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Z_4-Symmetric Factorized S-Matrix in Two Space-Time Dimensions

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Abstract. The factorized S-matrix with internal symmetry Z_4 is constructed in two space-time dimensions. The two-particle amplitudes are obtained by means of solving the factorization, unitarity and analyticity equations. The solution of factorization equations can be expressed in terms of elliptic functions. The S-matrix contains the resonance poles naturally. The simple formal relation between the general factorized S-matrices and the Baxter-type lattice transfer matrices is found. In the sense of this relation the Z_4-symmetric S-matrix corresponds to the Baxter transfer matrix itself.

1. Introduction

During the last years a number of examples were found of nontrivial and exactly calculable relativistic scattering theories in two space-time dimensions (see for the review [1] and references therein). These examples, the so-called factorized S-matrices, correspond to simplified scattering kinematics restricted by special selection rules. These forbid a change in the number of particles and also preserve the set of individual momenta. Therefore the scattering process reduces to the redistribution of momenta between the different particles of the same mass. Selection rules of this type are characteristic of the quantum dynamics of a completely integrable field-theoretic system such as the sine-Gordon model.

The presence of these selection rules in the scattering theory forces the remarkable property of the total S-matrix: it is factorized in the standard manner into two-particle S-matrices [1, 2, 8]. Any multiparticle S-matrix element can be expressed in terms of the two-particle amplitudes, the unitarity and analyticity of the total S-matrix being the consequence of the same properties of the two-particle one. Furthermore, the factorized form of the S-matrix requires special functional relations (the so-called factorization equations) for the two-particle amplitudes to be satisfied. In a number of interesting cases one can work out the two-particle S-matrix explicitly by solving the factorization equations together with the unitarity and analyticity conditions. In this way the factorized S-matrices having $O(N)$ [3],

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$SU(N)$ [4] isotopic symmetries and some other examples [5, 6] were obtained. The $O(2)$-symmetric S-matrix corresponds to the quantum sine-Gordon model while $O(N)$-symmetric (with $N \geqq 3$) ones correspond to the non-linear σ-model and the Gross-Neveu model [1].

In this paper we construct by the same method the relativistic factorized S-matrix with discrete internal symmetry Z_4. The general solution of the corresponding factorization equations can be expressed in terms of elliptic functions: the ratios of different amplitudes are double-periodic functions of θ – the rapidly difference of colliding particles. The two-particle S-matrix satisfying the unitarity and analyticity conditions depends on two parameters of coupling constant type [the $O(2)$ symmetric sine-Gordon S-matrix turns out to be a degenerate particular case]. It possesses some unusual properties for purely elastic S-matrices, in particular, it includes the resonance poles naturally.

The other point we discuss in this paper is the remarkable formal connection between two-dimensional factorized S-matrices and the exactly diagonalizable transfer matrices of plane-lattice statistical systems. The most interesting known example of them is the eight-vertex transfer matrix of Baxter [7]. If one considers the square lattice of N columns labelled by $J = 1, 2, \ldots, N$, the Baxter row-to-row transfer matrix is of the form (see (3.3) of [7]).

$$T_{\{i\}\{i'\}}(v) = \sum_{\substack{\alpha 1, \ldots, \alpha_N = \pm 1 \\ \alpha_{N+1} = \alpha_1}} \prod_{J=1}^{N} R_{i_J i'_J}^{\alpha_J \alpha_{J+1}}(v), \tag{1.1}$$

where $\{i\} = \{i_1, \ldots, i_N\}$; $i_J = \pm 1$ are the operator (vertical) indices of transfer-matrix and $R^{\alpha_J \alpha_{J+1}}$ is 2×2 operator matrix of the special form, analytically dependent on the "spectral" parameter v. The remarkable property of the parametric set of transfer matrices (1.1) is their commutativity at different values of spectral parameter:

$$T(v)\,T(v') = T(v')\,T(v). \tag{1.2}$$

This is the important condition for the exact diagonalizability; in fact it is the consequence of some special properties of the matrix $R(v)$ (see [7] for the details).

In principle, one can consider a more general case of $n \times n$ matrices $R^{\alpha\alpha'}$ and of n-dimensional space, labelled by the vertical indices i_J. It will be shown that, in general, the equations on a two-particle S-matrix necessary for the total S-matrix factorization coincide formally with the conditions on matrix R, given by the requirement of transfer matrix commutativity (1.2); while the rapidity difference θ (which is the argument of the two-particle amplitudes) plays the role of spectral parameter. This correspondence enables one to construct, given any factorized S-matrix, the related set of Baxter-type transfer matrices with property (1.2). The Z_4-symmetric S-matrix described in this paper is connected to the Baxter transfer matrix itself (in the formal sense explained above).

The paper is organized as follows. In Sect. 2 the definition and the general properties of the Z_4 symmetric factorized S-matrix are presented. The solution of factorization, unitarity and analyticity conditions is found in Sect. 3. The discussion of some physical properties of this solution is the matter of Sect. 4. In Sect. 5 we treat the connection between facorized S-matrices and a Baxter-type transfer-matrices; this section is relatively independent of the others. In Sect. 6 we discuss the results and point out some unsolved problems.

2. Z_4-Symmetric Factorized S-Matrix

Consider the factorized scattering theory including in the asymptotic states the degenerate doublet of the particles A and $\bar{A}$ which can be treated as particle and antiparticle. All the properties of the factorized S-matrix are determined completely by the structure of the two-particle one. As for the latter, we suppose it to be symmetric under exchange $A \leftrightarrow \bar{A}$ (C-invariance) and contain the following non-vanishing elements only

$$ {}_{\text{out}}\langle A(\theta_1)A(\theta_2)|A(\theta_1)A(\theta_2)\rangle_{\text{in}} = S(\theta_{12}), \tag{2.1a} $$

$$ {}_{\text{out}}\langle A(\theta_1)\bar{A}(\theta_2)|A(\theta_1)\bar{A}(\theta_2)\rangle_{\text{in}} = S_t(\theta_{12}), \tag{2.1b} $$

$$ {}_{\text{out}}\langle A(\theta_1)\bar{A}(\theta_2)|\bar{A}(\theta_1)A(\theta_2)\rangle_{\text{in}} = S_r(\theta_{12}), \tag{2.1c} $$

$$ {}_{\text{out}}\langle A(\theta_1)A(\theta_2)|\bar{A}(\theta_1)\bar{A}(\theta_2)\rangle_{\text{in}} = S_a(\theta_{12}), \tag{2.1d} $$

where $\theta_{12} = \theta_1 - \theta_2$, θ_1, and θ_2 are the rapidities of the colliding particles, i.e.

$$ p_k^0 = m \operatorname{ch} \theta_k; \qquad p_k^1 = m \operatorname{sh} \theta_k, \tag{2.2} $$

where p_k^μ is two-momentum of k-th particle and m is its mass. The dependence of two-particle amplitudes (2.1) on the rapidity difference θ_{12} is the consequence of relativistic invariance.

The presence of non-vanishing amplitude (2.1d) distinguishes our case from the known $O(2)$-symmetric S-matrix of quantum sine-Gordon solitons [1, 2]. Due to the open channel $AA \to \bar{A}\bar{A}$ the charge of a state is conserved only modulo 4, so the S-matrix (2.1) exhibits internal symmetry Z_4 [1].

In order to describe the total S-matrix it is convenient to use the algebraic representation of factorized S-matrix. Here we include a brief definition for our case; the general and more detailed consideration can be found in [1] (see also Sect. 5 of this paper).

Let us consider some non-commutative algebra with generators $A(\theta)$ and $\bar{A}(\theta)$ and identify the asymptotic scattering states with the ordered products of these generators, and generator $A(\theta_k)$ ($\bar{A}(\theta_k)$) in the product being in correspondence with the particle $A(\bar{A})$ of rapidity θ_k in the state; asymptotic in-(out-)states would be identified with the products in which the generators are arranged in the order of decreasing (increasing) values of rapidities θ_k. The two-particle S-matrix (2.1) corresponds to the following commutation relations for the generators

$$ A(\theta_1)A(\theta_2) = S(\theta_{12})A(\theta_2)A(\theta_1) + S_a(\theta_{12})\bar{A}(\theta_2)\bar{A}(\theta_1), \tag{2.3a} $$

$$ A(\theta_1)\bar{A}(\theta_2) = S_t(\theta_{12})\bar{A}(\theta_2)A(\theta_1) + S_r(\theta_{12})A(\theta_2)\bar{A}(\theta_1), \tag{2.3b} $$

$$ \bar{A}(\theta_1)A(\theta_2) = S_t(\theta_{12})A(\theta_2)\bar{A}(\theta_1) + S_r(\theta_{12})\bar{A}(\theta_2)A(\theta_1), \tag{2.3c} $$

$$ \bar{A}(\theta_1)\bar{A}(\theta_2) = S(\theta_{12})\bar{A}(\theta_2)\bar{A}(\theta_1) + S_a(\theta_{12})A(\theta_2)A(\theta_1). \tag{2.3d} $$

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1 Realization of the symmetry transformation on the particle wave functions can be chosen in the form: $A \to e^{i\frac{\pi}{2}}A$; $\bar{A} \to e^{-i\frac{\pi}{2}}\bar{A}$

These commutation relations are self-consistent provided the unitarity condition for the two-particle S-matrix (2.1) is satisfied:

$$S_t(\theta)S_t(-\theta)+S_r(\theta)S_r(-\theta)=1\,, \tag{2.4a}$$

$$S_t(\theta)S_r(-\theta)+S_r(\theta)S_t(-\theta)=0\,, \tag{2.4b}$$

$$S(\theta)S(-\theta)+S_a(\theta)S_a(-\theta)=1\,, \tag{2.4c}$$

$$S(\theta)S_a(-\theta)+S_a(-\theta)S(\theta)=0\,. \tag{2.4d}$$

The relations (2.3) allow us to recorder any in-state to out-states by performing ths subsequent pair commutations of neighbouring A's. To guarantee this procedure to give the unique result, independent of the order in which the pair commutations are made, one should require the associativity of the algebra. This requirement will be consistent with the commutation rules (2.3) provided the following functional relations for the two-particle amplitudes S are valid:

$$SS_aS_r+S_aS_tS_t=S_aSS\;+SS_rS_a\,, \tag{2.5a}$$

$$SS_aS_t+S_aS_tS_r=S_tS_aS\;+S_rS_tS_a\,, \tag{2.5b}$$

$$SS_tS_r+S_aS_aS_t=S_rS_rS_t+S_tSS_r\,, \tag{2.5c}$$

$$SS_rS\;+S_aSS_a=S_tS_rS_t+S_rSS_r\,, \tag{2.5d}$$

where the arguments of the first, the second and the third S in each term are implied to be θ, $\theta+\theta'$, and θ', respectively. The relations (2.5) are just the factorization equations; their physical meaning is discussed in [1].

If the Eqs. (2.4) and (2.5) are satisfied, the commutation rules enable us to expand uniquely any in-state into out-states, i.e., to express any multiparticle S-matrix element in terms of two-particle ones.

The relativistic scattering amplitudes (2.1) should possess certain analyticity properties. These amplitudes are meromorphic functions of θ, real at the imaginary θ-axis [1]. Furthermore the crossing-symmetry requirement for the S-matrix (2.1) provides the following equations

$$S_t(\theta)=S(i\pi-\theta)\,, \tag{2.6a}$$

$$S_r(\theta)=S_r(i\pi-\theta)\,, \tag{2.6b}$$

$$S_a(\theta)=S_a(i\pi-\theta)\,. \tag{2.6c}$$

Another equivalent description of the same S-matrix can be obtained by introducing the real doublet of particles A_1 and A_2;

$$\sqrt{2}A=A_1+iA_2\,;\qquad \sqrt{2}\bar{A}=A_1-iA_2\,. \tag{2.7}$$

It is easy to find, using (2.3), the following commutation relations for new generators $A_1(\theta)$ and $A_2(\theta)$

$$A_1(\theta_1)A_1(\theta_2)=\sigma(\theta_{12})A_1(\theta_2)A_1(\theta_1)+\sigma_a(\theta_{12})A_2(\theta_2)A_2(\theta_1)\,, \tag{2.8a}$$

$$A_1(\theta_1)A_2(\theta_2)=\sigma_t(\theta_{12})A_2(\theta_2)A_1(\theta_1)+\sigma_r(\theta_{12})A_1(\theta_2)A_1(\theta_2)A_2(\theta_1) \tag{2.8b}$$

[the relations for $A_2(\theta_1)A_2(\theta_2)$ and $A_2(\theta_1)A_1(\theta_2)$ differ from (2.8) only by the exchange $A_1 \leftrightarrow A_2$]. The new amplitudes σ are related to S as follows

$$\begin{aligned}2\sigma &= S_t + S_r + S + S_a \\ 2\sigma_a &= S_t + S_r - S - S_a \\ 2\sigma_t &= S + S_t - S_a - S_r \\ 2\sigma_r &= S + S_r - S_a - S_t\,.\end{aligned} \tag{2.9}$$

Since the commutation rules (2.8) coincide formally with (2.3), the unitarity conditions and factorization equations in terms of σ's can be obtained from (2.4) and (2.5) by the substitutions $S \to \sigma$, $S_a \to \sigma_a$, $S_t \to \sigma_t$, $S_r \to \sigma_r$. The crossing symmetry relations (2.6) become

$$\sigma_t(\theta) = \sigma_t(i\pi - \theta)\,, \tag{2.10a}$$

$$\sigma(\theta) = \sigma(i\pi - \theta)\,, \tag{2.10b}$$

$$\sigma_r(\theta) = \sigma_a(i\pi - \theta)\,. \tag{2.10c}$$

In the next Section we use the Eqs. (2.4)–(2.6) to construct explicitly the two-particle amplitudes (2.1).

3. The Solution of the Factorization, Unitarity, and Analyticity Equations

Obviously, the factorization Eqs. (2.5) provide restrictions on the ratios of the amplitudes $S(\theta)$ only. It is shown in Appendix A that the general solution of these equations has the form

$$S(\theta) = \frac{\mathrm{sn}(\mu\theta + 2\xi, l)}{\mathrm{sn}(2\xi, l)} S_r(\theta)\,, \tag{3.1a}$$

$$S_t(\theta) = -\frac{\mathrm{sn}(\mu\theta, l)}{\mathrm{sn}(2\xi, l)} S_r(\theta)\,, \tag{3.1b}$$

$$S_a(\theta) = -l\,\mathrm{sn}(\mu\theta, l)\,\mathrm{sn}(\mu\theta + 2\xi, l) S_r(\theta)\,, \tag{3.1c}$$

where l is the modulus of the elliptic functions, μ and ζ are arbitrary constants.

Here we concentrate on the real case l, $|l| \leqq 1$. In this case the elliptic sn has the real period $4\mathbf{K}_l$ and pure imaginary period $2i\mathbf{K}'_l$, where $\mathbf{K}_l$ and $\mathbf{K}'_l$ are complete elliptic integrals of the first kind of moduli l and $l' = (1 - l^2)^{1/2}$, respectively (see §8.112 of [10]).

Note that the commutation relations (2.3) are preserved under the rotation

$$A \to e^{i\frac{\pi}{4}} A\,; \qquad \bar{A} \to e^{-i\frac{\pi}{4}} \bar{A} \tag{3.2}$$

provided the simultaneous substitution $S_a \to -S_a$ is made. Therefore without loss of generality one can think of l as positive.

Alternatively, one could solve the factorization equations in terms of the amplitudes $\sigma(\theta)$. Since these equations can be obtained from (2.3) by the substitution $S \to \sigma$, the general solution can be represented in the form

$$\sigma_t(\theta) = -\frac{\mathrm{sn}(\lambda\theta, k)}{\mathrm{sn}(2\eta', k)}\sigma_r(\theta), \tag{3.3a}$$

$$\sigma(\theta) = \frac{\mathrm{sn}(\lambda\theta + 2\eta', k)}{\mathrm{sn}(2\eta', k)}\sigma_r(\theta), \tag{3.3b}$$

$$\sigma_a(\theta) = k\,\mathrm{sn}(\lambda\theta, k)\,\mathrm{sn}(\lambda\theta + 2\eta', k)\sigma_r(\theta), \tag{3.3c}$$

where the elliptic sn's are of some new modulus k. It can be shown from (2.9) that the parameters λ, η', and k are related to μ, ξ, and l as follows:

$$k = \left(\frac{1-\sqrt{l}}{1+\sqrt{l}}\right)^2, \tag{3.4a}$$

$$\lambda = -i(1+\sqrt{l})^2\mu, \tag{3.4b}$$

$$2\eta' = 2\eta + i\mathbf{K}' + 2\mathbf{K}; \qquad \eta = -i(1+\sqrt{l})^2\xi, \tag{3.4c}$$

where $\mathbf{K}$ and $\mathbf{K}'$ are complete elliptic integrals of moduli k and $k' = (1-k^2)^{1/2}$, respectively.

One can note the striking similarity between the formulae (3.3) [or (3.1)] and the ratios of vertex weights in Baxter's model (see (5.7) of [7]). The reasons for this similarity will be discussed in Sect. 5.

Consider now the crossing relations (2.6) or equivalently (2.10). They are consistent with Eqs. (3.1) [equivalently, (3.3)] provided the parameters ξ and μ (η and λ) are related as follows:

$$2\xi = -i\pi\mu; \qquad 2\eta = -i\pi\lambda. \tag{3.5}$$

The scattering amplitudes should satisfy the real analyticity condition, in particular, their ratios are to be real at pure imaginary θ. This implies the parameter ξ to be either real or pure imaginary. Note in this connection that the relations (3.3) can be represented in the form

$$\sigma_r(\theta) = \frac{\mathrm{sn}(\lambda\theta + 2\eta, k)}{\mathrm{sn}(2\eta, k)}\sigma(\theta), \tag{3.6a}$$

$$\sigma_a(\theta) = -\frac{\mathrm{sn}(\lambda\theta, k)}{\mathrm{sn}(2\eta, k)}\sigma(\theta), \tag{3.6b}$$

$$\sigma_t(\theta) = k\,\mathrm{sn}(\lambda\theta, k)\,\mathrm{sn}(\lambda\theta + 2\eta, k)\sigma(\theta). \tag{3.6c}$$

Therefore we can concentrate on the case of real ξ (imaginary η). The solution corresponding to the imaginary ξ can be obtained from this by the replacement

$$\begin{aligned} &\sigma_r \to S; \qquad \sigma_a \to S_t; \\ &\sigma \to S_r; \qquad \sigma_t \to S_a. \end{aligned} \tag{3.7}$$

After the substitution of (3.6) into the unitarity conditions (2.4) two of them, (2.4b) and (2.4d), are satisfied identically while the others, (2.4a) and (2.4b), lead to the equation

$$\sigma(\theta)\sigma(-\theta)=\frac{\mathrm{sn}^2(2\eta,k)}{\mathrm{sn}^2(2\eta,k)-\mathrm{sn}^2(\lambda\theta,k)}. \tag{3.8}$$

It is to be solved together with crossing-symmetry relation (2.10b), the solution being found in the class of meromorphic functions which are real at the imaginary θ-axis.

The Eqs. (3.8) and (2.10b) determine the function $\sigma(\theta)$ up to the multiplication by the arbitrary $2\pi i$ periodic function $\Phi(\theta)$ (from the same class) satisfying the relations

$$\begin{aligned}&\Phi(-\theta)=\Phi^{-1}(\theta)\\&\Phi(\theta)=\Phi(i\pi-\theta).\end{aligned} \tag{3.9}$$

This means it is possible to add some regular set of zeroes and poles. There is the unique solution which is regular, bounded and nonzero in the physical strip $0<\mathrm{Im}\,\theta<\pi$ [2] at $0\leqq\xi<\mathbf{K}_l$. This solution will be referred to as the "minimal" one. It is carried out in Appendix B:

$$\sigma(\theta)=\exp\left\{4\sum_{n=1}^{\infty}\frac{\mathrm{sh}^2\left[\frac{2\pi n(\pi-\gamma)}{\gamma'}\right]\sin\left[\frac{2\pi n}{\gamma'}\theta\right]\sin\left[\frac{2\pi n}{\gamma'}(i\pi-\theta)\right]}{n\,\mathrm{sh}\left[\frac{4\pi n\gamma}{\gamma'}\right]\mathrm{ch}\left[\frac{2\pi^2 n}{\gamma'}\right]}\right\}, \tag{3.10}$$

where the following notations are used

$$\begin{aligned}\gamma&=\frac{\pi\mathbf{K}_l}{\xi}=-i\frac{\pi\mathbf{K}'}{2\eta};\\ \gamma'&=\frac{\pi\mathbf{K}_l'}{\xi}=-i\frac{2\pi\mathbf{K}}{\eta}.\end{aligned} \tag{3.11}$$

It is the minimal solution which will be discussed in the next section.

4. Some Properties of the S-Matrix

It follows from (3.10) and (3.6) that

$$\begin{aligned}\sigma_t(\theta)&=\sigma_t(\theta+\gamma'/2)\\ \sigma_r(\theta)&=-\sigma_r(\theta+\gamma'/2)\\ \sigma_a(\theta)&=-\sigma_a(\theta+\gamma'/2).\end{aligned} \tag{4.1}$$

For the amplitudes S one obtains

$$\begin{aligned}S(\theta)&=S_t(\theta+\gamma'/2)\\ S_a(\theta)&=S_r(\theta+\gamma'/2).\end{aligned} \tag{4.2}$$

2 This strip corresponds to the physical sheet of the s-surface of the two-particle amplitudes

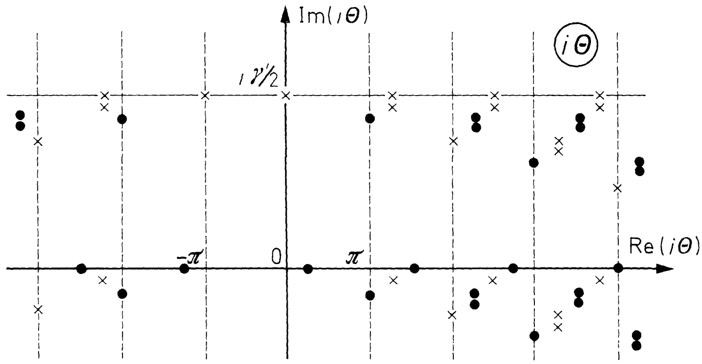

Fig. 1

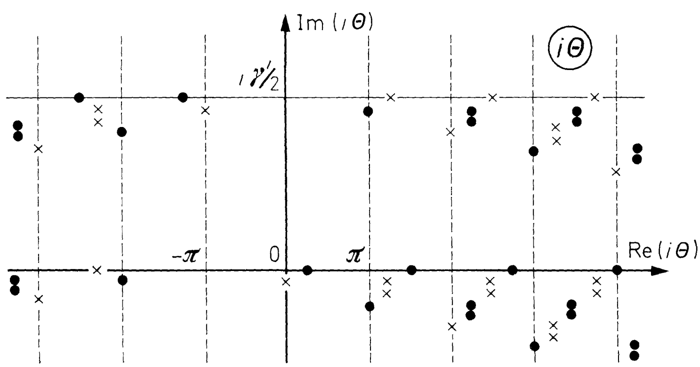

Fig. 2

Figs. 1 and 2. The locations of poles (points) and zeroes (crosses) of the amplitudes $S_r(\theta)$ (*1*) and $S_t(\theta)$ (*2*). Some of the points and crosses are slightly displaced from the axes $\mathrm{Re}\,\theta=0$ and $\mathrm{Re}\,\theta=\gamma'/2$ for the sake of transparency. The distance between the poles or zeroes in each row is γ. The picture should be periodically (with period γ') continued along the real θ-axis (in the vertical direction)

In particular, all the amplitudes are periodic functions of θ with the period γ'.

The locations of zeroes and poles of the amplitudes $S_r(\theta)$ and $S_t(\theta)$ are shown in Figs. 1 and 2. Note the series of poles

$$\theta_L = i\pi - i\gamma + L\gamma'; \qquad L=0, \pm 1, \pm 2, \ldots. \tag{4.3}$$

At $\gamma<\pi$ these poles come to the physical strip $0<\mathrm{Im}\,\theta<\pi$. If $\gamma'<\infty$ this indicates the existence of complex singularities on the physical s-sheet and, hence, contradicts casuality. Therefore we restrict the values of γ by

$$\gamma \geqq \pi. \tag{4.4}$$

In this case all the poles (4.3) are on the unphysical sheet. Then the pole with $L=0$ correspond to the virtual state while the others – to the resonance states in the channel $A\bar{A}\to A\bar{A}$. The complex masses of these resonances are

$$M_{A\bar{A}}^{(L)}=2m\,\mathrm{ch}(ig-L\gamma')\,; \qquad L=1,2,\ldots, \tag{4.5}$$

where $g=\gamma-\pi$. There is a similar set of resonances with complex masses

$$M_{AA}^{(L)}=2m\,\mathrm{ch}(ig-\gamma'/2-L\gamma')\,; \qquad L=1,2,\ldots \tag{4.6}$$

in the channels $AA\to AA$ and $AA\to\bar{A}\bar{A}$.

At $\gamma=\pi$ one obtains

$$\begin{aligned} S(\theta)&\equiv S_t(\theta)\equiv 1\,,\\ S_r(\theta)&\equiv S_a(\theta)\equiv 0\,, \end{aligned} \tag{4.7}$$

i.e., the total S-matrix is unity. Therefore the parameter g can be considered as the characteristic of the coupling strength.

Finally we turn to the degenerate case $l=0$, It corresponds to $\gamma'=\infty$, $S_a(\theta)\equiv 0$ and the expressions (3.1) and (3.10) come to

$$S(\theta)=-i\frac{\mathrm{sh}\left[\frac{\pi}{\gamma}(i\pi-\theta)\right]}{\sin\left[\frac{\pi^2}{\gamma}\right]}S_r(\theta)\,;$$

$$\begin{aligned} S_t(\theta)&=-i\frac{\mathrm{sh}\left[\frac{\pi}{\gamma}\theta\right]}{\sin\left[\frac{\pi^2}{\gamma}\right]}S_r(\theta)\\ &=\exp\left\{\frac{1}{2}\int_0^\infty\frac{\sin\left[\frac{2t}{\gamma}(i\pi-\theta)\right]\mathrm{sh}\left[\frac{t(\gamma-\pi)}{\gamma}\right]}{\mathrm{sh}(t)\,\mathrm{ch}\left(\frac{\pi t}{\gamma}\right)}\frac{dt}{t}\right\}. \end{aligned} \tag{4.8}$$

Thus, at $l=0$ the solution (3.1), (3.10) reduces to the sine-Gordon S-matrix [1, 2]. Obviously, the restriction (4.4) is not essential in this case.

5. The Factorized S-Matrices and the Baxter-Type Transfer Matrices

Consider the multiplet of particles A_i, $(i=1,\ldots,n)$ of equal mass m. The general factorized S-matrix describing the scattering of A_i's can be represented by some algebra with generators $A_i(\theta)$ satisfying the commutation relations (see [1] and Sect. 2)

$$A_i(\theta_1)A_\alpha(\theta_2)=S_{ij}^{\alpha\beta}(\theta_{12})A_\beta(\theta_2)A_j(\theta_1)\,, \tag{5.1}$$

where the summing over repeated indices from 1 to n is implies; $S_{ij}^{\alpha\beta}(\theta)$ is the two-particle S-matrix.

In this algebraic treatment the factorization equations for the two-particle S-matrix [1] arise as the requirement of associativity of production for the algebra of

A's. Consider any product of A's of the form $A_{i_1}(\theta_1)A_{i_2}(\theta_2)\ldots A_{i_r}(\theta_r)$. The commutation rules (5.1) enable one to recorder it to the products of the type $A_{j_1}(\theta_{P_1})A_{j_2}(\theta_{P_2})\ldots A_{j_r}(\theta_{P_r})$ where P is some transmutation of numbers $1, 2, \ldots, r$. The associativity requires the result of the recording to be independent on the way in which the pair commutations are performed. This can be expressed as certain functional equation on the matrix $S_{ij}^{\alpha\beta}(\theta)$. One can obtain it by reordering the in-product $A_i(\theta_1)A_j(\theta_2)A_k(\theta_3)$ to the out-products $A_l(\theta_3)A_p(\theta_2)A_q(\theta_1)$ in two possible successions of pair commutations and equalizing the results. This equation is

$$S_{ik}^{\alpha\mu}(\theta)S_{kj}^{\beta\nu}(\theta')S_{\mu q}^{\nu p}(\theta'-\theta)=S_{\alpha\nu}^{\beta\mu}(\theta'-\theta)S_{ik}^{\mu p}(\theta')S_{kj}^{\nu q}(\theta),\tag{5.2}$$

where again we imply the summing over the repeated indices. The Eq. (5.2) should be satisfied by the two-particle amplitudes of any factorized S-matrix.

Further we shall refer to the upper (lower) indices of the matrix $S_{ij}^{\alpha\beta}$ as horizontal (vertical) ones. The Eq. (5.2) means that two matrices $S_{ik}^{\alpha\mu}(\theta)S_{kj}^{\beta\nu}(\theta')$ and $S_{ik}^{\mu p}(\theta')S_{kj}^{\nu q}(\theta)$ are related by matrix conjugation in horizontal indices [the conjugating matrix being $S(\theta'-\theta)$]. Actually, such a relation is just the condition for the matrix R from (1.1) forcing the commutativity of transfer matrices (1.2). To see this, consider the square lattice with N columns and put the matrix $S_{i_J j_J}^{\alpha_J\alpha_{J+1}}(\theta)$ into the correspondence with J-th lattice vertex of a certain row. Then one can define the matrix

$$T_{\{i\}\{j\}}^{\alpha_1\alpha_{N+1}}(\theta)=S_{i_1j_1}^{\alpha_1\alpha_2}(\theta)S_{i_2j_2}^{\alpha_2\alpha_3}(\theta)\ldots S_{i_Nj_N}^{\alpha_N\alpha_{N+1}}(\theta).\tag{5.3}$$

It follows from (5.2) that

$$S_{\alpha\nu}^{\beta\mu}(\theta'-\theta)\,T_{\{i\}\{i'\}}^{\mu p}(\theta')\,T_{\{i'\}\{j\}}^{\nu q}(\theta)=T_{\{i\}\{j'\}}^{\alpha\mu}(\theta)\,T_{\{j'\}\{j\}}^{\beta\nu}(\theta')S_{\mu q}^{\nu p}(\theta'-\theta).\tag{5.4}$$

Consequently, for the transfer matrix

$$T(\theta)=T^{\alpha\alpha}(\theta)\tag{5.5}$$

the relation (1.2) is valid, with the variable θ replacing the parameter v.

Thus, each factorized S-matrix in two dimensions is connected to some lattice statistical model which is defined by the transfer matrix (5.5). The latter is of Baxter type [i.e., satisfies (1.2)] and the model is apparently exactly soluble. The S-matrix described in Sects. 2 and 3 corresponds to Baxter's transfer matrix [7] itself. This explains the coincidence between the ratios of the two-particle amplitudes (3.6) and the ratios of vertex weights in the Baxter model (see (5.7) of [7]):

$$a:b:c:d=\sigma_r:-\sigma_a:\sigma:\sigma_t\tag{5.6}$$

if

$$v=\lambda\theta+\eta.\tag{5.7}$$

The correspondence between the factorized S-matrices and transfer matrices could be enlarged to include, for instance, the cases of different spaces labelled by the horizontal and vertical indices of the vertex matrix $S_{ij}^{\alpha\beta}(\theta)$, but this point is beyond the scope of this paper.

6. Discussion

The relativistic S-matrix described in Sects. 2–4 provides a self-consistent scattering theory in two space-time dimensions, satisfying all the requirements of analyticity and unitarity. In principle, one can expect that there is some field-theoretic description of this scattering theory. The corresponding quantum field theory, if it exists, should possess the infinite number of conservation laws (to force the S-matrix factorization) and also Z_4 symmetry (explicit or dynamical).

The characteristic property of Z_4-symmetric S-matrix of Sects. 2–4 is its periodic (with real period) dependence on θ. At high energies ($s \gg m^2$) this implies the periodic behaviour of all the amplitudes and cross-sections with $\ln s$. Such a behaviour of some quantities (in fact, of effective charges) is just the property of the renormalizable field theories with ultraviolet limiting cycle [9]. Although there is no direct relation between the effective charges and the scattering amplitudes, we suggest this very type of field theory to correspond to the S-matrix (3.6), (3.10).

At present a considerable number of factorized S-matrizes in two dimensions have been explicitly constructed [3–6]. As it is shown in Sect. 5, each of them can be used to produce some Baxter-type lattice statistical model. We would like to note in this connection the following point. The factorization Eqs. (5.2) ensuring the commutativity of transfer matrices (5.5) determine the matrix $S_{ij}^{\alpha\beta}(\theta)$ up to an arbitrary θ-dependent factor. If considered in the S-matrix context, this factor should be chosen to satisfy the unitarity and crossing symmetry conditions for the S-matrix. One can note, comparing Eq. (3.10) of this paper and Eq. (7.7) of [7], that the inclusion of this "unitarizing" factor into the vertex weights of the Baxter model renormalizes the infinite-volume partition function to become unity. In other words, the partition function of Baxter model arises in solving the unitarity and analyticity equations for corresponding factorized S-matrix. At present we do not know any satisfactory explanation of this fact. It would be interesting to clarify this point.

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Appendix A

To examine the system of Eqs. (2.5) let us introduce the ratios

$$h(\theta)=\frac{S(\theta)}{S_r(\theta)}; \quad h_a(\theta)=\frac{S_a(\theta)}{S_r(\theta)}; \quad h_t(\theta)=\frac{S_t(\theta)}{S_r(\theta)}. \tag{A.1}$$

Then the equations take the form

$$h(\theta)h_a(\theta+\theta')+h_a(\theta)h_t(\theta+\theta')h_t(\theta')=h_a(\theta)h(\theta+\theta')h(\theta')+h(\theta)h_a(\theta'), \tag{A.2a}$$

$$h(\theta)h_a(\theta+\theta')h_t(\theta')+h_a(\theta)h_t(\theta+\theta')=h_t(\theta)h_a(\theta+\theta')h(\theta')+h_t(\theta+\theta')h_a(\theta'), \tag{A.2b}$$

$$h(\theta)h_t(\theta+\theta')+h_a(\theta)h_a(\theta+\theta')h_t(\theta')=h_t(\theta')+h_t(\theta)h(\theta+\theta'), \tag{A.2c}$$

$$h(\theta)h(\theta')+h_a(\theta)h(\theta+\theta')h_a(\theta')=h_t(\theta)h_t(\theta')+h(\theta+\theta'). \tag{A.2d}$$

Setting $\theta'=0$ in (A.2c) one obtains

$$[h_a^2(\theta)-1]h_t(0)=0.$$

Leaving the trivial possibility $h_a^2\equiv 1$ (which leads to $S(\theta)\equiv S_t(\theta)$, $S_r(\theta)\equiv \pm S_a(\theta)$) we find

$$h_t(0)=0. \tag{A.3}$$

At $\theta'=0$ the Eqs. (A.2b) and (A.2d) give

$$h_a(\theta)[1-h(0)]=h_a(0),$$
$$h_a(\theta)h_a(0)=1-h(0)$$

and hence

$$h_a(0)=0; \quad h(0)=1. \tag{A.4}$$

Furthermore, differentiating the Eqs. (A.2) with respect to θ and then setting $\theta=0$, one obtains

$$h_a'=\alpha_a[h^2-h_a^2], \tag{A.5a}$$
$$h_a'h_t+\alpha h_a h_t+\alpha_a h_t=h_t'h_a+\alpha_t h_a h, \tag{A.5b}$$
$$h_t'=\alpha_t h-\alpha h_t-\alpha_a h_a h_t, \tag{A.5c}$$
$$h'=\alpha h-\alpha_t h_t+\alpha_a h_a h, \tag{A.5d}$$

where the prime means the derivative and we have denoted

$$\alpha=h'(0); \quad \alpha_t=h_t'(0); \quad \alpha_a=h_a'(0).$$

From (A.5c) and (A.5d) it follows that

$$(hh_t)'=\alpha_t(h^2-h_t^2)=\frac{\alpha_t}{\alpha_a}h_a'$$

and one finds [using (A.3) and (A.4)]

$$\alpha_t h_a(\theta)=\alpha_a h(\theta)h_t(\theta). \tag{A.6}$$

Substituting the derivatives from (A.5a) and (A.5c) to (A.5b) and taking into account (A.6) one can obtain

$$\alpha_t(h^2+h_t^2-h_a^2-1)=2\alpha hh_t. \tag{A.7}$$

Now we can square Eq. (A.5c) and use (A.6) and (A.7) to exclude $h(\theta)$ and $h_a(\theta)$ from the right-hand side. This leads to

$$(h_t')^2=\alpha_t^2+(\alpha^2-\alpha_t^2-\alpha_a^2)h_t^2+\alpha_a^2h_t^4. \tag{A.8}$$

Similarly, one can find

$$(h')^2=\alpha_t^2+(\alpha^2-\alpha_t^2-\alpha_a^2)h^2+\alpha_a^2h^4. \tag{A.9}$$

The solution of (A.8) and (A.9) satisfying the conditions (A.3), (A.4) is given by formulae[3]

$$h(\theta)=\frac{\operatorname{sn}(\mu\theta+2\xi,l)}{\operatorname{sn}(2\xi,l)}, \tag{A.10a}$$

$$h_t(\theta)=-\frac{\operatorname{sn}(\mu\theta,l)}{\operatorname{sn}(2\xi,l)}, \tag{A.10b}$$

where the parameters μ, ξ and modulus l of Jacobian elliptic function $\operatorname{sn}(u,l)$ (see 8.14 of [10] for the definition) are related to α, α_a and α_t as follows

$$\alpha_t=-\frac{\mu}{\operatorname{sn}(2\xi,l)}; \quad \alpha=\frac{\mu\operatorname{cn}(2\xi,l)\operatorname{dn}(2\xi,l)}{\operatorname{sn}(2\xi,l)}; \quad \alpha_a^2=l^2\mu^2\operatorname{sn}^2(2\xi,l). \tag{A.11}$$

Finally, from (A.6), (A.10), and (A.11) we obtain

$$h_a(\theta)=\pm l\operatorname{sn}(\mu\theta,l)\operatorname{sn}(\mu\theta+2\xi,l). \tag{A.12}$$

Appendix B

It is useful to introduce the function

$$f(\theta)=\ln\sigma(\theta) \tag{B.1}$$

which should satisfy the equations

$$f(\theta)=f(i\pi-\theta), \tag{B.2}$$

$$f(\theta)+f(-\theta)=\ln\left[\frac{\operatorname{sn}^2(2\eta,k)}{\operatorname{sn}^2(2\eta,k)-\operatorname{sn}^2(\lambda\theta,k)}\right]. \tag{B.3}$$

At first we note that the argument of ln in the right-hand side of (B.3) can be represented in terms of elliptic theta functions of modulus k (see § 8.180 of [10]) as follows

$$\frac{\operatorname{sn}^2(2\eta,k)}{\operatorname{sn}^2(2\eta,k)-\operatorname{sn}^2(\lambda\theta,k)}=\frac{\Theta^2(2\eta+i\mathbf{K}')\Theta^2(\lambda\theta)}{\Theta(2\eta+\lambda\theta+i\mathbf{K}')\Theta(2\eta-\lambda\theta+i\mathbf{K}')\Theta^2(0)}. \tag{B.4}$$

Now one can use the Fourier expansion (see §§ 8.181, 8.192, 1.448 of [10])

$$\ln\Theta(u)=\ln\gamma-\sum_{n=1}^{\infty}\frac{2q^n}{n(1-q^{2n})}\cos\left(\frac{\pi n u}{\mathbf{K}}\right), \tag{B.5}$$

where $q=\exp(-\pi\mathbf{K}'/\mathbf{K})$, $\gamma=\prod_{n=1}^{\infty}(1-q^{2n})$, which is convergent in the strip $|\operatorname{Im}u|<\mathbf{K}'$, to represent the right-hand side of (B.3) in the form

$$\sum_{n=1}^{\infty}\frac{16q^n}{n(1-q^{2n})}\sin^2\left[\frac{\pi n(2\eta+i\mathbf{K}')}{2\mathbf{K}}\right]\sin^2\left[\frac{\pi n\lambda\theta}{2\mathbf{K}}\right]. \tag{B.6}$$

3 The sign minus in (A.10b) is chosen for the sake of the following convenience only. It is changed under translation $\xi\to\xi+\mathbf{K}_l$ which leaves (A.10a) inaffected

If η is pure imaginary and $2i\eta < \mathbf{K}'$ this series is convergent in the strip $|\mathrm{Im}\,\theta| < \pi$ [we recall (3.5)].

Let us introduce the following infinite set of functions

$$f_n(\theta) = -\frac{\sin\left(\frac{\pi n\lambda\theta}{2\mathbf{K}}\right)\sin\left(\frac{\pi n\lambda(i\pi-\theta)}{2\mathbf{K}}\right)}{2\cos\left(\frac{\pi n\eta}{\mathbf{K}}\right)}; \tag{B.7}$$

($n = 1, 2, \ldots$) which satisfies, as one can easiely verify, the relations

$$\begin{aligned} f_n(\theta) &= f_n(i\pi - \theta), \\ f_n(\theta) + f_n(-\theta) &= \sin^2\left(\frac{\pi n\lambda\theta}{2\mathbf{K}}\right). \end{aligned} \tag{B.8}$$

Then the expression

$$f(\theta) = 8\sum_{n=1}^{\infty} \frac{\sin^2\left[\frac{\pi n(2\eta + i\mathbf{K}')}{2\mathbf{K}}\right]}{n\,\mathrm{sh}[\pi n\mathbf{K}'/\mathbf{K}]} f_n(\theta) \tag{B.9}$$

satisfies (B.2) and (B.3), being regular and bounded in the physical strip $0 < \mathrm{Im}\,\theta < \pi$ at $2i\eta < \mathbf{K}'$.

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