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Solutions to the Ginzburg-Landau Equations for Planar Textures in Superfluid ^{3}He

V. L. Golo, M. I. Monastyrsky and S. P. Novikov

Institute of Theoretical and Experimental Physics, Academy of Sciences, Moscow, USSR

Abstract. The Ginzburg-Landau equations for planar textures of superfluid ^{3}He are proved to be equivalent to a completely integrable Hamiltonian system. General solutions to these equations are obtained by means of hyperelliptic integrals.

1. Introduction

Superfluid ^{3}He in the state of the p-pairing can be described in terms of a complex 3×3 matrix field A_{pi} (the order parameter), which minimizes the Ginzburg-Landau free energy, [1, 3, 5],

$$\mathscr{F} = \int d^3x [F_{\text{grad}} + F_b + F_h + F_d];$$

$$F_{\text{grad}} = \gamma_1 \partial_K A^*_{pi} \partial_K A_{pi} + \gamma_2 \partial_K A^*_{pi} \partial_i A_{pK} + \gamma_3 \partial_K A^*_{pK} \partial_i A_{pi}; \tag{1}$$

$$F_b = \alpha Tr(A^+ A) + \beta_1 |Tr(AA^t)|^2 + \beta_2 [Tr(A^+ A)]^2$$
$$+ \beta_3 Tr[(A^* A^t)(AA^t)^*] + \beta_4 Tr[(A^+ A)^2] + \beta_5 Tr[(A^+ A)(A^+ A)^*];$$

F_d is the dipole energy density,
F_h is the magnetic energy density.

For a uniform spacial configuration of the order parameter the F_{grad} terms are absent. Then the minimization of F_b gives values of the order parameter A_{pi} for the familiar A and B phases, which constitute smooth manifolds M_A and M_B.

In these two cases the order parameter is of the form:

(I) for the A phase

$$A_{pi} = \Delta \cdot d_p(\Delta'_i + \sqrt{-1}\Delta''_i), \qquad d^2 = 1, \qquad (\Delta'_i)^2 = (\Delta''_i)^2 = 1$$
$$\Delta = \text{const}, \qquad \Delta'_i \cdot \Delta''_i = 0$$

(II) for the B phase

$$A_{pi}=\frac{\Delta}{3^{1/2}}R_{pi}\times e^{i\varphi}, \qquad \Delta=\text{const}$$

$$M_B=SO(3)\times U(1)$$

where R_{pi} is a rotation matrix.

The density F_b is invariant under the transformations

$$A\to R_1^{-1}AR_2e^{i\varphi} \tag{2}$$

where R_1, R_2 are rotation matrices. Transformations (2) constitute the symmetry group $G=SO(3)\otimes SO(3)\otimes U(1)$. It should be noted that the order parameter for superfluid phases of ^{3}He takes its values in homogeneous spaces of the group G.

For space dependent states, $A_{pi}=A_{pi}(x)$, we have $F_{\text{grad}}\neq 0$. Then in the London (or hydrodynamic) limit, [3], we assume that the order parameter takes its values in the manifold of a superfluid phase, which lies in the space of complex 3×3-matrices, and the space dependence of the order parameter is determined by the minimization of free energy density (1) under appropriate boundary conditions.

This situation is similar to the problem of chiral fields which take values in a homogeneous space $M=G/H$ of the group G and generate a metric on M by the gradient terms of the Lagrangian. But it should be noticed that the gradient terms appearing in the theory of superfluid ^{3}He differ from the gradient terms of chiral theory in that they do not generate, in general, the standard-invariant metric on the manifold of the order parameter, while they do in chiral theory.

In this paper we assume that the order parameter depends only on a space variable z, i.e. we consider planar textures, which describe reasonably well superfluid ^{3}He confined between two parallel plates devided by a narrow gap [7].

We have found that for planar textures in the A phase of superfluid ^{3}He there exists a mechanical analogy with a top that substantially differs from the top of classical mechanics in having moments of inertia which can change. Until now there has been only the analogy with the top of classical rigid body, which was derived for the needs of NMR [19].

Using the $SO(3)\otimes SO(3)\otimes U(1)$-symmetry of superfluid ^{3}He we have obtained general solutions to the GL equations for the order parameter by means of hyperelliptic integrals. These solutions fulfill only the necessary conditions of the minimization problem and it is still necessary to select the true minima among them.

2. Equations for the A phase

We shall consider separately two cases: (i) $L\ll L_d$ and (ii) $L\gg L_d$, where L, L_d are the characteristic and dipole lengths respectively.

(i) $L\ll L_d$

In this case we may cancel out the dipole energy and take the order parameter in the most general form

$$A=e^{i\varphi}R_1^{-1}A_0R_2, \qquad A_0=\begin{pmatrix}0&0&0\\0&0&0\\1&i&0\end{pmatrix}\cdot\Delta. \tag{5}$$

We note that any value of the order parameter A for the A phase may be written in the form

$$A = R_1^{-1} A_0 R_2$$

since we have the formula

$$e^{i\varphi} A_0 = R_\varphi^{-1} A_0 R_\varphi, \qquad R_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

Following papers [8], [9], we shall introduce the velocities

$$v = i\partial_z R_1^{-1} \cdot R_1; \qquad w = R_2^{-1} \cdot i\partial_z R_2. \tag{7}$$

They take their values in the Lie algebra of the group $SO(3)$ and we may write the equations

$$v = v_a f_a, \qquad w = w_a f_a$$

where f_a, $a = 1, 2, 3$ are the generators of $SO(3)$ having matrix elements $(f_a)_{bc} = i\varepsilon_{abc}$. With the help of v, w and their coordinates v_a, w_a we may write down the density of the free energy given by Eq. (1) in the form

$$\begin{aligned} F_{\text{grad}} &= I_{ab}(A) w_a w_b + \chi_{ab}(A) v_a v_b, \\ I_{ab}(A) &= \gamma_1 (f_a A^+ A f_b)_{ii} + (\gamma_2 + \gamma_3)(f_a A^+ A f_b)_{33}, \\ \chi_{ab}(A) &= \gamma_1 (A^+ f_a f_b A)_{ii} + (\gamma_2 + \gamma_3)(A^+ f_a f_b A)_{33}. \end{aligned} \tag{8}$$

The velocities v, w are defined by Eq. (7) in the same way as the angular velocities of a three-dimensional rigid body. The matrices I_{ab} and χ_{ab} change for different values of A as is clearly seen from their explicit form

$$\chi_{ab}(A) = |\Delta|^2 (2\gamma_1 + (\gamma_2 + \gamma_3)|\Delta_3|^2)(\delta_{ab} - d_a d_b) \tag{8'}$$

$$I_{ab}(A) = \begin{pmatrix} \gamma_1 |\Delta_3|^2 + (\gamma_1 + \gamma_2 + \gamma_3)|\Delta|^2 & -(\gamma_1 + \gamma_2 + \gamma_3)\Delta_1 \Delta_2^* & -\gamma_1 \Delta_1 \Delta_3^* \\ -(\gamma_1 + \gamma_2 + \gamma_3)\Delta_1^* \Delta_2 & \gamma_1 |\Delta_3|^2 + (\gamma_1 + \gamma_2 + \gamma_3)|\Delta_1|^2 & -\gamma_1 \Delta_2 \Delta_3^* \\ -\gamma_1 \Delta_1^* \Delta_3 & -\gamma_1 \Delta_2^* \Delta_3 & \gamma_1 (|\Delta_2|^2 + |\Delta_3|^2) \end{pmatrix}.$$

Here Δ_i, d_p; i, $p = 1, 2, 3$ are the familiar complex and real vectors for the order parameter A_{pi}.

It is convenient to define a scalar product of two complex 3×3-matrices by the formula

$$\langle X | Y \rangle = \gamma_1 (X^+ Y)_{ii} + (\gamma_2 + \gamma_3)(X^+ Y)_{33}. \tag{9}$$

Then we have

$$I_{ab}(A) = \langle A f_a | A f_b \rangle, \qquad \chi_{ab}(A) = \langle f_a A | f_b A \rangle. \tag{10}$$

Now let us notice that under variations of the rotation matrices of the order parameter

$$R_{1,2} \to R_{1,2}(1 + i\vartheta^{(1,2)} + \ldots)$$

the velocities v_a, w_a and the order parameter are transformed as follows

$$v_a \to v_a + \partial_z \vartheta_a^{(1)} + \varepsilon_{abc} v_b \vartheta_c^{(1)} + \dots$$

$$w_a \to w_a - \partial_z \vartheta_a^{(2)} + \varepsilon_{abc} w_b \vartheta_c^{(2)} + \dots$$

$$A \to A - i\vartheta_a^{(1)} f_a A + i\vartheta_a^{(2)} A f_a + \dots.$$

Hence we may obtain the equations of motion (or the equations for texture) in the form

$$\nabla M^a_{\text{spin}} - (i\langle f_a f_b A | f_c A\rangle + \text{c.c.}) v_b v_c = 0, \tag{11}$$

$$\nabla M^a_{\text{orb}} - (i\langle A f_a f_b | A f_c\rangle + \text{c.c.}) w_b w_c = 0, \tag{12}$$

$$M^a_{\text{spin}} = \partial F_{\text{grad}}/\partial v_a; \qquad M^a_{\text{orb}} = \partial F_{\text{grad}}/\partial w_a,$$

$$\nabla M^a_{\text{spin}} = \partial_z M^a_{\text{spin}} - \varepsilon_{abc} v_b M^c_{\text{spin}},$$

$$\nabla M^a_{\text{orb}} = \partial_z M^a_{\text{orb}} + \varepsilon_{abc} w_b M^c_{\text{orb}}.$$

They are similar to the Euler equations for a top. Since there are no cross terms with respect to v_a, w_a in Eqs. (11–12) we may say that we have two 3-dimensional tops, which nonetheless do interact with each other as follows from Eq. (8′). The second terms in Eqs. (11–12) have appeared since our tops have changing inertia coefficients. We may transform Eqs. (11–12) into a Hamiltonian form by means of Poisson brackets as follows

$$H = F_{\text{grad}}; \qquad \partial_z M^a_{\text{spin(orb)}} = \{M^a_{\text{spin(orb)}};\ H\};$$

$$\{M^a_{\text{spin}};\ M^b_{\text{spin}}\} = \varepsilon_{abc} M^c_{\text{spin}}; \qquad \{A_{pi};\ A_{qj}\} = 0;$$

$$\{M^a_{\text{orb}};\ M^b_{\text{orb}}\} = -\varepsilon_{abc} M^c_{\text{orb}}; \qquad \{M^a_{\text{orb}};\ M^b_{\text{spin}}\} = 0;$$

$$\{M^a_{\text{spin}};\ A\} = -i f_a A; \qquad \{M^a_{\text{orb}};\ A\} = -iAf^a$$

We have six conserved quantities or integrals for our system:

(1) The spin currents j^a_{spin}, $a = 1, 2, 3$ generated by the rotations in the spin indices (since the dipole energy is cancelled out);

(2) The momentum along the axis OZ, i.e. the Hamiltonian H of the system;

(3) The superfluid current

$$j_m = \langle A | A f_a\rangle w_a + \text{c.c.}$$

generated by the gauge transformation $A \to e^{i\varphi} A$,

(4) M^3_{orb}, generated by the rotation in the orbital indices round the axis OZ

$$A \to AR_\psi; \qquad R_\psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We notice that the components j^a_{spin} of spin current coincide with the spin momenta M^a_{spin}, $a = 1, 2, 3$.

It is easy to see that the integrals

$$H = F_{\text{grad}},\ j_m,\ M^3_{\text{orb}},\ M^3_{\text{spin}},\ (M^a_{\text{spin}})^2$$

are in involution, i.e. the Poisson brackets among any two of them are zero. Hence our system is completely integrable. But it should be noticed that the actual integration requires additional consideration of the symmetry of the order parameter, (cf. below).

It is easy to incorporate the magnetic field in Eqs. (11–12), but since the magnetic energy is represented by the term

$$F_H = g_H |H_p A_{pi}|^2$$

we lose two spin current integrals and our system is no longer completely integrable.

(ii) $L \gg L_d$

In this case we shall use only the superfluid velocity $w = R^{-1} \cdot i\partial_z R$.

Now the function F_{grad} takes the form

$$F_{\text{grad}} = I_{ab}(A) w_a w_b ,$$
$$I_{ab}(A) = \langle [A; f_a] | [A; f_b] \rangle$$

where $\langle | \rangle$ means scalar product (9). First we suppose that the magnetic field is absent. Then by the considerations similar to those of the last subsection we obtain the Euler equations in the form

$$\begin{aligned} &\nabla M^a + (i\langle [A; f_b] | [[A; f_a] f_c] \rangle - i\langle [[A; f_a] f_b] | [A; f_c] \rangle) w_b w_c = 0 , \\ &M^a = \partial F_{\text{grad}} / \partial w_a , \\ &\nabla M^a = \partial_z M^a + \varepsilon_{abc} w_b M^c . \end{aligned} \tag{13}$$

The magnetic field can be incorporated in the free energy density through the additional term

$$F_H = g_H |H'_p A_{pi}|^2$$

which generates the right hand side in Eq. (13). We consider only the case of the magnetic field being parallel to the axis OZ. Then the arguments of the previous subsection go through for obvious reasons and we have three integrals in involution

$$H = F_{\text{grad}} + F_H, \; j_m, M^3$$

and our system is completely integrable.

(iii) Now we shall obtain explicit formulae for the solutions by symmetry considerations with the Euler angles. We consider separately the two cases: (i) $L \gg L_d$ and (ii) $L \ll L_d$.

(i) $L \gg L_d$

In this case the minimization of the dipole energy density reduces the order parameter to the form

$$A = R^{-1} A_0 R , \qquad A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & i & 0 \end{pmatrix} \Delta .$$

The free energy density is invariant under the transformation of the order parameter

$$A \to e^{i\varphi} R_\psi^{-1} A R_\psi \tag{14}$$

where R_ψ is a rotation round the OZ-axis by an angle ψ. We notice that

$$e^{i\varphi} R_\psi^{-1} A R_\psi = (R_\varphi^{-1} R R_\psi)^{-1} A_0 (R_\varphi^{-1} R R_\psi).$$

The whole point about the symmetry of the order parameter of the A-phase is that transformations (14) generate two commuting one-dimensional subgroups of $SO(3)$, which act on $SO(3)$ as follows:

$$R \to R_\varphi^{-1} R R_\psi .$$

From the Euler form of a rotation matrix

$$R = R_\varphi^{(z)} R_\vartheta^{(x)} R_\psi^{(z)}$$

where $R_\varphi^{(z)}$, $R_\psi^{(z)}$ are rotations round the axis OZ by the angles φ, ψ and $R_\vartheta^{(x)}$ is a rotation round the axis OX by the angle ϑ, we infer that the angles φ, ψ can be cancelled out by transformations (14) with a suitable choice of φ, ψ.

Therefore, the coefficients I_{ab} of the free energy density depend only on the angle ϑ[1]. Hence we obtain

$$\begin{aligned} F_{\text{grad}} &= I_{11}\dot{\vartheta}^2 + I_{22}\sin^2\vartheta\dot{\psi}^2 + I_{33}(\dot{\psi}\cos\vartheta + \dot{\varphi})^2 \\ &\quad - 2I_{23}\sin\vartheta(\dot{\psi}\cos\vartheta + \dot{\varphi})\dot{\psi}\,; \end{aligned} \tag{15}$$

$$I_{11} = |\Delta|^2(3\gamma_1 + \gamma_2 + \gamma_3)\,; \qquad I_{12} = I_{13} = I_{21} = I_{31} = 0$$

$$I_{22} = |\Delta|^2(2\gamma_1 + \gamma_2 + \gamma_3 + \gamma_1\cos^2\vartheta + (\gamma_2 + \gamma_3)\sin^2\vartheta\cos^2\vartheta)\,;$$

$$I_{33} = |\Delta|^2(2\gamma_1 + \gamma_1\sin^2\vartheta + (\gamma_2 + \gamma_3)\sin^4\vartheta)\,;$$

$$I_{23} = |\Delta|^2(\gamma_1 + (\gamma_2 + \gamma_3)\sin^2\vartheta)\sin\vartheta\cos\vartheta .$$

Here the dot denotes the derivative ∂_z. Using the cyclic variables φ, ψ we may put (15) in the form

$$\begin{aligned} F_{\text{grad}} &= I_{11}\dot{\vartheta}^2 + \frac{1}{4\sin^2\vartheta\cdot I(\vartheta)}\{j_m\cdot I_{33} - 2j_m M_3(I_{33}\cos\vartheta - I_{23}\sin\vartheta) \\ &\quad + M_3^2(I_{33}\cos^2\vartheta - 2I_{23}\sin\vartheta\cos\vartheta + I_{22}\sin^2\vartheta)\}\,, \end{aligned} \tag{16}$$

$$I(\vartheta) = I_{22}I_{33} - I_{23}^2 .$$

Since F_{grad} is an integral of motion we may write Eq. (16) in the form

$$I_{11}\cdot\dot{\vartheta}^2 + \frac{\Phi(\vartheta)}{4\sin^2\vartheta\cdot I(\vartheta)} = E\,, \tag{17}$$

where $\Phi(\vartheta)$ is the function in the brackets of Eq. (16).

We may incorporate the magnetic energy in the free energy density if the magnetic field is directed along the axis OZ, since in this case the symmetry

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1 This result was also obtained in papers [10, 11].

considerations of Eqs. (12–14) go through. We have

$$I_{11}\cdot\dot{\vartheta}^2+\frac{\Phi(\vartheta)}{4\sin^2\vartheta\cdot I(\vartheta)}+2g_H H^2|\Delta|^2\cos^2\vartheta=E. \tag{18}$$

We may write solutions to Eq. (18) as a hyperelliptic integral

$$\begin{aligned} z&=\pm I_{11}^{1/2}\int\frac{2\sin\vartheta\cdot I^{1/2}(\vartheta)d\vartheta}{(4\sin^2\vartheta\cdot I(\vartheta)(E-2g_H H^2|\Delta|^2\cos^2\vartheta)-\Phi(\vartheta))^{1/2}}\\ &=\mp 2I_{11}^{1/2}\int\frac{I(t)dt}{(\{4(1-t^2)(E-2g_H H^2|\Delta|^2t^2)I(t)-\Phi(t)\}\cdot I(t))^{1/2}}\\ &=\mp 2I_{11}^{1/2}\int\frac{I(t)dt}{(P_{12}(t))^{1/2}},\qquad t=\cos\vartheta, \end{aligned} \tag{19}$$

where $I(t)$ and $\Phi(t)$ are polynomials of the 4-th and 6-th order, respectively.

We may write solutions for φ,ψ in a similar form

$$\begin{aligned} \varphi&=\mp I_{11}^{1/2}\int\left[\frac{\tilde{K}_5(t)}{1-t^2}+K_4(t)\right]\frac{dt}{(P_{12}(t))^{1/2}},\\ \psi&=\mp I_{11}^{1/2}\int\frac{K_5(t)}{1-t^2}\cdot\frac{dt}{(P_{12}(t))^{1/2}} \end{aligned} \tag{20}$$

where $t=\cos\vartheta$, $P_{12}(t)$ is the polynomial under the radical in Eq. (19); $K_4(t)$, $K_5(t)$, $\tilde{K}_5(t)$ are polynomials in t of degree 4 and 5.

(ii) $L \ll L_d$

This case can be treated along the same lines as the last one. We shall consider textures in the absence of the magnetic field. We may eliminate all the Euler angles except ϑ and obtain the coefficients I_{ab}, X_{ab} in the form

$$X_{ab}=|\Delta|^2\{\gamma_1+(\gamma_2+\gamma_3)\sin^2\vartheta\}(\delta_{ab}-\delta_{a3}\delta_{b3});$$

$$I_{11}=\gamma_1+(\gamma_2+\gamma_3)\cos^2\vartheta;\qquad I_{23}=-\gamma_1\sin\vartheta\cos\vartheta;\qquad I_{12}=I_{21}=0;$$

$$I_{22}=\gamma_1+\gamma_2+\gamma_3+\gamma_1\sin^2\vartheta;\qquad I_{33}=\gamma_1+\gamma_1\cos^2\vartheta;\qquad I_{32}=I_{23}=0.$$

We have the conserved quantities

$$\varphi_{\text{spin}},\ \psi_{\text{spin}},\ M^2_{\text{spin}}=\text{const}.$$

With their help we can put the free energy density in the form

$$\begin{aligned} F&=I_{11}\dot{\vartheta}^2+I_{22}\sin^2\vartheta\cdot\dot{\psi}^2-2I_{23}\sin\vartheta\cdot\dot{\psi}(\dot{\psi}\cos\vartheta+\dot{\varphi})\\ &\quad+I_{33}(\dot{\psi}\cos\vartheta+\dot{\varphi})^2+\frac{M^2_{\text{spin}}}{4|\Delta|^2}\cdot[\gamma_1+(\gamma_2+\gamma_3)\sin^2\vartheta]^{-1}. \end{aligned}$$

We notice that φ,ψ are cyclic variables and we can derive an equation for ϑ. Here

We have

$$z = \pm \int \frac{(\gamma_1 + (\gamma_2 + \gamma_3)u) I(u) du}{(u(\gamma_1 + (\gamma_2 + \gamma_3)u) P_3(u))^{1/2}},$$

$$\psi = \mp \frac{j_m \gamma_1}{2} \int \frac{\gamma_1 + (\gamma_2 + \gamma_3)u}{((\gamma_1 + (\gamma_2 + \gamma_3)u) P_3(u))^{1/2}} \frac{du}{1-u},$$

$$\varphi = \pm \frac{j_m \gamma_1}{2} \int \frac{\gamma_1 + (\gamma_2 + \gamma_3)u}{1-u} \cdot \frac{du}{(u(\gamma_1 + (\gamma_2 + \gamma_3)u) P_3(u))^{1/2}}$$

$$\mp j_m \frac{2\gamma_1 + \gamma_2 + \gamma_3}{2} \int \frac{(\gamma_1 + (\gamma_2 + \gamma_3)u) du}{(u(\gamma_1 + (\gamma_2 + \gamma_3)u) P_3(u))^{1/2}},$$

where $u = \cos^2 \vartheta$, and

$$I(u) = |\Delta|^2 (\gamma_1 (2\gamma_1 + \gamma_2 + \gamma_3) + \gamma_1 (\gamma_2 + \gamma_3) u)$$

$$P_3(u) = I(u) [4E(1-u) I(u) - \Phi(u)]$$

$$\Phi(u) = (2\gamma_1 + \gamma_2 + \gamma_3) |\Delta|^2 j_m - (\gamma_2 + \gamma_3) |\Delta|^2 j_m \cdot u.$$

Appendix I. Equations for the *B* Phase

We assume that the characteristic length of a texture is much less than the dipole length L_d. Then the order parameter is of the form

$$A = \frac{\Delta}{\sqrt{3}} R e^{i\varphi}$$

where R is a rotation matrix and Δ is a complex number. Again we shall use the velocities

$$w = R^{-1} \cdot i\partial_z R, \qquad v = -\partial_z \varphi.$$

Then it is easy to see that the gradient part of the free energy density can be written as

$$F = \frac{I}{2}(w_1^2 + w_2^2) + \frac{J}{2} w_3^2 + \frac{m}{2} v^2,$$

$$J = \frac{2|\Delta|^2}{3} \cdot 2\gamma_1 ; \qquad I = \tfrac{2}{3} |\Delta|^2 (2\gamma_1 + \gamma_2 + \gamma_3), \tag{21}$$

$$m = \tfrac{2}{3} |\Delta|^2 (3\gamma_1 + \gamma_2 + \gamma_3).$$

We notice that Eq. (21) has the form of a Lagrangian for symmetric top with a mass m, inertia-coefficients I, J. The velocities $\mathbf{w} = (w_1, w_2, w_3)$, v mean the velocity of the center mass and the angular velocity of the top.

Now the integration technique for symmetric top must work for planar textures in the B phase. Here we want to notice an interesting example. Let us take into account the dipole energy contribution to the free-energy

$$F_d = g_d (\cos \vartheta + 2 \operatorname{os}^2 \vartheta),$$

where ϑ is a rotation angle of the order parameter. If we assume that the rotation axis is perpendicular to the plates confining the superfluid throughout the gap, then there exists a solution, first found by Maki, for which the Euler equations reduce to only one equation for ϑ. It has the form

$$J\ddot{\vartheta} + g_d \sin\vartheta(1 + 4\cos\vartheta) = 0$$

and can be solved by means of the elliptic functions.

Appendix II. Singularities of the Euler Angles

We shall demonstrate that for the rotation group the Euler angles φ, ψ, ϑ constitute a system of coordinates having a singularity at $\vartheta = 0, \pi$.

Since the Euler angles change within range

$$0 \leqq \varphi, \psi \leqq 2\pi, \qquad 0 \leqq \vartheta \leqq \pi$$

we may say using the geometrical language that they form a product Π of a two-dimensional torus and a segement

$$\Pi = S^1_\varphi \times S^1_\psi \times I_\vartheta$$

the angles φ, ψ taking values in the circles and the angle ϑ in the segment I_ϑ. The formulae, which express a rotation matrix by the Euler angles, give a map

$$\Pi = S^1_\varphi \times S^1_\psi \times I_\vartheta \rightarrow SO(3), \qquad R = R(\varphi, \psi, \vartheta) \tag{22}$$

of the space Π onto the rotation group. We notice that Π is a manifold with the boundary consisting of two tori

$$S^1_\varphi \times S^1_\psi, \qquad \vartheta = 0, \pi .$$

Map (22) has points of degeneracy at the boundary. To see this we may consider the formulae for the angular velocity

$$\boldsymbol{\omega} = C \begin{pmatrix} \dot{\varphi} \\ \dot{\psi} \\ \dot{\vartheta} \end{pmatrix}; \quad C = \begin{pmatrix} \sin\vartheta \sin\psi & 0 & \cos\psi \\ \sin\vartheta \cos\psi & 0 & -\sin\psi \\ \cos\vartheta & 1 & 0 \end{pmatrix}. \tag{23}$$

We may say that Eq. (23) gives the differential of smooth map (22) and $\dot{\varphi}, \dot{\psi}, \dot{\vartheta}$ and $\omega_1, \omega_2, \omega_3$ are coordinates of tangent vectors on Π and $SO(3)$ respectively. The matrix C in Eq. (23) is degenerate of rank 2 at $\vartheta = 0, \pi$ i.e. at the boundary which is mapped into the matrices

$$R(\varphi, \psi, \vartheta = 0) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad R(\varphi, \psi, \vartheta = \pi) = \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ \sin\beta & -\cos\beta & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

here

$$\alpha(\varphi, \psi) = \varphi + \psi, \qquad \beta(\varphi, \psi) = \varphi - \psi .$$

To put this in geometrical terms we may say that the tori of the boundary are contracted by map (22) into circles. On the other hand we have no contraction or singular points for $0<\vartheta<\pi$. Hence the Euler angles provide a means to obtain the manifold $SO(3)$ as follows:

(1) two pairs (φ, ψ) and $(\varphi+2\pi n,\ \psi+2\pi n)$, n is an integer, are equivalent;

(2) ϑ changes within the segment $0\leqq\vartheta\leqq\pi$;

(3) if $\vartheta=0$ the pairs (φ, ψ) and (φ', ψ') determine the same point of $SO(3)$ if

$$\varphi+\psi=\varphi'+\psi'$$

if $\vartheta=\pi$ the pairs (φ, ψ) and (φ', ψ') determine the same point of $SO(3)$ if

$$\varphi-\psi=\varphi'-\psi'.$$

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