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Local Theory of Solutions for the $0(2k+1)$ σ-Model

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Abstract. We develop a theory of solutions n for the Euclidean nonlinear $0(2k+1)\sigma$-model for arbitrary k and for a region $G \subset \mathbb{R}^2$. We consider a subclass of solutions characterized by a condition which is fulfilled, for $G=\mathbb{R}^2$, by those n that live on the Riemann sphere $\mathbb{S}^2 \supset \mathbb{R}^2$. We are able to characterize this class completely in terms of k meromorphic functions, and we give an explicit inductive procedure which allows us to calculate all $0(2k+1)$ solutions from the trivial $0(1)$ solutions.

Introduction

In this paper we want to investigate instanton solutions n of the Euclidean $0(N)$-invariant σ-model. This model is characterized by the Lagrangean density

$$L(n) = \sum_{\alpha=1}^{d} \sum_{l=1}^{N} (\partial_\alpha n_l)^2 \tag{0.1}$$

with constraint

$$n^2 = (n, n) := \sum_{l=1}^{N} (n_l)^2 = 1\,, \tag{0.2}$$

and by instanton solutions we understand stationary points of the action

$$S(n) := \int L(n)\, d^d x \tag{0.3}$$

that are *continuous* and for which the total action is finite:

$$S(n) < \infty\,. \tag{0.4}$$

It is known that for the σ-model, (nontrivial) instantons in $d>2$ dimensions do not exist. More precisely, the argument is as follows:

Stationary points of L are, as expected, weak solutions of the Euler-Lagrange equations associated with (0.1)

$$\Delta n_l + L(n) n_l = 0 \qquad l = 1, \ldots, N\,. \tag{0.5}$$

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Of course, the class of variations has to be specified; see [1] for details. Now, by a previous result of us [2], continuous weak solutions of (0.5) are real analytic. By a well-known argument, any twice differentiable solution of (0.5) with finite action is constant in $d>2$ dimensions [1, Theorem 5.1].

We may therefore restrict ourselves to $d=2$ in the following.

In the case of the $0(3)\sigma$-model, all instantons have been found and are most easily described by projecting the three components of n stereographically from the unit sphere [cf. (0.2)] onto the plane [with coordinates (w_1, w_2), say]:

$$w_i := n_i(1+n_3)^{-1}, \qquad i=1,2. \tag{0.6}$$

Then, instantons are precisely those functions $n(x_1, x_2)$ for which either $w := w_1 + iw_2$ or $\bar{w}$ is a rational function of $z = x_1 + ix_2$, see [1].

For the general two-dimensional $0(N)$ model, we give a complete characterization for those finite action solutions of (0.5) that fulfil (0.5) on the whole Riemann sphere $\mathbb{S}^2 \supset \mathbb{R}^2$, i.e. for which also the conformal transform

$$\hat{n}(z, \bar{z}) := n(z^{-1}, \bar{z}^{-1})$$

fulfils (0.5) in $\mathbb{R}^2 \cong \mathbb{C}$. We show in Sect. 6 that these solutions fulfil the orthogonality conditions

$$(\partial^i n, \partial^j n) = 0 \tag{0.7}$$

for all non negative integers i, j with $i+j \geqq 1$, where

$$\partial := \tfrac{1}{2}(\partial_1 - i\partial_2) \tag{0.8}$$

and the bilinear form (,) is defined as in (0.2) by

$$(f, g) := \sum_{m=1}^{N} f_m g_m .$$

By exploiting (0.7), we are able to characterize solutions n explicitly by rational mappings. In proving this characterization, we develop the local theory for (0.5): We consider an arbitrary region $G \subset \mathbb{R}^2$ and characterize completely the class $I_k(G)$ of solutions of (0.5) [with constraint (0.2)] obeying (0.7) for $i, j \leqq k$ where $2k+1 = N$. Section 3 deals with the regular case where the vectors $\partial^m n$ for $1 \leqq m \leqq k$ are linearly independent, and Sects. 2 and 5 with the remaining singular case.

The characterization obtained (by k meromorphic mappings), though explicit, still contains a constraint mirroring (0.7). In Sect. 4, however, we show how to obtain all local I_k solutions in the $0(N)$ model by starting from the known (trivial) solutions of the $0(1)$ model, using an explicit inductive construction in k.

1. Local Properties of Regular Solutions

In this section, we want to analyze, for a fixed region $G \subset \mathbb{C}$, real continuous weak solutions n of (0.5), i.e. solutions of

$$n = -(\bar{\partial} n, \partial n)n = -L(n)n \tag{1.1}$$

with (locally) finite action (0.3). (Note that we have redefined L by a factor of four.) By the results in [2], all such solutions obeying the constraint (0.2) are real analytic and, in particular, in $C^\infty(G)$. So we may consider the subclass $I_k^r = I_k^r(G)$ of solutions for which

$$\begin{aligned}&\text{(i) the space } H_k \text{ spanned by } \{\bar\partial^l n\},\ l\in\mathbb{N}, \text{ has dimension } k;\\ &\text{(ii) } (\partial^l n, \partial^m n) = 0;\ l+m \geqq 1,\ 0 \leqq l,\ m \leqq k.\end{aligned} \tag{1.2}$$

As the bilinear form $(\,,)$ is not necessarily definite for complex-valued functions, we use also the (definite) scalar product $\langle\,,\rangle$ defined by

$$\langle f, g\rangle := \sum_{l=1}^{N} \bar f_l g_l$$

and the conjugation K defined by $Kf := \bar f$ for which

$$\langle f, g\rangle = (Kf, g).$$

By (i), the space KH_k is also k-dimensional; by (ii), H_k and KH_k are orthogonal, and n is orthogonal to both. Hence, n, H_k, and KH_k form a $(2k+1)$-dimensional subspace of the N-dimensional space of the n_l, $l=1, \dots, N$. Thus, $2k+1 \leqq N$.

Since H_k is k-dimensional, there are $\lambda_l \in \mathbb{C}$, not all zero, with $\sum_{l=1}^{k+1} \lambda_l \bar\partial^l n = 0$. Let m be the highest index with $\lambda_m \neq 0$ and apply $\bar\partial^{k+1-m}$ to see that $\bar\partial^{k+1} n$ is a linear combination of $\{\bar\partial^l n\}_{l=1}^k$. By the invariance of H_k under $\bar\partial$ one sees that H_k is spanned by the first k vectors $\{\bar\partial^l n\}_{l=1}^k$.

Of course, H_k depends on $z \in G \subset \mathbb{C}$. More precisely, we have

1.1 Lemma. *For any region $G \subset \mathbb{C}$, the dimension of H_k is constant in G with the possible exception of a nowhere dense real analytic manifold.*

Proof. Consider $k := \max\{k(z)/z \in G\}$ where $k(z)$ is the rank of the matrix $B := (\bar\partial^l n_i)_{l,i=1,\dots,N}$. Then there is a point $z_0 \in G$ with $k = k(z_0)$ and a $k \times k$ subdeterminant D_k of $\det B$ with $D_k \neq 0$ in z_0, whereas any $(k+1)\times(k+1)$ subdeterminant D_{k+1} is zero. By the maximality of k, D_{k+1} is zero everywhere. By the analyticity of n, $D_k = 0$ describes a real analytic manifold which is nowhere dense since the real analytic D_k does not vanish in z_0. □

Suppose now that $G' \subset G$ is a connected region for which the dimension of H_k is constant throughout. Then the space spanned by H_k, KH_k, and n is a constant real linear subspace of $\mathbb{R}^N$ of dimension $2k+1$ independent of $z \in G'$. Thus, we are dealing in this case with the $0(2k+1)$-model trivially embedded into the $0(N)$-model. Hence, we may as well suppose

$$2k+1 = N$$

in the following.

We denote by $I_k(G)$ the subclass of solutions n fulfilling (1.2) (ii) for which $\max(\dim H_j(z)/z \in G) = k$. Then $I_k^r(G)$ is a subclass of $I_k(G)$ of regular solutions (for which the space spanned by $\{\bar\partial^l n\}$, $l \in \mathbb{N}$, has constant dimension k in G).

Our first goal is to associate with I_k^r an analytic vector function $f \in H_k$ (for which we will in fact show in the next section that it characterizes I_k^r). Consider the function $f \in \hat H_k$, $\hat H_k := H_k \cup \{n\}$, defined by

$$\langle f, \bar\partial^i n\rangle := \delta^{ik} \qquad i = 0, \dots, k. \tag{1.3}$$

The case $k=0$ is trivial: Then $\bar{\partial} n=\lambda n$, but $\lambda=\lambda(n,n)=\lambda(n,\bar{\partial} n)=0$ since $(n,\bar{\partial} n)=0$ by differentiating (0.2). Thus, $\bar{\partial} n=0=\partial n$, so n is constant, $f=\mu n$, and $\mu=1$ by (1.3) and (0.2). Hence, we may suppose $k>0$.

1.2. Lemma. *The vector function f, defined by (1.3), is analytic.*

Proof. Since $f\in\hat{H}_k$, $\bar{\partial} f\in\hat{H}_k$ by (1.2) (i). Hence, it is enough to show

$$\langle\bar{\partial} f,\bar{\partial}^i n\rangle=0 \qquad i=0,\ldots,k.$$

Suppose first $i>0$. Then

$$\langle\bar{\partial} f,\bar{\partial}^i n\rangle=\partial\langle f,\bar{\partial}^i n\rangle-\langle f,\bar{\partial}^{i-1}(\partial\bar{\partial} n)\rangle.$$

The first term on the right is zero by (1.3). For the second term, use (1.1) and the product rule to see that $\bar{\partial}^{i-1}(\partial\bar{\partial} n)$ is a linear combination of $\bar{\partial}^l n$ for $l=0,\ldots,i-1$. Thus, the second term vanishes by (1.3), too.

Now suppose $i=0$. By (1.2) (i), f can be written

$$f=\sum_{i=0}^{k}\lambda_i\bar{\partial}^i n. \tag{1.4}$$

But $\lambda_0=0$ by (1.2) and (1.3) so that $f\in H_k$ and

$$\bar{\partial} f=\sum_{i=1}^{k+1}\mu_i\bar{\partial}^i n$$

with suitable coefficients μ_i. Again by (1.2),

$$\langle n,\bar{\partial} f\rangle=(n,\bar{\partial} f)=\sum_{i=1}^{k+1}\mu_i(n,\bar{\partial}^i n)=\mu_{k+1}(n,\bar{\partial}^{k+1}n).$$

But this vanishes, too:

$$(n,\bar{\partial}^{k+1}n)=\bar{\partial}(n,\bar{\partial}^k n)-(\bar{\partial} n,\bar{\partial}^k n)=0,$$

by (1.2) for $l=0$, $m=k$ and $l=1$, $m=k$. □

We collect further properties of f in the following

1.3. Lemma

(i) $\langle\partial^l f,\bar{\partial}^i n\rangle=(-1)^l\delta^{i+l,k} \qquad i,l\geqq 0, 0\leqq i+l\leqq k$;

(ii) $(\partial^l f,\partial^i f)=\delta^{lk}\delta^{ik} \qquad i,l=0,\ldots,k$;

(iii) *the vectors* $\{\partial^l f\}_{l=0,\ldots,k}$ *are linearly independent.*

Proof.

(i) The proof is by induction on $i+l$. For $i+l=0$, (i) is just (1.3). Assume (i) for some value $i+l$ with $0<i+l<k$:

$$\langle\partial^l f,\bar{\partial}^i n\rangle=\bar{\partial}\langle\partial^{l-1}f,\bar{\partial}^i n\rangle-\langle\partial^{l-1}f,\bar{\partial}^{i+1}n\rangle.$$

The first term on the right is zero by the induction assumption so that

$$\langle\partial^l f,\bar{\partial}^i n\rangle=(-1)^m\langle\partial^{l-m}f,\bar{\partial}^{i+m}n\rangle$$

as one can prove by induction on m. Now choose $m=l$ and use (1.3).

(ii) Note that $\partial^i f$ has a representation

$$\partial^i f = \sum_{j=0}^{k} \lambda_j^i \bar{\partial}^j n \tag{1.5}$$

with $\lambda_0^i = (-1)^k \delta^{ik}$. For $i=0$, this is just (1.4). If (1.5) is true for $i \leqq k-1$, then

$$\partial^{i+1} f = \partial\left(\sum_{j=1}^{k} \lambda_j^i \bar{\partial}^j n\right) = \lambda_0^{i+1} n + \sum_{j=1}^{k} \lambda_j^{i+1} \bar{\partial}^j n$$

with suitable coefficients λ_j^{i+1}, if (1.1) is used. Form the scalar product of this equation with n and use (i) to conclude $\lambda_0^{i+1} = (-1)^{i+1} \delta^{i+1,k}$. Thus

$$(\partial^l f, \partial^i f) = \sum \lambda_j^l \lambda_m^i \overline{(\partial^j n, \partial^m n)} = \lambda_0^l \lambda_0^i$$

by (1.2) (ii). But

$$\lambda_0^l \lambda_0^i = \delta^{lk} \delta^{ik}.$$

(iii) Assume a relation

$$\sum_{l=0}^{k} \lambda_l \partial^l f = 0,$$

form the scalar product with $\bar{\partial}^k f$ and use (ii) to see that $\lambda_k = 0$. Then, differentiate once and again form the scalar product with $\bar{\partial}^k f$ to infer $\lambda_{k-1} = 0$. Continuing in this manner, $\lambda_l = 0$ for $l = 0, \ldots, k$. □

Note that (iii) is implied by (ii) only; no property of n has been used.

We are now ready to express n in terms of f:

1.4. Lemma. *Let n be a solution of (1.1) in $G \subset \mathbb{C}$ and consider the function f of (1.3). Define the $k \times k$ matrix M by*

$$M := (M_{li}), M_{li} := \langle \partial^l f, \partial^i f \rangle, \qquad i, l = 0, \ldots, k-1.$$

Then M is invertible, and

$$(-1)^k n = \partial^k f - \sum_{i,l=0}^{k-1} (M^{-1})_{il} \partial M_{l,k-1} \partial^i f. \tag{1.6}$$

Proof

(i) To see that M is invertible, note that M is, by definition, positive semi-definite. But the quadratic form associated with M is non-degenerate:

$$\sum_{i,l} \bar{\lambda}_l \lambda_i M_{li} = \left\langle \sum_i \lambda_i \partial^i f, \sum_i \lambda_i \partial^i f \right\rangle = 0$$

only for $\lambda_i = 0$ since the vectors $\partial^i f$ are linearly independent by Lemma 1.3 (iii). Thus, M is definite and hence invertible.

(ii) Consider the space L_k spanned by $\{\partial^l f\}$, $l = 0, \ldots, k-1$ which is k-dimensional by Lemma 1.3 (iii). The right hand side of (1.6) is orthogonal to L_k and KL_k: The orthogonality to KL_k follows by Lemma 1.3 (ii) ($l \neq k!$), and the orthogonality to L_k is true since

$$\langle \partial^j f, \partial^k f - \sum (M^{-1})_{il} \partial M_{l,k-1} \partial^i f \rangle = \langle \partial^j f, \partial^k f \rangle - \partial M_{j,k-1}$$
$$= \langle \partial^j f, \partial^k f \rangle - \partial \langle \partial^j f, \partial^{k-1} f \rangle = 0$$

by the analyticity of f. By Lemma 1.3 (ii), L_k and KL_k are orthogonal, hence the right hand side of (1.6) is orthogonal to the $2k$-dimensional subspace of $\hat{H}_k$ spanned by KL_k and L_k. By Lemma 1.3 (i), this one-dimensional orthogonal complement is spanned by n. Now form the scalar product of the right hand side of (1.6) with n to see that the proportionality factor is $(-1)^k$. □

We close this section by illustrating the results for the 0(3)-model:

Since $2k+1 \leqq N=3$, $k=0$ or 1. For $k=0$, n is constant. Suppose $k=1$. Then, by (1.4), $f=\lambda \bar{\partial} n$, and λ can be determined from the defining Eq. (1.3):

$$1 = \langle f, \bar{\partial} n \rangle = \langle \lambda \bar{\partial} n, \bar{\partial} n \rangle = \bar{\lambda} \langle \partial n, \partial n \rangle$$

so that λ is real and $\lambda^{-1} = \langle \partial n, \partial n \rangle = L(n)$, the Lagrangian. Hence, $f = L^{-1} \bar{\partial} n$, which is analytic:

$$\bar{\partial} f = \bar{\partial} L^{-1} \bar{\partial} n + L^{-1} \bar{\partial}^2 n\,;$$

but

$$\bar{\partial}^2 n = \mu_0 n + \mu_1 \bar{\partial} n\,,$$

and forming the scalar products with n and $\bar{\partial} n$ gives $\mu_0 = 0$ and

$$\langle \bar{\partial} n, \bar{\partial}^2 n \rangle = \bar{\partial} \langle \bar{\partial} n, \bar{\partial} n \rangle - \langle \partial \bar{\partial} n, \bar{\partial} n \rangle = \bar{\partial} L$$

by (1.1) so that $\mu_1 L = \bar{\partial} L$. Hence,

$$\bar{\partial} f = -L^{-2} \bar{\partial} L \bar{\partial} n + L^{-1} \cdot \bar{\partial} n \cdot \bar{\partial} L L^{-1} = 0\,.$$

2. Structure of Singular Sets

Suppose n is a solution of (1.1) in a region $G \subset \mathbb{C} \cong \mathbb{R}^2$. We know how to associate with n an analytic function f in case the rank of the matrix $(\bar{\partial}^l n)_l$ is constant. Examples of 0(3)-solutions show, however, that in general one must expect singular points where the rank of $(\bar{\partial}^l n)_l$ is not maximal, and we want to investigate what happens at those points. From Lemma 1.1 we know already that the set S of singular points is a nowhere dense real analytic manifold, i.e. a set of points that can be described as the set of common zeros of a finite number of analytic functions.

Real analytic manifolds in 2 dimensions are either one-dimensional or points (zero-dimensional). Knowing from the last section that we can associate with n an analytic function in one variable one can expect that the type of singular manifolds which can appear in n are governed by manifolds of functions in one variable which means that only isolated points can appear.

Let z_1, z_2 be the complexification of the variables x_1 and x_2. We introduce the new variables $u = z_1 + iz_2$, $v = z_1 - iz_2$; then the real manifold $z_1 = \bar{z}_1$, $z_2 = \bar{z}_2$ becomes $u = \bar{v}$. Since the above transformation is bi-holomorphic we see that the analytic extension of n can be written as an analytic function $n(u,v)$ where n is analytic in some domain $\tilde{G}\subset\mathbb{C}^2$ such that $G=\{(u,v)\in\tilde{G};\, u=\bar{v}\}$. With this notation the differential Eq. (1.1) can be extended into $\tilde{G}$ and reads

$$\partial_u\partial_v n+(\partial_u n,\partial_v n)n=0.$$

Our singular manifold can now also be extended into $\tilde{G}$ and reads $\tilde{S}:=\{(u,v)\in\tilde{G}/\mathrm{rank}\{\partial_v^l n\}_{l=0}^{\infty}<k\}$. Since the exceptional points are given by zeros of sub-determinants, this is an analytic manifold. The defining relations (1.3) for the function f can trivially be extended into $\tilde{G}\backslash\tilde{S}$ and we find as in Lemma 1.2 that f depends only on the variable u. Replacing the family $\{\partial_v^l n\}$ by $\{\partial_u^l n\}$ we obtain in a similar fashion an analytic vectorfield g which depends only on the variable v and which is also defined outside of $\tilde{S}$. On the subset $v=\bar{u}$ we find in addition the relation $g(v)=\overline{f(u)}=\bar{f}(\bar{v})$, so that the relation $g(v)=\bar{f}(\bar{v})$ holds everywhere in $\tilde{G}$ by analytic continuation.

When working in the complex $\mathbb{C}^2$ it has some advantage to avoid the positive scalar product. For instance the defining relations for $f(u)$ become

$$\begin{aligned}(\partial_u^i n, f)&=\delta^{ik}\\ (\partial_v^i n, f)&=0\end{aligned}\qquad i=0,\dots,k \tag{2.1}$$

which by using the product rule of differentiation can also be written as

$$(\partial_u^i n,\partial_u^j f)=(-1)^{k-i}\delta^{i+j,k},\qquad i+j=0,\dots,k. \tag{2.2}$$

From these equations together with the Eqs. (ii) from Lemma 1.3 which now read

$$(\partial_u^i f,\partial_u^j f)=\delta^{ik}\delta^{jk},\qquad i,j=0,\dots,k \tag{2.3}$$

we learn the following:

2.1. Lemma

(i) *In $\tilde{G}\backslash\tilde{S}$ the vector field f is regular and depends only on the variable u. Moreover the vectors $f,\partial_u f,\dots,\partial_u^k f$ are linearly independent there.*

(ii) *If f defined by (2.1) is regular and fulfils the Eqs. (2.3) then $f,\partial_u f,\dots,\partial_u^k f$ are linearly independent and hence $\partial_u n,\dots,\partial_u^k n$ are linearly independent also.*

Proof

(i) The manifold $\tilde{S}$ is defined as the set where the $2k+1$ vectors $n,\partial_u n,\dots,\partial_u^k n$, $\partial_v n,\dots,\partial_v^k n$ become linearly dependent. From this follows in particular that the matrix

$$L:=(\partial_u^i n,\partial_v^j n)_0^k=\begin{pmatrix}1&0\\0&\tilde{L}\end{pmatrix} \tag{2.4}$$

is invertible in $\tilde{G}\backslash\tilde{S}$ by the following argument. Insert the vectors $n,\partial_u n,\dots,\partial_v^k n$ as columns in a $(2k+1)$ by $(2k+1)$ matrix N. Then the linear independence tells us $\det N\neq 0$, and hence $\det N^t N\neq 0$. Using the Eqs. (1.2) we obtain

$$N^t N=\left(\begin{array}{c|c|c}1&0&\\\hline &0&\tilde{L}\\ 0&&\\\hline &\tilde{L}^t&0\end{array}\right)$$

from which it follows that $\det\tilde{L}\neq 0$ and hence $\det L\neq 0$. With this knowledge we can solve the Eqs. (2.1) by the ansatz:

$$f=\sum_{i=0}^{k}\lambda_i\partial_v^i n+\sum_{j=1}^{k}\mu_j\partial_u^j n$$

and obtain after multiplying with the vectors $n, \partial_u n, \ldots, \partial_v^k n$ a linear system for λ_i, μ_j which can be solved, and we obtain

$$f=\sum_{i=0}^{k}\partial_v^i n(L^{-1})_{ik}. \tag{2.5}$$

Since n is analytic in $\tilde{G}$ and $\det L \neq 0$ in $\tilde{G}\backslash\tilde{S}$ it follows that f is analytic in $\tilde{G}\backslash\tilde{S}$. That f depends only on u can be proved in the same way as in Lemma 1.2. Also the linear independence of the vectors $\{\partial_u^i f\}_0^k$ can be proved in the same way as statement (iii) of Lemma 1.3.

(ii) Using the product rule we can generalize Eqs. (2.3) to

$$\begin{aligned}(\partial_u^i f, \partial_u^j f)&=0, \qquad & i+j<2k\\ (\partial_u^{k+i} f, \partial_u^{k-i} f)&=(-1)^i, \qquad & i=0,\ldots,k.\end{aligned} \tag{2.6}$$

If f is now regular then it follows from these Eqs. (2.6) that $f, \partial_u f, \ldots, \partial_u^k f$ are linearly independent. Using now Eqs. (2.2) we get that the set of vectors $\{\partial_u^i n\}_1^k$ are linearly independent. □

Using formula (2.5) we obtain

2.2. Lemma. *The vector field f of (2.1), defined in $\tilde{G}\backslash\tilde{S}$, has an extension into all of $\tilde{G}$ as a meromorphic function of one variable.*

Proof. From formula (2.5) we see that f can be extended into all of $\tilde{G}$ as a meromorphic function. But since we know that f depends only on u in $\tilde{G}\backslash\tilde{S}$, the same is true in $\tilde{G}$ by analytic continuation. □

The last statement allows us to disentangle the structure of the singular set S. But first let us remark that we get for $g(v)$ similar statements as for $f(u)$.

Let us denote by $P_u\tilde{G}$ the projection of $\tilde{G}$ into the u-plane and similarly for v, and let us denote by $p(f)$ the pole-set of the extension of f into all of $\tilde{G}$. Then we obtain:

2.3. Lemma. *The singular set $\tilde{S}$ has the structure*

$$\tilde{S}=(p(f)\times P_v\tilde{G}\cup P_u\tilde{G}\times p(g))\cap\tilde{G}. \tag{2.7}$$

Proof. If (u,v) is not in $\tilde{S}$ then it follows from Lemma 2.1 (i) that f and g are both regular and hence $\tilde{S}$ contains the right hand side of (2.7). Assume next f, g are both regular at (u,v); then it follows from Lemma 2.1 (ii) and Eqs. (2.2) together with the analogous equations for g that $n, \partial_u n, \ldots, \partial_u^k n, \partial_v n, \ldots, \partial_v^k n$ are linearly independent. This shows that $(u,v)\in\tilde{G}\backslash\tilde{S}$. Hence the right hand side of (2.7) contains $\tilde{S}$. □

Putting now everything together we obtain

2.4. Theorem. *Let a solution of (1.1) be an element of $I_k(G)$, then the set of vectors $\{\bar{\partial}^i n\}$ has constant rank k except on a set of isolated points which have no accumulation point inside G.*

Proof. Using the fact that $G=\tilde{G}\cap\{v=\bar{u}\}$ and $g(v)=\bar{f}(\bar{v})$ we obtain from Lemma 2.3

$$S=\tilde{S}\cap\{v=\bar{u}\}=p(f).$$

Since f depends on one variable only and f was meromorphic in $\tilde{G}$ (Lemma 2.2), it follows that $p(f)$ consists of isolated points with no accumulation points in $P_u(\tilde{G})$. Identifying now x_1+ix_2 with u we see $G\subset P_u(\tilde{G})$. □

3. Characterization of Regular Solutions

We have seen in Sect. 1 that, to every solution n of (1.1) in $G\subset\mathbb{C}$ which obeys (1.2), an analytic vector function f can be associated which fulfils Lemma 1.3. For convenience, we denote the set of such functions f by a special name. For any region $G\subset\mathbb{C}$, define

$$A_k(G):=\{f:G\to\mathbb{C}^N/f \text{ analytic}; (\partial^l f,\partial^i f)=\delta^{lk}\delta^{ik}\}. \tag{3.1}$$

We will now show the converse: Any $f\in A_k(G)$ determines a solution n of (1.1) which obeys (0.2):

3.1. Theorem. *Suppose $f\in A_k(G)$. Then the $k\times k$ matrix*

$$M_{li}:=(\langle\partial^l f,\partial^i f\rangle)_{li};\qquad l,i=0,\dots,k-1$$

is invertible, and the function n defined by

$$(-1)^k n:=\partial^k f-\sum_{i,l}(M^{-1})_{il}\partial M_{l,k-1}\partial_i f \tag{3.2}$$

has the properties

(a) $(n,n)=1$,
(b) $n=\bar{n}$,
(c) $\partial\bar{\partial}n=-(\bar{\partial}n,\partial n)n$.

In short, any $f\in A_k(G)$ gives rise to a (real) solution of the $0(N)$-model. We show in the next theorems that the functions n defined by (3.2) exhaust the class $I_k^r(G)$.

In the proof, we will use the *summation convention* that repeated indices will be summed over with the sum extending from 0 to $k-1$.

Proof. (a, b) By the argument in the proof of Lemma 1.3 (iii), for any $z\in G$, the vectors $\{\partial^l f\}$, $l=0,\dots,k$ form a $(k+1)$-dimensional vector space $\hat{L}_k$ with scalar product $\langle\,,\rangle$ and conjugation K defined by

$$Kf:=\bar{f} \tag{3.3}$$

fulfilling

$$\langle f,g\rangle=(Kf,g). \tag{3.4}$$

By (3.1), the k vectors $\{K\partial^l f\}$, $l=0,\dots,k-1$, are orthogonal (in $\langle\,,\rangle$) to $\hat{L}_k$ and are linearly independent since the $\{\partial^l f\}$ are. Hence, they form, together with $\hat{L}_k$, a $(2k+1)$-dimensional space $\hat{L}_{2k+1}$.

Consider the $2k$-dimensional subspace $H_{2k}\subset\hat{L}_{2k+1}$ spanned by $\{\partial^l f\}$ and $\{K\partial^l f\}$ for $l=0,\dots,k-1$, and choose a vector $e\neq 0$ orthogonal to H_{2k}. Since KH_{2k}

$=H_{2k}$, Ke is orthogonal to H_{2k}, too. Hence, one can choose

$$e = Ke \tag{3.5}$$

and, since $e \neq 0$,

$$1 = \langle e, e \rangle = (Ke, e) = (e, e). \tag{3.6}$$

Next, the vector $\partial^k f \in \hat{L}_{2k+1}$ has the representation

$$\partial^k f = \lambda e + \mu_i \partial^i f, \qquad \lambda, \mu_i \in C^\infty(G) \tag{3.7}$$

from which one obtains, using (3.1) and (3.6)

$$1 = (\partial^k f, \partial^k f) = \lambda^2 (e, e) = \lambda^2$$

by the fact that $e = Ke$ is orthogonal (in $\langle\,,\rangle$) to H_{2k}. This implies $\lambda = \pm 1$ so that one may assume $\lambda = +1$ (if necessary, replace e by $-e$).

Now form the scalar product of (3.7) with $\partial^l f$:

$$\partial M_{l,k-1} = \mu_i M_{li}. \tag{3.8}$$

The matrix M is invertible as it is positive definite [cf. Part (i) of the proof of Lemma 1.4] so that $\mu_i = (M)^{-1}_{il} \partial M_{l,k-1}$. Compare (3.7) with (3.2) to see that

$$e = (-1)^k n \tag{3.9}$$

which proves (a, b) by (3.6), (3.5).

(c) By (3.9) and the construction of e,

$$\begin{aligned} &\langle \partial^i f, n \rangle = (-1)^k \delta^{ik} && i = 0, \ldots, k \\ &\langle K \partial^i f, n \rangle = 0 && i = 0, \ldots, k-1. \end{aligned} \tag{3.10}$$

We now show that these equations remain true, for $i < k$, if n is replaced by $\partial\bar{\partial} n$:

$$\langle \partial^i f, \bar{\partial}\partial n \rangle = \partial \langle \partial^i f, \bar{\partial} n \rangle = \partial\bar{\partial} \langle \partial^i f, n \rangle - \partial \langle \partial^{i+1} f, n \rangle = 0 \tag{3.11}$$

by (3.10), and the complex conjugate Eq. (3.11) is

$$\langle K \partial^i f, \partial\bar{\partial} n \rangle = 0.$$

Hence, $\bar{\partial}\partial n$ is orthogonal to the space H_{2k} and thus proportional to e or n:

$$\partial\bar{\partial} n = \lambda n.$$

The proportionality factor is easily determined. By (a),

$$\begin{aligned} 0 &= \partial\bar{\partial}(n, n) = 2\partial(\bar{\partial} n, n) = 2(\partial\bar{\partial} n, n) + 2(\bar{\partial} n, \partial n) \\ &= 2\lambda + 2(\bar{\partial} n, \partial n) \end{aligned}$$

so that

$$\lambda = -(\bar{\partial} n, \partial n)$$

which implies (c). □

As a tool in the singularity theory to be developed in Chap. 5, we will need a relation between the action density $L(n)$ and the matrix M. This relation requires several preparatory lemmas:

3.2. Lemma.
(i) *M is symmetric,* $\bar{M}_{il}=M_{li}$,
(ii) $(M^{-1})_{il}\partial M_{lj}=\delta_{i,j+1}$ *for* $j<k-1$.

Proof.
(i) follows from the symmetry of $\langle\,,\rangle$.
(ii) For $j<k-1$,

$$\partial M_{ij}=\partial\langle\partial^i f,\partial^j f\rangle=\langle\partial^i f,\partial\partial^j f\rangle=M_{i,j+1} \tag{3.12}$$

by the analyticity of f. But (3.12) implies

$$(M^{-1})_{li}\partial M_{ij}=(M^{-1})_{li}M_{i,j+1}=\delta_{l,j+1}\,. \quad \square$$

Part (ii) of the lemma and (3.2) suggest the definition of the matrix

$$N:=M^{-1}\partial M\,. \tag{3.13}$$

3.3. Lemma.

$$M_{im}\bar{\partial}N_{m,k-1}=\delta_{i,k-1}\,. \tag{3.14}$$

Proof. We have

$$(-1)^k n\delta_{i,k-1}=\partial^{i+1}f-\partial^j f(M^{-1})_{jl}\partial M_{l,i}\,. \tag{3.15}$$

For $i=k-1$, (3.15) is just (3.2), the definition of n. For $i<k-1$, the right hand side is zero by Lemma 3.2 (ii) so that (3.15) holds, too.

By Parts (a), (b) of Theorem 3.1, $\langle n,n\rangle=1$ so that (3.15) implies

$$\begin{aligned}\delta_{i,k-1}&=\langle\partial^{i+1}f-\partial^j f(M^{-1})_{jl}\partial M_{li},\partial^k f-\partial^m f(M^{-1})_{ms}\partial M_{s,k-1}\rangle\\&=\partial\bar{\partial}\langle\partial^i f,\partial^{k-1}f\rangle-M_{lj}^{-1}\bar{\partial}M_{il}\partial\langle\partial^j f,\partial^{k-1}f\rangle\\&\quad-(M^{-1})_{ms}\partial M_{s,k-1}\bar{\partial}\langle\partial^i f,\partial^m f\rangle\\&\quad+M_{lj}^{-1}\bar{\partial}M_{il}(M^{-1})_{ms}\partial M_{s,k-1}\end{aligned}$$

where we used the symmetry of M and M^{-1} [Lemma 3.2 (i)]. By the definition of M, the second and fourth term on the right cancel. Hence

$$\delta_{i,k-1}=\partial\bar{\partial}M_{i,k-1}-(M^{-1})_{ms}\partial M_{s,k-1}\bar{\partial}M_{im}\,. \tag{3.16}$$

Since $MM^{-1}=1$, $\bar{\partial}(MM^{-1})=0$ so that $-(\bar{\partial}M)M^{-1}=M(\bar{\partial}M^{-1})$:

$$\begin{aligned}\delta_{i,k-1}&=M_{im}(M^{-1})_{ms}\partial\bar{\partial}M_{s,k-1}+M_{im}(\bar{\partial}M^{-1})_{ms}\partial M_{s,k-1}\\&=M_{im}\bar{\partial}[(M^{-1})_{ms}\partial M_{s,k-1}]\\&=M_{im}\bar{\partial}N_{m,k-1}\,. \quad \square\end{aligned}$$

3.4. Lemma

$$\bar{\partial}N_{k-1,k-1}=(\bar{\partial}n,\partial n)\,. \tag{3.17}$$

Proof. By (3.2) and the analyticity of f,

$$\begin{aligned}(-1)^k\partial\bar{\partial}n&=-\partial^{i+1}f\bar{\partial}N_{i,k-1}-\partial^i f\partial\bar{\partial}N_{i,k-1}\\&=-L(n)(-1)^k n\\&=-L(n)\{\partial^k f-N_{i,k-1}\partial^i f\}\,,\end{aligned} \tag{3.18}$$

where we used Theorem 3.1 (c) and (3.2) again. Since the vectors $\{\partial^m f\}$ are independent, we may compare the coefficients of $\partial^k f$ in (3.18) to obtain (3.17). □

3.5. Lemma.

$$\langle \partial n, \partial n \rangle = M^{-1}_{k-1,k-1} \tag{3.19}$$

Proof. Multiply (3.14) by M^{-1}_{li} to get

$$\bar{\partial} N_{l,k-1} = M^{-1}_{l,k-1},$$

put $l = k-1$ and apply Lemma 3.5. □

We remark that it is, in fact, enough to require, in definition (3.1), only the weaker equations

$$(\partial^i f, \partial^i f) = \delta^{ik} \qquad i = 0, \ldots, k \tag{3.20}$$

The original equations will follow if we can show that, for all $i < k$,

$$(\partial^i f, \partial^{i+l} f) = 0 \quad \text{for} \quad 0 \leqq l < k - i.$$

Assume that the last equation is true for all i and all $l \leqq m-1$. To show it for all i and m, differentiate once to obtain

$$0 = (\partial^{i+1} f, \partial^{i+l} f) + (\partial^i f, \partial^{i+l+1} f).$$

But the first term is zero by the induction assumption for the term $i+1$.

To show that the solutions of the 0(N)-model constructed from f are of class I^r_k, we need the result corresponding to Lemma 1.3 (i):

3.6. Lemma. *For i, $l \geqq 0$, $0 \leqq i + l \leqq k$,*

$$\langle \partial^l f, \bar{\partial}^i n \rangle = (-1)^l \delta^{i+l,k}. \tag{3.21}$$

Proof. Form the scalar product of (3.2) with $\partial^l f$ and use (3.1):

$$(-1)^k (n, \partial^l f) = \delta^{lk} \tag{3.22}$$

which shows that (3.21) is true for $i + l = 0$, by the reality of n. Assume (3.21) for all l, i such that $l + i$ is fixed. Then

$$\langle \partial^l f, \bar{\partial}^i n \rangle = \bar{\partial} \langle \partial^l f, \bar{\partial}^{i-1} n \rangle - \langle \partial^{l+1} f, \bar{\partial}^{i-1} n \rangle,$$

where the first term is zero by the induction assumption so that

$$\langle \partial^l f, \bar{\partial}^i n \rangle = (-1)^m \langle \partial^{l+m} f, \bar{\partial}^{i-m} n \rangle$$

as one can prove by induction on m. Now choose $m = i$ and use (3.22). □

We are now ready to prove

3.7. Theorem. *Assume $f \in A_k(G)$, and define n by (3.2). Then n is of class I^r_k, i.e.*

(a) $(\partial^l n, \partial^m n) = 0$ *for* $l + m \geqq 1$; $0 \leqq l, m \leqq k$;

(b) *the vectors* $\{\bar{\partial}^l n\}$, $l = 0, \ldots, k$ *are linearly independent.*

Proof.

(b) Assume a relation

$$\lambda_i \bar{\partial}^i n = 0,$$

form the scalar product with $\partial^l f$ and use (3.21) to conclude that $\lambda_{k-l}=0$, for $l=0, \ldots, k$ successively.

(a) Differentiate (3.2) j-times and use the analyticity of f to obtain

$$\bar{\partial}^j n = \lambda_i^j \partial^i f. \tag{3.23}$$

Note that $\lambda_k^j = 0$ for $j>0$ as the coefficient of $\partial^k f$ in (3.2) is constant. By (3.22),

$$(\bar{\partial}^j n, \bar{\partial}^l n) = \lambda_i^j \lambda_m^l (\partial^i f, \partial^m f) = \lambda_k^j \lambda_k^l$$

using the fact that $f \in A_k(G)$. But $\lambda_k^j \lambda_k^l = 0$, as just remarked, if either $j>0$ or $l>0$. □

By the results of Sects. 1 and 3, solutions of the $0(N)$-model which are of class I_k^r can be described in two equivalent ways:

Either a solution is given (locally in a region $G \subset \mathbb{C}$) by a real analytic function $n: G \to \mathbb{R}^N$ with the properties:

(a) $\partial \bar{\partial} n = -(\bar{\partial} n, \partial n) n$;

(b) $\{\partial^l n\}$, $l=0, \ldots, k$ are linearly independent;

(c) $(\partial^l n, \partial^m n) = \delta^{l0} \delta^{m0}$.

Alternatively, a solution can be given (locally in $G \subset \mathbb{C}$) by an analytic function $f: G \to \mathbb{C}^N$ such that

(a′) $\bar{\partial} f = 0$;

(b′) $\{\partial^l f\}$, $l=0, \ldots, k$ are linearly independent;

(c′) $(\partial^l f, \partial^m f) = \delta^{lk} \delta^{mk}$.

Note that (b′) is implied by (c′).

The connection of f and n is given by

(d) $(-1)^k n = \partial^k f - N_{i,k-1} \partial^i f$,

$$N := M^{-1} \partial M, \qquad M_{il} := \langle \partial^i f, \partial^l f \rangle; \qquad i, l = 0, \ldots, k-1$$

or

(d′) $f = (L^{-1})_{ik} \bar{\partial}^i n$,

$$L_{li} := \langle \partial^l n, \partial^i n \rangle, \qquad i, l = 1, \ldots, k;$$

$$(\mathrm{e}) = (\mathrm{e}'): \langle \partial^l f, \bar{\partial}^i n \rangle = (-1)^l \delta^{i+l,k}.$$

In short, we have proven:

3.8. Theorem. *There is a bijective mapping from $I_k^r(G)$ onto $A_k(G)$ (given explicitly by* (d) *and* (d′)*).*

4. Construction of Solutions

In this section we want to obtain new solutions of the $0(N)$-model from known ones. This will be particularly convenient if the "f-language" developed in the last sections is used.

The first lemma is simple:

4.1. Lemma. *Let $f \in A_k(G)$, and suppose $w: \check{G} \to G$ is analytic with $\partial w \neq 0$ in $\check{G}$. Then*

$$g := (\partial w)^{-k} f \circ w \tag{4.1}$$

is in $A_k(\check{G})$.

Proof. The function g is analytic in $\check{G}$ by definition. Thus it is enough to check condition (3.1) for $\check{G}$. But $\partial^l g$ has the representation

$$\partial^l g = \sum_{i=0}^{l} \binom{l}{i} \partial^{l-1}[(\partial w)^{-k}] \partial^i f =: \sum_{i=0}^{l} \mu_i^l \partial^i f \tag{4.2}$$

so that $\mu_l^l = (\partial w)^{l-k}$, as one can immediately prove by induction on l.

Now, (3.1) is implied by (3.20). But from (4.2), for $l < k$,

$$(\partial^l g, \partial^l g) = \sum_{i=0}^{l} \sum_{j=0}^{l} \mu_i^l \mu_j^l (\partial^i f, \partial^j f) = 0,$$

and

$$(\partial^k g, \partial^k g) = \mu_k^k \mu_k^k = 1. \quad \square$$

Next, we want to construct, from those of the $0(N)$-model, solutions of the $0(N+2)$-model ($N = 2k+1$). This is particularly useful since solutions of the $0(3)$-model are explicitly known [1], and the only solution of the $0(1)$-model is $f = \pm 1$.

4.2. Lemma. *Let $f \in A_k(G)$, and let $\mathbb{C}^{N+2} = \mathbb{C}^N \oplus \mathbb{C}^2$ with a basis e_1, e_{N+2} of $\mathbb{C}^2$ fulfilling*

$$(e_1, e_{N+2}) = 1, (e_1, e_1) = 0 = (e_{N+2}\, e_{N+2}), Ke_1 = e_{N+2}.$$

Define $\hat{f}: G \to \mathbb{C}^{N+2}$ by

$$\hat{f} := ae_1 + F - a^{-1} h e_{N+2}, \qquad 0 \neq a \in \mathbb{C}$$

with

$$F(z) := \int_0^z f(\zeta) d\zeta$$

and

$$h(z) := (F(z), F(z)).$$

Then $\hat{f}$ is in $A_{k+1}(G)$.

Proof. All $N+2$ components of $\hat{f}$ are analytic, and

$$(\hat{f}, \hat{f})_{N+2} = (F, F)_N - h = 0,$$

where the index at the scalar product refers to the number of components. Furthermore,

$$(\partial^l \hat{f}, \partial^l \hat{f})_{N+2} = (\partial^l F, \partial^l F)_N = (\partial^{l-1} f, \partial^{l-1} f)_N = \delta^{l-1,k}$$

since f is in A_k. But $\delta^{l-1,k} = \delta^{l,k+1}$. $\square$

By combining the processes of the last two lemmas, we can generate a lot of solutions of various 0(N)-models. The surprising fact is that we get *all* solutions of every 0(N)-model (of the class considered before) in this way:

4.3. Theorem. *Let* $\hat{f}\in A_{k+1}(G)$, *and let* $\mathbb{C}^{N+2}=\mathbb{C}^N\oplus\mathbb{C}^2$ *with the same basis as in the last lemma. Then there is a function* $F:\check{G}\to\mathbb{C}^N$, $\check{G}\subset G$, *a number* $a\in\mathbb{C}$, $a\neq 0$, *a function* $w:\check{G}\to G$ *with* $\partial w\neq 0$ *in* $\check{G}$ *such that*

$$(\partial w)^{-k}\hat{f}\circ w=ae_1+F-a^{-1}he_{N+2}$$

with

$$h(z):=(F(z),F(z)).$$

Furthermore, ∂F *is in* $A_k(\check{G})$.

Proof

(i) Note that, with f, any translated f is in $A_k(G)$, too; thus we may suppose $0\in G$. Next, one can find by induction an orthonormal basis $\{\psi_i\}_{i=1,\dots,N+2}$ such that

$$\partial^l\hat{f}(0)=\sum_{i=1}^{l+1}a_i^l\psi_i. \tag{4.3}$$

Since the $\partial^l\hat{f}$ are independent, $\hat{f}(0)\neq 0$ so that $a:=a_1^0\neq 0$. Consider

$$\varphi(z):=a^{-2}\langle\hat{f}(0),\hat{f}(z)\rangle.$$

Then $\varphi(0)=1$; hence, there exists a simply connected domain $G_1\subset G$ in which $\varphi(z)\neq 0$ so that

$$v:=\varphi^{-1/k+1} \tag{4.4}$$

is well defined in G_1, and $v(0)=1$. Furthermore, the function

$$w(z):=\int_0^z v(\zeta)d\zeta$$

has $\partial w(0)=1$ and can be inverted in some region $\hat{G}\subset G_1\subset G$. Choose $\hat{G}$ so small that $\check{G}:=w(\hat{G})\subset G$. The inverse function has $\partial z\neq 0$ in $\check{G}$; define

$$g(w):=(\partial z)^{-(k+1)}\hat{f}(z(w)).$$

(ii) We now show that g can be written in the form $g=ae_1+F-a^{-1}he_{N+2}$ as asserted. Note that, since $w(0)=0$ which implies $z(0)=0$, and since $\partial z(0)=1$,

$$g(0)=\hat{f}(0)=a\psi_1, \tag{4.5}$$

by (4.3). This results in

$$\begin{aligned}\langle g(0),g(w)\rangle&=(\partial z)^{-(k+1)}\langle\hat{f}(0),\hat{f}(z(w))\rangle=a^2(\partial z)^{-(k+1)}\varphi(z(w))\\&=a^2\varphi(z)(\partial w)^{k+1}=a^2\end{aligned} \tag{4.6}$$

by (4.4). On the other hand, $g(w)$ has a representation

$$g(w)=\sum_i g_i(w)\psi_i \tag{4.7}$$

and inserting (4.7) and (4.5) in (4.6) gives

$$a^2 = \langle g(0), g(w)\rangle = \sum_i g_i(w) a \langle \psi_0, \psi_i\rangle = g_0(w)\cdot a\,,$$

i.e. $g_0(w) = a$ is constant. That is, g can be written

$$g = a\psi_1 + F - a^{-1}\hat{h}\psi_{N+2}\,, \tag{4.8}$$

where $F := \sum_{i=2}^{N+1} g_i\psi_i$, and $\hat{h} := -a g_{N+1}$.

To show that ψ_1, $\psi_{N+2} = \psi_{2k+3}$ have the properties of the basis stated in the last lemma, note that, since $\hat{f} \in A_{k+1}(G)$,

$$(\partial^l \hat{f}(0), \partial^i \hat{f}(0)) = \delta^{l,k+1}\delta^{i,k+1}$$

which implies, by induction on m, for $0 \leqq m \leqq i \leqq k+1$,

$$(\partial^{l+m} \hat{f}(0), \partial^{i-m} \hat{f}(0)) = (-1)^m \delta^{l,k+1}\delta^{i,k+1}$$

In the first equation, choose $l = i = 0$ to see that $(\psi_1, \psi_1) = 0$. Next, choose $l = 0$ to see by induction on i that $(\psi_1, \psi_i) = 0 = \langle K\psi_1, \psi_i\rangle$ for $i \leqq k+1$. In the second equation, choose $i = m$ to see by induction on m that $\langle K\psi_1, \psi_i\rangle = 0$ for $i \leqq 2k+2$, and $\langle K\psi_1, \psi_{2k+3}\rangle = (-1)^{k+1}$. Thus, $K\psi_1$ is a multiple of ψ_{2k+3}, $K\psi_1 = (-1)^{k+1} \cdot \psi_{2k+3}$. Now choose $e_1 := \psi_1$, $e_{2k+3} := (-1)^{k+1}\psi_{2k+3}$. Then $(e_1, e_1) = 0$, $Ke_1 = e_{2k+3}$, $(e_{2k+3}, e_{2k+3}) = (Ke_1, Ke_1) = (e_1, e_1) = 0$, and $(e_1, e_{2k+3}) = \langle e_{2k+3}, e_{2k+3}\rangle = 1$.

(iii) By (4.8), g can be written

$$g = ae_1 + F - a^{-1}he_{N+2}$$

with $h := (-1)^{k+1}\hat{h}$, and it remains to show that F, h and ∂F have the stated properties. By Lemma 4.1, $g \in A_{k+1}(\check{G})$, hence $(g, g) = 0$, i.e. $h = (F, F)$. Furthermore,

$$\delta^{l,k+1} = (\partial^l g, \partial^l g)_{N+2} = (\partial^{l-1} f, \partial^{l-1} f)_N\,,$$

which proves (3.20) for $f := \partial F$ so that $f \in A_k(\check{G})$. □

We will illustrate the procedure of the last theorem in the simplest case, the transition from $k=0$ to $k=1$, i.e. from the 0(1) to the 0(3) model:

For $k=0$, $f=n$ [see the argument after (1.3)]; hence, $f=1$ by $(n,n)=1$. Next, consider an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{C}^3 = \mathbb{C}^{2k+1}$ with $(e_1, e_1) = 0$, $Ke_1 = e_3$. Then

$$\hat{f} = e_1 + ze_2 - z^2 e_3$$

(where we have taken $a = 1$ in the last theorem). We remark that one can get all 0(3) instantons from $\hat{f}$ by applying to it the process described in Lemma 4.1.

5. On the Behaviour at Isolated Singular Points

The structure of regular solutions $n \in I_k^r(G)$, i.e. solutions of (0.5) for which the rank of $(\partial^l n)_{l=1}^k$ is constant and equal to k, has been fully described in Sects. 1 and 3 in terms of an analytic vector function f fulfilling

$$(\partial^i f, \partial^j f) = \delta^{ik}\delta^{jk} \quad i,j=0,\ldots,k. \tag{5.1}$$

In case the solution n is not regular, i.e. $n \in I_k(G)$ so that $\operatorname{rank}(\partial^l n)_{l=1}^k \leqq k$, it was shown in Sect. 2 that one could associate with n a meromorphic function f fulfilling (5.1), i.e. an element of the set

$$M_k(G) := \{f : G \to \mathbb{C}^N / f \text{ meromorphic}, (\partial^i f, \partial^j f) = \delta^{ik}\delta^{jk}\} \tag{5.2}$$

the poles of f being precisely the points for which the rank of $(\partial^l n)$ is not maximal.

Now we want to investigate the converse question: Suppose an arbitrary $f \in M_k(G)$ is given so that $f \in A_k(G \backslash S)$ where S is a set of isolated points. By (3.2), one finds a solution $n \in I_k^r(G \backslash S)$ of (1.1). The question is if n is continuous in all of G, and if it is whether it fulfils (1.1) in G [so that $n \in I_k(G)$]. It is the goal of this section to show that these questions have a positive answer. The strategy of attacking this problem will be a reduction procedure in the number of dimensions, similar to the technique described in Sect. 4.

As the set S consists of isolated points, it is no loss in generality to consider a single point which, by the translation invariance of (0.5) we may take to be $z=0$. We start with some preparations:

5.1. Lemma. *Let $\psi(z)$ be an analytic function holomorphic in a domain G containing $z=0$, and let $\alpha \geqq 0$. Define*

$$\chi(z) := \int_0^z \zeta^\alpha \psi(\zeta)\, d\zeta \tag{5.3}$$

then there exists a unique function $\gamma(z)$ holomorphic in G with

$$\chi(z) = (1+\alpha)^{-1} z^{1+\alpha} \gamma(z) \tag{5.4}$$

having the property

$$\gamma(0) = \psi(0). \tag{5.5}$$

Proof. Since $0 \in G$ there exists a power series

$$\psi(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$$

converging in some neighbourhood of zero.

From this we get

$$\chi(z) = \sum_{\nu=0}^{\infty} a_\nu (1+\nu+\alpha)^{-1} z^{1+\nu+\alpha} = (1+\alpha)^{-1} z^{1+\alpha} \sum_{\nu=0}^{\infty} (1+\alpha)(1+\alpha+\nu)^{-1} a_\nu z^\nu$$

so we obtain

$$\gamma(z) = \sum_{\nu=0}^{\infty} (1+\alpha)(1+\alpha+\nu)^{-1} a_\nu z^\nu. \tag{5.6}$$

Since $|(1+\alpha)(1+\alpha+\nu)^{-1}| \leqq 1$, the power series of $\gamma(z)$ has the same radius of convergence as that of χ. Since $(1+\alpha)z^{-(1+\alpha)}\int\limits_0^z \zeta^\alpha \psi(\zeta)d\zeta$ is uniquevalued in a neighbourhood of zero, the existence of γ in all of G follows. The second statement (5.5) follows directly from the power series of $\gamma(z)$. □

5.2. Lemma. *Let $f(z)\in A_k(G\backslash\{0\})$ and define $g(z)=z^n\cdot f(z)$. Then $g(z)$ fulfils the equations*

$$(\partial^i g(z), \partial^i g(z)) = \delta^{ik} z^{2n} \qquad i=0, \ldots, k. \tag{5.7}$$

On the other hand assume $g(z)$ is holomorphic in $G\backslash\{0\}$ and maps it into $\mathbb{C}^{2k+1}$. If in addition $g(z)$ fulfils the relations (5.7) *then*

$$f(z) = z^{-n} g(z)$$

belongs to $A_k(G\backslash\{0\})$.

Proof. The relations

$$(\partial^i f, \partial^i f) = \delta^{ik}, \qquad i=0, \ldots, k$$

are equivalent to (see Lemma 1.3)

$$(\partial^i f, \partial^j f) = \delta^{ik}\delta^{jk}, \qquad i,j=0, \ldots, k.$$

Inserting $f=z^{-n}g$, we see by induction with respect to i and j that these are equivalent to

$$(\partial^i g, \partial^j g) = \delta^{ik}\delta^{jk} z^{2n}.$$

But by the same argument as in Lemma 1.3 we see that the last equations are equivalent to (5.7). □

Now we are prepared to enter the reduction procedure.

5.3. Lemma. *Let $f\in A_k(G\backslash\{0\})$ and assume f is meromorphic in G. Choose n such that $g(z)=z^n f$ is holomorphic in G and $g(0)\neq 0$. Let $G_1\subset\mathbb{C}$ contain zero and consider a holomorphic map $w: G_1\to G$ such that $w(0)=0$, $\partial w(0)=1$, and $\partial w \neq 0$ in G_1. Define*

$$\tilde{f}(z) := (\partial w)^{-k} f\circ w = \left(\frac{dz}{dw}\right)^k \cdot f(w(z))$$

$$\tilde{g} := z^n \tilde{f}.$$

Then $f\in A_k(G_1\backslash\{0\})$ by Lemma 4.1, $\tilde{g}$ is holomorphic in G_1, and the relation

$$\tilde{g}(0) = g(0) \tag{5.8}$$

holds. In addition we can choose $w(z)$ in such a way that

$$\langle \tilde{g}(0), \tilde{g}(z)\rangle = \langle \tilde{g}(0), \tilde{g}(0)\rangle \tag{5.9}$$

holds in a suitable neighbourhood of zero.

Proof. From $w^n f(w)$ holomorphic in G_1 follows by assumption that $z^n\cdot\tilde{f}(z) = \frac{z^n}{w^n} \cdot\left(\frac{dz}{dw}\right)^k \cdot w^n f(w(z))$ is holomorphic in a neighbourhood of zero. Since $\tilde{f}$ is holomorphic in $G_1\setminus\{0\}$, we get by the last two statements taken together that $\tilde{g}=z^n\tilde{f}$ is holomorphic in G_1. From $\frac{z}{w}(0)=1$ and $\frac{dz}{dw}(0)=1$ follows $\tilde{g}(0)=g(0)$. This proves the first part.

Because of (5.8) we get

$$\langle\tilde{g}(0),\tilde{g}(z)\rangle=\left(\frac{z}{w}\right)^n\cdot\left(\frac{dz}{dw}\right)^k\cdot\langle g(0),g(w)\rangle\,. \tag{5.10}$$

From $g(0)\neq 0$ follows $\langle g(0),g(w)\rangle\neq 0$ in a suitable domain $\Gamma\subset G$. In this domain we can define $\psi(w)=\left\{\frac{\langle g(0),g(w)\rangle}{\langle g(0),g(0)\rangle}\right\}^{-1/k}$. Let now $\chi(w):=\int\limits_0^w\zeta^{n/k}\psi(\zeta)d\zeta$; then we have by Lemma 5.1

$$\chi(w)=\frac{w^{1+\frac{n}{k}}}{1+\frac{n}{k}}\gamma(w)$$

with γ holomorphic in Γ and

$$\gamma(0)=\psi(0)=\left\{\frac{\langle g(0),g(0)\rangle}{\langle g(0),g(0)\rangle}\right\}^{-1/k}=1\,.$$

Let $\Gamma_1\subset\Gamma$ be a subdomain containing zero with $\gamma(w)\neq 0$ in Γ_1, define $\delta(w):=\gamma(w)^{\frac{1}{1+n/k}}$ with $\delta(0)=1$ and put $z(w):=w\cdot\delta(w)$. We find $z(0)=0$ and $\left(\frac{dz}{dw}\right)(0)=1$. Choose now a simply connected subdomain $\Gamma_2\subset\Gamma_1$ containing zero such that $\frac{dz}{dw}\neq 0$. In this region we can invert the function and define $w(z)$. We now claim that this function has the desired property. By construction we obtain

$$\left(1+\frac{n}{k}\right)^{-1}z^{1+\frac{n}{k}}=\left(1+\frac{n}{k}\right)^{-1}w^{1+\frac{n}{k}}\delta^{1+\frac{n}{k}}=\left(1+\frac{n}{k}\right)^{-1}w^{1+\frac{n}{k}}\gamma$$

$$=\chi(w)=\int\limits_0^w\zeta^{n/k}\psi(\zeta)d\zeta\,.$$

Differentiating with respect to w we obtain:

$$z^{\frac{n}{k}}\cdot\frac{dz}{dw}=w^{\frac{n}{k}}\psi(w)$$

or

$$z^n\left(\frac{dz}{dw}\right)^k=w^n\psi^k(w)\,.$$

This shows by (5.10):

$$1=\left(\frac{z}{w}\right)^n\cdot\left(\frac{dz}{dw}\right)^k\psi^{-k}=\left(\frac{z}{w}\right)^n\left(\frac{dz}{dw}\right)^k\cdot\frac{\langle g(0),g(w)\rangle}{\langle g(0),g(0)\rangle}$$
$$=\frac{\langle\tilde{g}(0),\tilde{g}(z)\rangle}{\langle\tilde{g}(0),\tilde{g}(0)\rangle},$$

which shows the lemma. □

Knowing this result, we can set up a reduction- and induction-procedure in the same way as in the last section. The only difference is that we have to work with the functions $g(z)$ now which are again free of singularities.

5.4. Reduction Lemma. *Assume $f(z)\in A_k(G\backslash\{0\})$ and $f(z)=z^{-n}\cdot g(z)$ with $g(z)$ holomorphic in G and $g(0)\neq 0$. Assume in addition the relation* (5.9)

$$\langle g(0),g(z)\rangle=|a|^2>0$$

Then $\gamma'(z):=\partial g-|a|^{-2}\overline{g(0)}\langle g(0),\partial g(z)\rangle$ defines a mapping $\gamma':G\to\mathbb{C}^{2k-1}$. Define $g_1(z)$ and m by the relation

$$\gamma'(z)=z^m g_1(z)$$

with g_1 holomorphic in G and $g_1(0)\neq 0$. Then we obtain

$$(\partial^i g_1,\partial^i g_1)=\delta^{i,k-1}z^{2(n-m)},\qquad i=1,\dots,k-1$$

and hence

$$f_1:=z^{-(n-m)}g_1(z)\in A_{k-1}(G\backslash\{0\}$$

by Lemma 5.2.

Proof. By assumption follows $\langle g(0),\partial g\rangle=0$ and $\langle\gamma',g(0)\rangle=\langle\gamma',\overline{g(0)}\rangle=0$. Hence $\gamma'(z)$ is living in a $2k-1$ dimensional space. The mapping γ' might have a zero of order m. From the relation $(\partial^i g,\partial^i g)=z^{2n}\delta^{ik}$, $i=0,\dots,k$ it follows easily that $m\leq n$. Because of $\langle\gamma',g(0)\rangle=\langle\gamma',\overline{g(0)}\rangle=0$ we have $(\partial^i\gamma',\partial^i\gamma')=z^{2n}\delta^{i,k-1}$, $i=1,\dots,k-1$. Inserting now $\gamma'=z^m g_1$, the desired relation for g_1 follows by induction. □

5.5. Reconstruction Lemma. *Assume f_1 defines a solution of the* $0(2k-1)$ *σ-model in $G\backslash\{0\}$ and assume*

$$f_1(z)=z^{-n_{k-1}}g_1(z)$$

with $g_1(z)$ holomorphic in G and $g_1(0)\neq 0$. Let m_k be any positive integer and $a\neq 0$. Put

$$\gamma(z):=\int_0^z\zeta^{m_k}g_1(\zeta)d\zeta$$

and define

$$g(z):=\{a\cdot 2^{-1/2}(e_{2k}+ie_{2k+1}),\gamma(z),-a^{-1}\cdot 2^{-1/2}(e_{2k}-ie_{2k+1})(\gamma(z),\gamma(z))\}.$$

Then we find

$$(\partial^i g,\partial^i g)=z^{2(n_{k-1}+m_k)}\delta^{ik},\qquad i=1,\dots,k.$$

Hence

$$f(z):=z^{-(m_k+n_{k-1})}g$$

is an element of $A_k(G\backslash\{0\})$.

Proof. By construction we find $(g,g)=0$. The higher relations follow from the properties of g_1 (see also Lemma 5.2). □

5.6. Corollary. *Let* $f\in A_k(G\backslash\{0\})$ *and assume* f *is meromorphic in* G *with*

$$f(z)=z^{-n_k}g(z),$$

$g(z)$ *holomorphic in* G *and* $g(z)\neq 0$. *Then* $f(z)$ *can be reconstructed locally at* $z=0$ *from the case* $k=0$. *The order of the pole* n_k *can be written*

$$n_k=\sum_{i=1}^{k} m_i,$$

where m_i *is the order of the singularity we gain by the* i-th *reconstruction step.*

Proof. This follows immediately from Lemma 5.5 and the fact that the transformations in Lemma 5.3 do not change the order of the pole. □

It is our next goal to show that the solution $n(z,\bar{z})$ of (1.1) associated with $f(z)$ stays a solution at points where $f(z)$ has a pole, the only thing happening being a change in the number of linearly independent vectors. The key for the solution of this problem is the following

5.7. Lemma. *Let* $n(z,\bar{z})$ *be a solution of* (1.1) *in* $G\backslash\{0\}$ *and assume the action density* $L(n)$ *in* (0.1) *stays continuous at* $z=0$. *Then* $n(z,\bar{z})$ *is a solution of* (1.1) *in all of* G.

Proof. If L is continuous, then it is also bounded, i.e.

$$L(z)\leqq M\,;\ z\in G_1\ni\{0\}\,.$$

Assume G_1 is convex; then we get

$$\|n(z_1)-n(z_2)\|=\left\|\int_{z_1}^{z_2}(\operatorname{grad} n, d\mathbf{s})\right\|\leqq M^{1/2}\|z_1-z_2\|$$

which shows the continuity of n. Since we have $\Delta n=-L\cdot n$ it follows that Δn $=$continuous part + "δ". Since the equation $\Delta n=$"δ" leads to some n which is not continuous at $z=0$ it follows that also Δn is continuous at $z=0$. Hence we get $\Delta n=-Ln$ in all of G. □

Now it remains to show the continuity of $L(n)$. There we have to distinguish two cases, namely the induction step from $k=0$ to $k=1$ and the other cases. For $k=1$ we have by the remarks at the end of Sect. 3 the formula $f=L^{-1}\bar{\partial}n$ [cf. (2.5)] and hence $L=\langle f,f\rangle^{-1}$. Thus, if f has a pole z^{-n_1}, then L has a zero $|z|^{2n_1}$. For $k>1$ we get L by Lemma 3.6 and have

$$L=M^{-1}_{k-1,k-1}=\frac{\det(M_{ij})_0^{k-2}}{\det(M_{ij})_0^{k-1}}\,. \tag{5.11}$$

We are not able to compute these determinants exactly, but this is not necessary since we are only interested in the leading singularity of these determinants. The outcome of this investigation is the following:

5.8. Lemma. *Let $f \in A_k(G\backslash\{0\})$ and assume f has a pole of order $n_k = \sum_{i=1}^{k} m_i$. Then L behaves at zero as $|z|^{2m_1}$. Hence L is continuous in G and f defines a solution of Eq.* (1.1) *in all of G.*

Proof. The case $k=1$ has been treated separately and therefore we can proceed by induction with respect to k. This means we have to follow the steps described in Lemmas 5.3 and 5.5. We start with the analytic transformations.

Let $w(z)$ be an analytic map with $w(0)=0$ and $w'(0)=1$ and let $\tilde{f}(z) := (\partial w)^{-k} f(w(z))$. Then we get with $\tilde{M}_{ij} = \langle \partial^i \tilde{f}, \partial^j \tilde{f} \rangle$ and $w' = \partial w$

$$\det \begin{pmatrix} \langle \tilde{f}, \tilde{f} \rangle & \dots & \langle \tilde{f}, \partial^l \tilde{f} \rangle \\ \vdots & & \vdots \\ \langle \partial^l \tilde{f}, \tilde{f} \rangle & \dots & \langle \partial^l \tilde{f}, \partial^l \tilde{f} \rangle \end{pmatrix}$$

$$= \det \begin{pmatrix} \overline{(w')}^{-k} (w')^{-k} \langle f, f \rangle & \dots & \overline{(w')}^{-k} (w')^{-k+l} \langle f, \partial_w^l f \rangle \\ & & \vdots \\ \overline{(w')}^{-k+l} (w')^{-k} \langle \partial_w^l f, f \rangle & \dots & \end{pmatrix}$$

$$= \det \begin{pmatrix} \overline{(w')}^{-k} & & 0 \\ & \ddots & \\ 0 & & \overline{(w')}^{-k+l} \end{pmatrix} \cdot \begin{pmatrix} \langle f, f \rangle & \dots & \langle f, \partial_w^l f \rangle \\ \vdots & & \vdots \\ \langle \partial_w^l f, f \rangle & \dots & \end{pmatrix} \begin{pmatrix} (w')^{-k} & & 0 \\ & \ddots & \\ 0 & & (w')^{-k+l} \end{pmatrix}.$$

In this calculation, linearly dependent vectors have been subtracted. This means

$$\det(\tilde{M}_{ik})_0^l = \det(M_{ik})_0^l \cdot |(w')^{-k(l+1)+l(l+1)2^{-1}}|^2 .$$

Since $w'(0)=1$ it follows that $\det \tilde{M}$ and $\det M$ have the same behaviour at zero. So it remains to evaluate the determinant of $(M_{ij})_0^l$ where f has the special form described in Lemma 5.5. Inserting $f = z^{-(m_k + n_{k-1})} g$ we obtain with $n_{k-1} + m_k = n_k$

$$\det(\langle \partial^i f, \partial^j f \rangle)_0^l = |z|^{-2(l+1)n_k} \det(\langle \partial^i g, \partial^j g \rangle)_0^l . \tag{5.12}$$

Remember g was of the special form

$$g = (a 2^{-1/2}(e_{2k} + i e_{2k+1}), h)$$

with $a \neq 0$ so that we obtain

$$\langle \partial^i g, \partial^j g \rangle = \delta^{i0} \delta^{j0} |a|^2 + \langle \partial^i h, \partial^j h \rangle .$$

Using the expansion of the determinant and estimating $|\langle \partial^i h, \partial^j h \rangle|$ by $\langle \partial^i h, \partial^i h \rangle^{1/2} \cdot \langle \partial^j h, \partial^j h \rangle^{1/2}$ we obtain

$$\det(\langle \partial^i g, \partial^j g \rangle)_0^l \leqq |a|^2 \prod_{i=1}^{l} \langle \partial^i h, \partial^i h \rangle + \{(l+1)! + 1\} \prod_{i=0}^{l} \langle \partial^i h, \partial^i h \rangle$$

and

$$\det(\langle \partial^i g, \partial^j g \rangle)_0^l \geqq |a|^2 \prod_{i=1}^{l} \langle \partial^i h, \partial^i h \rangle - \{(l+1)! + 1\} \prod_{i=0}^{l} \langle \partial^i h, \partial^i h \rangle .$$

Remember that $\langle h, h \rangle$ has a zero of order $|z|^{2(m_k+1)}$, so that we obtain:

$$\det(\langle \partial^i g, \partial^j g \rangle)_0^l = \prod_{i=1}^{l} \langle \partial^i h, \partial^i h \rangle \{|a|^2 + 0(|z|^{2(m_k+1)})\} .$$

Inserting now the value of h we obtain

$$\langle \partial^i h, \partial^i h \rangle = \langle \partial^i \gamma, \partial^i \gamma \rangle + \left| \frac{\partial^i(\gamma, \gamma)}{a} \right|^2 .$$

From $(\partial^i g_1, \partial^j g_1) = 0$ we calculate

$$\begin{aligned}\langle \partial^i h, \partial^i h \rangle &= \langle \partial^{i-1}(z^{m_k} g_1), \partial^{i-1}(z^{m_k} g_1) \rangle + 4|a|^{-2} |(\gamma, \partial^{i-1}(z^{m_k} g_1)|^2 \\ &\leqq \langle \partial^{i-1}(z^{m_k} g_1), \partial^{i-1}(z^{m_k} g_1) \rangle (1 + 4|a|^{-2} \langle \gamma, \gamma \rangle) \\ &= \langle \partial^{i-1}(z^{m_k} g_1), \partial^{i-1}(z^{m_k} g_1) \rangle (1 + 0(|z|^{2(m_k+1)}) .\end{aligned}$$

Using the argument backwards, we can replace the product of the $\langle \partial^{i-1}(z^{m_k} g_1),$ $\partial^{i-1}(z^{m_k} g_1) \rangle$ again by the determinants. So we finally obtain

$$\begin{aligned}\det(\langle \partial^i g, \partial^j g \rangle)_0^l &= \det(\langle \partial^i(z^{m_k} g_1), \partial^j(z^{m_k} g_1) \rangle)_0^{l-1} \\ &\quad \cdot \{|a|^2 + 0(|z|^{2(m_k+1)}) + 0(|z|^{2(m_{k-1}+1)})\} \\ &= |z|^{2lm_k} \det(\langle \partial^i g_1, \partial^j g_1 \rangle)_0^{l-1} \\ &\quad \cdot \{|a|^2 + 0(|z|^{2(m_k+1)}) + 0(|z|^{2(m_{k-1}+1)})\} .\end{aligned}$$

This result inserted into (5.11) gives together with (5.12):

$$\begin{aligned}L(f) &= |z|^{2n_k} \frac{\det(\langle \partial^i g, \partial^j g \rangle)_0^{k-2}}{\det(\langle \partial^i g, \partial^j g \rangle)_0^{k-1}} = \frac{|z|^{2m_k + n_{k-1}} \det(\langle \partial^i g_1, \partial^j g_1 \rangle)_0^{k-3}}{|z|^{2m_k} \det(\langle \partial^i g_1, \partial^j g_1 \rangle)_0^{k-2}} \\ &\quad \cdot (1 + 0(|z|^{2(m_k+1)}) + 0(|z|^{2(m_{k-1}+1)})) \\ &= L(f_1)(1 + 0(|z|^{2(m_k+1)} + 0(|z|^{2(m_{k-1}+1)}) .\end{aligned}$$

Since we had $n_k = \sum m_l$ we obtain the desired result. □

Lemma 5.8 and Theorem 2.4 now combine to give

5.9. Theorem. *The bijective mapping from $I_k^r(G)$ onto $A_k(G)$ constructed in Theorem 3.8 extends to a bijective mapping from $I_k(G)$ onto $M_k(G)$.*

We have shown in Sect. 4 that every $f \in A_k(G)$ can be obtained by reconstructing it from $f_0 \in A_0(G)$. Note, however, that the induction procedure described in Lemma 4.2, since it involves integration, will only lead again to a meromorphic function if no poles of first order appear (which can always be achieved by starting with poles of high enough order).

6. Global Solutions

In this section, we study continuous finite action solutions of the $0(N)$ model in all of $\mathbb{R}^2 \cong \mathbb{C}$, i.e. continuous solutions n of (1.1) with $(n, n) = 1$ such that $L(n) \in L^1(\mathbb{R}^2)$. We denote this class by $I(\mathbb{R}^2)$.

If the orthogonality conditions (0.7) are imposed on these global solutions, we may obtain a characterization of them rather easily by choosing $G = \mathbb{C}$ in Theorem 5.9 and remembering Lemma 3.5: They are precisely given by those meromorphic mappings $f \in M_k(\mathbb{C})$ for which $M_{k-1,k-1}^{-1}$ is in $L^1(\mathbb{C})$.

This characterization is, however, deficient in two ways: The integrability condition on f is rather implicit, and the orthogonality conditions (0.7) on n have to be imposed. Thus, neither the class of solutions n to be characterized nor the class of characterizing mappings f are described in a very illuminating way.

Fortunately, the situation is completely transparent if we restrict ourselves to the class of solutions n of (1.1) which are solutions on the whole Riemann sphere $\mathbb{S}^2 \supset \mathbb{R}^2$, i.e. for which also the conformal transform

$$\hat{n}(z,\bar{z}) := n(z^{-1}, \bar{z}^{-1}) \tag{6.1}$$

is a continuous finite action solution of (1.1) in all of $\mathbb{R}^2$. As the action S [cf. (0.3)] is a conformal invariant, we merely assume by this that $\hat{n}$ is a solution of (1.1) for $z=0$ and is continuous there. We denote the class of these solutions by $I(\mathbb{S}^2)$. Note that there is no index k: We do *not* assume any orthogonality condition of the type (0.7). In fact, we are now able to derive (0.7) as a consequence:

6.1. Theorem. *Let n be a solution of the* $\mathrm{O}(N)$ *model on* $\mathbb{S}^2 \supset \mathbb{R}^2$. *Then, for all non negative integers i, j with $i+j \geqq 1$,*

$$(\partial^i n, \partial^j n) = 0\,. \tag{6.2}$$

Proof. We prove (6.2) by induction on $i+j$. For $i+j=1$, we may suppose $i=0, j=1$ by the symmetry of (6.2) in i, j. But $(n, \partial n)=0$ by differentiating $(n,n)=1$. Now assume (6.2) for all i, j with $1 \leqq i+j \leqq m$. As a first step in proving (6.2) for $i+j=m+1$, we show that $(\partial^i n, \partial^j n)$ is analytic.
We distinguish the cases (i) $i,j \geqq 1$ and (iii) $i=0, j \geqq 1$:

$$\text{(i)}\quad \bar{\partial}(\partial^i n, \partial^j n) = -(\partial^{i-1}(Ln), \partial^j n) - (\partial^i n, \partial^{j-1}(Ln)) = 0$$

by the induction assumption.

$$\text{(ii)}\quad \bar{\partial}(n, \partial^j n) = (\bar{\partial} n, \partial^j n) - (n, \partial^{j-1}(Ln))\,;$$

the last term is equal to $-\partial^{j-1}L$ by the induction assumption. We will prove

$$(\bar{\partial} n, \partial^j n) = \partial^{j-1} L \tag{6.3}$$

by a separate induction argument on j. Equation (6.3) is true for $j=1$ by the definition of L. For $j>1$,

$$(\bar{\partial} n, \partial^j n) = \partial(\bar{\partial} n, \partial^{j-1} n) - L(n, \partial^{j-1} n)\,,$$

where the last term vanishes by the induction assumption for (6.2) and the first is equal to $\partial\partial^{j-1}L$ by the induction assumption for (6.3).

Hence, in all cases, $(\partial^i n, \partial^j n)$ is analytic. Next, we want to prove

$$\|\partial^i n\| \in L^2(\mathbb{R}^2)\,. \tag{6.4}$$

This will finish the proof, since by Cauchy-Schwarz, we then have the majorization

$$|(\partial^i n, \partial^j n)| \leqq \|\partial^i n\|\ \|\partial^j n\|\,,$$

where the right hand side is in $L^1(\mathbb{R}^2)$, by (6.4). But an analytic function that is in $L^1(\mathbb{R}^2)$ is identically zero, by a variant of Liouville's theorem. (For a short proof, see e.g. (3.13) of [1].)

To prove (6.4), note first that, by induction on i,

$$(\partial^i n)(z,\bar{z}) = \sum_{m=1}^{i} c_{mi} z^{-(m+1)} (\partial^m \hat{n})(z^{-1}, \bar{z}^{-1})$$

with suitable constants c_{mi}. Hence, with $w=z^{-1}$,

$$\int_{|z|>1} \langle \partial^i n, \partial^i n \rangle dz d\bar{z} \leqq i \sum_{m=1}^{i} |c_{mi}|^2 \cdot \int_{|w|<1} (w\bar{w})^{m+i-2} \langle \partial^m \hat{n}, \partial^m \hat{n} \rangle dw d\bar{w}, \tag{6.5}$$

where we used the Cauchy-Schwarz inequality

$$\left| \sum_{m=1}^{i} a_m \right|^2 \leqq i \sum_{m=1}^{i} |a_m|^2 .$$

But the right hand side of (6.5) is finite: Since $\hat{n}$ is a continuous solution of (1.1) with $L(\hat{n})$ integrable for $|w|<1$ [put $i=1$ in (6.5)!], n is analytic there, by the results of [2]. Consequently, the integrals exist as m, $i \geqq 1$. This proves (6.4), since the integrals over $|z|<1$ are finite by the analyticity of n. □

At this point, one might speculate about the possibility that the orthogonality conditions (6.2) are automatically satisfied for any n which is merely in $I(\mathbb{R}^2)$. This is the case for the 0(3) model, and we believe it to be true in general but have been unable to prove it. However, not much is missing: We can show (by using the Calderon-Zygmund and Sobolev inequalities in tandem) that (6.2) is satisfied for any solution n of (1.1) on $\mathbb{R}^2$ for which $L(n) \in L^{1+\varepsilon}(\mathbb{R}^2)$ for some $\varepsilon>0$.

Our next goal is to give a simple characterization of solutions $n \in I(\mathbb{S}^2)$. Note that, for any such solution, there is an integer k such that $n \in I_k(\mathbb{R}^2)$: The rank of the matrix $(\partial^m n)_{m\in\mathbb{N}}$ is finite, and (1.2) is implied by (6.2).

6.2. Theorem. *The following statements are equivalent:*

(i) $n \in I(\mathbb{S}^2)$;

(ii) $f \in M_k(\mathbb{C})$ *is a rational function of* $z=x_1+ix_2 \in \mathbb{C}$;

(iii) $n \in I(\mathbb{R}^2)$ *is a rational function of* $(x_1, x_2) \in \mathbb{R}^2$.

Proof. (i)⇒(ii): By the definition of $I(\mathbb{S}^2)$, $n \in I(\mathbb{S}^2)$ implies $\hat{n} \in I(\mathbb{S}^2)$. By the arguments before the theorem, there are integers k, j such that $n \in I_k(\mathbb{R}^2)$, $\hat{n} \in I_j(\mathbb{R}^2)$. Since by (6.1), $\partial^i n$ and $\partial^i \hat{n}$ are linear combinations of each other, $j=k$ so that the functions $f, \hat{f}$ corresponding to $n, \hat{n}$ by Theorem 5.9 fulfil

$$\langle f, \partial^i n \rangle = \delta^{ik} = \langle \hat{f}, \partial^i \hat{n} \rangle . \tag{6.6}$$

Now simply note that

$$\hat{f}(z) := (-z^{-2})^k f(z^{-1}) \tag{6.7}$$

satisfies (6.6) by which $f, \hat{f}$ are defined uniquely. Since $f, \hat{f}$ are both meromorphic, (6.7) implies that f is rational.

(ii)⇒(iii): By (1.6) n is a rational combination of $\{\partial^m f\}$ for $m=0, \ldots, k$. But f and hence all derivatives are rational functions of x_1+ix_2.

(iii)⇒(i): Since $n \in I(\mathbb{R}^2)$, $\hat{n}$ fulfils (1.1) in all of $\mathbb{R}^2$ except possibly for $z=0$. Since n is rational and bounded, $((n,n)=1)$, so is $\hat{n}$. Thus n defines a distribution for which the combination $T=\partial\bar{\partial}\hat{n}+L(\hat{n})\hat{n}$ has support only in $z=0$ and is hence a combination of derivatives of the δ measure at $z=0$. But T is actually arbitrarily often differentiable at $z=0$ and hence zero [so that $\hat{n}$ fulfils (1.1) in all of $\mathbb{R}^2$]: Since $\hat{n}_m$ is rational, $\hat{n}_m = P_m(Q_m)^{-1}$; since $\hat{n}_m$ is bounded, $Q_m \neq 0$ so that $\hat{n}_m$ and hence T is infinitely often differentiable. □

We remark that in the 0(3) model, the conclusion that f is rational follows already from the assumption $n \in I(\mathbb{R}^2)$. Again we believe this to be true in general but have been unable to prove it.

As a last point we want to show that the previously known solutions of $0(2k+1)$ models (see Sect. 4 of [1]) fall into the subclass considered here. This will follow easily from condition (iii) of the last theorem.

We briefly recapitulate how the harmonic polynomial solutions of the $0(2k+1)$ model described in [1] were constructed:

By stereographic projection of $\mathbb{R}^2$

$$x_\mu = r_\mu (1+r_3)^{-1}, \qquad \mu = 1, 2 \tag{6.8}$$

into the unit sphere $r^2 = \sum_{\mu=1}^{3} x_\mu^2 = 1$, the equation of motion (1.1) can be written

$$\mathbf{L}^2 n - (\mathbf{L}^2 n, n) n = 0, \tag{6.9}$$

where $\mathbf{L}$ is the angular momentum operator on the sphere $r^2 = 1$. Hence, any eigenfunction of $\mathbf{L}^2$ to any fixed eigenvalue $k(k+1)$ will solve (6.9). Choose, in particular, the following combination of spherical harmonics Y_{km}:

$$n_i := c_k \sum_{m=-k}^{k} U_{i, m+k+1} Y_{km}, \tag{6.10}$$

where U is a constant unitary $(2k+1)$ by $(2k+1)$ matrix fulfilling

$$U_{i, m+k+1} = (-1)^m \bar{U}_{i, -m+k+1} \tag{6.11}$$

so that n_i is real. By the addition theorem for the spherical harmonics, n^2 is a constant, and c_k can be adjusted so that $n^2 = 1$. Thus, (6.10) are true $0(2k+1)$ solutions, not lying in any subspace of dimension smaller than $2k+1$, by the independence of the spherical harmonics. The action of these solutions is given by $4\pi k(k+1)$.

To see that the solutions (6.10) fall into the class $I(\mathbb{S}^2)$, we use Theorem 6.2 (iii): Note that the spherical harmonics are polynomials in $\cos\theta$, $\sin\theta$; $\cos\phi$, $\sin\phi$ where θ and ϕ are polar angles on $r^2 = 1$, and thus rational functions of the r variables. By (6.8), n_i as defined in (6.10) are rational functions of x_1, x_2.

Finally, we remark that the harmonic polynomial solutions, though stationary points of the action (0.3), are not minima for $k > 1$; this instability of the solutions (6.10) has been shown in [3]. This result is in sharp contrast with the case $k = 1$, the 0(3) model, see [1].

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