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The Infinite Cluster Method in the Two-Dimensional Ising Model*

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Abstract. By studying infinite clusters in the two dimensional ferromagnetic Ising model some new results on the problem of existence of non-translation invariant equilibrium states are obtained. Furthermore a new proof of a theorem by Abraham and Reed is given.

1. Introduction

The existence of non-translation invariant equilibrium states for the two-dimensional Ising model is a still open problem.

Dobrushin [1] first proved that the three-dimensional Ising model admits non-translation invariant equilibrium states. Gallavotti [2] proved that the state obtained by using boundary conditions analogous to those considered in [1] is translation invariant at low temperature in two dimensions.

More recently the same state was studied at any temperature by Abraham and Reed [3, 4]: they proved that its magnetization is everywhere zero.

By using this last result and Lebowitz inequalities [5] Messager and Miracle-Sole have shown that in the two-dimensional case a large class of boundary conditions (including the ones studied in [2, 3, 4]) give rise to translation invariant states [6].

These results strongly support the conjecture that all equilibrium states of the two-dimensional Ising model are translation invariant.

The motivation of the present work was an attempt to prove this conjecture. This goal has not been achieved, but some other related results have been obtained; in particular we prove here that if an equilibrium measure μ is translation invariant along one direction of the lattice, than μ is translation invariant; furthermore a new proof (which makes no use of direct computations) of the result by Abraham and Reed quoted above is given.

We study the equilibrium states of the model by finding the probability of suitable tail events. In other words we try to characterize pure phases by means of

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global features of their typical configurations. In our case, of n.n. interaction, the most useful tail events are those related to existence of infinite clusters.

We remark that this approach is not new, but it goes back to Peierls [7]; he proved the existence of phase transitions by showing that for sufficiently low temperature typical configurations are characterized by an infinite cluster to which the greater part of sites belong.

The method used here, in our opinion, has the advantage to be strictly related to the intuition; furthermore it allows to avoid direct computations in proving some structural features of the model.

In a previous joint paper [8] a description of typical configurations of the states μ_+ and μ_- was given; we complete this description in Sect. 4. In Sect. 5 we give a new proof of the statement, proved in [6], that once one has fixed the spins on one axis equal to 1 the state becomes independent from boundary conditions. In Sect. 6 we draw from the results of Sect. 5 some statements about infinite clusters in typical configurations of a generic equilibrium measure of the model. A representation of any equilibrium measure which is translation invariant along one direction of the lattice is given in Sect. 7. Main results are exposed in Sect. 8.

All the proofs are based on Markov property and FKG inequalities [9, 10].

2. Definitions and Notations

We consider the configuration space $\Omega = \{-1, 1\}^{Z^2}$.

We define in Ω the partial order $\leqq$ by putting $\omega_1 \leqq \omega_2$ if and only if $\forall x \in Z^2$ $\omega_1(x) \leqq \omega_2(x)$ and we call positive [negative] an event A if its characteristic function is non-decreasing [non-increasing]. We put:

$$E_x^+[E_x^-] = \{\omega \in \Omega \mid \omega(x) = 1[-1]\}.$$

For every $K \subset Z^2$ we call $\mathscr{B}_K$ the σ-algebra generated by the events E_x^+, $x \in K$. We put $\mathscr{B}_\infty = \bigcap_K \mathscr{B}_{\tilde{K}}$, where K runs over the class of all finite subsets of Z^2 (here and in the following "$\sim$" means complementation).

We are interested in some events in $\mathscr{B}_\infty$. In order to define them we fix our terminology as follows.

Two points in Z^2 which differ only by one unit in one coordinate are called adjacent; they are called *adjacent if they are adjacent or such that both their coordinates differ by one unit. A finite sequence $(x_1, \ldots, x_n)$ of distinct points in Z^2 is called a (self-avoiding) chain if x_i and x_j are adjacent if and only if $|i-j|=1$ and a circuit if for any $j \in (1, \ldots, n)$ $(x_j, x_{j+1}, \ldots x_n, x_1, \ldots x_{j-2})$ is a chain; *chains and *circuits are defined in an analogous way. A subset $Y \subset Z^2$ is connected [*connected] if, for all pairs x, y of points in Y, there is a chain [*chain] made up of points in Y having x, y as terminal points.

The boundary [*boundary] of a given subset $Y \subset Z^2$ is the set $\partial Y[\partial^* Y]$ of all points in $Z^2 \backslash Y$ that are adjacent [*adjacent] to at least one point in Y. Note that the external boundary of a connected set is *connected and the external *boundary of a *connected set is connected.

If $\omega \in \Omega$, the $(+)$cluster in ω are the maximal connected components of $\omega^{-1}(1)$; given $K \subset Z^2$ we call $(+)$clusters of K in ω the maximal connected components of $\omega^{-1}(1) \cap K$; $(-)$, $(+*)$ and $(-*)$clusters are defined in the same way. We put:

$$C^{\pm}[C^{\pm *}] = \{\omega \in \Omega \mid \text{in } \omega \text{ there is an infinite } (\pm)\text{cluster } [(\pm *)\text{cluster}]\}$$
$$C_K^{\pm}[C_K^{\pm *}] = \{\omega \in \Omega \mid \text{in } \omega \text{ there is an infinite } (\pm)\text{cluster } [(\pm *)\text{cluster}] \text{ of } K\}.$$

We shall consider, in particular, infinite clusters of

$$\pi = \{x \in Z^2 \mid x_2 \geqq 0\}, \quad \pi' = \{x \in Z^2 \mid x_2 \leqq 0\}, \quad Q = \{x \in Z^2 \mid x_1 \geqq 0, x_2 \geqq 0\}.$$

It is clear that all the events $C^{\pm}$, $C^{\pm *}$, $C_K^{\pm}$, $C_K^{\pm *}$, where $K = \pi, \pi', Q$, belong to $\mathscr{B}_{\infty}$.

Furthermore we shall consider the events:

$$V_j[V_j'] = \{\omega \in \Omega \mid (j,0) \text{ belongs to an infinite } (+)\text{cluster of } \pi[\pi']\}$$
$$V_j^*[V_j^{*\prime}] = \{\omega \in \Omega \mid (j,0) \text{ belongs to an infinite } (+*)\text{cluster of } \pi[\pi']\}.$$

We call $(+)$chain $[(+)$circuit$]$ in ω any chain [circuit] included in $\omega^{-1}(1)$; $(-)$, $(+*)$ and $(-*)$chains (and circuits) are defined in the same way. Note that $\omega \in \tilde{C}^+$ if and only if there are in ω infinitely many disjoint $(-*)$circuits surrounding the origin.

If s is a $(-)$chain $[(-*)$chain$]$ included in π with starting point $(a,0)$ and endpoint $(b,0)$ and $a \leqq j \leqq b$, we say that s is a $(-)$half-circuit $[(-*)$half-circuit$]$ surrounding $(j,0)$ in π. Note that $\omega \in \tilde{V}_j[\tilde{V}_j^*]$ if and only if there is in ω some $(-*)$ $[(-)]$half-circuit surrounding $(j,0)$ in π.

We consider the ferromagnetic Ising model at zero external field, i.e., for each finite $\Lambda \subset Z^2$ the energy function U_Λ is defined by

$$\forall \omega \in \Omega, \quad U_\Lambda(\omega) = \sum_{(x,y)} -j\omega(x)\omega(y) \tag{2.1}$$

where the sum is over the pairs of adjacent sites in Λ and j is a positive real number. We call M the set of Gibbs measures corresponding to the energy function (2.1). $\mu_+[\mu_-] \in M$ is the Gibbs measure obtained by using $+[-]$ boundary conditions.

We shall consider the phase coexistence region of the model, i.e. we shall suppose that j is great enough to have $\mu_+ \neq \mu_-$.

We list below some known statements on which the proofs of the following sections are based:

a) all measures $\mu \in M$ are one-step Markov,

b) all measures $\mu \in M$ are everywhere dense (i.e. if $\mu \in M$, $K \subset Z^2$ is a finite set and $\emptyset \neq A \in \mathscr{B}_K$, then $\mu(A) > 0$),

c) if μ is an extremal point of M, $\mathscr{B}_{\infty}$ measured by μ is trivial,

d) μ_+ and μ_- are extremal points of M (in particular this implies that μ_+ and μ_- are ergodic with respect to any non-trivial subgroup of the translation group),

e) μ_+ and μ_- are invariant under translation, rotations by right angles and reflections.

3. Some Preliminary Lemmas

In this and in the following sections we shall call Λ_n the square $\{x \in Z^2 | |x_1| \leqq n; |x_2| \leqq n\}$.

A first example of the usefulness of infinite clusters in characterizing the elements of M is given by the following lemma.

Lemma 1. *If* $\mu \in M$, $\mu(C^+)=0$, *then* $\mu = \mu_-$.

Proof. We consider a negative event $A \in \mathscr{B}_{\Lambda_n}$. By the hypothesis, μ-a.s. there is a $(-*)$ circuit surrounding Λ_n; then, given $\varepsilon > 0$, we can choose N such that the event "there is in $\Lambda_N \backslash \Lambda_n$ a $(-*)$circuit surrounding $\underline{0} = (0,0)$" has μ-probability greater than $1-\varepsilon$. We put:

$M_c^N = \{\omega \in \Omega |$in ω c is the maximal $(-*)$circuit surrounding $\underline{0}$ and contained in $\Lambda_N\}$.

We have:

$$\sum_{c \subset \Lambda_N \backslash \Lambda_n} \mu(M_c^N) > 1-\varepsilon .$$

On the other hand the Markov property and the FKG inequality imply that for any $c \subset \Lambda_N \backslash \Lambda_n$ $\mu(A|M_c^N) \geqq \mu_-(A)$. Hence:

$$\mu(A) \geqq \sum_{c \subset \Lambda_N \backslash \Lambda_n} \mu(A|M_c^N)\mu(M_c^N) \geqq (1-\varepsilon)\mu_-(A).$$

Another application of the FKG inequality shows that $\mu(A) \leqq \mu_-(A)$. Hence we have $\mu(A) = \mu_-(A)$. By observing that the measures of negative local events uniquely characterize μ we get the lemma.

We shall see in the Sect. 8 that if $\mu(C_\pi^+)=0$, then $\mu = \mu_-$. Here we prove the following weaker result:

Lemma 2. *If* $\mu \in M$, $\mu(C_\pi^+)=0$, *then* $\mu(E_{\underline{0}}^-) \geqq 1/2$.

Proof. μ-a.s. $\underline{0}$ is surrounded in π by a $(-*)$half-circuit. Given $\varepsilon > 0$, we choose an integer N such that the event "there is in Λ_N a $(-*)$half-circuit surrounding $\underline{0}$ in π" has μ-probability greater than $1-\varepsilon$. We put:

$M_{\pi,s}^N = \{\omega \in \Omega |$in ω s is the maximal $(-*)$half-circuit surrounding $\underline{0}$ in π and contained in $\Lambda_N\}$.

Then we have:

$$\sum_{s \subset \Lambda_N \cap \pi} \mu(M_{\pi,s}^N) > 1-\varepsilon . \tag{3.1}$$

We call s' the *half-circuit obtained by reflecting s with respect to the 1-axis and we consider the events

$E_s^* = \{\omega \in \Omega | \forall x \in s \ \omega(x) = -1 ; \forall x \in s' \backslash s \ \omega(x) = 1\}$,

$E_s' = \{\omega \in \Omega | \forall x \in s \backslash s' \ \omega(x) = -1 ; \forall x \in s' \ \omega(x) = 1\}$.

We have:

$$\mu(E_{\underline{0}}^{-}) \geqq \sum_{s \subset \Lambda_N \cap \pi} \mu(E_{\underline{0}}^{-} | M_{\pi,s}^{N}) \mu(M_{\pi,s}^{N})$$

$$\mu(E_{\underline{0}}^{-} | M_{\pi,s}^{N}) = \sum_{B} \mu(E_{\underline{0}}^{-} | M_{\pi,s}^{N} \cap B) \mu(B | M_{\pi,s}^{N})$$

where B runs over all possible spin assignments on $s' \backslash s$. By applying the Markov property and the FKG inequality we get

$$\begin{aligned} &\mu(E_{\underline{0}}^{-} | M_{\pi,s}^{N} \cap B) \geqq \mu(E_{\underline{0}}^{-} | E_s^*) \\ &\mu(E_{\underline{0}}^{-}) \geqq \sum_{s \subset \Lambda_N \cap \pi} \mu(E_{\underline{0}}^{-} | E_s^*) \mu(M_{\pi,s}^{N}). \end{aligned} \tag{3.2}$$

Another application of FKG inequality and a symmetry argument show that

$$\mu(E_{\underline{0}}^{-} | E_s^*) \geqq \mu(E_{\underline{0}}^{-} | E_s') = \mu(E_{\underline{0}}^{+} | E_s^*).$$

Hence

$$\mu(E_{\underline{0}}^{-} | E_s^*) \geqq 1/2. \tag{3.3}$$

The lemma is proved by collecting together (3.1), (3.2) and (3.3).

Corollary 1. $\mu_+(C_\pi^+) = 1$, $\mu_+(V_0) > 0$.

Proof. The phase coexistence region is characterized by spontaneous magnetization; hence from Lemma 2 and the extremality of μ_+ it follows that $\mu_+(C_\pi^+) = 1$. The second relation can be easily proved by using the FKG inequality and b).

4. Typical Configurations of the Measures μ_+ and μ_-

It is known [8] (and it follows from Lemma 1) that, in the phase coexistence region, μ_+-a.s. there is an infinite (+)cluster. Furthermore it was proved in [8] that μ_+-a.s. there is no infinite (−)cluster. In this section we complete the description of the typical configurations of the measures μ_+ and μ_- by proving the following proposition.

Proposition 1. $\mu_+(C^{-*}) = 0$.

Proposition 1 in particular implies that μ_+-a.s. the infinite (+)cluster is unique. The proof of Proposition 1 is based on the following lemma.

Lemma 3. $\mu_+(C_\pi^{-*}) = 0$.

Proof. We consider the events $V_j'' = V_j \cap V_j'$. Translation and reflection invariance of the measure μ_+, the FKG inequality and Corollary 1 imply

$$\mu_+(V_j'') \geqq \mu_+(V_0)^2 > 0.$$

Therefore, by Birkhoff's ergodic theorem, we have

$$\mu_+\left(\bigcup_{j<0} V_j''\right) = \mu_+\left(\bigcup_{j>0} V_j''\right) = 1. \tag{4.1}$$

Now we suppose that

$$\mu_+(C_Q^+)=0. \tag{A}$$

(A) implies that μ_+-a.s. any infinite $(+)$cluster of $\pi[\pi']$intersects the 2-axis. Hence μ_+-a.s. if $\omega\in V_j''$ $(j>0)$, there is in ω a $(+)$half-circuit surrounding $\underline{0}$ in the half-plane $\{x_1\geqq 0\}$. This, by rotation invariance of the measure μ_+, ends the proof in the case (A).

If (A) does not hold, the extremality of μ implies

$$\mu_+(C_Q^+)=1. \tag{B}$$

We consider the events:

$$W_j[W_j']=\{\omega\in\Omega\,|\,(j,0) \text{ belongs to an infinite } (+)\text{cluster of the quadrant } \{x_1\leqq j;\, x_2\geqq 0\}\,[\{x_1\leqq j;\, x_2\leqq 0\}]\}$$

$$W_j''=W_j\cap W_j'.$$

In the case (B) translation and reflection invariance of μ_+, the FKG inequality and b) imply $\mu_+(W_j'')\geqq\mu_+(W_0)^2>0$. Hence, if B) holds, μ_+-a.s. infinitely many of the events W_j'' occur. On the other hand it is easy to realize that if $\omega\in W_j''(j>0)$, then μ_+-a.s. there is a $(+)$half-circuit surrounding $\underline{0}$ in the half-plane $\{x_1\geqq 0\}$ (it suffices to observe that μ_+-a.s. there is no infinite $(+)$cluster in the strip $0\leqq x_1\leqq j$). This ends the proof in the case (B).

Proof of Proposition 1. It is enough to prove that $\mu_+(C_{\underline{0}}^{-}*)=0$, where $C_{\underline{0}}^{-}*=\{\omega\in\Omega\,|\,\text{in } \omega\ \underline{0} \text{ belongs to an infinite } (-*)\text{cluster}\}$. Lemma 3 implies that μ_+-a.s. the infinite $(+)$cluster of $\pi[\pi']$ is unique. It is easy to see that this implies that, for any pair of positive integers (j,k)

$$\mu_+(C_{\underline{0}}^{-}*\,|\,V_{-j}''\cap V_k'')=0.$$

On the other hand (4.1) implies that

$$\mu_+\Big(\bigcup_{j,k>0} V_{-j}''\cap V_k''\Big)=1 \tag{4.2}$$

and this proves Proposition 1.

5. Uniquenness of the Semi-Infinite State

We call $\hat\mu_n^+[\hat\mu_n^-]$ the measure on $\Omega_\pi=\{-1,1\}^\pi$ obtained by using the following "boundary conditions"

$$\omega(x)=1 \qquad \text{if} \quad x_2=-1$$

$$\omega(x)=1[\omega(x)=-1] \quad \text{if} \quad x\in\partial\Lambda_n\cap\pi.$$

Proposition 2. $\lim\limits_{n\to\infty}\hat\mu_n^+=\lim\limits_{n\to\infty}\hat\mu_n^-$.

Proposition 2, by FKG inequality, implies that once one has fixed equal to 1 the spins on the line $x_2=-1$ the state becomes independent from boundary conditions. A discussion on this point was announced by Dobrushin in [11]; in [6] the Proposition 2 is proved as a direct consequence of the translation invariance of the state $\mu^{\pm}$ defined in Sect. 8; in this paper the Proposition 2 is proved by using direct computations by Abraham and Reed [3, 4].

By following a reverse way, we shall use Proposition 2 in the sequel of the paper (in particular in proving the result by Abraham and Reed); in this section we give a direct proof of it based on the analysis of the infinite clusters.

We start by collecting in a lemma some statements which easily follow from the definition.

Lemma 4. *The limit*

$$\hat{\mu}^{-} = \lim_{n\to\infty} \hat{\mu}_n^{-} \tag{5.1}$$

exists; $\hat{\mu}^{-}$ is reflection invariant with respect to the 2-axis and translation invariant along the 1-axis. $\mathscr{B}_\infty$ measured by $\hat{\mu}^{-}$ is trivial.

Proof. For any positive local event A the sequence $\hat{\mu}_n^{-}(A)$ is eventually non-decreasing; this implies the existence of the limit 5.1 (in the vague topology of measures). By using FKG inequality it is easy to see that $\hat{\mu}^{-}$, as a measure on Ω_π, is an extremal equilibrium measure with respect to the energy function obtained from (2.1) by adding an external field $-j$ in the sites on the line $x_2=0$.

Taking account of this remark the other statements can be proved in the same way as the analogous statements for the measure μ_{-}. (See for example [12].)

Lemma 5. $\hat{\mu}^{-}(D_k) \geqq (1/2)\mu_{+}(V_0)$ *where*

$D_k = \{\omega \in \Omega \mid (0, k)$ *is* $(+*)$*connected with the 1-axis*$\}$

(here and in the following two sets $A, B \subset Z^2$ are said $(+)$connected $[(+*)$connected$]$ in ω if there is in ω a $(+)$chain$[(+*)$chain$]$ starting in A and ending in B).

Proof. For a given positive k we consider the events

$$B_j^k = \{\omega \in \Omega \mid \omega(j,0) = \omega(j,1) = \ldots = \omega(j,k) = 1\}.$$

It is easy to realize that, by ergodicity, $\hat{\mu}^{-}\left(\bigcup_{j,j'>0} (B_{-j}^k \cap B_{j'}^k)\right) = 1$. Hence for any $\varepsilon > 0$, we can choose M such that

$$\sum_{-M \leqq j < 0 < j' \leqq M} \hat{\mu}^{-}(\bar{B}_{jj'}^k) > 1 - \varepsilon \tag{5.2}$$

where

$$\bar{B}_{jj'}^k = \{\omega \in \Omega \mid (j,j') \text{ is the maximal interval containing } 0 \text{ and included in } (-M, M) \text{ such that } \omega \in B_j^k \cap B_{j'}^k\}.$$

We call $\gamma_{jj'}$ the chain $(j,k)\ (j,k-1) \ldots (j,-1)\ (j+1,-1) \ldots (j',-1)\ (j',0) \ldots (j',k)$ and we call $\gamma'_{jj'}$ the chain obtained from $\gamma_{jj'}$ by a reflection with respect to the line

$x_2 = k+1/2$. In the same way as in the proof of Lemma 2 we get

$$\begin{aligned}\hat{\mu}^-(D_k) &\geqq \sum_{-M\leqq j<0<j'\leqq M} \hat{\mu}^-(D_k|\bar{B}^k_{jj'})\hat{\mu}^-(\bar{B}^k_{jj'}) \\ &\geqq \sum_{-M\leqq j<0<j'\leqq M} \mu(D_k|E^*_{\gamma_{jj'}})\hat{\mu}^-(\bar{B}^k_{jj'}) \end{aligned} \tag{5.3}$$

where μ is a generic equilibrium measure (the conditional probability in (5.3) does not depend on μ) and $E^*_{\gamma_{jj'}} = \{\omega\in\Omega | \forall x\in\gamma'_{jj'}\ \omega(x) = -1; \forall x\in\gamma_{jj}\ \omega(x) = 1\}$.

We call $F^+[F^-]$ the event "$(0,k)$ is surrounded in the rectangle $\{j\leqq x_1<j';\ -1\leqq x_2\leqq 2k+2\}$ by a $(+*)$circuit[$(-*)$circuit] $(+*)$connected[$(-*)$connected] with $\gamma_{jj'}[\gamma'_{jj'}]$". It is easy to verify that $F^+\cup F^- = \Omega$ (note that $F^+\cap F^- \neq \emptyset$).

Furthermore we have

$$\mu(F^+|E^*_{\gamma_{jj'}}) \geqq \mu(F^-|E^*_{\gamma_{jj'}}).$$

The last inequality follows from an argument similar to the one used in Lemma 2 using the FKG inequality and the reflection and change of sign symmetries. Hence

$$\begin{aligned}&\mu(F^+|E^*_{\gamma_{jj'}}) \geqq 1/2;\ \mu(D_k|E^*_{\gamma_{jj'}}) = \frac{\mu(D_k\cap E^*_{\gamma_{jj'}})}{\mu(E^*_{\gamma_{jj'}})} \geqq \frac{\mu(D_k\cap E^*_{\gamma_{jj'}}\cap F^+)}{2\mu(E^*_{\gamma_{jj'}}\cap F^+)} \\ &\quad = \mu(D_k|E^*_{\gamma_{jj'}}\cap F^+)/2 \geqq \mu_+(V_0^*)/2 \geqq \mu_+(V_0)/2.\end{aligned}$$

The lemma is proved by collecting together (5.2), (5.3) and the last inequality.

Lemma 6. $\hat{\mu}^-(V_0^*) > 0$

Proof. We put:

$D^r_k[D^l_k] = \{\omega\in\Omega | (0,k)$ is $(+*)$connected in π with the non-negative [non-positive] 1-half-axis$\}$.

By Lemma 5, the reflection invariance of $\hat{\mu}^-$, and the FKG inequality we get:

$$\begin{aligned}&\hat{\mu}^-(D^r_k\cup D^l_k) \geqq \mu_+(V_0)/2; \qquad \hat{\mu}^-(D^r_k) = \hat{\mu}^-(D^l_k) \geqq \mu_+(V_0)/4 \\ &\hat{\mu}^-(D^r_k\cap D^l_k) \geqq \mu_+(V_0)^2/16.\end{aligned} \tag{5.4}$$

We consider the events

$$P_k = \{\omega\in\Omega | \underline{0} \text{ belongs to a } (+*)\text{cluster of } \pi \text{ of size greater than } k\}.$$

It is easy to check that, by the FKG inequality,

$$\hat{\mu}^-(P_k|D^r_k\cap D^l_k) \geqq \mu_+(V_0).$$

Hence we have:

$$\begin{aligned}&\hat{\mu}^-(P_k) \geqq \mu_+(V_0)^2\hat{\mu}^-(P_k|D^r_k\cap D^l_k)/16 \geqq \mu_+(V_0)^3/16 \\ &\hat{\mu}^-(V_0^*) = \lim_{k\to\infty}\hat{\mu}^-(P_k) \geqq \mu_+(V_0)^3/16.\end{aligned}$$

Lemma 7. $\hat{\mu}^-(C^-_Q) = 0$.

Proof. Lemma 6 and the ergodicity of $\hat{\mu}^-$ with respect to the translations along the 1-axis imply that $\hat{\mu}^-\left(\bigcup_{j=0}^{\infty} V_j^*\right)=1$.

Hence, given $\varepsilon>0$, we can choose N such that

$$\hat{\mu}^-\left(\bigcup_{j=0}^{N} V_j^*\right)>1-\varepsilon. \tag{5.5}$$

We consider the event

$$G_N=\{\omega\in\Omega \mid \text{there is in } \{0\leqq x_1\leqq N\}\cap\pi \text{ a } (+*)\text{chain connecting the two axes}\}.$$

We call G'_N the event obtained by reflecting G_N with respect to the line $x_1=N/2$. $\hat{\mu}^-$-a.s. no infinite $(+*)$cluster is contained in the strip $0\leqq x_1\leqq N$; hence

$$\hat{\mu}^-\left(G_N\cup G'_N \,\middle|\, \bigcup_{j=1}^{N} V_j^*\right)=1. \tag{5.6}$$

By using (5.5), (5.6), reflection invariance of $\hat{\mu}^-$ with respect to the line $x_1=N/2$ and the FKG inequality we get:

$$\hat{\mu}^-(G_N\cup G'_N)\geqq 1-\varepsilon\,; \quad \hat{\mu}^-(\tilde{G}_N)^2\leqq\hat{\mu}^-(\tilde{G}_N\cap\tilde{G}'_N)\leqq\varepsilon\,; \quad \hat{\mu}^-(G_N)\geqq 1-\varepsilon^{1/2}.$$

The last inequality proves the lemma.

Lemma 8. $\hat{\mu}^-(C_\pi^-)=0$.

Proof. Let n be a positive integer. By the Lemma 7 and the FKG inequality we can choose $k>n$ and $N>k$ such that the event "in both regions $(\Lambda_k\backslash\Lambda_n)\cap Q$ and $(\Lambda_N\backslash\Lambda_k)\cap Q$ there are $(+*)$chains connecting the two positive half-axes" has $\hat{\mu}^-$-probability greater than 1/2. We put:

$$E_s[E_S]=\{\omega\in\Omega \mid s[S] \text{ is the minimal [maximal] } (+*)\text{chain connecting the two positive half-axes contained in } (\Lambda_k\backslash\Lambda_n)\cap Q[(\Lambda_N\backslash\Lambda_k)\cap Q]\}$$

$$E_{sS}=E_s\cap E_S.$$

We have

$$\sum_{\substack{s\subset(\Lambda_k\backslash\Lambda_n)\cap Q\\ S\subset(\Lambda_N\backslash\Lambda_k)\cap Q}} \hat{\mu}^-(E_{sS})>1/2. \tag{5.7}$$

Now we consider the events

$$D^r_{k,n}[D^l_{k,n}]=\{\omega\in\Omega \mid (0,k) \text{ is } (+*)\text{connected in } \pi\backslash\Lambda_n \text{ with the non-negative [non positive] 1-half-axis}\}.$$

By using the same argument of the proof of Lemma 5 it can be proved that

$$\hat{\mu}^-(D^r_{k,n}\mid E_{sS})\geqq\mu_+(V_0)/2.$$

Then (5.7) yields

$$\hat{\mu}^-(D^r_{k,n})\geqq \sum_{\substack{s\subset(\Lambda_k\backslash\Lambda_n)\cap Q\\ S\subset(\Lambda_N\backslash\Lambda_k)\cap Q}} \hat{\mu}^-(D^r_{k,n}\mid E_{sS})\hat{\mu}^-(E_{sS})\geqq\mu_+(V_0)/4.$$

We consider the events:

$$R_n=\{\omega\in\Omega \mid \text{there is in } Z^2\backslash\Lambda_n \text{ a } (+*)\text{half-circuit surrounding } (\underline{0}) \text{ in } \pi\}.$$

By using the FKG inequality and the reflection symmetry of the measure $\hat{\mu}^-$ we get:

$$\hat{\mu}^-(R_n)\geqq\hat{\mu}^-(D^r_{k,n}\cap D^l_{k,n})\geqq\mu_+(V_0)^2/16.$$

Therefore

$$\hat{\mu}^-\left(\bigcap_{n=1}^{\infty} R_n\right)=\lim_{n\to\infty}\hat{\mu}^-(R_n)\geqq\mu_+(V_0)^2/16.$$

Since $\bigcap_{n=1}^{\infty} R_n\in\mathscr{B}_\infty$ we get $\hat{\mu}^-\left(\bigcap_{n=1}^{\infty} R_n\right)=1$ and this ends the proof.

Proof of Proposition 2. By using Lemma 8, Proposition 2 can be proved in the same way as Lemma 1.

6. Infinite Clusters in a Half-Plane

In this section we consider a generic measure $\mu\in M$ (not necessarily extremal) and we draw from Theorem 1 some statements about typical configurations of μ.

Proposition 3. *For any* $\mu\in M$ μ-a.s. *any infinite cluster*[**cluster*] *of* π *intersects infinitely many times* (i.m.t.) *the 1-axis.*

Proof. We consider infinite $(-)$clusters. The proof works in the same way for $(+)$, $(-*)$ or $(+*)$clusters. Let G be the event "there is an infinite $(-)$cluster of π non-intersecting the 1-axis". We have:

$$G=\bigcup_{x:x_2>0}\bigcap_{k=1}^{\infty}(P^k_x\cap H_x)$$

where

$$P^k_x=\{\omega\in\Omega \mid x \text{ belongs to a } (-)\text{cluster of size greater than } k\}$$

$$H_x=\{\omega\in\Omega \mid \text{there is in } \pi \text{ an infinite } (+*)\text{chain separating } x \text{ from the 1-axis}\}.$$

Therefore

$$\mu(G)\leqq\sum_{x:x_2>0}\lim_{k\to\infty}\mu(P^k_x\cap H_x). \tag{6.1}$$

In order to prove that $\mu(G)=0$ it is enough to show that if for some x $\mu(H_x)\neq 0$ then

$$\lim_{k\to\infty}\mu(P^k_x \mid H_x)=0. \tag{6.2}$$

For given x and k let n be such that $P^k_x\in\mathscr{B}_{\Lambda_n}$. We can write:

$$\mu(P^k_x \mid H_x)=\sum_B\mu(P^k_x \mid H_x\cap B)\,\mu(B \mid H_x)$$

where B runs over all boundary conditions on $\partial\Lambda_n$. By using the Markov property and the FKG inequality it is easy to verify that for any B

$$\mu(P_x^k | H_x \cap B) \leqq \hat{\mu}_n^-(P_x^k).$$

Hence:

$$\mu(P_x^k | H_x) \leqq \hat{\mu}^-(P_x^k).$$

Then the Proposition 2, the FKG inequality and the Proposition 1 imply

$$\lim_{k\to\infty} \mu(P_x^k | H_x) \leqq \lim_{k\to\infty} \hat{\mu}^+(P_x^k) \leqq \lim_{k\to\infty} \mu_+(P_x^k) \leqq \mu_+(C^{-*}) = 0.$$

Therefore $\mu(G)=0$. We consider the events

$$G_n = \{\omega \in \Omega | \text{there is in } \omega \text{ an infinite } (-)\text{cluster of } \pi \backslash \Lambda_n \text{ non-intersecting the 1-axis}\}$$

$$E_{\Lambda_n}^+ = \{\omega \in \Omega | \forall x \in \Lambda_n\ \omega(x) = 1\}.$$

Another application of the FKG inequality shows that

$$0 = \mu(G) \geqq \mu(G_n \cap E_{\Lambda_n}^+) = \mu(G_n)\mu(E_{\Lambda_n}^+ | G_n) \geqq \mu(G_n)\mu_-(E_{\Lambda_n}^+).$$

From the last inequality we have, for any n, $\mu(G_n)=0$ and this proves the proposition.

An useful corollary of Proposition 3 is the following

Proposition 4. *For any* $\mu \in M$ μ-a.s. *there is at most one infinite cluster* [**cluster*] *of* π *of each sign.*

Proof. We call Ω' the set of full measure for which Proposition 3 holds, and we prove that for any $\omega \in \Omega'$ there is in ω at most one infinite $(-)$cluster of π. If in ω there is no infinite $(+*)$cluster of π, then the statement is obviously true. Suppose that there is some infinite $(+*)$cluster of π. Then, by Proposition 3, we can suppose that there is an infinite $(+*)$cluster of π intersecting i.m.t. the negative 1-half-axis; then it is easy to realize that all infinite $(-)$clusters of π contain at most a finite number of points of the negative 1-half-axis. Hence all infinite $(-)$clusters of π intersect i.m.t. the positive 1-half-axis and this implies that they actually coincide.

7. 1-Invariant Equilibrium Measures

In this section we give a representation of the measures $\mu \in M$ which are translation invariant along one direction of the lattice.

We consider the following events:

$$A_+ = (C_\pi^+ \cap C_{\pi'}^+) \backslash (C_\pi^- \cup C_{\pi'}^-); \qquad A_- = (C_\pi^- \cap C_{\pi'}^-) \backslash (C_\pi^+ \cup C_{\pi'}^+)$$

$$A_1 = (C_\pi^+ \cap C_\pi^-) \cup (C_{\pi'}^+ \cap C_{\pi'}^-); \qquad A_0 = \Omega \backslash (A_+ \cup A_- \cup A_1).$$

Note that if $\mu \in M$, $A \in \mathscr{B}_\infty$, $\mu(A) > 0$, then $\mu_A = \mu(\cdot | A) \in M$.

The events A_+, A_-, A_1, A_0 belong to $\mathscr{B}_\infty$ and they form a partition of Ω; hence each measure $\mu \in M$ has an unique decomposition of the type

$$\mu = a_+ \nu_+ + a_- \nu_- + a_1 \nu_1 + a_0 \nu_0 \tag{7.1}$$

where the ν's belong to M and $\nu_+(A_+) = \nu_-(A_-) = \nu_1(A_1) = \nu_0(A_0) = 1$.

If μ is translation invariant along the 1-axis the decomposition (7.1) can be better specified; for this we need some lemmas.

Lemma 9. *If μ is translation invariant along the 1-axis, then $a_1 = 0$.*

Proof. We call $V_k^+ [V_k^-]$ the event "k is the least integer such that $(k, 0)$ belongs to an infinite $+[-]$cluster of π" and we put $V_k = V_k^+ \cup V_k^-$. The proof of Proposition 4 shows that

$$\mu(C_\pi^+ \cap C_\pi^-) = \mu\left(\bigcup_k V_k\right) = \sum_{k=-\infty}^{+\infty} \mu(V_k).$$

The 1-invariance of μ implies that $\mu(V_k)$ does not depend on k; hence, by the finiteness of the measure μ, we have $\mu(V_k) = 0$, $\mu(C_\pi^+ \cap C_\pi^-) = 0$. In the same way we get $\mu(C_{\pi'}^+ \cap C_{\pi'}^-) = 0$; hence $a_1 = \mu(A_1) = 0$.

Lemma 10. *If μ is translation invariant along the 1-axis then $\nu_+ = \mu_+$, $\nu_- = \mu_-$.*

Proof. By Lemma 1, it is enough to prove that $\nu_+(C^-) = 0$; on the other hand, by Proposition 3, ν_+-a.s. any infinite $(-)$cluster intersects i.m.t. at least one of the two 1-half-axes. Hence it suffices to prove that $\nu_+(N) = 0$, where

$N = \{\omega \in \Omega \mid$ there is in ω an infinite $(-)$cluster intersecting i.m.t. the positive 1-half-axis$\}$.

We have:

$$N = N_1 \cup N_2 \cup N_3$$

where

$N_1 = \{\omega \in \Omega \mid$ there is in ω an infinite $(-)$cluster intersecting i.m.t. both 1-half-axes$\}$

$N_2 = N \cap \{\omega \in \Omega \mid$ in ω at most a finite number of points of the negative 1-half-axis belong to an infinite $(-)$cluster intersecting i.m.t. the positive 1-half-axis$\}$

$N_3 = \{\omega \in \Omega \mid$ in ω infinitely many points of the negative 1-half-axis belong to different infinite $(-)$clusters intersecting i.m.t. the positive 1-half-axis$\}$.

Furthermore we consider the event

$N_4 = \{\omega \in \Omega \mid$ for any n there is in ω an infinite $(-)$cluster of $Z^2 \setminus \Lambda_n$ intersecting both 1-half-axes$\}$.

Note that ν_+, ν_- are obtained by conditioning μ with respect to 1-invariant events; therefore they are 1-invariant measures.

By using 1-invariance in the same way as in the proof of Lemma 9 we get $\nu_+(N_2)=0$; furthermore it is easy to verify that $N_3 \subset N_4 \subset \tilde{A}_+$; since $\nu_+(A_+)=1$ we get

$$\nu_+(N_2)=\nu_+(N_3)=\nu_+(N_4)=0. \tag{7.2}$$

Suppose $\nu_+(N)>0$; since N belongs to $\mathscr{B}_\infty$, $\nu'=\nu_+(\cdot|N)\in M$ and (7.2) implies $\nu'(N_1)=1$, $\nu'(N_4)=0$; on the other hand, by using b), one can prove that if $\nu'\in M$, $\nu'(N_4)=0$, then $\nu'(N_1)<1$. This contradiction shows that $\nu_+(N)=0$.

Lemma 11. *If $\mu\in M$, $\mu(C_\pi^+)=0$, then for each $x\in Z^2 \mu(E_x^-)\geqq \frac{1}{2}$.*

Proof. This lemma is a simple extension of Lemma 2. It suffices to prove that if $\mu\in M$, $\mu(C_\pi^+)=0$, then μ-a.s. there is no infinite $(+)$cluster of the half-plane $x_2\geqq -1$; then, by applying Lemma 2, the lemma follows from an inductive argument. Given a positive integer n, we consider the events:

$R'_n=\{\omega\in\Omega |$there is in ω a $(-*)$half-circuit surrounding $(0,-1)$ in $\{x_2\geqq -1\}\backslash \Lambda_n\}$

$R^n_s=\{\omega\in\Omega | s$ is the minimal $(-*)$half-circuit contained in $Z^2\backslash \Lambda_n$ surrounding $\underline{0}$ in $\pi\}$.

The hypothesis $\mu(C_\pi^+)=0$ implies $\sum_s \mu(R^n_s)=1$. Therefore, by using the Markov property and the FKG inequality we get

$$\mu(R'_n)=\sum_s \mu(R'_n | R^n_s)\mu(R^n_s)\geqq p^2>0 \tag{7.3}$$

where $p>0$ is the probability that $\omega(x)=-1$ conditioned to the event "$\omega(y)=1$ for any n.n. y of x".

Suppose $\mu\left(\bigcap_{n=1}^\infty R'_n\right)<1$ and put $\mu'=\mu\left(\cdot \Big| \widetilde{\bigcap_{n=1}^\infty R'_n}\right)$. Since $\bigcap_{n=1}^\infty R'_n\in\mathscr{B}_\infty$, μ' is an equilibrium measure such that $\mu'(C_\pi^+)=0$; therefore (7.3) holds for μ'; on the other hand the definition of μ' implies $\lim_{n\to\infty}\mu'(R'_n)=0$. This contradiction shows that $\mu\left(\bigcap_{n=1}^\infty R'_n\right)=1$ and this proves the lemma.

Lemma 12. *The magnetization of ν_0 is everywhere zero.*

Proof. We prove that for each $x\in Z^2$ $\nu_0(E_x^-)\geqq\frac{1}{2}$. It can be proved exactly in the same way that $\nu_0(E_x^+)\geqq\frac{1}{2}$. We put:

$$\nu_0=\nu_0(C_\pi^+)\nu_0(\cdot|C_\pi^+)+\nu_0(\tilde{C}_\pi^+)\nu_0(\cdot|\tilde{C}_\pi^+).$$

The magnetization of the measure $\nu_0(\cdot|\tilde{C}_\pi^+)$ is everywhere non-positive by Lemma 11. Consider the measure $\nu'_0=\nu_0(\cdot|C_\pi^+)$; since $\nu'_0(C_\pi^+)=\nu'_0(A_0)=1$ we have

$\nu'_0(C^+_{\pi'})=0$ (note that $C^+_\pi \cap C^+_{\pi'} \cap A_0 = \emptyset$). Hence by an obvious modification of Lemma 11, the magnetization of the measure ν'_0 is everywhere non-positive, too.

We collect the results of this section in the following proposition.

Proposition 5. *If $\mu \in M$ is translation invariant along the 1-axis, then μ has an unique decomposition of the type:*

$$\mu = a_+\mu_+ + a_-\mu_- + a_0\mu_0 \tag{7.4}$$

where $\mu_0(A_0)=1$; furthermore the magnetization of μ_0 is everywhere zero.

8. Main Results

Theorem 1 (Abraham and Reed). $\forall x \in Z^2 \lim_{s\to\infty} \mu(E^+_x | E^*_s) = 1/2$ *(where $\mu \in M$ and E^*_s is defined in Sect. 3).*

Proof. By using duplicated spin variables it can be shortly proved (see [6]) that the limit

$$\tau_x = \lim_{s\to\infty} \mu(E^+_x | E^*_s) \tag{8.1}$$

exists on the set of half-circuits ordered by "inclusion".

We put

$$R_{nk} = \{x \in Z^2 | \, |x_1| \leqq k;\ |x_2| \leqq n\};\ s_{nk} = (\partial R_{nk}) \cap \pi$$

the existence of the limit (8.1) implies

$$\tau_x = \lim_{n\to\infty} \lim_{k\to\infty} \mu(E^+_x | E^*_{s_{nk}}) = \lim_{n\to\infty} \mu^\pm_n(E^+_x)$$

where the measures $\mu^\pm_n$ are obtained by putting

$$\forall x : x_2 = n\ \omega(x) = 1; \qquad \forall x : x_2 = -n\ \omega(x) = -1.$$

Let ν be any limit point, in the vague topology of the measures, of the sequence $\mu^\pm_n$ (the existence of such a limit point follows from a compactness argument). It is clear that ν is translation invariant along the 1-axis; hence the decomposition (7.4) holds for ν. Furthermore a symmetry argument shows that the ν-mean value of $\omega(0,-1)+\omega(0,1)$ is zero and this implies that $a_+ = a_-$. Then, by Proposition 5, the magnetization of ν is everywhere zero and we get that, for any x, $\tau_x = \nu(E^+_x) = \frac{1}{2}$.

By using Theorem 1, it can be proved by general arguments (see [6]) that

$$\mu^\pm = \lim_{s\to\infty} \mu(\cdot | E^*_s) = (\mu_+ + \mu_-)/2. \tag{8.2}$$

Lemma 13. *If $\mu \in M$ and $\mu(C^+_\pi)=0$, then $\mu = \mu_-$.*

Proof. We consider the events:

$$A_n = \{\omega \in \Omega | \text{in } \omega \text{ there is a } (-)\text{circuit surrounding } \Lambda_n\}$$

$$A^m_n = \{\omega \in \Omega | \text{in } \omega \text{ there is in } \Lambda_m \text{ a } (-)\text{circuit surrounding } \Lambda_n\}$$

$$A_\infty = \bigcap_{n=1}^\infty A_n.$$

We suppose $\mu(A_\infty)<1$; then the measure $\mu'=\mu(\cdot|\tilde{A}_\infty)$ is an equilibrium measure and we have

$$\lim_{n\to\infty}\mu'(A_n)=\mu'(A_\infty)=0\,; \qquad \mu'(C_\pi^+)=0\,. \tag{8.3}$$

We choose n such that

$$\mu'(A_n)<\tfrac{1}{16}\,. \tag{8.4}$$

Then, by Proposition 1, we can choose $m>n$ such that

$$\mu_-(A_n^m)>\tfrac{1}{2} \tag{8.5}$$

(8.2) and (8.5) imply that there is $M>m$ such that, for any half-circuit $s\subset\pi\backslash\Lambda_M$

$$\mu'(A_n^m|E_s^*)>\tfrac{1}{8}\,. \tag{8.6}$$

On the other hand the second of (8.3) implies that we can choose $N>M$ such that

$$\sum_{s\subset(\Lambda_N\backslash\Lambda_M)\cap\pi}\mu'(M_{\pi s}^N)>\tfrac{1}{2} \tag{8.7}$$

(where the events $M_{\pi s}^N$ have been defined in the proof of Lemma 2).

By using the same arguments of the proof of Lemma 2 we get

$$\begin{aligned}\mu'(A_n^m)&\geqq\sum_{s\subset(\Lambda_N\backslash\Lambda_M)\cap\pi}\mu'(A_n^m|M_{\pi s}^N)\mu'(M_{\pi s}^N)\\&\geqq\sum_{s\subset(\Lambda_N\backslash\Lambda_M)\cap\pi}\mu'(A_n^m|E_s^*)\mu'(M_{\pi s}^N)\geqq\tfrac{1}{16}\,.\end{aligned} \tag{8.8}$$

Where we have used (8.6) and (8.7).

Since $A_n^m\subset A_n$ (8.4) and (8.8) are incompatible. Hence we have $\mu(A_\infty)=1$; this implies $\mu(C^+)=0$ and, by Lemma 1, $\mu=\mu_-$.

Theorem 2. *If $\mu\in M$ is translation invariant along the 1-axis, then μ is a linear convex combination of μ_+ and μ_-.*

Proof. By using Lemma 13 it can be easily verified that, for any $\mu\in M$ $\mu(A_0)=0$; then Theorem 2 follows from Proposition 4.

Lemma 13 in particular implies that if $\mu\in M$, $\mu(A_+)=1$, then $\mu=\mu_+$.

Hence if μ is an extremal equilibrium measure and $\mu\neq\mu_+$, $\mu\neq\mu_-$, then $\mu(A_+)=\mu(A_-)=0$, $\mu(A_1)=1$. By recalling the proof of Proposition 4 we get the following proposition.

Proposition 6. *If μ is an extremal equilibrium measure and $\mu\neq\mu_+$, $\mu\neq\mu_-$, then μ is neither translation invariant with respect to any direction of the lattice nor reflection invariant with respect to any axis of the lattice.*

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