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On the Decoupling of Massive Particles in Field Theory

J. Ambjørn

The Niels Bohr Institute, University of Copenhagen, DK-2100 Copenhagen Ø, Denmark

Abstract. The article examines the Feynman amplitude when some of the mass parameters are scaled to infinity. Contributions from diagrams containing the scaled mass go to zero provided no particles are massless and BPHZ subtractions are used.

I. Introduction

During the last few years there have been numerous applications of the decoupling theorem [1], [2]. These applications have made desirable a more stringent proof of the theorem. It will be shown that the heavy sector decouples perturbatively when BPH subtractions are used, in the case of a theory with two mass scales and momenta in the Euclidean regime.

As in Weinberg's power counting theorem the problems are mainly technical. The works of Appelquist [3], Anikin, Polivanov, and Zavialov [4], Bergere and Zuber [5], and Bergère and Lam [6] have shown that the α-parametric integral representation allows one to write down a closed expression for the renormalized Feynmann amplitude.

The rest of the paper is organized in the following way:

Section II gives the definition of the Feynman amplitude in the case of a scalar theory. In Sect. III a simple proof of the decoupling theorem is outlined for a scalar theory where technical problems are minimalized. The generalization to theories with spin and derivative couplings can be found in Sect. IV. Section V contains a short discussion of the results. Two appendices are devoted to technical questions.

II. Parametric Integral Representation of the Feynman Amplitude

To any connected Feynman graph G with L lines and V vertices corresponds the Feynman amplitude (in Euclidian space)

$$\tilde{F}(p) = \int \prod_{l \in L} \frac{d^4 k_l}{(2\pi)^4 (k_l^2 + m_l^2)} \cdot \prod_{v \in V} (2\pi)^4 \delta^{(4)}\left(p_v - \sum_{l \in L} \langle v, l \rangle\, k_l\right)$$

where p_v denotes the sum of the external momenta beginning at vertex v and $\langle v, l \rangle$ the incidence matrix on $V \times L$.

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There are several well-known ways of writing $\tilde{F}$ as a parametric integral. The following expression will be useful [7]

$$\tilde{F}(p)=(2\pi)^{4-2I}F(p)\delta^{(4)}\Big(\sum_{v\in V} p_v\Big)$$

$$F(p)=\int_0^\infty \prod_{l\in L} d\alpha_l\cdot\frac{\Gamma(L-2I)\sum_{l\in L}\alpha_l\exp\Big(-\sum_{l\in L}\alpha_l\Big)}{U^2(\alpha)\cdot\Big[\sum_{l\in L}\alpha_l m_l^2+E(\alpha,p_v)\Big]^{L-2I}}. \tag{1}$$

The functions involved are given by:

$$U(\alpha)\equiv\prod_{l\in L_G}\alpha_l\cdot\sum_{T\in\mathbb{T}_G}\prod_{l\in L_T}\alpha_l^{-1} \tag{2}$$

$$W_S(\alpha)\equiv\prod_{l\in L_G}\alpha_l\cdot\sum_{T^2\in\mathbb{T}^2(S)}\prod_{l\in L_{T^2}}\alpha_l^{-1} \tag{3}$$

$$E(\alpha,p)\equiv\sum_{S\in\mathbb{S}_{\mathrm{ex}}}(W_S/U)\cdot p^2(S) \tag{4}$$

where the notation is defined as follows [7]:

I denotes the number of loops in G, L both the set of lines and its cardinalnumber.

$\mathbb{T}_G(\mathbb{T}_G^2)$ denote the set of trees (2-trees) in G. A tree in G is a connected subgraph which has no loops and the same vertices as G. If one removes a line from a tree the corresponding subgraph is denoted a 2-tree.

A cut-set S is a minimal set of lines such that the subgraph of G where these lines are removed is not connected. $\mathbb{T}^2(S)$ denotes the set of 2-trees satisfying $T_2\cap S=\emptyset$. $\mathbb{S}_{\mathrm{ex}}$ denotes the set of cut-sets dividing G in two parts both containing external vertices. $p(S)$ is the sum of external momenta in one of the connected parts defined by S.

The parametric representations of F is not always well-defined. Ultraviolet divergencies can manifest themselves through the gamma function or a non-integrability of the integrand in regions where α_l goes to zero. The BPH procedure takes care of this. Following f.ex. APZ [5] the subtracted amplitude can be written in the closed form:

$$F^R=\sum_{n=z(0)}^{D_0+1}c_n\int_0^1\prod_{i=0}^{\kappa}\frac{d\zeta_i(1-\zeta_i)^{D_i}}{D_i!}\zeta_0^{q_n}\int_0^\infty\prod_{l\in L}d\alpha_l\sum_{l\in L}\alpha_l\exp\Big(-\sum_{l\in L}\alpha_l\Big)$$
$$\cdot\prod_{i=1}^{\kappa}\left(\frac{\partial}{\partial\zeta_i}\right)^{D_i+1}\cdot\left[\prod_{i=1}^{\kappa}\zeta_i^{4I_{\Gamma_i}}\cdot\frac{\Gamma(N_G+n)\cdot E^n(\beta,p)}{U^2(\beta)\cdot\Big[\sum_{l\in L}\alpha_l m_l^2+E(\beta,p)\cdot\zeta_0^2\Big]^{N_G+n}}\right] \tag{5}$$

$\Gamma_1,\dots,\Gamma_\kappa$ denotes the divergent subgraphs different from G. A subgraph is divergent if $D_i\equiv -2L_{\Gamma_i}+4I_{\Gamma_i}\geqq 0$. If $D_0<0$ (G superficial convergent) ζ_0 must be omitted and $n\equiv 0$. If $D_0\geqq 0$, the ζ_0 differentiation has given the terms $\sum_n c_n\zeta_0^{q_n}E^n$; $q_n\geqq 0$. $z(0)=-N_G+1$ when $D_G\geqq 0$ and $N_G=L-2I$.

$$\beta_l\equiv\pi_l^2(\zeta)\cdot\alpha_l,\qquad \pi_l(\zeta)\equiv\prod_{\substack{\Gamma_i\ni l\\ i\geqq 1}}\zeta_i. \tag{6}$$

(6) shows that if a line l belongs to a divergent subgraph Γ_i the corresponding parameter α_l is multiplied by a "subtraction parameter" ζ_i^2.

A glance at (5) makes it clear that the β-variables are the natural integration variables. For this reason we first change to the β-variables. Next the β-integration domain is decomposed into Hepp-sectors of the form:

$$\beta_{l_L} \geqq \ldots \geqq \beta_{l_1} \geqq 0 \quad (l_L > \ldots > l_1). \tag{7}$$

The reason is that in these sectors the singularities of $U^{-2}(\beta)$ are controlled by the introduction of Speers scaling variables [8]

$$\beta_{l_j} = \prod_{i \geqq j}^{L} t_i^2 \quad (0 \leqq t_i \leqq 1, \, i < L. \quad 0 \leqq t_L < \infty). \tag{8}$$

Speers lemma [8] states that in a sector (7), (8)

$$U(\beta) = \left(\prod_{i=1}^{L} t_i^{4 I_{G_i}} \right) \cdot \mathscr{P}(t_1^2, \ldots, t_{L-1}^2) \tag{9}$$

$$E(\beta, p_v) = t_L^2 \mathscr{F}(t_1^2, \ldots, t_{L-1}^2, p_v) / \mathscr{P}(t_1^2, \ldots, t_{L-1}^2) \tag{10}$$

where G_i consist of lines $l_1, \ldots, l_i$ and the vertices belonging to these lines. $\mathscr{P}$ is a polynomial >0 in (8). $\mathscr{F}$ is likewise a polynomial $\geqq 0$ in (8).

As a result of these rather trivial manipulations $F^R(p)$ can be written in the form:

$$\begin{aligned} F^R(p_v) = & \sum_{\text{sectors}} \sum_{n=z(0)}^{D_0+1} \int_0^1 d\zeta_0 \frac{(1-\zeta_0)^{D_0+1}}{D_0!} \zeta_0^{q_n} \int_0^1 \prod_{i=1}^{K} \frac{d\zeta_i (1-\zeta_i)^{D_i}}{\zeta_i D_i!} \\ & \cdot \int_0^\infty dt_L \int_0^1 \prod_{i=1}^{L-1} dt_i \cdot \sum_{l \in L} t_L^2 c_l \exp\left(-t_L^2 \sum_{l \in L} c_l \right) \\ & \cdot t_L^{-1} \cdot \left(\sum_{S \geqq N_G + n} \chi_S^{(n)}(t_1, \ldots, t_{L-1}, p_v) \cdot \left[\sum_{l \in L} c_l m_l^2 + \zeta_0^2 \mathscr{F}(t, p_v) / \mathscr{P}(t) \right]^{-S} \right) \end{aligned} \tag{11}$$

where $c_{l_i} \equiv c_{l_i}(\zeta, t) \equiv (\pi_{l_i}^2(\zeta))^{-1} \cdot \prod_{j \geqq i}^{L-1} t_j^2$ and $\chi_S^{(n)}$ is a polynomium in p_v of degree $D_0 + 2S$ and is analytic in $t_1, \ldots, t_{L-1}$ in the integration domain. The Taylor series in t_i starts with a power N_i higher than or equal to $\max(-D_{G_i} - 1, 0)$.

For the sake of completeness the details of this derivation are given in appendix A. The derivation involves only a minor change of the proof of APZ.

Formula (11) will be used to derive the decoupling theorem in the next section.

III. The Decoupling Theorem

Some of the masses are now scaled to infinity. Let us assume that there are two masses m and M and that M goes to infinity. The following estimate will be proved:

$$|F^R(p_v, m, M)| < c(\varepsilon) \cdot (M)^{-2\nu_M + \varepsilon} \quad \textit{for} \quad \varepsilon \in]0, 1[\tag{12}$$

$$\nu_M \equiv \max(1, \min_{H \subseteq G} (-D_H)) \tag{13}$$

where H runs over subgraphs (including G) containing all particles with mass M. p_v and m are allowed to vary in a compact domain not containing $m=0$.

Proof. Let l_{i_0} denote the largest line (relative to sector (7)) corresponding to particles with mass M. One can make the estimates:

$$\left[\sum_{l\in L} c_l m_l^2 + \zeta_0 \mathscr{F}/\mathscr{P}\right]^{-s} \leqq \left(M^2 \prod_{i\geqq i_0}^{L-1} t_i^2\right)^{-\nu_M+\varepsilon/2} (m^2)^{-s+\nu_M} \left(m^2 \sum_{l\in L} c_l\right)^{-\varepsilon/2}$$

$$|\chi_s^{(n)}(t_1, \ldots, t_{L-1}, p_v)| \leqq c_s^{(n)} \prod_{i=1}^{L-1} t_i^{N_i}.$$

The last line is correct because the variables belong to a compact domain. The t-integration in (11) is therefore dominated by a sum of terms of the form:

$$\int_0^\infty dt_L \int_0^1 \prod_{i=1}^{L-1} dt_i t_i^{N_i} \cdot \left(t_L^2 \sum_{l\in L} c_l\right)^{1-\varepsilon/2} \exp\left(-t_L^2 \sum_{l\in L} c_l\right)$$
$$\cdot t_L^{\varepsilon-1} \cdot \prod_{i\geqq i_0}^{L-1} t_i^{\varepsilon-2\nu_M} \cdot [(M^2)^{\nu_M-\varepsilon/2} (m^2)^{s-\nu_M+\varepsilon/2}]^{-1}. \tag{14}$$

By definition $N_i - 2\nu_M \geqq -1$ when $l_i \geqq l_{i_0}$ and the integral exists. Going back to the β-variables and taking advantage of the fact that in the sector considered $\beta_{l_1}^{1-\varepsilon/2} \beta_{l_2} \ldots \beta_{l_L} \geqq (\beta_{l_1} \cdot \beta_{l_2} \ldots \beta_{l_L})^{1-\varepsilon/2L}$ the t integration in (14) is dominated by the expression:

$$\int_0^\infty d\beta_{l_L} \int_0^{\beta_{l_L}} d\beta_{l_{L-1}} \ldots \int_0^{\beta_{l_2}} d\beta_{l_1} \cdot \frac{\left(\sum_{l\in L} \beta_l/\pi_l(\zeta)\right)^{1-\varepsilon/2} \exp\left(-\sum_{l\in L} \beta_l/\pi_l(\zeta)\right)}{(\beta_{l_1} \ldots \beta_{l_L})^{1-\varepsilon/2L_{G_i}}}$$

or going to α-variables

$$\left(\prod_{i=1}^{\kappa} \zeta_i^{L_{\Gamma_i}}\right)^{\varepsilon/L_G} \cdot \frac{1}{L_G!} \int_0^\infty \prod_{l\in L} d\alpha_l \cdot \frac{\left(\sum_{l\in L} \alpha_l\right)^{1-\varepsilon/2} \exp\left(-\sum_{l\in L} \alpha_l\right)}{(\alpha_{l_1} \ldots \alpha_{l_L})^{1-\varepsilon/2L_{G_i}}}$$

(11) finally gives:

$$|F^R| \leqq (M)^{-2\nu_M+\varepsilon} \cdot \left[\sum_{\text{sectors}} \sum_n \sum_s (m^{2s-2\nu_M+\varepsilon})^{-1} \right.$$
$$\left. \cdot \frac{\Gamma(\varepsilon/(2L_G))^{L_G}}{L_G!\,\Gamma(\varepsilon/2)} \cdot \prod_{i=1}^{\kappa} \frac{\Gamma(\varepsilon L_{\Gamma_i}/L_G)}{D_i!}\right]. \tag{15}$$

This completes the proof.

IV. Generalization to Theories with Spin and Derivative Couplings

The Feynman amplitude for graphs appearing in such theories can be written (in Euclidean space)

$$\tilde{F}(p) = \int \prod_{l\in L} \frac{d^4k_l R_l(k_l, m_l)}{k_l^2 + m_l^2} \cdot \prod_{v\in V_G} \left(S_v(p_v, k_l) \cdot \delta^{(4)}(p_v - \sum_{l\in L} \langle v, l\rangle k_l)\right) \tag{16}$$

where numerical constants and γ matrices have been left out. R_l is a polynomium in k_l. Its degree d_l depends on the spin of the particle at l. If the particle is a fermion, m_l appears in a positive power. S_v is a homogeneous polynomium of degree d_v in the internal and external momenta ending at vertex v.

Following Zimmermann, the superficial degree of a subgraph H is defined as:

$$D_H = 4I_H - 2L_H + v_H\,; \qquad v_H = \sum_{l\in L_H} d_l + \sum_{v\in V_H} d_v\,. \tag{17}$$

It will be useful to introduce the following notation:

$$\prod_{l\in L_G} R_l(k_l, m_l) \prod_{v\in V_G} S_v(p_v, k_l) = \sum_\sigma \prod_{i=1}^{\lambda(\sigma)} p_{v_i} \prod_{j=1}^{v(\sigma)} k_{l_j} \prod_{r=1}^{\mu(\sigma)} m_{l_r} \tag{18}$$

where Lorentz indices, constant coefficients and masses appearing in a negative power are suppressed.

$$D_H(\sigma) = 4I_H - 2L_H + v_H(\sigma) \tag{19}$$

where $v_H(\sigma)$ denotes $\sum_{v\in V_H} d_v$ + (number of factors in $\prod_{i=1}^{v(\sigma)} k_{l_i}$ which come from $\prod_{l\in L_H} R_l(k_l)$). $\Delta_H(\sigma) \equiv D_H - D_H(\sigma)$ and $\delta_H(\sigma) \equiv e(\Delta_H(\sigma)/2)$. Here $e(X)$ denotes the integral part of X.

$\tilde{F}$ can be written as a parametric integral (numerical constants and Lorentz indices are omitted as above):

$$\begin{aligned}
\tilde{F}(p) &= \delta^{(4)}\Big(\sum_{v\in V_G} p_v\Big)\cdot F(p)\\
F(p) &= \sum_\sigma \sum_{a=0}^{e(v(\sigma)/2)} \sum_{\mathrm{Div}(a)} \cdot \prod_{i=1}^{\lambda(\sigma)} p_{v_i} \prod_{r=1}^{\mu(\sigma)} m_{l_r}\\
&\quad \cdot \int \prod_{l\in L} d\alpha_l \Big(\sum_{l\in L}\alpha_l\Big) e^{-\sum_{l\in L}\alpha_l} \cdot \left[\frac{\Gamma(N_G - a)\prod_{i=1}^{a} \chi_{l_i' l_i''}(\alpha) \prod_{l\in J_a} Q_l(\alpha, p)}{U^2(\alpha)\cdot \Big[\sum_{l\in L} \alpha_l m_l^2 + E(\alpha, p)\Big]^{N_G - a}}\right]
\end{aligned} \tag{20}$$

The $\chi_{lm}(\alpha)$'s are homogeneous in α of degree -1, the $Q_l(\alpha, p_v)$'s are homogeneous in α of degree 0 and linear in p_v. The definitions are as follows:

$$\chi_{l,m} = -\sum_{S\in\mathbb{S}} \langle S, l\rangle \langle S, m\rangle \frac{W_S(\alpha)}{\alpha_l \alpha_m U_G(\alpha)} \tag{21}$$

$$Q_l(\alpha, p_v) = \sum_{S\in\mathbb{S}} \langle S, l\rangle \frac{W_S(\alpha)}{\alpha_l U_G(\alpha)} \cdot p(S)\,. \tag{22}$$

$\mathbb{S}$ denotes the set of cut-sets, and $\langle S, l\rangle$ the incidence matrix on $\mathbb{S}\times L$. Div(a) denotes the division of the $v(\sigma)$ lines appearing in $\prod_{i=1}^{v(\sigma)} k_{l_i}$ into $a+1$ parts: (l_1', l_1''), (l_2', l_2''), ..., (l_a', l_a''), J_a. $\chi_{l,m}$ should not be mistaken as the $\chi_s^{(n)}$ in (11) and (24).

The subtracted amplitude can now be written:

$$F^R(p)=\sum_{\sigma,a,\mathrm{Div}(a)}\sum_{n=z(a)}^{D_0+1}\prod_{i=1}^{\lambda(\sigma)}p_{v_i}\prod_{r=1}^{\mu(\sigma)}m_{l_r}$$

$$\cdot\int_0^1 d\zeta_0(1-\zeta_0)^{D_0}\zeta_0^{q_n}\cdot\int_0^1\prod_{i=1}^{\kappa}d\zeta_i(1-\zeta_i)^{D_i}\int_0^\infty\prod_{l\in L}d\alpha_l\Big(\sum_{l\in L}\alpha_l\Big)e^{-\sum_{l\in L}\alpha_l}$$

$$\cdot\prod_{i=1}^{\kappa}\left(\frac{\partial}{\partial\zeta_i}\right)^{D_i+1}\left[\frac{\prod_{i=1}^{\kappa}\zeta_i^{4I_{\Gamma_i}+v_{\Gamma_i}(\sigma)}E(\beta,p)^n\prod_{j=1}^{a}\chi_{l_j'l_j''}(\beta)\prod_{l\in J_a}Q_l(\beta,p)}{U^2(\beta)\Big(\sum_{l\in L}\alpha_l m_l^2+\zeta_0^2E(\beta,p)\Big)^{N_G-a+n}}\right] \tag{23}$$

If $D_0\geqq 0$ $z(a)=-N_G+a+1$ and $N_G-a+n\geqq 1+\delta_G(\sigma)$. If $D_0<0$, ζ_0 must be omitted and $z(a)=n=0$, but still $N_G-a+n\geqq 1+\delta_G(\sigma)$.

The same manipulations that led to (11) now give:

$$F^R(p)=\sum_{\text{sectors}}\sum_{\sigma,a,\mathrm{Div}(a),n}\prod_{i=1}^{\lambda(\sigma)}p_{v_i}\prod_{r=1}^{\mu(\sigma)}m_{l_r}$$

$$\cdot\int_0^1 d\zeta_i(1-\zeta_i)^{D_0}\zeta_0^{q_n}\cdot\int_0^1\prod_{i=1}^{\kappa}\frac{d\zeta_i(1-\zeta_i)^{D_{\Gamma_i}}}{\zeta_i^{1+\Delta_{\Gamma_i}(\sigma)}}\cdot\int_0^\infty dt_L\int_0^1\prod_{j=1}^{L-1}dt_j$$

$$\cdot t_L\prod_{j=1}^{L-1}t_j^{\Delta_{G_j}(\sigma)}\cdot\sum_{l\in L}c_l(\zeta,t)e^{-t_L^2\sum_{l\in L}c_l(\zeta,t)}$$

$$\cdot\left(\sum_{s\geqq N_G+n-a}\chi_s(t_1,\dots,t_{L-1},p)\Big(\sum_{l\in L}c_lm_l^2+\zeta_0^2\mathscr{F}/\mathscr{P}\Big)^{-s}\right). \tag{24}$$

Some details are given in Appendix B.

The following estimate can now be proved:

$$F^R(p,m,M)\leqq h(p,m,\varepsilon)\cdot M^{-2v_M+\varepsilon} \tag{25}$$

where v_M is defined as $\mathrm{Max}\Big(1/2,\underset{H}{\mathrm{Min}}(-D_H/2)\Big)$. H is defined in (13).

The proof follows the proof in Sect. III in all essentials, although one has to be more careful when making the estimates. Details are given in Appendix B.

It should be pointed out that the decoupling theorem is not valued if one uses the "minimal" subtractions defined in [5] instead of Zimmermann's subtraction (17).

V. Conclusion

The article has proven that the massive sector decouples at least as fast as $M^{-2v_M+\varepsilon}$, $\varepsilon\in]0,1[$, where $v_M(G)$ is given by (13) or (25).

A more detailed study using the Mellin transformation or the asymptotic expansion of the Laplace transformation would presumably improve the estimate so it could be written in the form $M^{-2v_M}[\ln(M/m)]^n$. This would, however, require considerably more work and is not needed in the context of the decoupling theorem.

It would be interesting to produce a stringent proof to all orders in perturbation theory in the case of massive-massless particles, e.g. in gauge theories. This, however, seems to be a difficult task.

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Appendix A

The purpose of this appendix is to present some details of the transformation from (5) to (12).

The change of variables from (α_l, ζ_i) to $(\beta_l, \zeta_i)=(\pi_l^2(\zeta_i)\cdot\alpha_l, \zeta_i)$ is described by the following equations:

$$\operatorname{Det}\left\{\frac{\partial(\alpha,\zeta)}{\partial(\beta,\zeta)}\right\}=\prod_{l\in L}\pi_l(\zeta)^{-2}=\prod_{i=1}^{\kappa}\zeta_i^{-2L_{\Gamma_i}}$$

$$\left(\frac{\partial}{\partial\zeta_i}\right)^{D_i+1}[\zeta_i^{4I_{\Gamma_i}}f(\alpha,\zeta)]=\zeta_i^{2L_{\Gamma_i}-1}\mathscr{L}_i\hat{f}(\beta,\zeta)$$

$$\mathscr{L}_i\equiv\prod_{s=0}^{D_i}\left(4I_{\Gamma_i}-s+2\left(\zeta_i^2\frac{\partial}{\partial\zeta_i^2}+\sum_{l\in L_{\Gamma_i}}\beta_l\frac{\partial}{\partial\beta_l}\right)\right).$$

Equation (5) considered in a Hepp sector (7) takes the form:

$$\begin{aligned}&\int_0^1 d\zeta_0\frac{(1-\zeta_0)^{D_0}}{D_0!}\zeta_0^{q_n}\int_0^1\prod_{i=1}^{\kappa}\frac{d\zeta_i(1-\zeta_i)^{D_i}}{\zeta_iD_i!}\cdot\int_0^{\infty}d\beta_{l_L}\int_0^{\beta_{l_L}}d\beta_{l_{L-1}}\dots\int_0^{\beta_{l_2}}d\beta_{l_1}\\&\cdot\left(\sum_{l\in L}\frac{\beta_l}{\pi_l^2(\zeta)}\exp\left(-\sum_{l\in L}\frac{\beta_l}{\pi_l^2(\zeta)}\right)\right)\cdot\prod_{i=1}^{\kappa}\mathscr{L}_i\\&\cdot\frac{\Gamma(N_G+n)\cdot E(\beta,p_v)^n\cdot U^{-2}(\beta)}{\left[\sum_{l\in L}\frac{\beta_l}{\pi_l(\zeta)}m_l^2+\zeta_0E(\beta,p_v)\right]^{N_G+n}}.\end{aligned}\tag{A.1}$$

Transforming to Speers variables (8), the Jacobians are:

$$\operatorname{Det}\left\{\frac{\partial\beta}{\partial t}\right\}=\prod_{i=1}^{L}t_i^{2i-1},\qquad\operatorname{Det}\left\{\frac{\partial t}{\partial\beta}\right\}=(\beta_1^{1/2}\beta_2\dots\beta_L)^{-1}$$

and (A.1) becomes

$$\begin{aligned}&\int_0^1 d\zeta_0\frac{(1-\zeta_0)^{D_0}}{D_0!}\zeta_0^{q_n}\int_0^1\prod_{i=1}^{\kappa}\frac{d\zeta_i(1-\zeta_i)^{D_i}}{\zeta_iD_i!}\cdot\int_0^{\infty}dt_L\int_0^1 dt_i\\&\cdot\left(\sum_{l\in L}t_L^2c_l(\zeta,t)\exp\left(-t_L^2\sum_{l\in L}c_l(\zeta,t)\right)\right)\\&\cdot\left(\prod_{i=1}^{L}t_i^{2L_{G_i}-1}\cdot\prod_{i=1}^{\kappa}\mathscr{L}_i'(t_L^{-2L}\prod_{i=1}^{L-1}t_i^{-4I_{G_i}}f(t,p))\right)\end{aligned}\tag{A.2}$$

where

$$c_l \equiv c_l(\zeta, t) \equiv \pi_l(\zeta)^{-2} \prod_{l_i \geqq l}^{L-1} t_i^2$$

$$\mathscr{L}_i' \equiv \prod_{s=0}^{D_i} \left(I_{\Gamma_i} - s + 2\zeta_i^2 \frac{\partial}{\partial \zeta_i^2} + \sum_{l_j \in L_{\Gamma_i}} t_j \frac{\partial}{\partial t_j} - t_{j-1} \frac{\partial}{\partial t_{j-1}} \right) \tag{A.3}$$

$$f(t,p) = \Gamma(N_G + n)(\mathscr{F}/\mathscr{P})^n \mathscr{P}^{-2} \cdot \left(\sum_{l \in L} c_l m_l^2 + \zeta_0^2 \mathscr{F}/\mathscr{P} \right)^{-N_G - n}.$$

Note that $\frac{\partial}{\partial \zeta_i} \alpha_l = 0$ implies $\left(2\zeta_i^2 \frac{\partial}{\partial \zeta_i^2} + \sum_{l_j \in L_{\Gamma_i}} t_j \frac{\partial}{\partial t_j} - t_{j-1} \frac{\partial}{\partial t_{j-1}} \right) \cdot t_L^2 c_l(\zeta, t) = 0$. $2\zeta_i^2 \frac{\partial}{\partial \zeta_i^2}$ can therefore be omitted from $\mathscr{L}_i'$ if we agree that $t_l \partial/\partial t_l$ *does not* act on $c_l(\zeta, t)$.

Note also that if G_{i_0} is a divergent subgraph

$$\sum_{l_j \in L_{G_{i_0}}} t_j \frac{\partial}{\partial t_j} - t_{j-1} \frac{\partial}{\partial t_{j-1}} = t_{i_0} \frac{\partial}{\partial t_{i_0}}.$$

Using these facts

$$\prod_{i=1}^{L} t_i^{2L_{G_i}-1} \prod_{i=1}^{K} \mathscr{L}_i' \left(t_L^{-2L} \prod_{j=1}^{L-1} t_j^{-4I_{G_j}} \cdot f(t,p) \right)$$

is seen to be of the form stated in Sect. II. The only problems come from t_i belonging to divergent subgraphs G_i. In this case, however, $\mathscr{L}_i'$ annihilates the troublesome term $t_i^{2L_{G_i} - 4I_{G_i} - 1}$ because:

$$t_i^{2L_{G_i}-1} \mathscr{L}_i'(t_i^{-4I_{G_i}} f) = \left(\frac{\partial}{\partial t_i} \right)^{D_{i_0}+1} f, \quad (G_i \equiv \Gamma_{i_0}).$$

This shows that the function in question is analytic in each t_i variable separately. Hartog's theorem [9] then implies that the function is analytic in all variables together. The specific form of $f(p,t)$, (A.3), results in (11) in Sect. II.

Appendix B

The purpose of this appendix is to present some details in the derivation of (24) and (25).

The change of variables from (α_l, ζ_i) to (β_l, ζ_i) is described in Appendix A. The only difference is the following equations:

$$\left(\frac{\partial}{\partial \zeta_i} \right)^{D_i+1} [\zeta_i^{4I_{\Gamma_i} + \nu_{\Gamma_i}(\sigma)} f(\alpha, \zeta^2)] = \zeta_i^{2L_{\Gamma_i} - \Delta_{\Gamma_i}(\sigma) - 1} \mathscr{L}_i(\sigma) \hat{f}(\beta, \zeta^2)$$

$$\mathscr{L}_i(\sigma) = \prod_{s=0}^{D_i} \left(4I_{\Gamma_i} + \nu_{\Gamma_i}(\sigma) - s + 2\left(\zeta_i^2 \frac{\partial}{\partial \zeta_i^2} + \sum_{l \in L_{\Gamma_i}} \beta_l \frac{\partial}{\partial \beta_l} \right) \right).$$

The transformation to t-variables in a β-Hepp sector can be performed as in Appendix A.

Lemma 1.

$$\prod_{i=1}^{a} \chi_{l_i' l_i''}(\beta) \prod_{l\in J_a} Q_l(\beta,p) = \frac{A(t,p)}{t_L^{2a}\cdot \prod_{i=1}^{L-1} t_i^{\nu_{G_i}(\sigma)}}$$

where $A(t,p)$ is a function of $t_1,\dots,t_{L-1}$, analytic in the t domain.

Proof. $Q_l(\beta)$ is bounded and creates no difficulties. $|\chi_{lm}(\beta)| \leqq \sum_{H\ni l,m} (\beta_l+\beta_m)^{-1}$ ([7] Theorem 10–3). Therefore it is sufficient to prove that

$$\prod_{i=1}^{a} (\beta_{l_i'}+\beta_{l_i''}) \geqq t_L^{2a} \prod_{j=1}^{L-1} t_j^{\nu_{G_j}(\sigma)}.$$

Let $l_i'' \geqq l_i'$ and let n_j denote the number of $l_i'' \leqq j$ we have:

$$\prod_{i=1}^{a} \beta_{l_i''} = \prod_{j=1}^{L} \beta_j^{n_j-n_{j-1}} = \prod_{j=1}^{L} (\beta_j/\beta_{j+1})^{n_j} = \prod_{j=1}^{L} t_j^{2n_j}.$$

$2n_j \leqq \nu_{G_j}(\sigma)$ because $l_i'' < j$ implies $l_i', l_i'' \in G_j$. This gives:

$$\prod_{j=1}^{L} t_j^{2n_j} = t_L^{2a} \prod_{j=1}^{L-1} t_j^{2n_j} \geqq t_L^{2a}\cdot \prod_{j=1}^{L-1} t_j^{\nu_{G_j}(\sigma)}.$$

As a consequence of Lemma 1, the integrals in question have the form:

$$\int_0^1 d\zeta_0 (1-\zeta_0)^{D_0}\zeta_0^{q_n}\cdot \int_0^1 \prod_{i=1}^{K} d\zeta_i (l-\zeta_i)^{D_i}\zeta_i^{-1-\Delta_{\Gamma_i}(\sigma)}\cdot \int_0^\infty dt_L \int_0^1 \prod_{i=1}^{L-1} dt_i$$

$$\cdot\left(t_L^2 \sum_{l\in L} c_l\right) e^{-t_L^2 \sum_{l\in L} c_l} \left\{\prod_{i=1}^{L} t_i^{2L_{G_i}-1}\cdot \prod_{i=1}^{K} \mathscr{L}_i'(\sigma)\cdot \left(t_L^{-2L}\cdot \prod_{i=1}^{L-1} t_i^{-4I_{G_i}-\nu_{G_i}(\sigma)}\cdot f\right)\right\} \quad \text{(B.1)}$$

$$f(t,p) = (\mathscr{F}/\mathscr{P})^n \mathscr{P}^{-2} A(t,p)\cdot \left(\sum_{l\in L} c_l m_l^2 + \zeta_0^2 \mathscr{F}/\mathscr{P}\right)^{-N_G+a-n}.$$

As before $\{\cdot\}$ in (B.1) can be written as:

$$t_L^{-1}\cdot \prod_{i=1}^{L-1} t_i^{\Delta_{G_i}(\sigma)}\cdot \sum_{s\geqq N_G+n-a} \chi_s^{(n)}(t,p)\left(\sum_{l\in L} c_l m_l^2 + \zeta_0^2 \mathscr{F}/\mathscr{P}\right)^{-s} \quad \text{(B.2)}$$

where $\chi_s^{(n)}(t,p)$ is defined as in (11).

One can make the estimate:

$$\left(\sum_{l\in L} c_l m_l^2\right)^s \geqq (t_L^2 \dots t_{i_0}^2 M^2)^{\nu_M-\varepsilon/2} \left(\sum_{l\in F(\sigma)} c_l M^2\right)^{F(\sigma)/2}$$

$$\cdot\left(\sum_{l\in L} c_l\right)^{\varepsilon/2}\cdot \left(\sum_{l\in L} c_l\right)^{\Delta_G(\sigma)/2 - F(\sigma)/2}\cdot (m^2)^{s-\nu_M-F(\sigma)/2+\varepsilon/2} \quad \text{(B.3)}$$

where ν_M is given by (25), $F(\sigma)$ denotes the number or the set of massive particles appearing in $\prod_{r=1}^{\mu(\sigma)} m_{l_r}$ and i_0 is the largest line corresponding to particles with mass M.

Using (B.2) and (B.3), the t-integrand in (B.1) is dominated by an expression of the form:

$$\frac{(\sum t_L^2 c_l) e^{-t_L^2 \sum c_l} \cdot \prod_{i \geq i_0}^{L-1} t_i^{N_i - 2v_M + \varepsilon} \cdot \prod_{i=1}^{L-1} t_i^{\Delta_{G_i}(\sigma)} t_L^{\Delta_{G_L}(\sigma)}}{t_L \left(\sum_{l \in L} c_l\right)^{\varepsilon/2} \left(t_L^2 \sum_{l \in L} c_l\right)^{\Delta_G/2 - F(\sigma)/2} \cdot \left(t_L^2 \sum_{l \in F(\sigma)} c_l\right)^{F(\sigma)/2}}$$

or

$$\frac{\left(\sum_{l \in L} \alpha_l\right)^{1-\varepsilon/2} \cdot e^{-\sum_{l \in L} \alpha_l} \cdot \prod_{i=1}^{L} t_i^{\Delta_{G_i}(\sigma)}}{\left(\prod_{i=1}^{L} t_i\right)^{1-\varepsilon} \left(\sum_{l \in L} \alpha_l\right)^{\Delta_G(\sigma)/2 - F(\sigma)/2} \cdot \left(\sum_{l \in F(\sigma)} \alpha_l\right)^{F(\sigma)/2}} \tag{B.4}$$

In deriving (B.4), we have used the facts: $N_i - 2v_M + 1 \geqq 0$ for $i \geqq i_0$, $N_i \geqq 0$ and $\left(\prod_{i \geqq i_0}^{L} t_i\right)^{1-\varepsilon} \geqq \left(\prod_{i=1}^{L} t_i\right)^{1-\varepsilon}$.

Transforming to β variables, using $\beta_1^{1-\varepsilon/2} \beta_2 \ldots \beta_L \geqq (\beta_1 \ldots \beta_L)^{1-\varepsilon/2L}$, transforming further to the α-variables and using $\left(\sum_{l \in F(\sigma)} \alpha_l\right)^{F(\sigma)/2} \geqq \left(\prod_{l \in F(\sigma)} \alpha_l\right)^{1/2}$ the t-integral in (B.1) is dominated by

$$\begin{aligned} &\int_0^\infty \prod_{l \in L} d\alpha_l \left(\sum_{l \in L} \alpha_l\right)^{1-\varepsilon/2} \cdot e^{-\sum_{l \in L} \alpha_l} \cdot \prod_{l \in L} \alpha_l^{-1+\varepsilon/2L} \\ &\cdot \prod_{l \in L} \alpha_l^{\Delta_{G_l}(\sigma)/2 - \Delta_{G_{l-1}}(\sigma)/2} \cdot \prod_{l \in F(\sigma)} \alpha_l^{-1/2} \cdot \left(\sum_{l \in L} \alpha_l\right)^{-(\Delta_G(\sigma)/2 - F(\sigma)/2)} \\ &\cdot \prod_{l \in L} (\pi_l(\zeta))^{\Delta_{G_l}(\sigma) - \Delta_{G_{l-1}}(\sigma) + \varepsilon/L} . \end{aligned} \tag{B.5}$$

Lemma 2. (B.5) *contains* ζ *(from* Γ*) in a power larger than or equal* $\Delta_\Gamma(\sigma) + \varepsilon/L$.

Proof. $\pi_l(\zeta)$, $l \in \Gamma$ contains ζ. $\Delta_\Gamma(\sigma) = \sum_{l \in L_\Gamma} a_l$ where a_l is the power from $R_l(k_l)$ not present in $\prod_{j=1}^{v(\sigma)} k_{l_j}$. Consequently

$$\sum_{l \in \Gamma} \Delta_{G_l}(\sigma) - \Delta_{G_{l-1}}(\sigma) = \sum_{l \in \Gamma} a_l = \Delta_\Gamma(\sigma) .$$

The integral in (B.5) exists: $l \in F(\sigma)$ implies $\Delta_{G_l}(\sigma) - \Delta_{G_{l-1}}(\sigma) \geqq 1$. It follows that $\alpha_l^{-1/2}$, $l \in F(\sigma)$ is cancelled by $(\alpha_l)^{(\Delta_{G_l}(\sigma)/2 - \Delta_{G_{l-1}}(\sigma)/2}$. The integral can be written:

$$\int_0^\infty \prod_{l \in L} d\alpha_l \delta\left(1 - \sum_{l \in L} \alpha_l\right) \cdot \prod_{l \in L} \alpha_l^{-1+\varepsilon/2L} \cdot \left[\frac{\prod_{l \in L} \alpha_l^{\Delta_{G_l}(\sigma) - \Delta_{G_{l-1}}(\sigma)}}{\prod_{l \in F(\sigma)} \alpha_l}\right]^{1/2}$$

which exists.

Lemma 2 shows that the ζ_i integrations in (B.1) can be performed. Consequently (24) is dominated by a sum of terms of the form

$$h(p, m, \varepsilon) M^{F(\sigma)}/(M^2)^{\nu_M + F(\sigma)/2 - \varepsilon/2}.$$

As $2\nu_M - \varepsilon \geqq 1 - \varepsilon > 0$ we have the decoupling (25).

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