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The Borel Transform in Euclidean φ_ν^4 Local Existence for $\mathrm{Re}\,\nu < 4$

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Abstract. We consider the φ^4 theory in Euclidean space of complex dimension ν and prove that, for $\mathrm{Re}\,\nu<4$ the renormalized Feynman amplitudes grow at worst exponentially in the number of vertices in the graph. This implies that the Borel transform of any Schwinger function may be defined in a neighborhood of the origin in the Borel plane.

1. In this paper we prove the existence, in a neighborhood of the origin, of the Borel transform of the perturbative series for any Schwinger function of the Euclidean φ_ν^4 model, when the (possibly complex) dimension ν of space time satisfies $\mathrm{Re}\,\nu<4$.

We do not discuss the more difficult problem of extending the domain of analyticity of the transform and proving the Borel summability of the theory (a problem which has been solved by constructive quantum field theory in the integer dimensions $\nu=1,2,3$). A discussion of the background and motivation for this study is given in [1] (see also [2]), to which we refer the interested reader; here we summarize the problem briefly.

We consider then a truncated Schwinger function of $N \geqq 2$ fields. Such a function has a formal power series development in the coupling constant λ:

$$\delta\left(\sum_1^N p_i\right) \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} a_n(p, \nu),$$

where a_n is a sum of Feynman amplitudes which are associated with connected Feynman graphs and in which ν appears as a parameter. Since the number of such graphs is obviously smaller than the total number of graphs appearing in the development of the full Schwinger function, and since this number $K_{N,n}=(4n+N-1)!!$ is certainly bounded by $(n!)^2 \cdot (16)^n\ (4n+N-1)^{N/2}$, the con-

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vergence of the series for the Borel transform $B(t)$:

$$B(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} a_n(p, v)$$

in a neighborhood of the origin ($|t| < t_0$) will thus follow from our main result:

Theorem. *Let G_0 be a connected graph in the φ_v^4 theory with n internal and $N \geqq 2$ external vertices, and let $\delta(\sum p_i) I_{G_0}^R(p, v)$ be a renormalized Feynman amplitude for G_0. Then for $0 < \mathrm{Re}\, v < 4$, there is a constant A_v independent of n and of the graph G_0, such that:*

$$|I_{G_0}^R(v)| \leqq A_v^n. \tag{1}$$

Moreover, A_v may be chosen uniformly on compact subsets of $\{0 < \mathrm{Re}\, v < 4\}$.

We note although we do not deal here with this question, that a precise numerical calculation of A_v would provide a rigorous bound on the behavior of large orders of perturbation theory for φ_v^4, that could be compared with the more precise behavior obtained by Lipatov's method [3, 4].

We remark also (see [1]) that it suffices to establish the theorem for any one choice of renormalization method, since a finite renormalization will not affect the estimate (1). The generalization of the theorem to the region $\mathrm{Re}\, v < 4$ is trivial (see [1]).

The remainder of the paper is organized as follows. In Sect. 2 we introduce our notation, recall briefly the method of dimensional renormalization, and reduce the proof of the theorem above to an estimate on unrenormalized amplitudes. Section 3 is devoted to two preliminary estimates, on convergent graphs with generalized propagators, and on self-energy graphs. Section 4 gives an integral representation of an arbitrary amplitude in terms of the amplitudes considered in Sect. 3 and proves the final estimate.

2. We establish our notation for an arbitrary connected Feynman graph G in which line l has generalized propagator $(p^2 + \zeta_l)^{-\lambda_l}$, with $\lambda_l \geqq 1$, $\mathrm{Re}\, \zeta_l > 0$.

Let L_G, n_G, and h_G denote the number of lines, vertices, and loops, respectively, and write

$$\delta_G(v) = v/2 \cdot h_G - \sum_{l \in G} \lambda_l$$

for the superficial divergence in dimension v. Some vertices $\{v_a\}$ of G are external and formally associated to them are Euclidean momenta $\{p_a\}$ satisfying overall momentum conservation; the complex dimensional amplitude is a function of the invariants $\{s_{ab}\}$ formally given by $s_{ab} = p_a \cdot p_b$. The standard combinatoric functions are:

$$\begin{aligned} M_G(\alpha, \zeta) &= \sum \alpha_l \zeta_l \\ U_G(\alpha) &= \sum_T \prod_{l \notin T} \alpha_l \\ V_G(\alpha, s) &= U_G^{-1}(\alpha) \sum_{T_2} \Big(\prod_{l \notin T_2} \alpha_l\Big) \Big(\sum_{a \in E} p_a\Big)^2, \end{aligned} \tag{2}$$

where T is a (spanning) tree in G, T_2 a 2-tree which separates the external vertices into two non-empty sets, one of which is E. Finally the unrenormalized Feynman amplitude [5] of G is, omitting irrelevant factors:

$$I_G(\nu,s)=\int_0^\infty\dots\int_0^\infty\prod_{l\in G}[\Gamma(\lambda_l)^{-1}\alpha_l^{\lambda_l-1}d\alpha_l]\,U_G^{-\nu/2}\exp[-(V_G+M_G)]\,. \tag{3}$$

This integral converges absolutely for Re $\nu<2$.

To study renormalization of the φ^4 theory, we fix $\nu_0<4$ and restrict ν to the domain $\Omega=\{\nu/0<\mathrm{Re}\,\nu<\nu_0\}$.

A graph G_0 in the φ^4 theory has $\zeta_l=m^2$, $\lambda_l=1$ for each line l; the amplitude I_{G_0} is meromorphic in ν and is analytic in the domain $\Omega'=\left\{\nu/0<\mathrm{Re}\,\nu<\nu_0,\ \nu\neq 4-\frac{2}{n}\right.$ for any $\left.n\geqq 1\right\}$ [5]. A one-particle irreducible (1PI) graph G_0 is superficially divergent for some $\nu\in\Omega$ [i.e. $\delta_G(\nu_0)>0$] if and only if G_0 has two external lines (i.e. is a self energy graph, obtained by amputating a graph in the Schwinger function of two fields) and satisfies $n_{G_0}<[2-(\nu_0/2)]^{-1}$.

Thus there is a finite collection $\mathscr{J}$ of divergent 1PI self energy graphs and all other graphs are superficially convergent.

Suppose then that G_0 is superficially convergent in $\Omega(\delta_{G_0}(\nu_0)\leqq 0)$. G_0 contains a (possibly empty) collection $\{H_1,\dots,H_I\}$ of divergent connected subgraphs, each isomorphic to an element of $\mathscr{J}$. Then the dimensionally renormalized amplitude $I_{G_0}^R$ has the form [5]:

$$I_{G_0}^R(\nu;s)=\sum_S\left[\prod_{i\in s}f_{H_i}(\nu)\right]I_{G_S}(\nu,s)\,, \tag{4}$$

where the sum is over subsets $S\subset\{1,\dots,I\}$ such that $\{H_i, i\in S\}$ are disjoint, G_S is obtained from G by replacing each H_i, $i\in S$, by a single φ^2 vertex, and $f_{H_i}(\nu)$ depends only on the structure of H_i and is analytic in Ω'. Moreover, $I_{G_0}^R$ is analytic in Ω.

Now in Sect. 4 we will prove:

Lemma 1. *For any compact subset $K\subset\Omega'$ there is a constant $B_K\geqq 1$ such that, for $\nu\in K$ and for any Feynman graph G containing φ^4 and φ^2 vertices and at least two external lines:*

$$|I_G(\nu,s)|\leqq B_K^{L_G}\,. \tag{5}$$

Here we note that the main theorem of Sect. 1 follows immediately. For the right side of (4) contains fewer than 2^{L_G} terms; thus if $K\subset\Omega'$ is compact and $b_K=\max_{H\in\mathscr{J}}\left\{\sup_{\nu\in K}f_H(\nu)\right\}$ then for $\nu\in K$, $|I_G^R(\nu,s)|\leqq(2b_KB_K)^{L_G}$.

A similar bound follows on a compact subset $\bar{K}\subset\Omega$ by writing I_G^R in terms of Cauchy's integral formula on a Jordan curve in Ω' enclosing $\bar{K}$. Since $L_G<2n_G$, (1) follows.

3. We need two preliminary estimates for the proof of Lemma 1. The first controls the behavior of *convergent* graphs; the key idea is to use a convexity argument (as in [1, 2]) together with a standard result from linear programming to estimate $U_G^{-\nu/2}$ as a product.

Lemma 2. *For any compact $K \subset \Omega$ and any $a>0$, there is a constant $C_{K,a} \geqq 1$ such that, if $\bar{G}$ is a connected Feynman graph whose propagators $(p^2+\zeta_l)^{-\lambda_l}$ satisfy $\lambda_l \geqq 1$, $\operatorname{Re}\zeta_l \geqq a$, and for which $\delta_J(\nu_0) \leqq 0$ for every subgraph J, then for $\nu \in K$:*

$$|I_{\bar{G}}(\nu, s)| \leqq (C_{K,a})^{\sum_{l\in G} \lambda_l}. \tag{6}$$

Proof. Let $\mu = \operatorname{Re}\nu$. We claim that there exist weights W_T for the (spanning) trees T of $\bar{G}$ such that:

$$\begin{aligned} &W_T \geqq 0 \\ &\sum W_T = 1 \\ &\sum_{T \not\ni l} W_T \leqq \frac{2\lambda_l}{\nu_0} \qquad \forall l \in \bar{G}. \end{aligned} \tag{7}$$

Suppose this is true; then:

$$U_{\bar{G}}(\alpha) \geqq \sum_T W_T \prod_{l \notin T} \alpha_l \geqq \prod_T \Big(\prod_{l\notin T} \alpha_l\Big)^{W_T} = \prod_l \alpha_l^{\left(\sum_{T \not\ni l} W_T\right)} \tag{8}$$

by the standard inequality between geometric and arithmetic means. Hence from (3), since $V_{\bar{G}} \geqq 0$,

$$\begin{aligned} |I_{\bar{G}}| &\leqq \int_0^\infty \dots \int_0^\infty \prod_l [\Gamma(\lambda_l)^{-1} \alpha_l^{\lambda_l - 1} d\alpha_l]\, U_{\bar{G}}^{-\mu/2} \exp[-a \textstyle\sum \alpha_l] \\ &\leqq \prod_l \left[\int_0^\infty \Gamma(\lambda_l)^{-1} \alpha_l^{\varrho_l - 1} \exp(-a\alpha_l) d\alpha_l\right] \\ &= \prod_l \{a^{\varrho_l} \Gamma(\varrho_l)/\Gamma(\lambda_l)\}, \end{aligned}$$

where $\varrho_l = \lambda_l - \frac{1}{2}\mu \sum_{T \not\ni l} W_T$ satisfies $\lambda_l \geqq \varrho_l \geqq 1 - \mu/\nu_0$.

Then (6) follows: for example, we may take $C_{K,a} = (\max\{1, a\})\Gamma(r)$ where $r = \inf_{\nu \in K}\left(1 - \dfrac{\operatorname{Re}\nu}{\nu_0}\right)$.

It remains to prove the claim. Now (7) may be reformulated as follows: there exist weights $\{W_T\}$ and $\{u_l | l \in \bar{G}\}$ such that:

$$\begin{aligned} &W_T \geqq 0, \quad u_l \geqq 0 \\ &\sum W_T = 1 \\ &\sum_{T \ni l} W_T - u_l = 1 - 2\frac{\lambda_l}{\nu_0} \equiv \chi_l \qquad \forall l \in \bar{G}. \end{aligned} \tag{9}$$

By a lemma of Farkas [6, 7], (9) has a solution if there is no solution (y, x_l) to the dual problem:

$$\begin{aligned} x_l &\leqq 0 \qquad \forall l \in G \\ y+\sum_{l\in T} x_l &\geqq 0 \qquad \forall T \\ y+\sum_{l\in\bar{G}} \chi_l x_l &< 0 . \end{aligned} \tag{10}$$

Suppose then that a solution of (10) exists, we number the lines of $\bar{G}$ so that $x_{l_1} \leqq x_{l_2} \leqq \ldots \leqq x_{l_L} \leqq x_{l_{L+1}} \equiv 0$ and let J_k be the graph formed by $l_1, \ldots, l_k$.

The condition $\delta_{J_k}(v_0) \leqq 0$ is precisely:

$$\sum_{j=1}^{k} \chi_{l_j} \leqq L_{J_k} - h_{J_k} = n_{J_k} - c_{J_k}, \tag{11}$$

where c_{J_k} is the number of connected components of J_k.

Multiply (11) by $x_{l_{k+1}} - x_{l_k}$ and sum over $k=1, \ldots, L$ to obtain:

$$\sum_{\bar{G}} \chi_l x_l \geqq \sum_{k=1}^{L} x_{l_k}[(n_{J_k} - c_{J_k}) - (n_{J_{k-1}} - c_{J_{k-1}})] = \sum_{l\in T} x_l, \tag{12}$$

where $T = \{l_k / n_{J_k} - c_{J_k} \neq n_{J_{k-1}} - c_{J_{k-1}}\}$ (set $n_{J_0}=0$, $c_{J_0}=0$, by convention).

But T is clearly a tree in $\bar{G}$, so that (10) implies:

$$\sum_{T} x_l \geqq -y > \sum_{\bar{G}} \chi_l x_l$$

contradicting (12). Thus (10) has no solution, and a solution exists for (9). □

The second estimate we need controls the high energy behavior of self energy graphs. Such results are known in greater generality [8] and we give a proof in our somewhat different case primarily for completeness.

For a self energy graph we write z for the external energy $(z=p^2)$ and the notation $V(\alpha, z) = v(\alpha) z$.

Lemma 3. *Let H be a* 1-PI *self energy graph in the φ^4 theory. Then for* Re$z > -\frac{m^2}{2}$, *the defining integral:*

$$I(\nu, z) = \int_0^\infty \ldots \int_0^\infty \prod d\alpha_l U^{-\nu/2} \exp[-(vz + (\sum \alpha) m^2)] \tag{13}$$

converges absolutely if Re$\nu<2$, *may be analytically continued to the domain Ω', and on compact subsets $K \subset \Omega'$ satisfies*

$$|I(\nu, z)| \leqq D_K |z+m^2|^{\bar{\delta}} \tag{14}$$

for some constant D_K, with $\bar{\delta} = \max\{\mathrm{Re}\,\delta_H(\nu), 0\}$.

Proof. We leave the case $v(\alpha)=0$ (in which the external vertices of H coincide) to the reader. In the general case we first note that, from (2), $v(\alpha) \leqq \sum_H \alpha_l$ and hence:

$$\mathrm{Re}\left[vz + \left(\sum_H \alpha_l\right) m^2\right] \geqq \tfrac{1}{2}\left(\sum_H \alpha_l\right) m^2 . \tag{15}$$

Now following [5], we write $I=\sum I_\xi$ where the sum is over all s-families ξ for H, and in I_ξ introduce scaling variables $\{t_J|J\in\xi\}$ by $\alpha_l=\prod_{J\ni l} t_j$. Then:

$$I_\xi(v,z)=\int_0^\infty t_H^{-\delta_H(v)-1}dt_H\int_0^1\dots\int_0^1\prod_{\substack{J\in\xi\\ J\neq H}} t_J^{-\delta_J(v)-1}dt_J e(t)^{-v/2}$$
$$\cdot\exp\left\{-t_H\left[\left(\prod_{J\in\xi'} t_J\right)f(t)z+g(t)m^2\right]\right\},\tag{16}$$

where $c(t)$, $f(t)$, and $g(t)$ are independent of t_H, continuous and positive in the integration region, and $\xi'=\{J\in\xi/J\neq H$ and J connects the external vertices of $H\}$. The absolute convergence of (16) for $\mathrm{Re}\,v<2$ now follows from (15) and the inequality $\mathrm{Re}\,\delta_J(v)<0$, $J\in\xi$.

To continue I_ξ to all of Ω' we use the identity, valid for $\mathrm{Re}\,v<2$,

$$\int_0^1 t_J^{-\delta_J(v)-1}\varphi(t_J)dt_J=(-\delta_J(v))^{-1}\varphi(o)+\int_0^1\int_0^1 t_J^{-\delta_J(v)-1}dt_J\,d\tau_J\left[\frac{d}{d\tau_J}\varphi(t_J\tau_J)\right]\tag{17}$$

for each $J\in\mathfrak{J}=\{J\in\xi/J\neq H,\ \delta_J(v_0)>0\}$ carry out the τ_J derivatives, then do the t_H integral explicitly. The result is a finite sum of terms which we label by a partition $\mathfrak{J}=\mathfrak{J}'\cup\mathfrak{J}''$ describing which term of (17) occurs for each J, and non negative integers c,d, describing the number of times the z and m^2 terms respectively are pulled down from the exponential in (16) by τ derivatives. Thus (using $\mathfrak{J}\cap\xi'=\emptyset$):

$$I_\xi=\sum_{\mathfrak{J}',c,d}\Gamma(c+d-\delta_H(v))\prod_{\mathfrak{J}'}[-\delta_J(v)]^{-1}\int_0^1\dots\int_0^1\prod_{\mathfrak{J}''}t_J^{-\delta_J(v)}$$
$$\cdot dt_J d\tau_J\prod_{J\notin\mathfrak{J},J\neq H}t_J^{-\delta_J(v)-1}dt_J\left(z\prod_{\xi'}t_J\right)^c(m^2)^d R(t,\tau)S(t,\tau,z)^{\delta_H(v)-c-d},$$

where R is a continuous function and:

$$S(t,\tau,z)=\left[\prod_{\xi'}t_J f(t)z+g(t)m^2\right]\Bigg|_{\substack{t_J\to 0\ J\in\mathfrak{J}'\\ t_J\to\tau_J t_J,J\in\mathfrak{J}''}}$$
$$=\left[\prod_{\xi'}t_J\bar{f}(t,\tau)z+\bar{g}(t,\tau)m^2\right].$$

By (15), $\mathrm{Re}\,S\geqq\bar{g}(t,\tau)\frac{m^2}{2}$; hence the integral in (18) is convergent for $v\in\Omega$ and (18) provides an analytic continuation of I to Ω'.

To verify (14), note that for appropriate C_1, C_2:

$$C_1|z+m^2|\geqq|S|\geqq C_2\left[\prod_{\xi'}t_J|z|+m^2\right].$$

Hence since $|\arg S|\leqq\pi/2$,

$$|S^{\delta_H(v)-c-d}|=|S^{\delta_H(v)}|\,|S^{-c}|\,|S^{-d}|$$
$$\leqq e^{\pi/2|\mathrm{Im}\,\delta_H(v)|}\left[C_2\prod_{\xi'}t_J|z|\right]^{-c}[C_2m^2]^{-d}$$
$$\cdot\begin{cases}[C_1|z+m^2|]^{\mathrm{Re}\,\delta_H(v)} & \text{if}\quad \mathrm{Re}\,\delta_H(v)\geqq 0\\ [C_2m^2]^{\mathrm{Re}\,\delta_H(v)} & \text{if}\quad \mathrm{Re}\,\delta_H(v)\leqq 0.\end{cases}$$

Inserting these estimates in (18) yields (14).

4. Our goal in this section is to prove Lemma 1.

Thus, let G be a graph with φ^4 or φ^2 vertices and propagators $(p^2+m^2)^{-1}$; since Lemma 1 is an asymptotic result we may assume $\delta_G(v_0)\leqq 0$.

We identify in G a collection $G_1,\ldots,G_p$ of subgraphs as in Fig. 1; G_j is a maximal chain of $r(j)$ subgraphs $G_{jk}\in\mathscr{J}$, joined by lines $l_{j0}, l_{j1},\ldots,l_{jr(j)}$.

Moreover, we suppose that G_{jk} is a maximal subgraph of G belonging to $\mathscr{F}$ and that every such maximal subgraph is a G_{jk} for some j, k. Feynman parameters in G_j are denoted α_{jk} (for l_{jk}) or α_{jki} (in G_{jk}); other Feynman parameters in G are denoted γ_i.

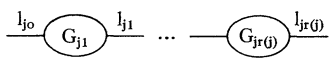

Fig. 1. The subgraphs G_j

We consider also the graph $\bar{G}$ obtained by replacing G_j in G with a single line l_j having Feynman parameters β_j and propagators $(p^2+\zeta_j)^{-\lambda_j}$ where

$$\lambda_j=-\delta_{G_j}(v_0)=1+\sum_{k=1}^{r(j)}(1-\delta_{G_{jk}}(v_0)). \tag{19}$$

Note that if J is a subgraph of $\bar{G}$ and J' the corresponding subgraph of G obtained by replacing l_j by G_j throughout, then $\delta_j(v_0)\leqq\delta_{J'}(v_0)$; thus Lemma 2 applies to $\bar{G}$. We will write $U,\bar{U},U_j$ etc. ... for $U_G, U_{\bar{G}}, U_{G_j}$ etc. ... and from (2) note that:

$$\begin{aligned} U(\alpha,\gamma)&=\bar{U}(\beta,\gamma)/\beta_j=v_j\prod_1^p U_j\\ V(\alpha,\gamma,s)&=\bar{V}(\beta,\gamma,s)/\beta_j=v_j. \end{aligned} \tag{20}$$

Also:

$$\begin{aligned} U_j(\alpha)&=\prod_{k=1}^{r(j)}U_{jk}(\alpha)\\ v_j(\alpha)&=\sum_{k=1}^{r(j)}v_{jk}(\alpha)+\sum_{k=0}^{r(j)}\alpha_{jk} \end{aligned} \tag{21}$$

and hence:

$$I_{G_j}(v,z)=(z+m^2)^{-r(j)-1}\prod_{k=1}^{r(j)}I_{G_{jk}}(v,z). \tag{22}$$

We now give a purely formal derivation of a new representation of I_G; a similar representation is used in [9]. From (3) and (20),

$$\begin{aligned} I_G(v,s)=\int_0^\infty\ldots\int_0^\infty\prod d\alpha d\gamma\prod_{j=1}^p\{\delta(\beta_j-v_j(\alpha))\\ \cdot[\beta_j/v_j(\alpha)]^{\lambda_j-1}U_j^{-v/2}d\beta_j\}\bar{U}^{-v/2}\exp[-(\bar{V}+M)]. \end{aligned} \tag{23}$$

Now write

$$\delta(\beta_j - v_j) = \frac{1}{2\pi i} \int_{\mathrm{Re}\zeta_j = m^2/2} d\zeta_j \exp[\zeta_j(v_j - \beta_j)], \tag{24}$$

$$v_j^{-(\lambda_j - 1)} = \Gamma(\lambda_j - 1)^{-1} \int_0^\infty d\eta_j \eta_j^{\lambda_j - 2} \exp(-\eta_j v_j) \tag{25}$$

so that (23) becomes [using (3) for $\bar{G}$ and G_j, then (22)]:

$$I_G(v, s) = \int_{\mathrm{Re}z_j > -m^2/2} \prod_{j=1}^{P} d\mu_j(z_j) I_{\bar{G}}(v, s) \prod_{j=1}^{P} \left\{(z_j + m^2)^{-r(j)-1} \prod_k I_{G_{jk}}(v, z_j)\right\}, \tag{26}$$

where $z_j = \eta_j - \zeta_j$ and $d\mu_j(z_j) = \frac{1}{2\pi i}(\lambda_j - 1)\eta_j^{\lambda_j - 2} d\eta_j d\zeta_j$ $\left[\text{recall that } I_{\bar{G}} \text{ contains a factor } \prod_j \Gamma(\lambda_j)^{-1}\right]$.

Lemma 4. *The formula* (26) *is valid for all* $v \in \Omega'$.

Proof. We first establish (26) for $\mathrm{Re}\,v$ sufficiently small. Let φ be a smooth, even, non-negative function of compact support on $\mathbb{R}$ for which $\int_{\mathbb{R}} \varphi(x)dx = 1$, and let $\tilde{\varphi}(t) = \int_{\mathbb{R}} \varphi(x) e^{ixt} dx$. For $\varepsilon > 0$, we may regularize the δ-function above by inserting a factor $\tilde{\varphi}(\varepsilon \,\mathrm{Im}\,\zeta_j)$ in the integral (24); substitution of (25) and the modified (24) into (23) leads us to define

$$I^\varepsilon(v, s) = \int_{\mathrm{Re}z_j > -m^2/2} \prod_{j=1}^{P} \tilde{\varphi}(\varepsilon \,\mathrm{Im}\, z_j) \Gamma(\lambda_j)^{-1} d\mu_j(z_j) \int_0^\infty \prod d\alpha\, d\gamma$$

$$\cdot \prod_{j=1}^{P} \beta_j^{\lambda_j - 1} d\beta_j [\bar{U}_{(\beta,\gamma)}]^{-\nu/2} \exp[-(\bar{V} + \bar{M})] \prod_{j=1}^{P} U_j(\alpha)^{-1/2} \exp[-(v_j z_j + M_j)]. \tag{27}$$

This integral is absolutely convergent, since taking the absolute value of the integrand replaces v by $\mathrm{Re}\,v$ and ζ_j by $m^2/2$, thus

$$\int |\ldots| d\eta\, d\zeta\, d\alpha\, d\gamma\, d\beta = \int \prod_j |\tilde{\varphi}(\varepsilon \,\mathrm{Im}\, z_j)| d\mu_j(z_j) I_{\bar{G}}(\mathrm{Re}\,v, s)|_{\zeta_j = m^2/2}$$

$$\cdot \prod_j I_{G_j}(\mathrm{Re}\,v, \mathrm{Re}\,z_j)$$

which converges by (22) and Lemmas 2 and 3.

Thus we may evaluate (27) by doing the α, β, and γ integrals to obtain

$$I^\varepsilon(v, s) = \int \prod_j \tilde{\varphi}(\varepsilon \,\mathrm{Im}\, z_j) d\mu_j(z_j) I_{\bar{G}}(v, s) \prod_j [(z_j + m^2)^{-r(j)-1} \prod_k I_{G_{jk}}(v, z_j)]. \tag{28}$$

Since $\tilde{\varphi}(o) = 1$ the Lebesgue dominated convergence theorem and Lemmas 2, 3 yield:

$$\lim_{\varepsilon \to 0^+} I^\varepsilon(v, s) = [\text{right hand side of (26)}].$$

[The use of Lemmas 2 and 3 to estimate integrals such as (28) is given in detail in the proof of Lemma 1, below.]

It remains to show that

$$\lim_{\varepsilon\to 0^+} I^\varepsilon(\nu,s)=I_G(\nu,s)\,. \tag{29}$$

For this, we do the z integrals in (27) to obtain

$$I^\varepsilon(\nu,s)=\int\prod d\alpha\prod d\gamma\prod_{j=1}^{P}\left[v_j^{-(\lambda_j-1)}U_j^{-\nu/2}e^{v_jm^2/2}\right]\exp(-M)G^\varepsilon(\alpha,\gamma)\,, \tag{30}$$

where $G^\varepsilon(\alpha,\gamma)=\int\limits_0^\infty\prod\limits_j\{\beta_j^{\lambda_j-1}\varepsilon^{-1}\varphi[(\beta_j-v_j)/\varepsilon]\exp(-\beta_jm^2/2)d\beta_j\}\,\bar U^{-\nu/2}\exp(-\bar V)$.

Clearly

$$\lim_{\varepsilon\to 0^+}G^\varepsilon(\alpha,\gamma)=(v_j)^{\lambda_j-1}\exp(-v_jm^2/2)[\bar U^{-\nu/2}\exp(-\bar V)]|_{\beta_j=v_j} \tag{31}$$

and if we can take this limit outside the integral in (30), (29) is established. Now choosing any term in $\bar U$ we may estimate

$$\bar U\geqq\prod\beta_j^{b_j}\prod\gamma_i^{c_i},$$

where b_j, c_i are zero or one; using $\bar V\geqq 0$ we have

$$|G^\varepsilon|\leqq\prod_i\gamma_i^{-c_i\mathrm{Re}\,\nu/2}\prod_j\left\{\sup_{\beta_j\geqq 0}\beta_j^{\lambda_j-1-b_j\mathrm{Re}\,\nu/2}\exp\left(-\beta_j\frac{m^2}{2}\right)\right\}. \tag{32}$$

Since $\lambda_j>1$ the supremum is finite for Reν sufficiently small. From (21),

$$v_j^{-\lambda_j-1}\leqq\left(\sum_{k=1}^{r(j)}\alpha_{jk}\right)^{\sum_k(\delta_{G_{jk}}(\nu_0)-1)}\leqq\prod_{k=1}^{r(j)}\alpha_{jk}^{(\delta_{G_{jk}}(\nu_0)-1)}. \tag{33}$$

Using (32), (33), and (15), we see that the integrand in (30) is in absolute value less than

$$(\mathrm{const})\cdot\prod_i\left[\gamma_i^{-c_i\mathrm{Re}\,\nu/2}e^{-\gamma_im^2}\right]\prod_j\left\{e^{-\alpha_{j0}m^2}\prod_{k=1}^{r(j)}\left[\alpha_{jk}^{(\delta_{G_{jk}}(\nu_0)-1)}e^{-\alpha_{jk}m^2}\right]\right.$$
$$\left.\cdot\prod U_{jk}^{-\nu/2}e^{-1/2M_{jk}}\right\}$$

and for Reν small the dominated convergence theorem yields (29).

To complete the proof we must verify (26) for all $\nu\in\Omega'$. But by Lemmas 2 and 3 the right hand side is absolutely convergent for all $\nu\in\Omega'$, (again see the proof of Lemma 1, below) and hence analytic in Ω'. The result follows by the principle of analytic continuation.

Proof of Lemma 1. We apply Lemmas 2 and 3 to estimate (26) for $\nu\in K\subset\Omega'$, K compact. Let $L'=L_G-\sum\limits_j L_{G_j}=L_{\bar G}-P$ and set $t_j=\mathrm{Im}\,z_j$.

Then

$$|I_G(v,s)| \leqq C_{K,m^2/2}^{(L'+\sum\lambda_j)} \prod_{j=1}^{P} \left\{ \frac{D_K(\lambda_j-1)}{2\pi} \int_0^\infty \eta_j^{\lambda_j-2}\, d\eta_j \int_{-\infty}^{+\infty} dt_j |z_j+m^2|^{-\mu_j} \right\} \tag{34}$$

with $\mu_j = r(j)+1-\sum_k \max[0, \operatorname{Re}\delta_{G_{jk}}(v)]$.

We introduce polar coordinates $\eta_j = r_j \sin\theta_j$, $t_j = r_j\cos\theta_j$, and note

$$|z_j+m^2| = |\eta_j + it_j + m^2/2| \geqq A(r_j+1)$$

for an appropriate constant A.

Thus we estimate the bracketed term in (34) by:

$$\{\ \} \leqq D_K \frac{(\lambda_j-1)}{2\pi} 2 \int_0^{\pi/2} (\sin\theta_j)^{\lambda_j-2}\, d\theta_j \int_0^\infty r_j^{\lambda_j-1}[A(r_j+1)]^{-\mu_j}\, dr_j$$

$$\leqq \frac{D_K}{\pi}\left(\frac{\pi}{2}\right)^{\lambda_j-1} A^{-\mu_j} B(\lambda_j, \mu_j-\lambda_j),$$

where B is the beta function. (We have used $\sin\theta_j \leqq \theta_j$.)

Since $\lambda_j > 1$ and $\mu_j - \lambda_j$ is uniformly bounded away from zero on K [see (19)], the beta function is uniformly bounded by some K-dependent constant.

Thus (34) becomes:

$$|I_G(v,s)| \leqq (C_{K,m^2/2})^{(L'+\sum\lambda_j)} \left(\frac{\pi}{2}\right)^{\sum(\lambda_j-1)} A^{-\sum\mu_j} E_K^P\,;$$

since $L' \leqq L_G$, $\sum\lambda_j \leqq \sum[r(j)+1] \leqq L_G$ by (19), $P < L_G$ and $\sum\mu_j \leqq \sum[r(j)+1] \leqq L_G$, this yields a bound of the form (5).

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