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On the Cook-Kuroda Criterion in Scattering Theory*

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Abstract. A new criterion of the Cook-Kuroda type for the existence of the wave operator in the two-space scattering theory is introduced. The condition is quite simple, but it generalizes not only the original Cook-Kuroda condition but also its generalization recently given by Schechter. Specialized to the one-space case, it is actually equivalent to Schechter's condition for an optimal choice of factorization. An application to potential scattering leads to a new result.

1. Introduction

Recently Schechter [1] and Simon [2] generalized the 20-year-old Cook-Kuroda criterion [3, 4] for the existence of the wave operator in scattering theory. The purpose of the present paper is to contribute another generalization in the context of *two-space scattering theory* [5]. Our condition (Theorem I) has several advantages. First, it is formally simpler than others [1–4], involving only bounded operators. Second, it has a simple, purely time-dependent proof. Third, it is valid in the two-space setting without any extra assumptions on the identification operator J except that J is bounded. Fourth, Schechter's theorem can easily be reduced to ours, so that our results contain a simplified proof of a two-space version of his theorem. At the same time, this shows that our result is in general stronger than Schechter's.

On the other hand, Schechter's condition is extremely flexible, involving a (formal) factorization of the perturbation that can be chosen in many different ways. In fact we shall show that some favorable choices of the factorization lead to a result equivalent to ours (Theorem III).

Let us first state our theorems. In two-space scattering theory, one considers two selfadjoint operators H_j, $j=1,2$, each acting in its Hilbert space $\mathfrak{H}_j$, and a bounded linear operator J (the *identification operator*) on $\mathfrak{H}_1$ to $\mathfrak{H}_2$. We denote by $U_j(t)=\exp(-itH_j)$ the unitary group generated by $-iH_j$. The associated *wave*

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operator $W_+ = W_+(H_2, H_1; J)$ will be defined by

$$W_+ f = \lim_{t\to\infty} W(t) f, \quad W(t) = U_2(-t) J U_1(t), \tag{1.1}$$

whenever the limit exists. Obviously the domain $\mathfrak{D}(W_+)$ of W_+ is a (closed) subspace of $\mathfrak{H}_1$. In this paper we do *not* extend W_+ beyond this domain. Another wave operator W_- defined with $-\infty$ instead of ∞ in (1.1) can be handled in the same way.

Denoting by $R_j(z) = (H_j - z)^{-1}$ the resolvent, and by $\varrho(H_j)$ the resolvent set, of H_j, we set

$$C(z) = R_2(z) J - J R_1(z), \quad z \in \varrho \equiv \varrho(H_1) \cap \varrho(H_2). \tag{1.2}$$

Our main result now reads

Theorem I. *Let* $z \in \varrho$. *If* $f \in \mathfrak{H}_1$ *satisfies*

$$\int_0^\infty \| C(z) U_1(t) f \| dt < \infty, \tag{1.3}$$

then $f \in \mathfrak{D}(W_+)$.

It is convenient to state Theorem I in a different form by introducing the set $\mathfrak{M}(z)$ of all $f \in \mathfrak{H}_1$ satisfying (1.3). $\mathfrak{M}(z)$ is obviously a linear manifold in $\mathfrak{H}_1$. Let us denote its closure by $[\mathfrak{M}(z)]$. Then Theorem I is equivalent to

Theorem I′. $[\mathfrak{M}(z)] \subset \mathfrak{D}(W_+)$.

Regarding the dependence of $\mathfrak{M}(z)$ and $[\mathfrak{M}(z)]$ on z, we have

Theorem I″. *For* $\operatorname{Im} z > 0$, $\mathfrak{M}(z) = \mathfrak{M}$ *is independent of* z, *while* $\mathfrak{M}(z) \subset \mathfrak{M}$ *for* $\operatorname{Im} z < 0$. $[\mathfrak{M}(z)] = [\mathfrak{M}]$ *is independent of* $z \in \varrho$.

We shall prove Theorem I′ in Sect. 2, and study its relationship to other criteria of the Cook-Kuroda type, in particular Schechter's, in Sects. 3 and 4. Theorem I″ will be proved also in Sect. 4. These sections contain other results related to the Schechter factorization. Sect. 5 contains an application to potential scattering in R^3.

2. Proof of Theorem I′

First we note some obvious facts regarding $\mathfrak{M}(z)$ and $[\mathfrak{M}(z)]$. Since f satisfies (1.3) if and only if $U_1(s) f$ does, where s is any real number, $\mathfrak{M}(z)$ is invariant under the map $U_1(s)$. Hence the same is true of $[\mathfrak{M}(z)]$, which therefore reduces H_1. This implies, in particular, that $R_1(z')\mathfrak{M}(z)$ is a dense subset of $[\mathfrak{M}(z)]$ for any $z' \in \varrho(H_1)$.

A simple calculation gives (we write $R_j(z) = R_j$, $C(z) = C$ for simplicity)

$$(d/dt)(R_2 W(t) R_1 f, g) = -i(U_2(-t) C U_1(t) f, g)$$

for any $f \in \mathfrak{H}_1$ and $g \in \mathfrak{H}_2$. Hence for $t' < t''$

$$\| R_2 W(t'') R_1 f - R_2 W(t') R_1 f \| \leqq \int_{t'}^{t''} \| C U_1(t) f \| dt. \tag{2.1}$$

Assume now that $f \in \mathfrak{M}(z)$ so that (1.3) is true. Then (2.1) shows that $\lim R_2 W(t) R_1 f$ exists. (Here and in what follows lim refers to $t \to \infty$.) Since $R_2 W(t)$ is uniformly bounded in t and $R_1 \mathfrak{M}(z)$ is dense in $[\mathfrak{M}(z)]$ as noted above, it follows that $s\text{-}\lim R_2 W(t)$ exists on $[\mathfrak{M}(z)]$. Then Lemma 2.1 given below shows that $s\text{-}\lim W(t)R_1$ exists on $[\mathfrak{M}(z)]$. Since $W(t)$ is uniformly bounded, a similar argument shows that $s\text{-}\lim W(t)$ exists on $[\mathfrak{M}(z)]$. This means that $[\mathfrak{M}(z)] \subset \mathfrak{D}(W_+)$.

Lemma 2.1. *$s\text{-}\lim [R_2(z)W(t) - W(t)R_1(z)] = 0$ on $[\mathfrak{M}(z)]$.*

Proof. Let $\phi \in C_0^\infty(-\infty, \infty)$ and set

$$g = \int_{-\infty}^{\infty} \phi(s) U_1(s) f ds, \qquad f \in \mathfrak{M}(z). \tag{2.2}$$

The set of all such g's is dense in $\mathfrak{M}(z)$, hence in $[\mathfrak{M}(z)]$ too, since (2.2) tends to f if ϕ tends to the delta function. On the other hand, we have

$$\|CU_1(t)g\| \leqq \int_{-\infty}^{\infty} |\phi(s)| \, \|CU_1(t+s)f\| ds \qquad (C = C(z)).$$

Since $\|CU_1(t)f\|$ is integrable in $t \in (0, \infty)$ and since ϕ has compact support, it follows that $\lim \|CU_1(t)g\| = 0$. Since $CU_1(t)$ is uniformly bounded and the g's are dense in $[\mathfrak{M}(z)]$ as noted above, we conclude that $s\text{-}\lim CU_1(t) = 0$ on $[\mathfrak{M}(z)]$. The lemma then follows from the identity

$$R_2 W(t) - W(t) R_1 = U_2(-t) C U_1(t).$$

3. Relation to Schechter's Theorem

We now compare our results with other criteria of the Cook-Kuroda type [1–4]. Since Schechter's theorem [1] is the strongest one among them, it suffices to consider it.

Schechter's condition involves a "factorization" of the perturbation which, in the context of two-space theory, takes the form

$$(Ju, H_2 v) - (JH_1 u, v) = (Au, Bv), \tag{3.1}$$

assumed to be true for every $u \in \mathfrak{D}(H_1)$ and $v \in \mathfrak{D}(H_2)$. Here A is a linear operator from $\mathfrak{H}_1$ to a Banach space $\mathfrak{K}$ with $\mathfrak{D}(A) \supset \mathfrak{D}(H_1)$, and B is a linear operator from $\mathfrak{H}_2$ to $\mathfrak{K}^*$ (the adjoint space of $\mathfrak{K}$) with $\mathfrak{D}(B) \supset \mathfrak{D}(H_2)$. B is assumed to be H_2-bounded.

Any operator A that appears in this *Schechter factorization* (together with some B) will be called a *Schechter operator* (for the triplet $\{H_2, H_1, J\}$).

Schechter's theorem now reads, with a slight generalization,

Theorem II. *Let A be a Schechter operator. If $f \in \mathfrak{D}(H_1)$ satisfies the condition*

$$\int_{t_f}^{\infty} \|AU_1(t)f\| dt < \infty \tag{3.2}$$

for some real number t_f, then $f \in \mathfrak{D}(W_+)$.

Remark 3.1. (a) In [1] the condition $f \in \mathfrak{D}(H_1)$ is assumed, though not stated explicitly.

(b) Even for $f \in \mathfrak{D}(H_1)$, $\|AU_1(t)f\|$ may not be measurable in t, since A is not assumed to be H_1-bounded or closable. Thus the integral in (3.2) should be taken in the sense of an upper integral.

(c) In [1] only the single-space case ($\mathfrak{H}_1=\mathfrak{H}_2$, $J=1$) is considered. In a lecture at the Utah Conference (July 1978), Schechter generalized the theorem to the two-space case under certain additional conditions on J. In Theorem II, however, we need no extra conditions on J.

As before, it is convenient to rewrite Theorem II by introducing the set $\mathfrak{M}(A)$ of all $f\in\mathfrak{D}(H_1)$ satisfying (3.2). Again it is obvious that $\mathfrak{M}(A)$ is a linear manifold in $\mathfrak{H}_1$ invariant under $U_1(t)$, and its closure $[\mathfrak{M}(A)]$ reduces H_1. Theorem II is equivalent to

Theorem II′. $[\mathfrak{M}(A)]\subset\mathfrak{D}(W_+)$.

We shall show that Theorem II′ can be reduced to Theorem I′. We achieve this by showing not $\mathfrak{M}(A)\subset\mathfrak{M}(z)$ (which is probably untrue) but $[\mathfrak{M}(A)]\subset[\mathfrak{M}(z)]$. More precisely:

Theorem III. *For any Schechter operator A, we have $[\mathfrak{M}(A)]\subset[\mathfrak{M}]$ (where $[\mathfrak{M}]$ is the common space $[\mathfrak{M}(z)]$, see Theorem I″). On the other hand, for any triplet $\{H_2,H_1,J\}$ there are Schechter operators A with $[\mathfrak{M}(A)]=[\mathfrak{M}]$.*

Remark 3.2. Any triplet $\{H_2,H_1,J\}$ has infinitely many Schechter factorizations. A simple and useful one is given by

$$A=A(z)=C(z)(H_1-z),\qquad B=B(z)=-(H_2-\bar{z}) \tag{3.3}$$

where $z\in\varrho$ and $C(z)$ is as before (1.2). Here we take $\mathfrak{K}=\mathfrak{H}_2$, $\mathfrak{D}(A)=\mathfrak{D}(H_1)$ and, of course, $\mathfrak{D}(B)=\mathfrak{D}(H_2)$. We shall refer to $A(z)$ as an *optimal Schechter operator*, since it gives the optimal result stated in Theorem III.

4. Proof of Theorem III

Lemma 4.1. $[\mathfrak{M}(A)]\subset[\mathfrak{M}(z)]$ *for all* $z\in\varrho$.

Proof. As is easily seen, (3.1) implies

$$C=R_2J-JR_1=-(BR_2^*)^*AR_1, \tag{4.1}$$

where we have again written $R_j=R_j(z)$, $C=C(z)$. Here $(BR_2^*)^*$ is a bounded operator on $\mathfrak{K}^{**}$ to $\mathfrak{H}_2$ because B is H_2-bounded. Since A has range in $\mathfrak{K}$, (4.1) makes sense by the canonical embedding of $\mathfrak{K}$ in $\mathfrak{K}^{**}$.

Let $f\in\mathfrak{M}(A)$. Then

$$\int_{t_f}^{\infty}\|CU_1(t)(H_1-z)f\|\,dt=\int_{t_f}^{\infty}\|(BR_2^*)^*AU_1(t)f\|\,dt<\infty$$

by (4.1) and (3.2) because BR_2^* is bounded. It follows that $(H_1-z)f\in\mathfrak{M}(z)$; the t_f on the left is irrelevant since C is bounded. Since $[\mathfrak{M}(z)]$ reduces H_1, we conclude that $f\in[\mathfrak{M}(z)]$. This proves that $\mathfrak{M}(A)\subset[\mathfrak{M}(z)]$, and hence $[\mathfrak{M}(A)]\subset[\mathfrak{M}(z)]$ for any $z\in\varrho$.

Lemma 4.2. *The Schechter operator $A(z)=C(z)(H_1-z)$ is optimal in the sense that $[\mathfrak{M}(A(z))]=[\mathfrak{M}(z')]$ for any $z'\in\varrho$.*

Corollary 4.3. $[\mathfrak{M}(z)]$ *is independent of* $z\in\varrho$.

Proof of Lemma 4.2. We have the identity

$$C(z')=(H_2-z)R_2(z')C(z)(H_1-z)R_1(z'),\qquad z,z'\in\varrho. \tag{4.2}$$

Hence, writing $A=A(z)$,

$$\|AU_1(t)R_1(z')f\| = \|C(z)(H_1-z)R_1(z')U_1(t)f\|$$
$$\leqq \|(H_2-z')R_2(z)\| \, \|C(z')U_1(t)f\|, \quad f\in\mathfrak{H}_1 .$$

Since $\|(H_2-z')R_2(z)\|<\infty$, this inequality shows that $f\in\mathfrak{M}(z')$ implies $R_1(z')f\in\mathfrak{M}(A)$. Since $[\mathfrak{M}(A)]$ reduces H_1 as remarked above, it follows that $f\in[\mathfrak{M}(A)]$. Thus $\mathfrak{M}(z')\subset[\mathfrak{M}(A)]$, hence $[\mathfrak{M}(z')]\subset[\mathfrak{M}(A)]$. Since the opposite inclusion is known (Lemma 4.1), we have proved that $[\mathfrak{M}(A)]=[\mathfrak{M}(z')]$.

It remains to complete the proof of Theorem I″. First we prove

Lemma 4.4. *Let* $z, z'\in\varrho$, *with* $\operatorname{Im} z'>0$. *Then* $f\in\mathfrak{M}(z)$ *implies* $R_1(z')f\in\mathfrak{M}(z)$.

Proof. Since $R_1(z')=i\int_0^\infty \exp(iz's)U_1(s)ds$, we have for $f\in\mathfrak{M}(z)$

$$\int_0^\infty \|C(z)U_1(t)R_1(z')f\|dt \leqq \int_0^\infty e^{-y's}ds \int_0^\infty \|C(z)U_1(t+s)f\|dt$$
$$\leqq K_f \int_0^\infty e^{-y's}ds = K_f/y' < \infty ,$$

where $y'=\operatorname{Im} z'>0$ and K_f is the finite number (1.3). Hence $R_1(z')f\in\mathfrak{M}(z)$.

Lemma 4.5. *Let* z, z' *be as in Lemma 4.4. Then* $\mathfrak{M}(z)\subset\mathfrak{M}(z')$.

Proof. (4.2) gives

$$\|C(z')U_1(t)f\| \leqq k\|C(z)U_1(t)[1+(z'-z)R_1(z')]f\|, \tag{4.3}$$

where $k=\|(H_2-z)R_2(z')\|<\infty$. Suppose now that $f\in\mathfrak{M}(z)$. Then $R_1(z')f\in\mathfrak{M}(z)$ by Lemma 4.4, so that $[1+(z'-z)R_1(z')]f\in\mathfrak{M}(z)$ too. Hence the right member of (4.3) is integrable in $t\in(0,\infty)$, and the same is true of the left member. This means that $f\in\mathfrak{M}(z')$.

Obviously Lemma 4.5 completes the proof of Theorem I″.

5. An Application to Potential Scattering

Consider potential scattering in $\mathbb{R}^3$:

$$H_1=-\Delta, \quad H_2=-\Delta+V(x), \quad x\in\mathbb{R}^3. \tag{5.1}$$

For simplicitly we assume that

$$V=V_1+V_2, \tag{5.2}$$

where the V_j are real-valued and

$$0\leqq V_1\in\mathfrak{L}^1_{\text{loc}}(\mathbb{R}^3), \quad V_2\in\mathfrak{L}^2(\mathbb{R}^3)+\mathfrak{L}^\infty(\mathbb{R}^3). \tag{5.3}$$

Then H_1 and H_2 are selfadjoint in $\mathfrak{H}=\mathfrak{L}^2(\mathbb{R}^3)$. Here H_2 should be taken as the *maximal realization* in $\mathfrak{H}$ of the formal differential operator $-\Delta+V$ or, equivalently, as the *form sum* of H_1+V_2 and V_1, both of which are semibounded (see [6]). Note that V_2 is H_1-bounded with H_1-bound 0.

We shall show that if in addition

$$V_j\in\mathfrak{L}^j(\mathbb{R}^3,(1+|x|)^{-k}dx), \quad j=1,2, \quad \text{for some} \quad k<1, \tag{5.4}$$

then the wave operators $W_\pm(H_2,H_1;1)$ exist on all of $\mathfrak{H}$.

To this end we use the optimal Schechter operator (3.3), which should give the same result as our theorem. In our case with $J=1$, (3.3) becomes

$$A=R_2(z)(H_1-z)-1, \quad \mathfrak{D}(A)=\mathfrak{D}(H_1). \tag{5.5}$$

We choose z real and sufficiently negative that $H_j-z\geqq 1$ for $j=1,2$. One might want to write $A=-R_2(z)V$, but this is hard to justify in general due to the singularity of V. But it is not difficult to show that

$$Au=-(V_1^{1/2}R_2(z))^*V_1^{1/2}u-R_2(z)V_2u \tag{5.6}$$

provided $u\in\mathfrak{D}(H_1)$ and $V_1^{1/2}u\in\mathfrak{H}$. Note that $V_1^{1/2}R_2(z)\in\mathscr{B}(\mathfrak{H})$ because H_2 is the form sum of H_1+V_2 and V_1, and that $u\in\mathfrak{D}(H_1)$ implies $V_2u\in\mathfrak{H}$. It follows that

$$\|Au\|\leqq \text{const}(\|V_1^{1/2}u\|+\|V_2u\|). \tag{5.7}$$

Now let $f(x)=\exp(-|x-a|^2/2)$ with a constant $a\in\mathbb{R}^3$, and $u(t)=U_1(t)f=\exp(-itH_1)f$. It is well known (see e.g. [7, pp. 536, 7]) that

$$\|(1+|x|)^{k/2}u(t)\|_{\mathfrak{L}^\infty}=O(t^{-(3-k)/2}), \quad t\to\infty. \tag{5.8}$$

In view of the assumption (5.4), it follows that $\|V_1^{1/2}u(t)\|$ and $\|V_2u(t)\|$ are $O(t^{-(3-k)/2})$. Hence the same is true of $\|Au(t)\|$ by (5.7), verifying condition (3.2).

Since the set of f with varying $a\in\mathbb{R}^3$ spans a dense set in $\mathfrak{H}$, we have proved that $\mathfrak{D}(W_+)=\mathfrak{H}$. The same is true of W_-.

Remark 5.1. (a) A more obvious "factorization" with $A=|V|^{1/2}$, $B=(\operatorname{sign}V)|V|^{1/2}$ does not work, since this violates the condition $\mathfrak{D}(A)\supset\mathfrak{D}(H_1)$ (except when $k=0$).

(b) If $V_1=0$, (5.4) reduces to Kuroda's condition [4].

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