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Computation of Quantum Fluctuations Around Multi-Instanton Fields from Exact Green's Functions: The CP^{n-1} Case

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Abstract. We calculate exactly the contribution of instanton fields to the partition function of $\mathbb{CP}^{n-1}$ models in two dimensions. For $n=2$, the pure instanton gas is infrared finite, infinitely dense and generates a mass dynamically. For $n \geqq 3$, the gas corresponds to a system with complicated n-body interactions, whose properties are yet to be explored.

1. Introduction

Previous work [1] based on the $1/n$ expansion of $\mathbb{CP}^{n-1}$ models [2] in two dimensions revealed that topologically non-trivial fields make a significant contribution to the low energy dynamics of the fundamental particles in these theories. How much of these effects is due to instantons was not clear, however, because their contribution appeared to be infrared divergent. In fact, the single instanton contribution is proportional to

$$\int_0^\infty d\lambda \lambda^{n-3}; \quad \lambda: \text{scale size of the instanton,} \tag{1}$$

which diverges for large λ. Assuming the instanton gas is dilute does not help but merely exponentiates the divergence. In other words, the thermodynamic limit of the dilute instanton gas does not exist.

Equation (1) says that large instantons are more probable than small ones so that the instanton gas may well be dense, i.e. the average thickness of an instanton might be much larger than the mean separation between instantons. Once the diluteness assumption is dropped, it may turn out that the exact instanton gas has a thermodynamic limit. In particular, for the partition function Z of the instanton gas in a volume V this would mean that

$$\ln Z = p.V \quad (V \to \infty) \tag{2}$$

where p is proportional to the pressure of the gas. To investigate the question of

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whether or not the instanton gas behaves thermodynamically, one has to compute the contribution of dense multi-instanton configurations to the functional integral.

The main difficulty in the derivation of the exact instanton gas is the explicit computation of the determinant of the fluctuation operator Δ in a multi-instanton background field. In outline, our strategy to solve this problem looks simple. Let P_0 be the projector onto the zero modes of Δ and

$$\Gamma = \mathrm{Tr}\ln(\Delta + P_0). \tag{3}$$

Varying Γ with respect to the parameters of the general multi-instanton solution, we obtain

$$\delta\Gamma = \mathrm{Tr}\{(G + P_0)(\delta\Delta + \delta P_0)\} \tag{4}$$

where G is the Green's function for Δ:

$$\Delta G = 1 - P_0; \quad P_0 G = G P_0 = 0. \tag{5}$$

G and P_0 can be computed explicitly so that Γ can be calculated from Eq. (4) up to a number independent of the instanton parameters. Finally, this number is determined by considering a special symmetric background field, where the eigenvalues of $\ln(\Delta + P_0)$ are known and can be summed by standard methods (see e.g. [3]).

The computations sketched above are complicated by the fact that Γ as given in Eq. (3) is ill defined and must be regularized. To cope with the ultraviolet divergences we shall introduce Pauli-Villars regulator fields (details are given in Sect. 4). To make the spectrum of Δ discrete, one could put the system in a finite box with appropriate boundary conditions. However, a far more convenient way is to formulate the theory in a spherical space-time with world radius R and to take $R \to \infty$ at the end of all calculations. Due to conformal invariance, the classical theory is independent of R, but after quantization, R can no longer be scaled away. This R dependence is governed by the conformal anomaly, see [4].

The one instanton contribution to the partition function of the $\mathbb{CP}^1$ model has previously been calculated [5] following 't Hooft's method [3]. While completing this paper, we received a preprint by Fateev, Frolov and Schwarz [6], where they derive the exact instanton gas for the $\mathbb{CP}^1$ case. Their method is different from ours, but the final formulae are the same.

Except for a few remarks in Sects. 7 and 8, we shall not elaborate on the physical significance of our results, but discuss in great detail the derivation of the exact instanton gas. We assume the reader is familiar with the definition and basic properties of the classical $\mathbb{CP}^{n-1}$ model [1, 2]. In Sect. 2 we project these models onto a spherical universe and discuss the instanton solutions in the new formulation. Fluctuations around multi-instanton fields are considered in Sect. 3 and the full regularized instanton gas is set up in Sect. 4. We then proceed to compute the regularized determinant of the fluctuation operator Δ. We first do this for the $\mathbb{CP}^1$ case (Sect. 5), which is much simpler than the general case (Sect. 6). Our results are summarized and discussed in Sect. 7, and conclusions are drawn in the final Sect. 8.

2. $\mathbb{CP}^{n-1}$ models in a Spherical Universe

$\mathbb{CP}^{n-1}$ models in flat Euclidean space-time $\mathbb{R}^2$ describe fields

$$z_\alpha(x); \quad \alpha = 1, \ldots, n; \quad x = (x_1, x_2); \quad |z|^2 = 1 \tag{6}$$

of complex unit vectors. Not all degrees of freedom of z_α are considered physical: fields z_α and z'_α related by a gauge transformation

$$z'_\alpha(x) = e^{i\Lambda(x)} z_\alpha(x) \tag{7}$$

should be identified. The gauge invariant action is

$$S = \frac{n}{2f} \int d^2x \overline{D_\mu z} \cdot D_\mu z; \quad D_\mu = \partial_\mu - \bar{z} \cdot \partial_\mu z. \tag{8}$$

Fields $z_\alpha(x)$ approaching the classical vacuum as $|x| \to \infty$,

$$z_\alpha(x) = h(x) v_\alpha; \quad |h| = 1; \quad v_\alpha = \text{constant} \tag{9}$$

fall into topological classes characterized by the winding number

$$Q = \frac{i}{2\pi} \int_{|x| = \infty} dx_\mu \cdot h^{-1}(x) \partial_\mu h(x). \tag{10}$$

The action S is conformally invariant so that the whole theory can be projected onto the conformal compactification S^2 of space-time $\mathbb{R}^2$. Points of S^2 are labelled by

$$r_a; \quad a = 1, 2, 3; \quad r_a r_a = R^2. \tag{11}$$

$\mathbb{R}^2$ is imbedded into S^2 via the stereographic projection

$$x_\mu = \frac{R r_\mu}{R + r_3} \quad (\mu = 1, 2) \tag{12}$$

$$r_\mu = \frac{2R^2 x_\mu}{R^2 + x^2}; \quad r_3 = R \frac{R^2 - x^2}{R^2 + x^2}.$$

Correspondingly, the field $z_\alpha(x)$ projects onto a field $\mathfrak{z}_\alpha(r)$ by

$$\mathfrak{z}_\alpha(r) = z_\alpha(x(r)).$$

From Eqs. (9) and (10) we see that for $Q \neq 0$, $\mathfrak{z}_\alpha(r)$ is necessarily discontinuous at infinity ($r = (0, 0, -R)$). This discontinuity can be moved around on the sphere by making gauge transformations, but it cannot be transformed away. It can properly be handled by choosing patches on the sphere, setting up a principal $U(1)$-bundle and identifying $\mathfrak{z}_\alpha$ with a smooth cross section of this bundle. For calculational purposes, this is not very practical and we therefore resort to another method borrowed from the theory of induced representations (see e.g. [7, § 5.3.3]). It is based on the observation that

$$S^2 \cong \mathrm{SU}(2)/U(1) \tag{13}$$

where $U(1)$ is the subgroup of SU(2) generated by σ^3 (σ^a denote the three Pauli matrices). More explicitly, the isomorphism (13) identifies the coset $g \cdot U(1) \in SU(2)/U(1)$ with the point $r_a \in S^2$ given by

$$r_a \sigma^a = R g \sigma^3 g^{-1}. \tag{14}$$

When $r \neq (0,0,-R)$, g can be decomposed uniquely and differentiably into

$$\begin{aligned} g &= u(x) \cdot e^{i(\tau/2)\sigma^3}; \quad x \in \mathbb{R}^2; \quad -2\pi \leqq \tau \leqq 2\pi \\ u(x) &= (R^2 + x^2)^{-1/2}(R + ix_2\sigma^1 - ix_1\sigma^2) \end{aligned} \tag{15}$$

where x and r are related by the stereographic map (12).

Instead of fields $\mathfrak{z}_\alpha(r)$ we now consider homogeneous fields $\mathfrak{z}_\alpha(g)$ $(g \in SU(2))$

$$|\mathfrak{z}|^2 = 1; \quad \mathfrak{z}_\alpha(g e^{i\omega\sigma^3}) = e^{-ik\omega}\mathfrak{z}_\alpha(g) \tag{16}$$

where k is some integer. $\mathfrak{z}_\alpha(g)$ projects onto the original field $z_\alpha(x)$ by

$$z_\alpha(x) = \mathfrak{z}_\alpha(u(x)). \tag{17}$$

Provided $\mathfrak{z}_\alpha(g)$ is smooth, it follows from

$$u(x) = -i\sigma^2 e^{i\varphi\sigma^3}(|x| \to \infty); \qquad x = |x|(\cos\varphi, \sin\varphi)$$

and the homogeneity property of $\mathfrak{z}_\alpha(g)$ that the topological charge of $z_\alpha(x)$ is equal to k. Conversely, any smooth field $z_\alpha(x)$ with charge k is gauge equivalent to a field obtained via Eq. (17) from a smooth field $\mathfrak{z}_\alpha(g)$ homogeneous of degree k. Thus, homogeneous fields $\mathfrak{z}_\alpha(g)$ on the group SU(2) provide a complete and non-singular description of $\mathbb{CP}^{n-1}$ fields on the sphere.

We next proceed to formulate the action S in terms of $\mathfrak{z}_\alpha(g)$. A convenient set of differential operators acting on functions $f(g)$, $g \in SU(2)$, is

$$(I_a f)(g) = \frac{1}{i}\frac{d}{dt} f(g \cdot e^{i(t/2)\sigma^a})\big|_{t=0}; \quad a = 1,2,3. \tag{18}$$

More explicitly, when g is parametrized by x and τ as in Eq. (15), $I_\pm = I_1 \pm iI_2$ and I_3 take the form

$$\begin{aligned} I_+ &= -\frac{e^{i\tau}}{R}\{(R^2 + |s|^2)\partial_s - i\bar{s}\partial_\tau\} \\ I_- &= \frac{e^{-i\tau}}{R}\{(R^2 + |s|^2)\partial_{\bar{s}} + is\partial_\tau\} \\ I_3 &= \frac{1}{i}\partial_\tau. \end{aligned} \tag{19}$$

Here, s denotes the complex variable $x_1 - ix_2$. Note that the operators I_a are self-adjoint with respect to the natural scalar product

$$(f_1, f_2) = \int dg \bar{f}_1(g) f_2(g) \tag{20}$$

dg: Haar measure on SU(2); $\int dg = 1$

and that they satisfy the angular momentum algebra

$$[I_a, I_b] = i\varepsilon_{abc} I_c$$

(they are the generators of the right regular representation of SU(2)). The action Eq. (8) can now be written in the symmetric form

$$S = \frac{2\pi n}{f} \int dg \, |J_a \mathfrak{z}|^2; \quad J_a = I_a - \bar{\mathfrak{z}} \cdot I_a \mathfrak{z}. \tag{21}$$

S is invariant under smooth gauge transformations

$$\mathfrak{z}'_\alpha(g) = e^{i\Lambda(g)} \mathfrak{z}_\alpha(g); \quad \Lambda(g \cdot e^{i\omega\sigma^3}) = \Lambda(g). \tag{22}$$

Instanton solutions are easily described within the new formulation. The projected form of the most general instanton configuration is

$$z_\alpha = \frac{p_\alpha}{|p|} \tag{23}$$

where p_α is a vector of polynomials of $s = x_1 - ix_2$ with no common root. The instanton number k is equal to the maximal degree of the p_α's. It is not difficult to show that the field $w_\alpha(g)$, $g \in SU(2)$, defined by

$$\begin{aligned} &w_\alpha(g \cdot e^{i\omega\sigma^3}) = e^{-ik\omega} w_\alpha(g) \\ &\quad w_\alpha(u(x)) = p_\alpha(x)(R^2 + x^2)^{-k/2} \end{aligned} \tag{24}$$

is smooth and satisfies

$$I_- w_\alpha = 0. \tag{25}$$

From Eqs. (16), (17), (23) and (24) we then conclude that

$$\mathfrak{z}_\alpha(g) = \frac{w_\alpha(g)}{|w(g)|} \tag{26}$$

so that

$$J_- \mathfrak{z}_\alpha = 0. \tag{27}$$

Of course, this is just the selfduality equation in the new language.

3. Fluctuations Around Multi-Instanton Fields

Let $\mathfrak{z}_\alpha(g)$ be a k instanton configuration as described in the preceding section. An arbitrary field $\tilde{\mathfrak{z}}_\alpha(g)$ with topological charge k can always be written in the form

$$\begin{aligned} &\tilde{\mathfrak{z}}_\alpha = e^{i\Lambda}\{(1 - |\eta|^2)^{1/2} \mathfrak{z}_\alpha + \eta_\alpha\} \\ &\bar{\mathfrak{z}} \cdot \eta = 0; \quad \eta_\alpha(g \cdot e^{i\omega\sigma^3}) = e^{-ik\omega} \eta_\alpha(g). \end{aligned} \tag{28}$$

Noting

$$\begin{aligned} &\tilde{J}_3 \tilde{\mathfrak{z}} = 0; \quad \tilde{J}_a = I_a - \bar{\tilde{\mathfrak{z}}} \cdot I_a \tilde{\mathfrak{z}} \\ &\int dg \{ |\tilde{J}_+ \tilde{\mathfrak{z}}|^2 - |\tilde{J}_- \tilde{\mathfrak{z}}|^2 \} = k \end{aligned}$$

the action of $\tilde{\mathfrak{z}}$ becomes

$$S = \frac{n\pi}{f} k + \frac{2n\pi}{f} \int dg \, |\tilde{J}_- \tilde{\mathfrak{z}}|^2.$$

Because $J_- \mathfrak{z}_\alpha = 0$, $\tilde{J}_- \bar{\mathfrak{z}}$ is of order η:

$$\tilde{J}_- \bar{\mathfrak{z}}_\alpha = (\delta_{\alpha\beta} - \mathfrak{z}_\alpha \bar{\mathfrak{z}}_\beta) J_- \eta_\beta + O(|\eta|^2)$$

i.e. up to higher orders in η

$$S = \frac{n\pi}{f} k + \frac{2n\pi}{f} \int dg \, |(1 - \mathfrak{z} \otimes \bar{\mathfrak{z}}) J_- \eta|^2. \tag{29}$$

It is helpful to introduce some more notation. Let $\mathscr{H}_l$, $l \in \mathbb{Z}$, be the Hilbert space of complex wave functions $\psi_\alpha(g)$, $g \in \mathrm{SU}(2)$, such that

$$\begin{aligned} &\bar{\mathfrak{z}} \cdot \psi = 0; \quad \psi(g \cdot e^{i\omega\sigma^3}) = e^{-il\omega} \psi(g) \\ &\| \psi \|^2 = \int dg \, |\psi|^2 < \infty. \end{aligned} \tag{30}$$

$\mathscr{H}_k$ is simply the space of all normalizable fluctuations η_α around $\mathfrak{z}_\alpha$. Next, define an operator

$$\begin{aligned} &T : \mathscr{H}_k \to \mathscr{H}_{k+2} \\ &(T\psi)_\alpha = (\delta_{\alpha\beta} - \mathfrak{z}_\alpha \bar{\mathfrak{z}}_\beta) J_- \psi_\beta \quad (\psi \in \mathscr{H}_k). \end{aligned} \tag{31}$$

The adjoint

$$(T^\dagger \varphi)_\alpha = (\delta_{\alpha\beta} - \mathfrak{z}_\alpha \bar{\mathfrak{z}}_\beta) J_+ \varphi_\beta \quad (\varphi \in \mathscr{H}_{k+2}) \tag{32}$$

maps $\mathscr{H}_{k+2}$ into $\mathscr{H}_k$ so that $T^\dagger T$ is an operator acting in $\mathscr{H}_k$.
The Gaussian approximation (29) to the action of $\bar{\mathfrak{z}}_\alpha$ can now be written compactly:

$$\begin{aligned} S &= \frac{n\pi}{f} k + \frac{2n\pi}{f} \| T\eta \|^2 \\ &= \frac{n\pi}{f} k + \frac{2n\pi}{f} (\eta, \Delta\eta) \end{aligned} \tag{33}$$

where

$$\Delta = T^\dagger T. \tag{34}$$

The fluctuation operator Δ has zero modes. They arise from fluctuations η, which are tangential to the k instanton manifold and can therefore be computed explicitly. The general solution of the zero mode equation

$$T\eta = 0$$

has the form

$$\eta_\alpha = (\delta_{\alpha\beta} - \mathfrak{z}_\alpha \bar{\mathfrak{z}}_\beta) \psi_\beta; \quad J_- \psi_\beta \propto \mathfrak{z}_\beta. \tag{35}$$

There are precisely $n(k+1) - 1$ linearly independent normalizable zero modes corresponding to the equal number of complex parameters of the general k instanton solution. A convenient choice of parameters is (cf. Eqs. (23)–(26))[1]

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[1] Setting $c_n = 1$ eliminates a complex overall factor, whose modulus drops out, when forming the ratio (23) and whose phase corresponds to a constant gauge rotation. The parametrization (36) is valid not for all but almost all k instanton fields

$$p_\alpha(s) = c_\alpha \prod_{j=1}^{k} (s - a_\alpha^j); \quad \alpha = 1, \ldots, n; \quad c_n = 1. \tag{36}$$

Labelling the complex parameters c_α, a_β^j by $\lambda_i, i = 1, \ldots, n(k+1) - 1$, a basis of zero modes is

$$\eta_\alpha^i = (\delta_{\alpha\beta} - \mathfrak{z}_\alpha \bar{\mathfrak{z}}_\beta) \frac{\partial}{\partial \lambda_i} \mathfrak{z}_\beta \tag{37}$$

Correspondingly, the projector P_0 onto the null space of Δ can be represented by the following integral operator

$$\begin{aligned} (P_0\psi)_\alpha(g) &= \int dg' P_0^{\alpha\beta}(g, g')\psi_\beta(g'); \quad (\psi \in \mathcal{H}_k) \\ P_0^{\alpha\beta}(g, g') &= \eta_\alpha^i(g) J_{ij}^{-1} \bar{\eta}_\beta^j(g') \end{aligned} \tag{38}$$

where

$$J_{ij} = (\eta^i, \eta^j) = \int dg \, \overline{\frac{\partial \mathfrak{z}}{\partial \lambda_i}} \cdot (1 - \mathfrak{z} \otimes \bar{\mathfrak{z}}) \cdot \frac{\partial \mathfrak{z}}{\partial \lambda_j}. \tag{39}$$

4. The Regularized Pure Instanton Gas

This section is devoted to the derivation of the pure instanton gas, the computation of the regularized determinants being postponed to the subsequent sections. We begin by writing down the functional integral for the expectation of a (gauge invariant) observable $\mathcal{O}$:

$$\langle \mathcal{O} \rangle = Z^{-1} \int \mathcal{D}[\mathfrak{z}] \mathcal{O} e^{-S}. \tag{40}$$

The partition function Z normalizes the expectations: $\langle 1 \rangle = 1$. The pure instanton gas arises from integrating (40) by the saddle point method, the saddle points being all the instanton solutions. The full semi-classical approximation to Eq. (40) requires that all finite action solutions of the second order field equations (in particular: anti-instantons) are taken into account, and possibly other "almost exact" solutions such as dilute instanton anti-instanton configurations. Here, we concentrate on the pure instanton gas (see, however, Sect. 8).

The saddle point approximation to Eq. (40) in the k instanton sector goes as follows. Let $\mathfrak{z}_\alpha(g, \lambda)$ be the general k instanton solution parametrized as in the preceding section (Eqs. (23), (24), (26) and (36)). Denote by $e_\alpha^i(g, \lambda), i = 1, 2, \ldots,$ a complete set of orthonormal eigenvectors with eigenvalues $E_i \neq 0$ of the fluctuation operator Δ. Any field $\tilde{\mathfrak{z}}_\alpha(g)$ with topological charge k, which is sufficiently close to the instanton manifold can be parametrized by (cp. Eq. (28))

$$\tilde{\mathfrak{z}}_\alpha(g) = e^{iA(g)} \left\{ (1 - |\xi|^2)^{1/2} \mathfrak{z}_\alpha(g, \lambda) + \sum_{i=1}^{\infty} \xi_i e_\alpha^i(g, \lambda) \right\} \tag{41}$$

the (complex) parameters being λ_i and ξ_i. Briefly, $\mathfrak{z}_\alpha(g, \lambda)$ is the instanton solution closest to $\tilde{\mathfrak{z}}_\alpha(g)$ and the parameters ξ_i measure the displacement of $\tilde{\mathfrak{z}}_\alpha$ away from the instanton manifold.

We next insert the parametrization (41) into Eq. (40) and make two approximations: firstly, the action S is replaced by the terms of zeroth and second order in ξ (cp. Eq. (33))

$$S=\frac{n\pi}{f}k+\frac{2n\pi}{f}\sum_{i=1}^{\infty}E_i|\xi_i|^2. \tag{42}$$

Secondly, the observable $\mathcal{O}$ is replaced by its value $\mathcal{O}(\lambda)$ for the instanton field $\mathfrak{z}_\alpha(g,\lambda)$. The pure instanton gas expectation of $\mathcal{O}$ then becomes

$$\begin{aligned}\langle\mathcal{O}\rangle_{\text{inst.}}=Z^{-1}\sum_{k=0}^{\infty}(k!)^{-n}\int\prod_{j,i}d^2\lambda_j d^2\xi_i J(\lambda,\xi)\mathcal{O}(\lambda)\\ \cdot\exp-\left\{\frac{n\pi}{f}k+\frac{2n\pi}{f}\sum_{i=1}^{\infty}E_i|\xi_i|^2\right\}.\end{aligned} \tag{43}$$

The Jacobian $J(\lambda,\xi)$ comes from

$$\mathscr{D}[\mathfrak{z}]=\prod_{j,i}d^2\lambda_j d^2\xi_i J(\lambda,\xi). \tag{44}$$

To compute it, we have to make precise what $\mathscr{D}[\mathfrak{z}]$ means. Let M_k be the manifold of all gauge equivalence classes of fields $\mathfrak{z}_\alpha(g)$ with topological charge k. M_k carries a natural metric: given a path $\mathfrak{z}_\alpha(g,t)$ in M_k, the length $\left\|\frac{d}{dt}\right\|$ of the tangential vector $\frac{d}{dt}$ is

$$\left\|\frac{d}{dt}\right\|^2=\int dg\,\frac{\partial\bar{\mathfrak{z}}}{\partial t}\cdot(1-\mathfrak{z}\otimes\bar{\mathfrak{z}})\cdot\frac{\partial\mathfrak{z}}{\partial t}.$$

Note that this is independent of the gauge chosen along the path. The metric described here naturally induces a measure $\mathscr{D}[\mathfrak{z}]$ on M_k. More explicitly, when a portion of M_k is parametrized by a set of real coordinates $\omega_1,\omega_2,\ldots$, we have

$$\mathscr{D}[\mathfrak{z}]=\prod_i d\omega_i[\det(G_{ij})]^{1/2}$$

where G_{ij} is the metric tensor in this coordinate system. In our case the coordinates ω_i are the real and imaginary parts of λ_j and ξ_i. A short calculation then yields

$$J(\lambda,\xi)=\det(J_{ij}) \tag{45}$$

with J_{ij} being the zero mode matrix (39).

Since J is independent of ξ_i, the integrals over ξ in Eq. (43) are Gaussian so that

$$\langle\mathcal{O}\rangle_{\text{inst.}}=Z^{-1}\sum_{k=0}^{\infty}(k!)^{-n}\int\prod_j d^2\lambda_j J(\lambda)\mathcal{O}(\lambda)e^{-(n\pi/f)k}\left[\det\left(\frac{2n}{f}\Delta\right)\right]^{-1}.$$

(the zero modes of Δ should be omitted in the determinant). At this stage, we have to regularize the theory, because det Δ is divergent. To this end, we introduce complex, scalar Pauli-Villars regulator fields $\phi_\alpha^i(g)$, $i=1,\ldots,v$, with large masses

M_i and alternating "metric" e_i to be chosen such that

$$\sum_{i=1}^{\nu} e_i = -1; \quad \sum_{i=1}^{\nu} e_i M_i^{2p} = 0 \quad (p = 1, \ldots, \nu - 1). \tag{46}$$

The regulator fields interact with the fundamental field $\mathfrak{z}_\alpha$ via the constraints (in the k instanton sector)

$$\bar{\mathfrak{z}} \cdot \phi^j = 0; \quad \phi^j_\alpha(g e^{i\omega\sigma^3}) = e^{-ik\omega} \phi^j_\alpha(g)$$

and the action[2]

$$S_{\text{reg}} = \frac{2\pi n}{f} \sum_{i=1}^{\nu} \int dg \bar{\phi}^i \cdot (\Delta(\mathfrak{z}) + (M_i R)^2) \cdot \phi^i.$$

Taking into account the contribution from the regulator fields we arrive at

$$\begin{aligned} \langle \mathcal{O} \rangle_{\text{inst.}} = Z^{-1} \sum_{k=0}^{\infty} (k!)^{-n} \int \prod_j \left(\frac{2n}{f} d^2 \lambda_j \right) J(\lambda) \mathcal{O}(\lambda) \\ \cdot \exp - \left\{ \frac{n\pi}{f} k + \Gamma_{\text{reg}} \right\} \end{aligned} \tag{47}$$

where

$$\Gamma_{\text{reg}} = \operatorname{Tr} \left\{ \ln(\Delta + P_0) + \sum_{i=1}^{\nu} e_i \ln(\Delta + (M_i R)^2) \right\}. \tag{48}$$

When $\nu \geqq 2$, this expression is finite and will be computed in the subsequent sections. Equation (47) is our final formula for the regularized pure instanton gas. After computing Γ_{reg} we shall remove the UV cutoff while renormalizing the coupling constant f and also take the world radius R to infinity.

5. Computation of Γ_{reg}: the $\mathbb{CP}^1$ Case

For $n = 2$, the computation of Γ_{reg} can be reduced to the calculation of the determinant of a Dirac operator in the presence of an *Abelian* external gauge potential. This problem has been solved long ago by Schwinger [8] in the case $k = 0$. When $k \neq 0$, some complications arise from the zero modes, but the calculations are still fairly simple.

The simplifications referred to root in the fact that any fluctuation $\eta_\alpha(g) \in \mathscr{H}_k$ (cf. Sect. 3) can be written as

$$\eta_\alpha(g) = \varepsilon_{\alpha\beta} \bar{\mathfrak{z}}_\beta(g) \chi(g); \quad \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}, \varepsilon_{12} = 1 \tag{49}$$

where χ is some *unconstrained* scalar amplitude. More technically speaking, Eq. (49) identifies $\mathscr{H}_k$ with $L^2(\text{SU}(2), 2k)$, the Hilbert space of all wave functions

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[2] The mass term is chosen proportional to $(M_i R)^2$ rather than M_i^2 so that when projected onto the plane, it becomes $4M_i^2 R^4 (R^2 + x^2)^{-2}$. As $R \to \infty$, this smoothly approaches $4M_i^2$

$\chi(g)$ with

$$\int dg|\chi|^2 < \infty; \quad \chi(ge^{i\omega\sigma^3}) = e^{-i2k\omega}\chi(g).$$

Similarly, $\mathscr{H}_l$ can be identified with $L^2(\mathrm{SU}(2), k+l)$. The operator T then maps $L^2(\mathrm{SU}(2), 2k)$ into $L^2(\mathrm{SU}(2), 2k+2)$,

$$T = I_- + I_-(\ln|w|^2)_- \tag{50}$$

and

$$T^+ = I_+ - I_+(\ln|w|^2) \tag{51}$$

maps $L^2(\mathrm{SU}(2), 2k+2)$ into $L^2(\mathrm{SU}(2), 2k)$. In what follows, we need not know that $|w|^2$ comes from an instanton solution and consider therefore the general case, where $|w|^2$ is replaced by any smooth positive function $\rho(g)$, which is homogeneous of degree zero:

$$\rho(g\cdot e^{i\omega\sigma^3}) = \rho(g).$$

A basis of zero modes of T is then given by (cf. Eq. (15))

$$\chi^j(u(x)\cdot e^{i(\tau/2)\sigma^3}) = \frac{s^j e^{-ik\tau}}{\rho(u(x))(R^2+x^2)^k} \quad (j=0,1,\ldots,2k) \tag{52}$$

so that the projector P_0 can be represented by

$$\begin{aligned} P_0(g,g') &= \chi^i(g)N_{ij}^{-1}\bar{\chi}^j(g') \\ N_{ij} &= \int dg\bar{\chi}^i(g)\chi^j(g). \end{aligned} \tag{53}$$

We now proceed to calculate Γ_{reg} along the lines sketched in the introduction. The computations are broken up into several steps.

5.1. The Exact Green's Function for Δ

The Green's function G(cf. Eq. (5)) can be written in the form

$$G = G_- G_+ \tag{54}$$

$$TG_- = 1; \quad P_0 G_- = 0 \tag{55}$$

$$T^+ G_+ = 1 - P_0. \tag{56}$$

The operator G_- maps $L^2(\mathrm{SU}(2), 2k+2)$ into $L^2(\mathrm{SU}(2), 2k)$ and G_+ is the adjoint of G_-. Equation (55) is solved by the Ansatz

$$G_- = (1-P_0)\rho^{-1}\check{G}_-\rho; \quad I_-\check{G}_- = 1. \tag{57}$$

Because I_- is invariant under left multiplications on the group SU(2), the integral kernel $\check{G}_-(g,g')$ of $\check{G}_-$ can be chosen left invariant:

$$\check{G}_-(g,g') = \gamma_{2k}(g'^{-1}\cdot g). \tag{58}$$

As $\check{G}_-$ maps $L^2(\mathrm{SU}(2), 2k+2)$ into $L^2(\mathrm{SU}(2), 2k)$, γ_{2k} must be homogeneous

$$\gamma_{2k}(e^{-i\omega'\sigma^3}\cdot g\cdot e^{i\omega\sigma^3}) = e^{i2k(\omega'-\omega)}e^{i2\omega'}\gamma_{2k}(g) \tag{59}$$

γ_{2k} therefore depends on essentially only one variable so that the equation $I_-\check{G}_- = 1$ reduces to an ordinary first order differential equation for γ_{2k}. This equation is easily solved, when written out in coordinates x and τ (cp. Eq. (15)). The outcome is

$$\gamma_{2k}(e^{i(\beta/2)\sigma^2}) = -\frac{\left(\cos\frac{\beta}{2}\right)^{2k+1}}{\sin\frac{\beta}{2}} \tag{60}$$

which completes the description of γ_{2k} and therefore of the full Green's function G.

For later use, note that

$$\check{P}_0\check{G}_- = 0, \tag{61}$$

$\check{P}_0$ being the projector onto the zero modes of I_- in $L^2(\mathrm{SU}(2), 2k)$. Also, the adjoint $\check{G}_+$ of $\check{G}_-$ satisfies

$$I_+\check{G}_+ = 1 - \check{P}_0. \tag{62}$$

5.2. Computation of the Variation $\delta\Gamma_{\mathrm{reg}}$

When the external field ρ is varied, Γ_{reg} changes according to

$$\delta\Gamma_{\mathrm{reg}} = \mathrm{Tr}\left\{\delta\Delta\left[(\Delta + P_0)^{-1} + \sum_{i=1}^{\nu} e_i(\Delta + (M_iR)^2)^{-1}\right] + \delta P_0(\Delta + P_0)^{-1}\right\}. \tag{63}$$

From the explicit form (53) of the projector P_0 it follows that

$$\mathrm{Tr}\{\delta P_0(\Delta + P_0)^{-1}\} = \mathrm{Tr}\{\delta P_0 P_0\} = 0.$$

Noting $\delta\Delta = \delta T^+ T + T^+\delta T$, Eq. (63) can be somewhat simplified:

$$\delta\Gamma_{\mathrm{reg}} = \mathrm{Tr}\left\{\delta T^+\left[G_+ + \sum_{i=1}^{\nu} e_i T(\Delta + (M_iR)^2)^{-1}\right]\right\} + \text{c.c.} \tag{64}$$

where c.c. means complex conjugate.

$\delta T^+ = -I_+(\delta \ln \rho)$ is a local operator so that the trace operation amounts to evaluate $G_+(g, g')$ at coinciding arguments and then to integrate over g. The short distance singularities of G_+ are cancelled by the corresponding singularities of the Green's functions of the regulator fields. The short distance expansion of $G_+^M = T(\Delta + (MR)^2)^{-1}$ can be calculated perturbatively (Appendix A):

$$G_+^M(u(x), 1) = -R(1 + i\varepsilon_{\mu\nu}x_\mu\partial_\nu \ln \rho)\frac{x_1 - ix_2}{x^2} \quad (x \to 0). \tag{65}$$

Here, terms vanishing at $x = 0$ have been neglected and the coefficients of the non-zero terms are given for $M = \infty$.

The short distance behaviour of the zero mass Green's function G_+ follows from the exact expression derived in the preceding subsection 5.1:

$$G_+(u(x),1) = -R(1 + x_\mu\partial_\mu \ln\rho)\frac{x_1 - ix_2}{x^2} - \int dg'\rho(1)\check{G}_+(1,g')\rho^{-1}(g')P_0(g',1) + O(|x|). \tag{66}$$

Inserting Eqs. (65) and (66) into Eq. (64), we see that all short distance singularities cancel indeed and we are left with

$$\delta\Gamma_{\mathrm{reg}} = 2\delta\int dg(I_+ \ln\rho)(I_- \ln\rho) - \mathrm{Tr}\{\delta T^+\rho\check{G}_+\rho^{-1}P_0\} + \mathrm{c.c.} \tag{67}$$

Equation (67) can be simplified noting that

$$\begin{aligned} \delta T^+ &= -[T^+, \delta\ln\rho] \\ P_0T^+ &= 0; \quad T^+\rho\check{G}_+ = \rho I_+\check{G}_+ = \rho(1-\check{P}_0) \\ \check{P}_0\rho^{-1}P_0 &= \check{P}_0\rho^{-1}. \end{aligned} \tag{68}$$

We then arrive at

$$\delta\Gamma_{\mathrm{reg}} = 2\delta\int dg(I_+ \ln\rho)(I_- \ln\rho) + 2(2k+1)\delta\int dg \ln\rho - 2\,\mathrm{Tr}(P_0\delta\ln\rho).$$

Finally, from the explicit form (53) of the zero mode projector P_0, we deduce

$$-2\,\mathrm{Tr}(P_0\delta\ln\rho) = \delta(\ln\det N)$$

which leads to the result

$$\Gamma_{\mathrm{reg}} = \alpha(k, M_j, R) + \ln\det N + 2\int dg\{(I_+ \ln\rho)(I_- \ln\rho) + (2k+1)\ln\rho\}. \tag{69}$$

The constant α is independent of ρ and will be calculated in the next subsection.

5.3. Computation of $\alpha(k, M_j, R)$

The spectrum of Δ can be calculated explicitly in the case $\rho = 1$. Denote by

$$\langle j,m|g|j,m'\rangle; \quad j = 0, \tfrac{1}{2}, 1, \ldots \qquad m,m' = -j, -j+1, \ldots, j$$

the matrix elements of $g \in \mathrm{SU}(2)$ in the irreducible representation of SU(2) with angular momentum j. It follows from the Peter-Weyl theorem (e.g. [7, Sect. 2.8]) that the functions

$$\psi_{jm} = \langle j,m|g|j,-k\rangle; \quad j = k, k+1, \ldots$$

form a complete orthogonal basis in $L^2(\mathrm{SU}(2), 2k)$. They are also the eigenfunctions of $\Delta = I_+I_-$, the eigenvalues being

$$E_j = (j-k)(j+k+1); \quad \text{multiplicity: } 2j+1.$$

Defining $\mu = j - k - 1, e_0 = 1$ and $M_0 = 0, \Gamma_{\mathrm{reg}}$ becomes

$$\Gamma_{\mathrm{reg}} = (2k+1)\sum_{i=1}^{\nu} e_i \ln(M_iR)^2$$

$$+\sum_{\mu=0}^{\infty}\left\{\sum_{i=0}^{\nu} e_i(2\mu+2k+3)\ln\left[(\mu+1)(\mu+2k+2)+(M_iR)^2\right]\right\}. \tag{70}$$

The master formula for sums of this type is

$$\begin{aligned}
&\sum_{\mu=0}^{\infty}\left\{\sum_{i=0}^{\nu} e_i(a\mu+b)\ln\left[(\mu+\alpha_1)(\mu+\alpha_2)+(M_iR)^2\right]\right\}\\
&\quad=\sum_{i=1}^{\nu} e_i\left\{-a(M_iR)^2\ln(M_iR)+\frac{\pi}{2}\left[2b-a(\alpha_1+\alpha_2)\right]M_iR\right.\\
&\quad\left.+\left[\frac{a}{2}(\alpha_1^2+\alpha_2^2)-\frac{a}{6}-b(\alpha_1+\alpha_2-1)\right]\ln(M_iR)\right\}\\
&\quad+\frac{a}{4}(\alpha_1-\alpha_2)^2+(a\alpha_1-b)\xi'(0,\alpha_1)+(a\alpha_2-b)\xi'(0,\alpha_2)\\
&\quad-a\left[\xi'(-1,\alpha_1)+\xi'(-1,\alpha_2)\right].
\end{aligned} \tag{71}$$

Here, $\zeta(z,g)$ is Riemann's zeta function [9, Sect. 9.5]. Eq. (71) has been derived following 't Hooft's [3] method. Since this method is fairly standard, we do not give the details of our computation here.

Applying Eq.(71) to the case at hand, we obtain

$$\begin{aligned}
\Gamma_{\text{reg}}&=2k\sum_{i=1}^{\nu} e_i\ln(M_iR)+2k(k+1)+(2k+1)\ln\Gamma(2k+2)\\
&\quad-2\sum_{\lambda=0}^{2k-1}(\lambda+2)\ln(\lambda+2)+\beta(M_i,R).
\end{aligned}$$

The number β is independent of k and is therefore irrelevant.

To compare with Eq. (69) we note that for $\rho=1$

$$N_{ij}=\delta_{ij}(R^2)^{j-2k}\frac{(2k-j)!j!}{(2k+1)!};\quad j=0,1,\ldots,2k$$

so that

$$\begin{aligned}
\Gamma_{\text{reg}}&=\alpha(k,M_j,R)-2k(2k+1)\ln R+(2k+1)\ln\Gamma(2k+2)\\
&\quad-2\sum_{\lambda=0}^{2k-1}(\lambda+2)\ln(\lambda+2).
\end{aligned}$$

It follows that

$$\alpha(k,M_j,R)-\alpha(0,M_j,R)=2k\left\{\sum_{i=1}^{\nu} e_i\ln(M_iR)+(k+1)+(2k+1)\ln R\right\}. \tag{72}$$

Our results Eqs. (69) and (72) are valid for arbitrary smooth external fields ρ. However, in case ρ stems from an instanton solution, the formulae can be simplified even further, a task, to which we turn now.

5.4. More Explicit Evaluation of Γ_{reg} in Case $\rho = |w|^2$

Recall from Eqs. (24) and (36) that

$$w_\alpha(u(x)) = (R^2 + x^2)^{-k/2} p_\alpha(x)$$

$$p_1(x) = c \prod_{j=1}^{k} (s - a^j); \quad p_2(x) = \prod_{j=1}^{k} (s - b^j).$$

For $\rho = |w|^2$, the integral in Eq. (69) can be computed explicitly in terms of c, a^i and b^j. Furthermore, det N can be related to the Jacobian J (Eqs. (39), (45), (66)), which will then cancel in the instanton gas expectation values $\langle \mathcal{O} \rangle_{\text{inst}}$.

In x-coordinates, the integral in Eq. (69) reads

$$A \stackrel{\text{def}}{=} 2\int dg\{(I_+ \ln|w|^2)(I_- \ln|w|^2) + (2k+1)\ln|w|^2\}$$

$$= \frac{2}{\pi}\int d^2x \left\{ -\left| \partial_s \ln \frac{|p|^2}{(R^2+x^2)^k} \right|^2 + \frac{(2k+1)R^2}{(R^2+x^2)^2} \ln \frac{|p|^2}{(R^2+x^2)^k} \right\}.$$

Partially integrating and noting that

$$\partial_{\bar{s}}\partial_s \ln(R^2+x^2)^{-k} = -kR^2(R^2+x^2)^{-2}$$

we obtain

$$A = -2k^2(1 + \ln R^2) + 2k\ln(1 + |c|^2)$$

$$+ \frac{2}{\pi}\int d^2x \left\{ \ln|p|^2 \partial_{\bar{s}}\partial_s \ln|p|^2 + \frac{R^2}{(R^2+x^2)^2} \ln \frac{|p|^2}{(R^2+x^2)^k} \right\}.$$

Define a complex variable

$$u = \frac{p_1}{p_2}.$$

As s runs through the complex plane, u assumes almost all values k times. Thus, excluding small disks around the zeros of p_2, we find

$$\int d^2x \ln|p|^2 \partial_{\bar{s}}\partial_s \ln|p|^2 = \int d^2x \{\ln(1+|u|^2)\partial_{\bar{s}}\partial_s \ln(1+|u|^2)$$
$$+ \ln|p_2|^2 \partial_{\bar{s}}\partial_s \ln|p|^2 - (\partial_{\bar{s}}\partial_s \ln|p_2|^2)\ln|p|^2\}$$
$$= k\int d^2u (1+|u|^2)^{-2} \ln(1+|u|^2)$$
$$+ \tfrac{1}{4}\oint dx_\mu \varepsilon_{\mu\nu}\{\ln|p|^2 \partial_\nu \ln|p_2|^2 - \ln|p_2|^2 \partial_\nu \ln|p|^2\}.$$

The line integral is to be taken over a large circle at infinity and small circles around the zeros of p_2. The result is

$$\int d^2x \ln|p|^2 \partial_{\bar{s}}\partial_s \ln|p|^2 = \pi k(1 - \ln(1+|c|^2) + \ln|c|^2)$$
$$+ \pi \sum_{i,j=1}^{k} \ln|a^i - b^j|^2.$$

The remaining integral in A can easily be computed for $R \to \infty$ making the substitution $x = Ry$. Summing up, we have

$$A = -2k^2(1 + \ln R^2) + \ln(1 + |c|^2)^2 + 2k \ln|c|^2 + 2\sum_{i,j=1}^{k} \ln|a^i - b^j|^2 + O\left(\frac{1}{R}\right). \tag{73}$$

In order to relate det N (Eq. (53)) to det J (Eq. (39)) we note that

$$J_{ij} = \frac{1}{\pi}\int d^2x \frac{R^2}{(R^2 + x^2)^2}|p|^{-4}\overline{\left(\varepsilon_{\alpha\beta}p_\alpha \frac{\partial}{\partial \lambda_i} p_\beta\right)}\left(\varepsilon_{\gamma\delta}p_\gamma \frac{\partial}{\partial \lambda_j} p_\delta\right)$$

$$\varepsilon_{\alpha\beta}p_\alpha \frac{\partial}{\partial \lambda_j} p_\beta = s^i L_{ij}; \quad (i,j = 0, \dots, 2k)$$

where the $(2k+1) \times (2k+1)$ matrix L_{ij} depends on the parameters λ_l only. Thus,

$$\det J = |\det \mathrm{L}|^2 \det N.$$

Because det L is a homogeneous polynomial in a^i, b^j of degree $k(2k-1)$ vanishing whenever two roots coincide, it follows that

$$\det L \propto \left[\prod_{i<j}(a^i - a^j)(b^i - b^j)\right]\prod_{i,j}(a^i - b^j).$$

The proportionality constant can be computed inductively giving the result

$$\det J = (\det N)|c|^{4k}\left[\prod_{i<j}|a^i - a^j|^2|b^i - b^j|^2\right] \times \prod_{i,j}|a^i - b^j|^2. \tag{74}$$

We finally collect Eqs. (69), (72), (73) and (74) to obtain the completely explicit formula

$$\Gamma_{\mathrm{reg}} = 2k\sum_{i=1}^{\nu} e_i \ln M_i + 2k + \ln(1 + |c|^2)^2 + \sum_{i,j}\ln|a^i - b^j|^2 - \sum_{i<j}[\ln|a^i - a^j|^2 + \ln|b^i - b^j|^2] + \ln\det J \tag{75}$$

(terms vanishing for $R \to \infty$ or $M_i \to \infty$ as well as k-independent constants have been dropped).

6. Computation of Γ_{reg}: the General Case

The scheme outlined in the introduction works in the $\mathbb{CP}^{n-1}(n \geqq 3)$ case, too. The details of the calculations, however, are more involved than in the $\mathbb{CP}^1$ case.

6.1. The Exact Green's Function for Δ

Again, the general structure Eqs. (54)–(56) holds with T, T^+ and P_0 given by Eqs. (31), (32) and (38) respectively. G_- maps $\mathscr{H}_{k+2}$ into $\mathscr{H}_k$ and can be written in the form

$$G_- = (1 - P_0)|w|^{-1}\check{G}_-|w|. \tag{76}$$

Here, the integral kernel $\check{G}_-^{\alpha\beta}(g, g')$ of $\check{G}_-$ satisfies

$$(\delta_{\alpha\beta} - \mathfrak{z}_\alpha(g)\bar{\mathfrak{z}}_\beta(g))I_-\check{G}_-^{\beta\gamma}(g, g') = (\delta_{\alpha\gamma} - \mathfrak{z}_\alpha(g)\bar{\mathfrak{z}}_\gamma(g)) \int_{-\pi}^{\pi} \frac{d\omega}{2\pi} e^{i(k+2)\omega}\, \delta(ge^{i\omega\sigma^3}, g')$$

where $\delta(g, g')$ is the δ-function on the group SU(2) (the right hand side of this equation is the integral kernel of the unit operator in $\mathscr{H}_{k+2}$). One solution is

$$\check{G}_-^{\alpha\beta}(g, g') = (\delta_{\alpha\gamma} - \mathfrak{z}_\alpha(g)\bar{\mathfrak{z}}_\gamma(g))\gamma_k(g'^{-1}\cdot g)(\delta_{\gamma\beta} - \mathfrak{z}_\gamma(g')\bar{\mathfrak{z}}_\beta(g')) \tag{77}$$

with γ_k defined by Eqs. (59) and (60). For brevity, we shall use the symbolic notation

$$\check{G}_- = (1 - \mathfrak{z}\otimes\bar{\mathfrak{z}})\cdot\gamma_k\cdot(1 - \mathfrak{z}\otimes\bar{\mathfrak{z}})$$

6.2. Computation of $\Gamma_{\rm reg}$

Consider a curve $\mathfrak{z}(g, t)$, $0 \leqq t \leqq 1$, of instanton fields. We would like to compute

$$\delta\Gamma_{\rm reg} = \frac{d}{dt}\Gamma_{\rm reg}\big|_{t=0}$$

Equation (64) is not straightforwardly applicable here, because the Hilbert spaces $\mathscr{H}_k$ depend on $\mathfrak{z}_\alpha$. This difficulty can be overcome as follows. Define a unitary matrix $U_{\alpha\beta}(g, t)$ by

$$\dot{U} = \{\dot{\mathfrak{z}}\otimes\bar{\mathfrak{z}} - \mathfrak{z}\otimes\dot{\bar{\mathfrak{z}}} + (\dot{\bar{\mathfrak{z}}}\cdot\mathfrak{z})(\mathfrak{z}\otimes\bar{\mathfrak{z}})\}U; \quad U(g, 0) = 1$$

(the dot denotes differentiation with respect to t). Obviously,

$$U(g\cdot e^{i\omega\sigma^3}, t) = U(g, t); \quad U(g, t)\cdot\mathfrak{z}(g, 0) = \mathfrak{z}(g, t)$$

so that $U(g, t)$ can be used to identify the Hilbert spaces $\mathscr{H}_l$ along the curve $\mathfrak{z}(g, t)$: if $\psi\in\mathscr{H}_l$ at "time" t, $U_{\alpha\beta}^{-1}(g, t)\psi_\beta(g)$ is an element of $\mathscr{H}_l$ at $t = 0$. Correspondingly, any operator O acting in some $\mathscr{H}_l$ at time t can be pulled back to an operator $U^{-1}OU$ acting at $t = 0$. Because U is unitary, Eq. (64) holds, provided we define

$$\delta T^+ = \frac{d}{dt}(U^{-1}T^+U)\big|_{t=0}$$
$$= -(1 - \mathfrak{z}\otimes\bar{\mathfrak{z}})\cdot\{(I_+\mathfrak{z})\otimes\delta\bar{\mathfrak{z}} + I_+(\delta\ln|w|)\} \tag{78}$$

As in the $\mathbb{CP}^1$ case, the short distance singularities in Eq. (64) cancel. We shall not repeat this calculation here, but merely state the result (cp. Eq. (67)):

$$\delta\Gamma_{\rm reg} = 2\int dg(I_-\ln|w|)\{\delta\bar{\mathfrak{z}}\cdot(1 - \mathfrak{z}\otimes\bar{\mathfrak{z}})\cdot I_+\mathfrak{z} + (n-1)(I_+\delta\ln|w|)\} - \mathrm{Tr}\{\delta T^+|w|\check{G}_+|w|^{-1}P_0\} + \text{c.c.} \tag{79}$$

Unfortunately, the reduction of $\delta\Gamma_{\rm reg}$ to a local expression requires a fair amount of algebra, which is deferred to Appendix B. The outcome is (cp. Eqs. (37), (39))

$$\delta\Gamma_{\mathrm{reg}} = \delta \int dg 2n\{(I_- \ln|w|)(I_+ \ln|w|) + (k+1)\ln|w|\}$$
$$- J_{ij}^{-1} \int dg \left\{ (\bar{\eta}^j \cdot \delta \mathfrak{z})|w|^{-1}\left(\bar{\mathfrak{z}} \cdot \frac{\partial w}{\partial \lambda_i}\right) + |w|^{-1}\left(\overline{\frac{\partial w}{\partial \lambda_j}} \cdot \mathfrak{z}\right)(\delta \bar{\mathfrak{z}} \cdot \eta^i) \right\} \tag{80}$$
$$- 2\,\mathrm{Tr}(P_0 \delta \ln|w|).$$

The terms in Eq. (80) involving the zero modes η^i can be related to the variation of det J:

$$\delta \ln \det J = J_{ij}^{-1} \delta J_{ji}$$
$$\delta J_{ji} = -\int dg \left\{ \bar{\eta}^j \eta^i 2\delta \ln|w| + (\bar{\eta}^j \cdot \delta \mathfrak{z})|w|^{-1}\left(\bar{\mathfrak{z}} \cdot \frac{\partial w}{\partial \lambda_i}\right) \right.$$
$$\left. + |w|^{-1}\left(\overline{\frac{\partial w}{\partial \lambda_j}} \cdot \mathfrak{z}\right)(\delta \bar{\mathfrak{z}} \cdot \eta^i) \right\} + \int dg |w|^{-1} \left\{ \delta\left(\overline{\frac{\partial w}{\partial \lambda_j}}\right) \cdot \eta^i + \bar{\eta}^j \cdot \delta\left(\frac{\partial w}{\partial \lambda_i}\right) \right\}.$$

The first integral here matches with the zero mode contribution to Eq. (80). To evaluate the second integral we note that

$$\delta w = \kappa_l \frac{\partial}{\partial \lambda_l} w\,; \quad \kappa_l = \text{constant}$$
$$\delta\left(\frac{\partial w}{\partial \lambda_i}\right) = \kappa_l \frac{\partial^2 w}{\partial \lambda_l \partial \lambda_i} = \kappa_l X_{li}^m \frac{\partial w}{\partial \lambda_m}$$

where the coefficients $X_{li}^m(\lambda)$ are independent of g. We then find

$$J_{ij}^{-1} \int dg |w|^{-1} \bar{\eta}^j \cdot \delta\left(\frac{\partial w}{\partial \lambda_i}\right) = \kappa_l X_{li}^i$$

which is not difficult to compute with the choice (36) of parameters:

$$\kappa_l X_{li}^i + \text{c.c.} = \delta \sum_{\alpha=1}^{n} \left\{ k \ln|c_\alpha|^2 + \sum_{i<j} \ln|a_\alpha^i - a_\alpha^j|^2 \right\}.$$

Summing up, we have

$$\delta\Gamma_{\mathrm{reg}} = \delta \left\{ 2n \int dg [(I_- \ln|w|)(I_+ \ln|w|) + (k+1)\ln|w|] \right.$$
$$\left. - \sum_{\alpha=1}^{n} \left[k \ln|c_\alpha|^2 + \sum_{i<j} \ln|a_\alpha^i - a_\alpha^j|^2 \right] + \ln \det J \right\}. \tag{81}$$

This equation determines Γ_{reg} up to a constant which can be computed by considering a $\mathbb{CP}^1$ instanton imbedded into $\mathbb{CP}^{n-1}$ and comparing with the results obtained in the previous section. The final formula is

$$\Gamma_{\mathrm{reg}} = nk \sum_{i=1}^{\nu} e_i \ln M_i + k + n\left(\frac{k}{2} + 1\right) \ln(\bar{c}_\alpha c_\alpha)$$
$$\sum^{n} \Bigg[\qquad \Bigg]$$

$$+\frac{n}{2\pi}\int d^2x \ln|p|^2 \partial_{\bar{s}}\partial_s \ln|p|^2 + \ln\det J.$$

7. Summary and Discussion

Composing Eqs. (47) and (82), we find the following expression for the instanton grand canonical ensemble

$$\langle \mathcal{O} \rangle_{\text{inst}} = Z^{-1} \sum_{k=0}^{\infty} (k!)^{-n} z^{nk} \int \frac{\prod_{\alpha=1}^{n-1} d^2c_\alpha}{(\bar{c}_\gamma c_\gamma)^n} \prod_{\beta=1}^{n} \prod_{j=1}^{k} d^2a_\beta^j \cdot \mathcal{O}(c_\alpha, a_\beta^j) \exp - U(c_\alpha, a_\beta^j). \tag{83}$$

Here, z is the fugacity,

$$z = \frac{2n}{f} \exp - \left\{ \frac{\pi}{f} + \sum_{i=1}^{v} e_i \ln M_i + \frac{n+2}{2n} \right\}, \tag{84}$$

U is the many body potential

$$U = \frac{n}{2\pi}\int d^2x \ln|p|^2 \partial_{\bar{s}}\partial_s \ln|p|^2 + \tfrac{1}{2}nk(\ln(\bar{c}_\alpha c_\alpha) - 1) - \sum_{\alpha=1}^{n}\left[k \ln|c_\alpha|^2 + \sum_{i<j} \ln|a_\alpha^i - a_\alpha^j|^2 \right] \tag{85}$$

and the instanton parameters c_α, a_β^j are defined in Eqs. (23), (36). In Eq. (83), the world radius R has already been taken to infinity. When the ultraviolet cutoff is removed,

$$\ln \Lambda \overset{\text{def}}{=} - \sum_{i=1}^{v} e_i \ln M_i \to \infty$$

the coupling constant f must be renormalized according to

$$\frac{2\pi}{f} = \ln \Lambda^2/\mu^2 + \frac{2\pi}{f_R(\mu)}. \tag{86}$$

Here, μ is the normalization point and $f_R(\mu)$ the renormalized coupling constant. Defining the renormalization group invariant mass

$$m^2 = \mu^2 \exp - \frac{2\pi}{f_R(\mu)} \tag{87}$$

Equation (84) reads[3]

$$z = m\frac{2n}{f} e^{-(n+2)/2n} \tag{88}$$

Equation (86) completely agrees with the renormalization of the $1/n$ expansion [1].

[3] The coupling constant down-stairs is not renormalized to this order in the semiclassical approximation

For large n, the mass m is just the mass of the fundamental z_α particles.

For $n \geqq 3$ we were not able so far to explicitly compute the integral appearing in the potential U. On the other hand, for the $\mathbb{CP}^1$ case we have (cf. Sect. 5.4)

$$U = \sum_{i,j} \ln |a^i - b^j|^2 - \sum_{i<j} \{\ln |a^i - a^j|^2 + \ln |b^i - b^j|^2\} \tag{89}$$

where $c = c_1, a^i = a_1^i$ and $b^j = a_2^j$. This is precisely the Coulomb interaction energy of k positive charges and k negative charges sitting at a^i respectively b^j. Thus, the pure $\mathbb{CP}^1$ instanton gas is equivalent to the two dimensional classical Coulomb gas, whose partition function in a volume V and at a temperature T is

$$Z_V(T,z) = \sum_{k=0}^{\infty} (k!)^{-2} z^{2k} \int_{a^i, b^j \in V} \prod_{i,j=1}^{k} d^2a^i d^2b^j \cdot \left(\prod_{i,j} |a^i - b^j|^{-2/T} \right) \left(\prod_{i<j} |a^i - a^j|^{2/T} |b^i - b^j|^{2/T} \right) \tag{90}$$

(in our case $T = 1$).

Fortunately, a number of interesting properties of the Coulomb gas have been established rigorously by Fröhlich [10]. First of all, he showed that the thermodynamic limit of the pressure p exists, provided $T > 1$:

$$p(T,z) = \lim_{V \to \infty} V^{-1} \ln Z_V(T,z). \tag{91}$$

The number ρ of instantons per unit area is[4]

$$\rho = \tfrac{1}{2} z \frac{\partial}{\partial z} p(T,z) = \frac{T}{2T-1} p(T,z) \tag{92}$$

which is therefore finite for $T > 1$. Qualitatively, in this range of temperature, the Coulomb gas is in a plasma phase. In terms of instanton thickness and position variables, this would be a dense phase, the instantons overlapping each other more and more as the temperature rises.

As T approaches 1, the Coulomb gas condenses. It follows from Fröhlich's work (Sect. 4.c) that the pressure $p(T,z)$ diverges as $T \to 1$ while keeping the fugacity z fixed. The equation of state (92) then implies that the density ρ is infinite at $T = 1$, i.e. in the $\mathbb{CP}^1$ model we have an instanton fluid rather than a gas. Superficially, the condensation of the Coulomb gas can be anticipated from the fact that $|a^i - b^j|^{-(2/T)}$ becomes singular as $T \to 1$ so that the charges preferably form neutral dipoles, which can be densely packed. This argument reveals that the pressure p diverges at $T = 1$, because of the *ultraviolet* rather than the infrared divergencies in the instanton scale size integrations (cp. Eq. (1) for $n = 2$; for a single instanton $\lambda = \frac{1}{2}|a - b|$). In particular, we do not expect such a condensation to take place for $n \geqq 3$.

At first sight, one might fear that the $\mathbb{CP}^1$ instanton fluid does not make sense

[4] Equation (92) is a simple consequence of the scaling properties of ZV(T, z), and Eq. (91)

at all, because its density is infinite. This is not the case, however, because the expectations $\langle \mathcal{O} \rangle_{\text{inst}}$ usually have a limit as $T \to 1$. In other words, the UV divergencies factorize and cancel in the ratio (83). This follows from the fact that the expectation value of $\mathcal{O}$ in the Coulomb gas can be represented by the expectation of a corresponding observable $\tilde{\mathcal{O}}$ in the massive Thirring model when $T > 1$ [11], [10]. This model is formally defined by the Largrangian density

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m')\psi - \tfrac{1}{2}g(\bar{\psi}\gamma_\mu\psi)^2 \tag{93}$$

where the mass m' is proportional to m and $g = \pi(T-1)$ (see Coleman's article [11], in particular Sect. IV, for more details). As $T \to 1, g \to 0$ so that the expectation $\langle \mathcal{O} \rangle_{\text{Cb. gas}} = \langle \tilde{\mathcal{O}} \rangle_{\text{MTM}}$ converges to an expectation value in the free massive Dirac field.

The equivalence of the $\mathbb{CP}^1$ instanton fluid and the free massive Dirac field not only shows there are no infrared divergencies but also reveals that a dynamical mass generation much like in the $1/n$ expansion has taken place. The equivalence could be exploited to compute correlation functions of the spin field [1]

$$q^a = \bar{z}_\alpha \sigma^a_{\alpha\beta} z_\beta$$

but we are not going into this here.

We have little to say concerning the physics of the $\mathbb{CP}^{n-1}$ instanton gas for $n \geqq 3$. We only remark that for $k = 1$, the potential energy U is

$$U = \frac{n}{2} \ln \left\{ \sum_{\alpha<\beta} |c_\alpha|^2 |c_\beta|^2 |a^1_\alpha - a^1_\beta|^2 \right\} - \sum_{\alpha=1}^{n} \ln |c_\alpha|^2 \tag{94}$$

This expression is already much more complicated than in the $\mathbb{CP}^1$ case, in particular, the parameters c_α no longer play a passive rôle. With respect to the "particle" positions a^1_α, U is a Coulomb like n-body potential. It is singular when all positions are equal, but for $n \geqq 3$, this singularity is harmless, because it is integrable (cp. Eq. (1)).

8. Conclusions

The most important insight provided by our investigation is that the infrared divergence of the one instanton contribution to the path integral not necessarily implies that the whole instanton gas is divergent. More pointedly expressed, the divergent scale size integral, Eq. (1), merely means that the instanton gas is dense. The dilute gas intuition, which has been gathered by studying models with built in mass scale (such as the two dimensional Higgs model), may be rather misleading in the σ-model case. For example, there is no dynamical mass generation in the cutoff dilute instanton gas approximation.

Technically, our computations were rather involved but the overall strategy was simple. We hope to apply our method to the general Yang-Mills instanton solution in four dimensions. A crucial ingredient of our computational scheme, the relevant exact Green's function, has already been found [12].

In the semi-classical approximation we did not consider contributions from solutions and "almost" solutions of the full field equations other than instantons.

As the instanton gas is dense, however, we doubt whether instanton anti-instanton configurations can be safely distinguished from vacuum fluctuations. Neglecting some exact solutions may be a more serious mistake. Fortunately, in the $\mathbb{CP}^1$ case there are no solutions of the full field equations with finite action other than instantons and anti-instantons [13]. The situation is different for $n \geqq 3$. For example, an infinite set of solutions for the $\mathbb{CP}^2$ field equations is given by choosing $z_\alpha(x)$, $\alpha = 1, 2, 3$, to be any real $O(3)$ instanton or anti-instanton. The $\mathbb{CP}^2$ topological density of these solutions is identically zero and the action is quantized. It is an interesting problem to classify and describe all solutions of the full field equations of non-linear σ-models and to clarify their rôle in the semi-classical approximation.

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Appendix A Short Distance Expansion of $T(\Delta + (MR)^2)^{-1}$

The integral kernel $D(g, g')$ of $(\Delta + (MR)^2)^{-1}$ is homogeneous

$$D(ge^{i\omega\sigma^3}, g'e^{i\omega'\sigma^3}) = e^{i2k(\omega' - \omega)}D(g, g').$$

We are interested in the behaviour of $D(g, g')$ as $g \to g'e^{i\omega'\sigma^3}$. Provided the field ρ is rotated as well, $D(g, g')$ is invariant under left rotations. It is therefore sufficient to consider the case $g' = 1$. Define (cf. Eq. (15))

$$D(x) = D(u(x), 1).$$

The differential equation for $D(x)$ is

$$\left[- D_\mu D_\mu - \varepsilon_{\mu\nu}\partial_\mu A_\nu + \frac{4M^2R^4}{(R^2 + x^2)^2} \right] D(x) = 4\pi\delta(x)$$

$$D_\mu = \partial_\mu + iA_\mu; \quad A_\mu = - \varepsilon_{\mu\nu}\partial_\nu \ln[(R^2 + x^2)^k \rho(u(x))].$$

We compute $D(x)$ perturbatively:

$$D(x) = 4\pi \sum_{\nu=0}^{\infty} \int d^2x_1 \dots d^2x_\nu D_0(x - x_1) V(x_1) D_0(x_1 - x_2) \dots V(x_\nu) D_0(x_\nu).$$

The perturbation $V(x)$ is

$$V = i(A_\mu \vec{\partial}_\mu - \overleftarrow{\partial}_\mu A_\mu) - A_\mu A_\mu + \varepsilon_{\mu\nu}\partial_\mu A_\nu + 4M^2[1 - R^4(R^2 + x^2)^{-2}]$$

and the "free" propagator D_0 is taken to be

$$D_0(x) = \int \frac{d^2p}{(2\pi)^2} e^{ipx}(p^2 + 4M^2)^{-1} = \frac{1}{2\pi} K_0(2M|x|)$$

(K_0 is a Bessel function, cp. [9, Sect. 8.432]).
The operator T acting on $D(x)$ reads

$$T = \frac{1}{2R}(R^2 + x^2)(D_1 - iD_2).$$

From power counting it follows that the terms of order $\nu \geqq 2$ in the perturbation expansion for $TD(x)$ give rise to a convergent one loop integral when $x=0$. As $M \to \infty$ these integrals vanish. The terms of order $\nu = 0$ and $\nu = 1$ are easily computed in the limit where first $x \to 0$ and then $M \to \infty$:

$$TD(x) = -R(1 - iA_\mu x_\mu)\frac{x_1 - ix_2}{x^2}.$$

Appendix B. Proof of Eq. (80)

Let

$$-\mathrm{Tr}\{\delta T^+ |w| \check{G}_+ |w|^{-1} P_0\} = A_1 + A_2$$

$$A_1 = \mathrm{Tr}\{I_+(\delta \ln|w|)|w|\check{G}_+|w|^{-1}P_0\}$$

$$A_2 = \mathrm{Tr}\{(I_+\mathfrak{z})\otimes\delta\bar{\mathfrak{z}}\cdot|w|\check{G}_+|w|^{-1}P_0\}.$$

We first compute A_1. Noting

$$(1 - \mathfrak{z}\otimes\bar{\mathfrak{z}})I_+(\delta \ln|w|) = [T^+, \delta \ln|w|]$$

we obtain

$$\begin{aligned} A_1 &= -\mathrm{Tr}\{(\delta \ln|w|)T^+|w|\check{G}_+|w|^{-1}P_0\} \\ &= -\mathrm{Tr}\{(\delta \ln|w|)|w|(1-\check{P}_0)|w|^{-1}P_0\} \\ &\quad + \mathrm{Tr}\{(\delta \ln|w|)(I_+\mathfrak{z})\otimes\bar{\mathfrak{z}}\cdot|w|\gamma_k^+|w|^{-1}P_0\}. \end{aligned}$$

Here, $\check{P}_0$ projects onto the zero modes of I_- (cp. Sect. 5.3):

$$\begin{aligned} \check{P}_0^{\alpha\beta}(g,g') = (\delta_{\alpha\gamma} - \mathfrak{z}_\alpha(g)\bar{\mathfrak{z}}_\gamma(g))(k+1)\left\langle \frac{k}{2}, -\frac{k}{2}\right| g'^{-1}g \left|\frac{k}{2}, -\frac{k}{2}\right\rangle \\ \cdot(\delta_{\gamma\beta} - \mathfrak{z}_\gamma(g')\bar{\mathfrak{z}}_\beta(g')). \end{aligned}$$

In particular

$$\check{P}_0|w|^{-1}P_0 = \check{P}_0|w|^{-1}$$

so that

$$\begin{aligned} A_1 &= -\mathrm{Tr}(P_0\delta \ln|w|) + (n-1)(k+1)\int dg\delta \ln|w| \\ &\quad + \mathrm{Tr}\{(\delta \ln|w|)(I_+\mathfrak{z})\otimes\bar{\mathfrak{z}}\cdot|w|\gamma_k^+|w|^{-1}P_0\}. \end{aligned}$$

Next, observe that

$$T(1 - \mathfrak{z}\otimes\bar{\mathfrak{z}})|w|^{-1}\gamma_k|w|\cdot\mathfrak{z} = 0$$

and therefore

$$\bar{\mathfrak{z}}\cdot|w|\gamma_k^+|w|^{-1}P_0 = \bar{\mathfrak{z}}\cdot|w|\gamma_k^+|w|^{-1}(1 - \mathfrak{z}\otimes\bar{\mathfrak{z}}).$$

Thus, we finally obtain

$$\begin{aligned} A_1 &= -\mathrm{Tr}(P_0\delta \ln|w|) + (n-1)(k+1)\int dg\delta \ln|w| \\ &\quad - \int dg(\delta \ln|w|)(I_-\bar{\mathfrak{z}})\cdot(1 - \mathfrak{z}\otimes\bar{\mathfrak{z}})\cdot(I_+\mathfrak{z}) \end{aligned}$$

$$= \delta \int dg \{[n(k+1)-1]\ln|w| + (I_- \ln|w|)(I_+ \ln|w|)\}$$
$$- \mathrm{Tr}(P_0 \delta \ln|w|).$$

We next compute A_2. Because $(1 - \mathfrak{z} \otimes \bar{\mathfrak{z}}) \cdot \delta\mathfrak{z}$ is a zero mode of T, we have

$$T\{|w|^{-1}\check{G}_- |w| \cdot \delta\mathfrak{z} - (1 - \mathfrak{z} \otimes \bar{\mathfrak{z}}) \cdot \delta\mathfrak{z}\gamma_0\} = 0.$$

Therefore,

$$\bar{A}_2 = -\mathrm{Tr}\{P_0 |w|^{-1} G_- |w| \cdot \delta\mathfrak{z} \otimes I_- \bar{\mathfrak{z}}\}$$
$$= -\int dg (I_- \bar{\mathfrak{z}}) \cdot (1 - \mathfrak{z} \otimes \bar{\mathfrak{z}}) |w|^{-1} I_+ [|w|(1 - \mathfrak{z} \times \bar{\mathfrak{z}}) \cdot \delta\mathfrak{z}]$$
$$- \mathrm{Tr}\{P_0 \cdot \delta\mathfrak{z}\gamma_0 \times I_- \bar{\mathfrak{z}}\}.$$

Recalling the explicit form Eq. (38) of P_0, it is easy to show that

$$(I_- \bar{\mathfrak{z}}_\alpha)(g) P_0^{\alpha\beta}(g, g') = I_- \left(|w|^{-1} \bar{\mathfrak{z}} \cdot \frac{\partial w}{\partial \lambda_i} \right)(g) J_{ij}^{-1} \bar{\eta}_\beta^j(g').$$

Partial integration then yields

$$\mathrm{Tr}\{P_0 \cdot \delta\mathfrak{z}\gamma_0 \otimes I_- \bar{\mathfrak{z}}\} = J_{ij}^{-1} B_{ji}$$

$$B_{ji} = \int dg (\bar{\eta}^j \cdot \delta\mathfrak{z}) |w|^{-1} \left(\bar{\mathfrak{z}} \cdot \frac{\partial w}{\partial \lambda_i} \right) - \int dg (\bar{\eta}^j \cdot \delta\mathfrak{z}) \int dg' |w|^{-1} \left(\bar{\mathfrak{z}} \cdot \frac{\partial w}{\partial \lambda_i} \right)$$

and consequently

$$A_2 = -\int dg I_- [|w| \delta\bar{\mathfrak{z}} \cdot (1 - \mathfrak{z} \otimes \bar{\mathfrak{z}})] \cdot |w|^{-1} (1 - \mathfrak{z} \otimes \bar{\mathfrak{z}}) \cdot I_+ \mathfrak{z}$$
$$- J_{ij}^{-1} B_{ji}^+.$$

Combining the results so far obtained with the other terms in Eq. (79) gives (after some trivial algebra):

$$\delta\Gamma_{\mathrm{reg}} = \delta \int dg 2\{n(I_- \ln|w|)(I_+ \ln|w|) + [n(k+1)-1]\ln|w|\}$$
$$- 2\,\mathrm{Tr}(P_0 \delta \ln|w|) - J_{ij}^{-1}(B_{ij}^+ + B_{ij}).$$

A further simplification is achieved by noting that

$$\delta\mathfrak{z} = -(\delta \ln|w|)\mathfrak{z} + |w|^{-1} \kappa_l \frac{\partial w}{\partial \lambda_l}$$

with some complex constants κ_l. Then

$$J_{ij}^{-1} \int dg (\bar{\eta}^j \cdot \delta\mathfrak{z}) = \kappa_i$$

and therefore

$$J_{ij}^{-1} \int dg (\bar{\eta}^j \cdot \delta\mathfrak{z}) \int dg' |w|^{-1} \left(\bar{\mathfrak{z}} \cdot \frac{\partial w}{\partial \lambda_i} \right) + \text{c.c.} = \delta \int dg 2 \ln|w|$$

which, when inserted above, yields Eq. (80).

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