2}\\ &\cdot\sup_\xi(1+\xi)^{\gamma+1/2}e^{\alpha\xi^2}|Kf(\xi,s)|.\end{aligned} \tag{12.3}$$

The first factor on the right is estimated according to the following lemma.

Lemma 12.1.

$$\sup_\xi e^{-t(1+\xi)^{-\gamma}}(1+\xi)^{-\gamma'}\leqq c(1+t)^{-\gamma'/\gamma} \tag{12.4}$$

for $t>0$, $\gamma>0$, $\gamma'>0$.

The second factor is recognized as $\|Kf(s)\|_{\alpha,\gamma+1/2}$. By splitting up this norm into two terms with $\xi<\xi_0$ and $\xi>\xi_0$, in which ξ_0 will be chosen later, we find

$$\begin{aligned}\|Kf(s)\|_{\alpha,\gamma+1/2}&\leqq e^{\alpha\xi_0^2}\|Kf(s)\|_{0,\gamma+1/2}+(1+\xi_0)^{-3/2}\|Kf(s)\|_{\alpha,\gamma+2}\\ &\leqq e^{\alpha\xi_0^2}\|f(s)\|+(1+\xi_0)^{-3/2}\|f(s)\|_\alpha,\end{aligned} \tag{12.5}$$

according to Proposition 6.1. Combining this with Lemma 12.1 we change (12.3) to

$$\|e^{-(t-s)\gamma}Kf(s)\|_\alpha\leqq c(1+t-s)^{-1-1/2\gamma}\{e^{\alpha\xi_0^2}\|f(s)\|+(1+\xi_0)^{-3/2}\|f(s)\|_\alpha\}. \tag{12.6}$$

Next we substitute this into (12.2). Use the facts that $\|f(s)\|\leqq c\|f_0\|_\alpha$ and that $\int_0^t(1+t-s)^{-1-1/2\gamma}ds\leqq c$ independent of t, to find

$$\|f(t)\|_\alpha\leqq\|f_0\|_\alpha+ce^{\alpha\xi_0^2}\|f_0\|_\alpha+c(1+\xi_0)^{-3/2}\sup_{0\leqq s\leqq t}\|f(s)\|_\alpha. \tag{12.7}$$

Choose ξ_0 large enough that $c(1+\xi_0)^{-3/2}\leqq\frac{1}{2}$. Then

$$\sup_{0\leqq s<t}\|f(s)\|_\alpha\leqq c\|f_0\|_\alpha, \tag{12.8}$$

which proves (3.3) in Theorem 3.1.

b) Next we show the decay of $\|f\|_\infty$. As before we first look at a fixed time interval $[T,T+1]$. Define w as in (8.3) and denote

$$g=\chi_w f \qquad h=\bar{\chi}_w f, \tag{12.9}$$

Use (3.3) in Theorem 3.1 and (8.7) in Proposition 8.1 to estimate

$$\begin{aligned}\|h(t)\|_\infty &\leqq ce^{-\alpha w^2}\|f(t)\|_\alpha \\ &\leqq ce^{-\alpha w^2}\|f_0\|_\alpha \\ &< ce^{-\lambda_1 t^\beta}\|f_0\|_\alpha .\end{aligned} \tag{12.11}$$

By rearranging the Boltzmann equation, we write

$$g(t) = e^{-(t-s)v} g(T) + \int_T^t e^{-(t-\sigma)v} \chi_w K f(\sigma) d\sigma . \tag{12.12}$$

Now

$$\|g(T)\|_\infty \leqq Q(T) e^{-\lambda_1 T^\beta} . \tag{12.13}$$

Since $\lambda_1 < \lambda_0$, the statement (3.1) of Theorem 3.1 is also true with λ_1 instead of λ. Using that and Proposition 6.1, we find

$$\begin{aligned}\|\chi_w K f(\sigma)\|_\infty &\leqq c\|f\| \\ &\leqq ce^{-\lambda_1 \sigma^\beta}\|f_0\|_\alpha .\end{aligned} \tag{12.14}$$

Since $v \geqq v(w)$ for $\xi < w$,

$$\|g(t)\|_\infty \leqq e^{-(t-T)v(w)} e^{-\lambda_0 T^\beta} Q(T) + c \int_T^t e^{-(t-\sigma)v(w)} e^{-\lambda_1 \sigma^\beta} d\sigma \|f_0\|_\alpha . \tag{12.15}$$

It follows from (8.5) in Proposition 8.1 that

$$-(t-\sigma)v(w) - \lambda_1 \sigma^\beta \leqq -\lambda_1 t^\beta , \tag{12.16}$$

and

$$-(t-T)v(w) - \lambda_0 T^\beta \leqq -\lambda_1 t^\beta . \tag{12.17}$$

Therefore

$$\|g(t)\|_\infty \leqq e^{-\lambda_1 t^\beta}(Q(T) + c\|f_0\|_\alpha) . \tag{12.18}$$

Combine this with (12.11) to find that

$$Q(t) \leqq Q(T) + c\|f_0\|_\alpha , \tag{12.19}$$

for $T \leqq t \leqq T+1$.

As before the statement (12.19) is true for large T, but can be made into a statement for all t. By adding up the contributions in each time interval we get

$$Q(t) \leqq ct\|f_0\|_\alpha . \tag{12.20}$$

As before the factor t can be absorbed into the exponential $e^{\lambda_1 t^\beta}$ in Q, by replacing the coefficient λ_1 by λ. This results in the desired inequality (3.2). Finally the proof of Theorem 3.1 is finished.

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Communicated by J. L. Lebowitz

Received October 25, 1979