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Time Evolution of Gibbs States for an Anharmonic Lattice

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Abstract. In this paper we study the time evolution of a regular class of states of an infinite classical system of anharmonic oscillators. The conditional probabilities are investigated and an explicit form for these is given.

1. Introduction

One of the main problem in non equilibrium Statistical Mechanics is to study the time evolution of states (i.e. probability measures on the phase space) of infinite interacting classical systems. A natural way is to consider the time evolution as described by a flow on the phase space arising from the Newton law of the motion.

The problem of constructing such a flow was solved in a satisfactory way for some classes of particle systems in [1], [2] and for anharmonic oscillators in [3]. Other results which are specifically related to the equilibrium situation were obtained in [4–8].

The next step is to study the time evolution of states, implemented by the flow on the phase space. An approach proposed in [9] and [10] is based on the hypothesis that a class of physically interesting states, the Gibbs states with respect to some Hamiltonian is preserved in the course of the evolution (the equilibrium states are precisely those states which are Gibbs with respect to the Hamiltonian governing the motion). The main advantage of this approach is that the change in time of the Hamiltonian of a given Gibbs state is described in a simple way, directly referred to finite-volume dynamics.

Such an approach was studied in [9] in the case of one dimensional hard core system interacting via a two body, bounded, short range potential. One of the main points in [9] is the use of the cluster dynamics that, roughly speaking, says that such systems behave in time as if they were formed by non interacting groups consisting of a finite number of particles.

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In this paper we study the time evolution of Gibbs states for a special class of anharmonic lattices in which the self-energy dominates over the interacting part. The interest of the anharmonic systems in the solid state physics is well known (see e.g. [11]). The dynamical flow we study here was investigated in [3]. Even if the dynamics of such systems is not, obviously, of cluster type, nevertheless it exhibits for many respects a simpler behaviour than the dynamics of continuous particle systems.

The plan of this work will be the following: in Sect. 2 we introduce the notations and formulate the results; in Sect. 3 the proofs are given; the Appendix is devoted in underlining some dynamical property we need.

2. Notation and Result

We consider the system of anharmonic oscillators on the ν-dimensional cubic lattice $\mathbb{Z}^\nu$. The phase space of a single oscillator is assumed to be $\mathbb{R}^1 \times \mathbb{R}^1$.

Definition 2.1. The *phase space* of the system under consideration is $\mathfrak{X} = \{(p_i q_i)_{i\in\mathbb{Z}^\nu} | q_i \in \mathbb{R}^1,\ p_i \in \mathbb{R}^1\}$. Points of $\mathfrak{X}$ are denoted x, y, etc. For every $\Lambda \subset \mathbb{Z}^\nu$, $\mathfrak{X}_\Lambda$ denotes the phase associated with the region Λ; $\mathfrak{X}_\Lambda = \{(p_i, q_i)_{i\in\Lambda} | q_i \in \mathbb{R}^1, p_i \in \mathbb{R}^1\}$. Points of $\mathfrak{X}_\Lambda$ will be denoted by x_Λ, y_Λ, etc. The space $\mathfrak{X}$ and $\mathfrak{X}_\Lambda$ are equipped with the natural Tychonov topologies. The corresponding Borel σ-algebras are denoted by $\mathscr{B}$ and $\mathscr{B}_\Lambda$ and are the σ-algebras generated by the variables $(p_i, q_i)_{i\in\mathbb{Z}^\nu}$ and $(p_i, q_i)_{i\in\Lambda}$.

Definition 2.2. A state μ of the system of oscillators on $\mathbb{Z}^\nu$ is a probability measure on $\mathscr{B}$. Since $\mathfrak{X}$ is a Polish space, $(\mathfrak{X}, \mathscr{B}^*, \mu)$, where $\mathscr{B}^*$ is the completion of $\mathscr{B}$ w.t.r. μ. is a Lebesgue space. [12].

Given a *partition* ξ of $\mathfrak{X}$, we say that ξ is *measurable* if there exists a countable family $\{f_1 \dots f_n \dots\}$ of measurable functions, such that every atom a_α of the partition ξ is labelled by $\alpha \in \mathbb{R}^{\mathbb{N}}$, and $a_\alpha = \{x \in \mathfrak{X} | f_n(x) = \alpha_n \dots \alpha \equiv (\alpha_i \dots \alpha_n \dots)\}$.

Given a measurable partition ξ, let us consider the factor space $\mathfrak{X}/\xi$ whose points are the atoms of the partition. The canonical map $\Pi : \mathfrak{X} \to \mathfrak{X}/\xi$ which associates at every point of $\mathfrak{X}$ its atom, determines the measurable sets on $\mathfrak{X}/\xi$ as the sets whose inverse image is measurable in $\mathfrak{X}$.

Let μ_ξ be the measure on $\mathfrak{X}/\xi$ defined by $\mu_\xi(B) = \mu(\Pi^{-1}(B))$. There exists a family $\{\mu(\cdot|a), a \in \xi\}$ of a measures on atoms $a \in \xi$ such that each $(a, \mu(\cdot|a))$ is a Lebesgue space. Furthermore, for all $A \in \mathscr{B}^*$, $A \cap a$ is $\mu(\cdot|a)$-measurable for μ_ξ-a.a. $a \in \xi$ and:

$$\mu(A) = \int_{\mathfrak{X}/\xi} \mu_\xi(da)\mu(A \cap a|a). \tag{2.1}$$

Such a family $\{\mu(\cdot|a)\}$ is unique mod 0 (see [12]) and is called the *system of conditional probabilities* of the state μ w.r.t. ξ.

In the sequel we shall use the following property of the conditional probabilities.

Let us consider two partitions, ξ whose atoms we denote by a, and η whose atoms we denote by b. Assume that η is a refinement of ξ that means that every b is contained in some a. Then denoting by $a(x)$ (or $b(x)$) the atom of type a (or b) which contains the point $x \in \mathfrak{X}$, it is not hard to prove that for μ-a.a. $x \in \mathfrak{X}$, the following equality holds:

$$\mu(A|a(x)) = \int_{a(x)} \mu(A|b(y))\mu(dy|a(x)). \tag{2.2}$$

Let us fix a bounded region Λ, and consider the following atoms

$$a_\Lambda(x) = \{y \in \mathfrak{X} | (x)_{\Lambda^c} = (y)_{\Lambda^c}\}.$$

Here $(x)_\Omega$ denote the restriction of x to $\mathfrak{X}_\Omega$, $\Omega \subset \mathbb{Z}^\nu$. Let us consider ξ_Λ, the partition given by the atoms $a_\Lambda(x)$. ξ is obviously measurable. Every atom $a_\Lambda(x)$ may be thought as $\mathfrak{X}_\Lambda$, hence $\mu(\cdot|a_\Lambda(x))$ induces a measure on $\mathfrak{X}_\Lambda$ still denoted

$$\mu(dx_\Lambda|a_\Lambda(x)) = \mu(dx_\Lambda|(x)_{\Lambda^c}).$$

Then for any bounded measurable function $f: \mathfrak{X} \to \mathbb{R}$ (2.1) reads as

$$\int_{\mathfrak{X}} \mu(dx) f(x) = \int_{\mathfrak{X}} \mu(dx) \int_{\mathfrak{X}_\Lambda} \mu(dx_\Lambda|(x)_{\Lambda^c}) f(x_\Lambda \cup (x)_{\Lambda^c}). \tag{2.3}$$

Definition 2.3. Let h be a real valued function on the set $\mathfrak{X}^{(0)} = \bigcup_{\Lambda \subset \mathbb{Z}^\nu} \mathfrak{X}_\Lambda$ (Λ finite) such that $h|_{\mathfrak{X}_\Lambda}$ is $\mathscr{B}$-measurable for all finite $\Lambda \subset \mathbb{Z}^\nu$. Given finite subsets $\Lambda, \Lambda' \subset \mathbb{Z}^\nu$, $\Lambda \cap \Lambda' = \Phi$ and a pair $(x_\Lambda, x_{\Lambda'})$, $x_\Lambda \in \mathfrak{X}_\Lambda$, $x_{\Lambda'} \in \mathfrak{X}_{\Lambda'}$, we set

$$h(x_\Lambda|x'_{\Lambda'}) = h(x_\Lambda \cup x'_{\Lambda'}) - h(x_\Lambda) - h(x'_{\Lambda'}). \tag{2.4}$$

We say that μ is a *Gibbs state* corresponding to the generating function h, if for any finite $\Lambda \subset \mathbb{Z}^\nu$

i) the limit

$$h(x_\Lambda|(x)_{\Lambda^c}) = \lim_{n \to \infty} h(x_\Lambda|(x)_{\Lambda n \setminus \Lambda}), \qquad x_\Lambda \in \mathfrak{X}_\Lambda, \qquad x \in \mathfrak{X} \tag{2.5}$$

exists in the sense of convergence in measure $(\lambda \times \mu)$ over the cartesian product $\mathfrak{X}_\Lambda \times \mathfrak{X}$, where Λ_n is the cube $[-n, n]^\nu$, $n \in \mathbb{N}$, and λ denotes the Lebesgue measure on $\mathfrak{X}_\Lambda$;

ii) the integral

$$\Xi_\Lambda(h; x) = \int_{\mathfrak{X}_\Lambda} d\lambda(x_\Lambda) \exp[-h(x_\Lambda) - h(x_\Lambda|(x)_{\Lambda^c})] \tag{2.6}$$

is finite for μ-a.a. $x \in \mathfrak{X}$ and any finite $\Lambda \subset \mathbb{Z}^\nu$;

iii) For any $\Lambda \subset \mathbb{Z}^\nu$ the conditional probability system for μ w.r.t. ξ_Λ is given by

$$\mu(dx_\Lambda|(x)_{\Lambda^c}) = \frac{d\lambda(x_\Lambda)}{\Xi_\Lambda(h; x)} \exp[-h(x_\Lambda) - h(x_\Lambda|(x)_{\Lambda^c})]. \tag{2.7}$$

Definition 2.4. The function $H: \mathfrak{X}^{(0)} \to \mathbb{R}^1$ defined by

$$H(x_\Lambda) = \sum_{i \in \Lambda} \left(p_i^2/2 + k q_i^2 + \lambda q_i^4 - J q_i \sum_{j \in \Lambda \cap v_i} q_d \right) \tag{2.8}$$

where $k, \lambda > 0$, $J \in \mathbb{R}^1$, $v_i = \{j \in \mathbb{Z}^\nu | |i-j| = 1\}$, $|i-j| = \sum_{\alpha=1}^{\nu} |i_\alpha - j_\alpha|$ is called the

Hamiltonian. The Hamiltonian (2.8) describes a physical model of anharmonic oscillators (with unitary mass). We choose a particular Hamiltonian for the sake of simplicity, but all our considerations are straightforwardly valid in the case of all Hamiltonians where the self-energy dominates the interacting part (see [3]).

Definition 2.5. By $\{\tilde{S}_t, t\in\mathbb{R}^1\}$ we denote the group of transformations $\mathfrak{X}^{(0)}\to\mathfrak{X}^{(0)}$ representing the motion of a finite system of oscillators with Hamiltonian H. Clearly, $\tilde{S}_t\mathfrak{X}_\Lambda=\mathfrak{X}_\Lambda$ for every t and finite $\Lambda\subset\mathbb{Z}^\nu$.

Definition 2.6. For every $n\in\mathbb{N}$ put $\Lambda_n=[-n,n]^\nu$ and denote by $\{S_t^{(n)}, t\in\mathbb{R}^1\}$ the following group of transformation $\mathfrak{X}\to\mathfrak{X}$. Given $x\in\mathfrak{X}$, $(S_t^{(n)}x)_{\Lambda_n^c}=x_{\Lambda_n^c}$ and $(S_t^{(n)}x)_{\Lambda_n}$ represents the solution of the Hamilton equations for the oscillators in Λ_n interacting via the Hamiltonian H and moving in the external field generated by the frozen oscillators outside Λ_n.

Definition 2.7. Let $\varphi:\mathbb{N}\to[1,\infty)$ be an arbitrary increasing function such that $\varphi(k)>c_\varphi\varphi(k+1)$ for some constant c_φ, $0<c_\varphi<1$.

We denote $\mathscr{L}_\varphi:\mathfrak{X}\to[1,\infty)$ the function given by

$$\mathscr{L}_\varphi(x)=\sup_{k\in\mathbb{N}}\frac{1}{\varphi(k)}\sup_{i\in\Lambda_k}\mathscr{L}^{(i)}(x) \tag{2.9}$$

where

$$\mathscr{L}^{(i)}(x)=p_i^2/2+kq_i^2+\lambda q_i^4+1\,. \tag{2.10}$$

We put $\mathfrak{X}_\varphi=\{x\in\mathfrak{X}|\mathscr{L}_\varphi(x)<+\infty\}$.

All the dynamical properties we need in the sequel can be summarized in the following

Theorem 1. i) *For all* $x\in\mathfrak{X}_\varphi$ and $t\in\mathbb{R}^1$ the limit

$$S_tx=\lim_{n\to\infty}S_t^{(n)}x \tag{2.11}$$

exists in the product topology on $\mathfrak{X}$. S_tx *is one parameter group of transformations on* $\mathfrak{X}_\varphi$ *and moreover*

$$\left[\mathscr{L}_\varphi(S_tx)\vee\sup_n\mathscr{L}_\varphi(S_t^{(n)}x)\right]\leqq e^{a|t|}\mathscr{L}_\varphi(x) \tag{2.12}$$

and for all bounded $\Omega\subset\mathbb{Z}^\nu$.

$$\mathscr{L}_\varphi[\tilde{S}_t(x)_\Omega\cup(x)_{\Omega^c}]\leqq e^{a|t|}\mathscr{L}_\varphi(x) \tag{2.12'}$$

where a *does not depend on* x, t *and* Ω.

ii) *If* $\varphi':\mathbb{N}\to[1,\infty)$ *is an increasing function such that*

$$\varphi'(k)>c_{\varphi'}\varphi'(k+1)\quad \textit{for some}\quad c_{\varphi'},\qquad 0<c_{\varphi'}<1$$

and

$$\varphi'(k)/\varphi(k)\xrightarrow[k\to\infty]{}0,\quad \textit{then for all}\quad x\in\mathfrak{X}_{\varphi'}$$

$$\lim_{n\to\infty}\mathscr{L}_\varphi(S_t^{(n)}x)=\mathscr{L}_\varphi(S_tx) \tag{2.13}$$

and for every $\varkappa>0$ *the convergence in* (2.13) *is uniform for such* x *that* $\mathscr{L}_{\varphi'}(x)\leqq\varkappa$.

iii) *Given* $x \in \mathfrak{X}_\varphi$, *the following bounds hold. For any* n *and* $i \in \Lambda_k$

$$|q_i^{(n)}(t,x)-\bar{q}_i(t,x)| \vee |p_i^{(n)}(t,x)-\bar{p}_i(t,x)|$$

$$\leqq \frac{(a_1 \varphi(n) \mathscr{L}_\varphi(x))^{n-k+1}}{(n-k)!} \qquad n>k. \tag{2.14}$$

Here $(q_i^{(n)}(t,x),\ p_i^{(n)}(t,x))$ denotes the coordinate and the momentum of the i-oscillator in $S_t^{(n)}x$, $(\bar{q}_i(t,x), \bar{p}_i(t,x))$ the coordinate and the momentum of the i-oscillator either in $S_t^{(n')}x$ or in $\tilde{S}_t(x)_{\Lambda_{n'}} \cup x_{\Lambda_{n'}^c}$, with $n' \geqq n$ and a_1 is constant for any fixed t.

For any k, s, $n \in \mathbb{N}$ with $s<k<n$ and x_{Λ_s}, $x'_{\Lambda_s} \in \mathfrak{X}_{\Lambda_s}$, $i \in \Lambda_n \setminus \Lambda_k$:

$$|\bar{q}_i^{(n)}(t, x_{\Lambda_s} \cup (x)_{\Lambda_s^c}) - \bar{q}_i^{(n)}(t, x'_{\Lambda_s} \cup (x)_{\Lambda_s^c})|$$

$$\vee\ |\bar{p}_i^{(n)}(t, x_{\Lambda_s} \cup (x)_{\Lambda_s^c}) - \bar{p}_i^{(n)}(t, x'_{\Lambda_s} \cup (x)_{\Lambda_s^c})|$$

$$\leqq \frac{(a_2[\mathscr{L}_\varphi(x_{\Lambda_s} \cup (x)_{\Lambda_s^c}) \vee \mathscr{L}_\varphi(x'_{\Lambda_s} \cup (x)_{\Lambda_s^c})]\varphi(2k))^{k-s+1}}{(k-s)!}. \tag{2.15}$$

Here $\bar{q}_i^{(n)}(t,x)$ and $\bar{p}_i^{(n)}(t,x)$ denote the coordinate and the momentum of the i-oscillator either in $S_t^{(n)}x$ or in $\tilde{S}_t(x)_{\Lambda_n} \cup x_{\Lambda_n^c}$ and a_2 is constant for any fixed t.

For any k, n, $n' \in \mathbb{N}$ such that $k<n$, n' denoting $(q_i^{(m,\Lambda)}(t,x),\ p_i^{(m,\Lambda)}(t,x))$ the coordinate and the momentum of the i-oscillator in $(\tilde{S}_t x_{\Lambda_m \setminus \Lambda}) \cup (x_{[\Lambda_m \setminus \Lambda]^c})$, such that $i \in \Lambda_k \setminus \Lambda_{k-1}$ and $\Lambda_k \supset \Lambda$, Λ' then:

$$|q_i^{(n,\Lambda)}(t,x)-q_i^{(n',\Lambda')}(t,x)| \vee |p_i^{(n,\Lambda)}(t,x)-p_i^{(n',\Lambda')}(t,x)|$$

$$\leqq \frac{(a_3 \varphi(n \vee n') \mathscr{L}_\varphi(x))^{d+1}}{d!} \tag{2.16}$$

where

$$d=\begin{cases} \min(n-k, n'-k) & \text{if} \quad \Lambda'=\Lambda \quad n \neq n' \\ \min(k-s, k-s') & \text{if} \quad \Lambda' \neq \Lambda \quad n=n' \\ \min(k-s, k-s', n-k, n'-k) & \text{if} \quad \Lambda \neq \Lambda' \quad n \neq n' \\ \infty & \text{if} \quad n=n' \quad \Lambda=\Lambda'. \end{cases} \tag{2.17}$$

a_3 is constant for any fixed t, and $s, s' = \min\{l \in \mathbb{N} | \Lambda_l \supset \Lambda, \Lambda'\}$, $s, s' < k$.

The ideas of the Theorem 1 are essentially contained in [3]. We outline the proof in the Appendix.

In this paper we study the evolution

$$\mu_t = S_t^* \mu = \mu(S_{-t} \cdot) \tag{2.18}$$

of a Gibbs μ w.r.t. a generating function h which satisfies the following conditions:

1) There exists a constant $c>0$ such that for every finite $\Lambda \subset \mathbb{Z}^\nu$ and $x_\Lambda \subset \mathfrak{X}_\Lambda$

$$h(x_\Lambda) \geqq c H(x_\Lambda). \tag{2.19}$$

2) There exists an integer r and a function $\psi:[0,r)\to\mathbb{R}^+$ such that for any finite $\Lambda,\Lambda'\subset\mathbb{Z}^\nu$ with $\Lambda\cap\Lambda'=\Phi$ and all $x_\Lambda\in\mathfrak{X}_\Lambda$ and $x'_{\Lambda'}\in\mathfrak{X}_{\Lambda'}$ one has:

$$h(x_\Lambda|x'_{\Lambda'})=h(x_\Lambda|x_{\partial\Lambda\cap\Lambda'}) \tag{2.20}$$

$$|h(x_\Lambda|x'_{\Lambda'})|\leqq\sum_{i\in\Lambda}\sum_{i'\in\Lambda'}\psi(|i-i'|)[\mathscr{L}^{(i)}(x_\Lambda)+\mathscr{L}^{(i')}(x'_{\Lambda'})] \tag{2.20'}$$

where $\partial\Lambda=\{i\in\mathbb{Z}^\nu\backslash\Lambda|\exists j\in\Lambda:|i-j|\leqq r\}$.

Here $\mathscr{L}^{(i)}(x_\Omega)$ is defined via (2.10) replacing x by x_Ω.

3) Finally, we require that there exists a constant $\bar{c}$ such that:

$$|h(x_\Lambda)-h(x'_\Lambda)|\leqq\bar{c}|\Lambda|\max_{i\in\Lambda}[\mathscr{L}^{(i)}(x_\Lambda)\vee\mathscr{L}^{(i)}(x'_\Lambda)]\cdot\max_{i\in\Lambda}[|q_i-q'_i|\vee|p_i-p'_i|]\,. \tag{2.21}$$

In the Appendix we prove:

Theorem 2. *Let h satisfy the condition 1)–3) above. Then there exists at least one Gibbs state μ corresponding to h such that*

$$\mu(\mathfrak{X}_{\varphi'})=1 \tag{2.22}$$

for $\varphi'(k)=(\log k)\vee 1$.

Theorem 2 allows us to define the time evolved state μ_t via equality (2.18).

Now we can formulate the main result of this paper.

Theorem 3. *Let μ be a Gibbs state corresponding to a generating function h satisfying the condition 1)–3), and such that* (2.22) *holds. Then μ_t is a Gibbs state corresponding to the generating function h_t given by*

$$h_t(x_\Lambda)=h(\tilde{S}_{-t}x_\Lambda) \tag{2.23}$$

where $\Lambda\subset\mathbb{Z}^\nu$ is finite.

Remark. A natural question arising in the examination of Theorem 3 is if the condition 1)–3) on the initial μ are preserved during the motion. Condition 1 which means superstability for h is obviously preserved by the conservation of energy, with the same coefficient c. A sort of local Lipschitz condition as (2.21) can also exhibited for h_t, by the use of Theorem 1.

Furthermore one can prove, by the use of the same ideas of Lemma 3.2 below, that condition (2.20′) is preserved for h_t with a function ψ_t (of course no more with compact support because dynamics destroys locality) more than exponentially decreasing at infinity. This will imply that the superstable estimates (Ref. [3] and A.II below) hold for μ_t. Obviously condition (2.20) is no more preserved.

3. Proofs

In order to prove Theorem 3 we have to give good estimates on the quantities $h_t(x_\Lambda|(x)_{\Lambda^c})$ and $\Xi_\Lambda(h_t;x)$ for a sufficiently large set of $x\in\mathfrak{X}$. While the first quantity may be estimated by the use of Theorem 1, it seems hard to have a good control of $\Xi_\Lambda(h_t;x)$ in terms of x by brute force using the dynamical properties we know. We do not approach the problem directly, but we shall prove an analog of Theorem 3 (see Theorem 4 below) where the conditional probabilities are taken not w.r.t. ξ_Λ but w.r.t. its refinement ξ_Λ^m whose atoms may be identificated with proper subsets of $\mathfrak{X}_\Lambda$ of finite Lebesgue measure, which will allow us to avoid problems on the convergence of the normalization factor. Once Theorem 4 is proved, it is not hard to prove Theorem 3. We shall start by giving the new partition and a precise formulation of Theorem 4. Then we shall show the passage from Theorem 4 to Theorem 3. The rest of this section will be devoted to the proof of Theorem 4.

For all $m\in\mathbb{N}$ we define the partition ξ_Λ^m of $\mathfrak{X}_\varphi$ by giving its atoms:

$$a_\Lambda^m(x)=\{y|(y)_{\Lambda^c}=(x)_{\Lambda^c};\quad \mathscr{L}_\varphi(y)<m \quad \text{if} \quad \mathscr{L}_\varphi(x)<m \quad [\mathscr{L}_\varphi(y)]=[\mathscr{L}_\varphi(y)] \quad \text{otherwise}\} \tag{3.1}$$

where

$$\varphi(k)=1\vee\log^2 k \qquad k\in\mathbb{N} \tag{3.1'}$$

where $[x]$ denotes the integer part of $x\in\mathbb{R}$. ξ_Λ^m is obviously measurable. Each atom $a_\Lambda^m(x)$ induces the subset $\tilde{a}_\Lambda^m(x)\subset\mathfrak{X}_\Lambda$:

$$\tilde{a}^m(x)=\{y_\Lambda|\mathscr{L}_\varphi(y_\Lambda\cup(x)_{\Lambda^c})<m \quad \text{if} \quad \mathscr{L}_\varphi(x)<m \quad [\mathscr{L}_\varphi((y_\Lambda)\cup(x)_{\Lambda^c})]=[\mathscr{L}_\varphi(x)] \quad \text{otherwise}\} \tag{3.2}$$

and for each state μ, the family $\{\mu(\cdot|a_\Lambda^m(x))\}$ induces the family of measures (still denoted by $\mu(\cdot|a_\Lambda^m(x))$ supported in corresponding $\tilde{a}_\Lambda^m(x)$).

The main point of the proof of Theorem 3 in the following:

Theorem 4. *Let h be a function satisfying the condition 1)–3) and μ be a Gibbs state corresponding to h and satisfying* (2.20). *Then for any $m\in\mathbb{N}$ we have:*

$$\mu_t(A|a_\Lambda^m(x))=\frac{\int\limits_A d\lambda(\tilde{x})\exp[-h_t(\tilde{x}_\Lambda)-h_t(\tilde{x}_\Lambda|(x)_{\Lambda^c})]}{\int\limits_{\tilde{a}_\Lambda^m(x)} d\lambda(\tilde{x})\exp[-h_t(\tilde{x}_\Lambda)-h_t(\tilde{x}_\Lambda)(x)_{\Lambda^c})]} \tag{3.3}$$

where $A\subset\tilde{a}_\Lambda^m(x)$ is a Borel set and h_t is defined by (2.21).

Proof of Theorem 3. Let us consider two bounded measurable sets A and B in $\mathfrak{X}_\Lambda$ such that $\lambda(A)$, $\lambda(B)$ are different from zero, and a sufficiently large m such that $A, B\subset\tilde{a}_\Lambda^m(x)$ for all $x\in\mathfrak{X}_\varphi$ with $\mathscr{L}_\varphi(x)<m$. Since ξ_Λ^m is a refinement of ξ_Λ, it results by (2.2):

$$\mu_t(A|a_\Lambda^m(x))\mu_t(\tilde{a}_\Lambda^m(x)|(x)_{\Lambda^c})=\mu_t(A|(x)_{\Lambda^c}) \tag{3.4}$$

and an analogous expression for B which hold for μ_t – a.a. x such that $\mathscr{L}_\varphi(x)<m$. By (3.3), both $\mu_t(A|a_\Lambda^m(x))$ and $\mu_t(B|a_\Lambda^m(x))$ are different from zero and:

$$\frac{\mu_t(A|(x)_{\Lambda^c})}{\mu_t(A|a_\Lambda^m(x))}=\frac{\mu_t(B|(x)_{\Lambda^c})}{\mu_t(B|a_\Lambda^m(x))}. \tag{3.5}$$

By the use of (3.3), we obtain:

$$\mu_t(A|(x)_{\Lambda^c})\int\limits_B e^{-[h_t(\tilde{x}_\Lambda)+h_t(\tilde{x}_\Lambda|(x)_{\Lambda^c})]}d\lambda(\tilde{x}_\Lambda) =\mu_t(B|(x)_{\Lambda^c})\int\limits_A e^{-[h_t(\tilde{x}_\Lambda)+h_t(\tilde{x}_\Lambda|(x)_{\Lambda^c})]}d\lambda(\tilde{x}_\Lambda) \tag{3.6}$$

for μ_t – a.a. $x\in\mathfrak{X}_\varphi$.

Consider now an increasing countable family of B's invading $\mathfrak{X}_A$. There exists a μ_t full measure set $\tilde{\mathfrak{X}}_\varphi$ such that (3.6) holds simultaneously for all elements of the family. Fixed $x \in \tilde{\mathfrak{X}}_\varphi$, it is possible to find a sufficiently large B for which the r.h.s. of (3.6) is different from 0. This implies that $\mu_t(A|(x)_{A^c}) \neq 0$. Finally, by taking the limit $B \to \mathfrak{X}_A$ we obtain simultaneously:

$$\mu_t(A|(x)_{A^c}) = \frac{\int_A d\lambda(\tilde{x}_A) \exp[-h_t(\tilde{x}_A) - h_t(\tilde{x}_A|(x)_{A^c})]}{\Xi_A(h_t; x)}$$
$$\Xi_A(h_t; x) = \int_{\mathfrak{X}_A} d\lambda(\tilde{x}_A) \exp[-h_t(\tilde{x}_A) - h_t(\tilde{x}_A|(x)_{A^c})] < +\infty \tag{3.7}$$

that proves Theorem 3.

Proof of Theorem 4. We shall consider the following measure:

$$\mu_t^n = S_t^{(n)*}\mu = \mu(S_{-t}^n \cdot) \tag{3.8}$$

where μ from now on is the same measure as in Theorem 3.

We denote by $\mu_t^n(\cdot|a_A^m(x))$ its conditional probability w.r.t. ξ_A^n. Then:

Lemma 3.1. *For every $x \in \mathfrak{X}_\varphi$ and n such that $\Lambda_n \supset \Lambda$, there exists the Radon-Nicodym derivative:*

$$P_t^n(x_A|a_A^m(x)) = \frac{d\mu_t^n(x_A|a_A^m(x))}{d\lambda(x_A)} \tag{3.9}$$

which is given by:

$$P_t^n(x_A|a_A^m(x)) = \frac{\exp[-h((S_{-t}^{(n)}(x_A \cup (x)_{A^c})_{\Lambda_{n+r}})]}{\int_{\tilde{a}_A^m(x)} \exp[-h((S_{-t}^{(n)}(x_A \cup (x)_{A^c})_{\Lambda_{n+r}})]} \tag{3.10}$$

where r is the same as in condition 2).

Proof. The estimate (2.19) and the conservation of energy

$$H[(S_{-t}^{(n)}(x_A \cup (x)_{A^c}))_{\Lambda_{n+r}}] = H[(x_A \cup (x)_{A^c})_{\Lambda_{n+r}}] \tag{3.11}$$

imply that the integral in r.h.s. of (3.10) is uniformly bounded on m. Furthermore, by condition 2) one can prove that:

$$P_t^n(x_{\Lambda_n}|(x)_{\Lambda_n^c}) = \frac{e^{-h((S_{-t}^{(n)}(x_{\Lambda_n} \cup (x)_{\Lambda_n^c}))_{\Lambda_n}) - h((S_{-t}^{(n)}(x_{\Lambda_n} \cup (x)_{\Lambda_n^c}))_{\Lambda_n}|x_{\Lambda_n^c})}}{\text{normalization factor}}$$
$$= \frac{\exp[-h((S_{-t}^{(n)}(x_{\Lambda_n} \cup (x)_{\Lambda_n^c})_{\Lambda_{n+r}})]}{\int_{\mathfrak{X}_{\Lambda_n}} d\lambda(\tilde{x}_{\Lambda_n}) \exp[-h((S_{-t}^{(n)}(x_{\Lambda_n} \cup (x)_{\Lambda_n^c})_{\Lambda_{n+r}})]} \tag{3.12}$$

where $P_t^n(x_{\Lambda_n}|(x)_{\Lambda_n^c})$ is the Radon-Nikodym derivative $\dfrac{d\mu_t^n(x_A|(x)_{\Lambda_n^c})}{d\lambda(x_A)}$.

Since ξ_A is a refinement of ξ_{Λ_n}, we apply (2.2) and deduce that the density $P_t^n(x_A|(x)_{A^c})$ has the same form as $P_t^n(x_{\Lambda_n}|(x)_{\Lambda_n^c})$ with the normalization factor, obtained by integrating the coordinates in $\mathfrak{X}_A$. Still using (2.2), we obtain that the density $P_t^n(x_A|a_A^m(x))$ and momenta has the same form as $P_t^n(x_A|(x)_{A^c})$ with the normalization factor obtained by integrating on $\tilde{a}_A^m(x)$. So (3.10) is proved.

Lemma 3.2. *For any $x\in\mathfrak{X}_\varphi$ and $x_A\in\tilde{a}_A^m(x)$ there exists the limit:*

$$P_t^{(\infty)}(x_A|a_A^m(x))=\lim_{n\to\infty}P_t^n(x_A|a_A^m(x)). \tag{3.13}$$

Moreover there exists functions γ_m, $\tilde{\gamma}_m:\mathbb{N}\times\mathfrak{X}\to\mathbb{R}$ with

$$\lim_{n\to\infty}\gamma_m(n;x)=\lim_{n\to\infty}\tilde{\gamma}_m(n;x)=0 \tag{3.14}$$

such that for all $n'\geqq n$

$$\exp[-\gamma_m(n;x)]\leqq\frac{P_t^n(x_A|a_A^m(x))}{P_t^{n'}(x_A|a_A^m(x))}\leqq\exp[\gamma_m(n;x)] \tag{3.15}$$

and $\gamma_m(n;S_t^{(n')}x)\leqq\tilde{\gamma}_m(n,x)$.

Proof. We check that

$$\lim_{\substack{n\to\infty\\ n'>n}}\frac{P_t^n(x_A|a_A^m(x))}{P_t^{n'}(x_A|a_A^m(x))}=1. \tag{3.16}$$

Suppose $n'>n$, $\tilde{x}_A$, $\tilde{x}'_A\in\mathfrak{X}_A$ are fixed, and consider the ratio:

$$\frac{\exp[-h((S_{-t}^{(n)}(\tilde{x}_A\cup(x)_{A^c}))_{A_{n+r}})]\exp[-h((S_{-t}^{(n')}(\tilde{x}'_A\cup(x)_{A^c}))_{A_{n'+r}})]}{\exp[-h((S_{-t}^{(n)}(\tilde{x}'_A\cup(x)_{A^c}))_{A_{n+r}})]\exp[-h((S_{-t}^{(n')}(\tilde{x}_A\cup(x)_{A^c}))_{A_{n'+r}})]}. \tag{3.17}$$

It is convenient to consider a sublattice of $\mathbb{Z}^\nu$ constructed by cells of side r. We denote these cells I_i, $i\in\mathbb{Z}^\nu$. We put:

$$\tilde{Y}_{h,i}=(S_{-t}^{(h)}(\tilde{x}_A\cup(x)_{A^c}))_{I_i\cap A_{h+r}};\qquad \tilde{Y}'_{n,i}=(S_{-t}^{(h)}(\tilde{x}'_A\cup(x)_{A^c}))_{I_i\cap A_{h+r}} \tag{3.18}$$

where $h=n,n'$.

In virtue of Condition 2), (3.17) become:

$$\frac{\exp-\left[\sum_i h(\tilde{Y}_{n,i})+\sum_{i_1,i_2}{}' h(\tilde{Y}_{n,i_1}|\tilde{Y}_{n,i_2})\right]\exp-\left[\sum_i h(\tilde{Y}'_{n',i})+\sum_{i_1,i_2}{}' h(\tilde{Y}'_{n',i_1}|\tilde{Y}'_{n',i_2})\right]}{\exp-\left[\sum_i h(\tilde{Y}'_{n,i})+\sum_{i_1,i_2}{}' h(\tilde{Y}'_{n,i_1}|\tilde{Y}'_{n,i_2})\right]\exp-\left[\sum_i h(Y_{n,i})+\sum_{i_1,i_2}{}' h(\tilde{Y}_{n',i_1}|\tilde{Y}_{n',i_2})\right]} \tag{3.19}$$

where $\sum'$ means the sum on the nearest neighbours.

Now we compare:

a) $h(\tilde{Y}_{n,i})$ with $h(\tilde{Y}'_{n,i})$ and
$h(\tilde{Y}_{n',i})$ with $h(\tilde{Y}'_{n',i})$ for $I_i\not\subset A_{n/2}$

b) $h(\tilde{Y}_{n,i})$ with $h(\tilde{Y}_{n,i})$ and
$h(\tilde{Y}'_{n,i})$ with $h(\tilde{Y}'_{n',i})$ for $I_i\subset A_{n/2}$.

Then, by condition 3) (see (2.19)) and Theorem 1, iii) (see (2.15)), we have for $I_i \not\subset \Lambda_{n/2}$:

$$|h(\tilde{Y}_{n,i}) - h(\tilde{Y}'_{n,\varepsilon})| v |h(\tilde{Y}_{n',i}) - h(\tilde{Y}'_{n',i})|$$
$$\leqq c' \max_{j \in I_1} [\mathscr{L}^{(j)}(\tilde{Y}_{n,i}) v \mathscr{L}^{(j)}(\tilde{Y}'_{n,i}) v \mathscr{L}^{(j)}(\tilde{Y}_{n',i}) v \mathscr{L}^{(j)}(\tilde{Y}'_{n',i})]$$
$$\cdot \frac{[a_2(\mathscr{L}_\varphi(\tilde{x}_\Lambda \cup (x)_{\Lambda^c}) v \mathscr{L}_\varphi(\tilde{x}'_\Lambda \cup (x)_{\Lambda^c}))\varphi(n+2r)]^{\frac{n}{2}+d(\Lambda)-r+1}}{\left(\frac{n}{2} - d(\Lambda) - r\right)!} \tag{3.20}$$

where $c' > 0$ is a constant and $d(\Lambda) = \min\{k | \Lambda_k \supset \Lambda\}$.

Analogously, by Theorem 1 iii) (see (2.14), we have for $I_i \subset \Lambda_{n/2}$:

$$|h(\tilde{Y}_{n,i}) - h(\tilde{Y}_{n',i})| v |h(\tilde{Y}'_{n,i}) - h(\tilde{Y}'_{n',i})|$$
$$\leqq c' \max_{j \in I_1} [\mathscr{L}^{(j)}(\tilde{Y}_{n,i}) v \mathscr{L}^{(j)}(\tilde{Y}'_{n,i}) v \mathscr{L}^{(j)}(\tilde{Y}_{n',i}) v \mathscr{L}^{(j)}(\tilde{Y}'_{n',i})]$$
$$\cdot \frac{[a_1(\mathscr{L}_\varphi((\tilde{x}_\Lambda \cup (x)_{\Lambda^c})) v \mathscr{L}_\varphi((\tilde{x}'_\Lambda \cup (x)_{\Lambda^c}))) \cdot \varphi(n)]^{n/2+1}}{\left(\frac{n}{2}\right)!}. \tag{3.21}$$

Furthermore, Theorem 1, i) (see (2.12)) gives:

$$\mathscr{L}^{(j)}(\tilde{Y}_{n,i}) \leqq \varphi(j) \mathscr{L}_\varphi(S^{(n)}_{-t}(\tilde{x}_\Lambda \cup (x)_{\Lambda^c})) \leqq e^{a|t|} \varphi(j) \mathscr{L}_\varphi(\tilde{x}_\Lambda \cup (x)_{\Lambda^c}) \tag{3.22}$$

and similar bounds on the other $\mathscr{L}^{(j)}_\varphi$'s comparing in (3.20) and (3.21).

Hence, there exists a function $\gamma_1(n, \tilde{x}_\Lambda, \tilde{x}'_\Lambda, x)$ such that:

$$\left|\sum_i h(\tilde{Y}_{n,i}) - \sum_i h(\tilde{Y}'_{n,i}) + \sum_i h(\tilde{Y}'_{n',i}) - \sum_i h(\tilde{Y}_{n',i})\right| \leqq \gamma_1(n, \tilde{x}_\Lambda, \tilde{x}'_\Lambda, x) \tag{3.23}$$

and, because of the definition of φ (see (3.1')),

$$\gamma_1(n, \mathscr{x}) \equiv \sup \{\gamma_1(n, \tilde{x}_\Lambda, \tilde{x}'_\Lambda, x) | x \in \mathfrak{X}_\varphi, \tilde{x}_\Lambda, \tilde{x}'_\Lambda \in \mathfrak{X}_\Lambda,$$
$$\cdot \mathscr{L}_\varphi(\tilde{x}_\Lambda \cup (x)_{\Lambda^c}) v \mathscr{L}_\varphi(\tilde{x}'_\Lambda \cup (x)_{\Lambda^c}) \leqq \mathscr{x}\} \tag{3.24}$$

verifies:

$$\lim_{n \to \infty} \gamma_1(n, \mathscr{x}) = 0. \tag{3.25}$$

A similar estimate may be obtained for:

$$\left\| \left[\sum_{i_1, i_2}{}' h(\tilde{Y}_{n,i_1} | \tilde{Y}_{n,i_2}) - \sum_{i_1, i_2}{}' h(\tilde{Y}'_{n,i_1} | \tilde{Y}'_{n,i_2})\right]\right.$$
$$\left. + \left[\sum_{i_2, i_2}{}' h(\tilde{Y}^i_{n',i_1} | \tilde{Y}'_{n',i_2}) - \sum_{i_1, i_2}{}' h(\tilde{Y}_{n',i_1} | \tilde{Y}_{n',i_2})\right]\right\|. \tag{3.26}$$

Hence, ratio (3.17) is bounded from above by $e^{2\gamma_1(n,\mathscr{x})}$ and from below by $e^{-2\gamma_1(n,\mathscr{x})}$ where $\tilde{x}_\Lambda$, $\tilde{x}'_\Lambda$ and x are chosen so that:

$$\mathscr{L}_\varphi(\tilde{x}_\Lambda \cup (x)_{\Lambda^c}) v \mathscr{L}_\varphi(\tilde{x}'_\Lambda \cup (x)_{\Lambda^c}) \leqq \mathscr{x}.$$

Now if $\tilde{x}_\Lambda, \tilde{x}'_\Lambda \in \tilde{a}^{(m)}_\Lambda(x)$, then there exists a function $\gamma_m(n;x)$ satisfying (3.14) and such that $e^{\pm\gamma_m(n;x)}$ bounds the ratio (3.17).

Integrating these estimate on $d\lambda(\tilde{x}'_\Lambda)$ over $\tilde{a}^m_\Lambda(x)$, one finally gets:

$$\exp[-\gamma_m(n;x)] \leqq \frac{P^{(n)}_t(\tilde{x}_\Lambda|a^m_\Lambda(x))}{P^{(n')}_t(\tilde{x}_\Lambda|a^m_\Lambda(x))} \leqq \exp[\gamma_m(n;x)] \tag{3.27}$$

and hence (3.13) and (3.15). The existence of $\tilde{\gamma}_m(n;x)$ easily follows by Theorem 1, i).

Lemma 3.3. *The function* $P^{(\infty)}_t(\cdot|a^m_\Lambda(x))$ *defined by* (3.13) *satisfies the equality:*

$$P^{(\infty)}_t(x_\Lambda|a^m_\Lambda(x)) = \frac{\exp[-h_t(x_1) - h_t(x_\Lambda|(x)_{\Lambda^c})]}{\int\limits_{\tilde{a}^m_\Lambda(x)} d\lambda(x'_\Lambda)\exp[-h_t(x'_\Lambda) - h_t(x'_\Lambda|(x)_{\Lambda^c})]}. \tag{3.28}$$

In particular the r.h.s. *of* (3.28) *makes sense.*

Proof. First of all we show the existence of $h_t(x_1|(x)_{\Lambda^c})$. To this purpose it is enough to prove:

$$\lim_{\substack{n\to\infty\\ n'>n}} |h_t(x_{\Lambda_n}) - h_t(x_{\Lambda_{n'}}) - h_t(x_{\Lambda_n\setminus\Lambda}) + h_t(x_{\Lambda_{n'}\setminus\Lambda})| = 0. \tag{3.29}$$

We only sketch the proof of (3.29) since it uses the same ideas as the proof of Lemma 3.1. We define $Z_{n,i} = (\tilde{S}_{-t}x_{\Lambda_n})_{I_i\cap\Lambda_n}$, $\bar{Z}_{n,i} = (\tilde{S}_{-t}x_{\Lambda_n\setminus\Lambda})_{I_i\cap\Lambda_n\setminus\Lambda}$, $Z_{n',i}$ and $\bar{Z}_{n',i}$ are defined analogously. Then:

$$\begin{aligned}
&|h_t(x_{\Lambda_n}) - h_t(x_{\Lambda_{n'}}) - h_t(x_{\Lambda_n\setminus\Lambda}) + h_t(x_{\Lambda_{n'}\setminus\Lambda})| \\
&\leqq \left|\sum_i h(Z_{n,i}) - \sum_i h(Z_{n',i}) - \sum_i h(\bar{Z}_{n,i}) + \sum_i h(\bar{Z}_{n',i})\right| \\
&\quad + \left|\sum_{i_1,i_2}{}' h(Z_{n,i_1}|Z_{n,i_2}) - \sum_{i_1,i_2}{}' h(Z_{h',i_1}|Z_{n',i_2})\right. \\
&\quad \left.\sum_{i_1,i_2}{}' h(\bar{Z}_{n,i_1}|\bar{Z}_{n,i_2}) + \sum_{i_1,i_2}{}' h(\bar{Z}_{h',i_1}|\bar{Z}_{n',i_2})\right|.
\end{aligned}$$

We compare

a) $h(Z_{n,i})$ with $h(Z_{n',i})$ if $I_i \subset \Lambda_{n/2}$
a)′ $h(\bar{Z}_{n,i})$ with $h(\bar{Z}_{n',i})$

b) $h(Z_{n,i})$ with $h(\bar{Z}_{n,i})$ if $I_i \not\subset \Lambda_{\frac{n}{2}}$

b)′ $h(Z_{n',i})$ with $h(\bar{Z}_{n',i})$.

Analog comparison are made for the terms $h(\cdot|\cdot)$'s.

By the use of the estimates (2.16) of Theorem 1, (2.21) of Condition 3, and (2.12)′ the statement (3.29) can be proved.

The second step is to prove the equality

$$\lim_{n\to\infty} \frac{\exp[-h(S^{(n)}_{-t}(\tilde{x}_\Lambda \cup (x)_{\Lambda^c})_{\Lambda_{n+r}}]}{\exp[-h(\tilde{S}_{-t}(\tilde{x}_\Lambda \cup (x)_{\Lambda_{n+r}\setminus\Lambda}))]} \cdot \frac{\int\limits_{\tilde{a}^m_\Lambda(x)} d\lambda(\tilde{x}'_\Lambda)\exp[-h(\tilde{S}_{-t}(\tilde{x}'_\Lambda 0(x)_{\Lambda_{n+r}\setminus\Lambda}))]}{\int\limits_{\tilde{a}^m_\Lambda(x)} d\lambda(\tilde{x}_\Lambda)\exp[-h((S^{(n)}_{-t}(\tilde{x}_\Lambda \cup (x)_{\Lambda^c}))_{\Lambda_{n+r}})]} = 1. \tag{3.30}$$

As in Lemma 3.2, we consider the ratio

$$\frac{\exp[-h((S_{-t}^{(n)}(\tilde{x}_\Lambda \cup (x)_{\Lambda^c}))_{\Lambda_{n+r}})]\exp[-h(\tilde{S}_{-t}(\tilde{x}'_\Lambda \cup (x)_{\Lambda_{n+r}\setminus\Lambda}))]}{\exp[-h((S_{-t}^{(n)}(\tilde{x}'_\Lambda \cup (x)_{\Lambda^c}))_{\Lambda_{n+r}})]\exp[-h(\tilde{S}_{-t}(\tilde{x}_\Lambda \cup (x)_{\Lambda_{n+r}\setminus\Lambda}))]} \tag{3.31}$$

and the repeating the construction utilized in Lemma 3.2 we obtain for (3.21) the same bound from above and below by $\exp[\pm\bar{\gamma}_m(n;x)]$ where $\bar{\gamma}_m$ has the same properties as γ_m. This gives (3.30).

Finally, we observe that in the ratio (3.30) the numerator of the first term being divided by, the denominator of the second gives in the limit $P^\infty(\cdot|a_\Lambda^m(x))$ by Lemma 3.2 and the remaining term may be written as:

$$\left(\frac{\exp-[h_t(\tilde{x}_\Lambda)+h_t(\tilde{x}_\Lambda|(x)_{\Lambda_{n+r}\setminus\Lambda})]}{\text{normalization factor}}\right)^{-1}.$$

Hence, by the existence of $h_t(x_\Lambda|(x)_{\Lambda^c})$, the Lemma 3.3 is proved.

Now we are able to prove Theorem 4 by showing the following equality:

$$\int_{\mathfrak{X}} \mu_t(dx)F(x) = \int_{\mathfrak{X}} \mu_t(dx) \int_{\tilde{a}_\Lambda^m(x)} d\lambda(x_\Lambda)P_t^\infty(x_\Lambda|a_\Lambda^m(x))F(x_\Lambda \cup (x)_{\Lambda^c}). \tag{3.32}$$

It suffices to consider the case where F is cylindrical (i.e. F depends explicitely only on the coordinates and momenta in some finite $\Lambda' \subset \mathbb{Z}^\nu$), continuous and bounded. The general case may be obtained by standard approximation arguments. For the brevity of notations we take $\Lambda' = \Lambda$ in the calculations below; the reader can easily extend them for arbitrary finite Λ'. Since for every bounded continuous cylindrical F

$$\int_{\mathfrak{X}} \mu_t(dx)F(x) = \lim_{n\to\infty} \int_{\mathfrak{X}} \mu_t^{(n)}(dx)F(x) \tag{3.33}$$

to prove (3.32) it suffices to prove:

$$\begin{aligned}\lim_{n\to\infty} \int_{\mathfrak{X}} \mu_t^{(n)}(dx) \int_{\tilde{a}_\Lambda^m(x)} d\lambda(x_\Lambda)P_t^{(n)}(x_\Lambda|a_\Lambda^m(x))\tilde{F}(x_\Lambda)\\ = \int_{\mathfrak{X}} \mu_t(dx) \int_{\tilde{a}_\Lambda^m(x)} d\lambda(x_\Lambda)P_t^\infty(x_\Lambda|a_\Lambda^m(x))\tilde{F}(x_\Lambda)\end{aligned} \tag{3.34}$$

where $\tilde{F}:\mathfrak{X}_\Lambda \to \mathbb{R}$ is defined by $F(x) = \tilde{F}((x)_\Lambda)$.

By the use of an $\varepsilon/3$ argument we show (3.34) by proving that the following three terms below are arbitrarily small for n_0 and n large enough with $n > n_0$.

$$\begin{aligned}&\left|\int_{\mathfrak{X}} \mu_t(dx) \int_{\tilde{a}_\Lambda^m(x)} d\lambda(x_\Lambda)P_t^{(\infty)}(x_\Lambda|a_\Lambda^m(x))\tilde{F}(x_\Lambda)\right.\\ &\quad \left.-\int_{\mathfrak{X}} \mu_t(dx) \int_{\tilde{a}_\Lambda^m(x)} d\lambda(x_\Lambda)P_t^{n_0}(x_\Lambda|a_\Lambda^m(x))\tilde{F}(x_\Lambda)\right|\end{aligned} \tag{3.35}$$

$$\begin{aligned}&\left|\int_{\mathfrak{X}} \mu_t(dx) \int_{\tilde{a}_\Lambda^m(x)} d\lambda(x_\Lambda)P_t^{n_0}(x_\Lambda|a_\Lambda^m(x))\tilde{F}(x_\Lambda)\right.\\ &\quad \left.-\int_{\mathfrak{X}} \mu_t^{(n)}(dx) \int_{\tilde{a}_\Lambda^m(x)} d\lambda(x_\Lambda)P^{n_0}(x_\Lambda|a_\Lambda^m(x))\tilde{F}(x_\Lambda)\right|\end{aligned} \tag{3.36}$$

$$\begin{aligned}&\left|\int_{\mathfrak{X}} \mu_t^{(n)}(dx) \int_{\tilde{a}_\Lambda^m(x)} d\lambda(x_\Lambda)P_t^{n_0}(x_\Lambda|a_\Lambda^m(x))\tilde{F}(x_\Lambda)\right.\\ &\quad \left.-\int_{\mathfrak{X}} \mu_t^{(n)}(dx) \int_{\tilde{a}_\Lambda^m(x)} d\lambda(x_\Lambda)P_t^n(x_\Lambda|a_\Lambda^m(x))\tilde{F}(x_\Lambda)\right|.\end{aligned} \tag{3.37}$$

We start with estimating (3.37). Changing variables one obtain

$$(3.37) \leqq \|F\|_{\infty} \int_{\mathfrak{X}} \int_{\tilde{a}_A^m(S_{-t}^{(n)}x)} d\lambda(x_A) \cdot |P_t^{n_0}(x_A|a_A^m(S_{-t}^{(n)}x)) - P_t^n(x_A|a_A^m(S_{-t}^{(n)}x))| . \tag{3.38}$$

According to the Lemma 3.2 (see (3.15)),

$$\begin{aligned} &|P_t^{n_0}(x_A|a_A^m(S_{-t}^{(n)}x)) - P_t^n(x_A|a_A^m((S_{-t}^{(n)}x))| \\ &\quad \leqq (\exp[\gamma_m(n_0;(S_{-t}^{(n)}x))] - \exp[-\gamma_m(n_0;(S_{-t}^{(n)}x))]) \\ &\quad\quad P_t^{n_0}(x_A|a_A^m(S_{-t}^{(n)}x)) \\ &\quad \leqq (\exp[\tilde{\gamma}_m(h_0,x)] - \exp[-\tilde{\gamma}_m(n_0;x)]) \\ &\quad\quad P_t^{n_0}(x_A|a_A^m(S_{-t}^{(n)}x)) . \end{aligned} \tag{3.39$$

Hence

$$(3.37) \leqq \|F\|_{\infty} \int_{\mathfrak{X}} \mu(dx) \min[2, e^{\tilde{\gamma}_m(n_0;x)} - e^{-\tilde{\gamma}_m(n_0,x)}] .$$

By the use of the Lebesgue theorem, for any $\varepsilon > 0$ we can find a sufficiently large h_0 such that (3.37) is smaller than ε for all $n \geqq n_0$. The same arguments show that the term (3.35) vanishes as $n_0 \to \infty$.

So (3.34) will be proven if we prove that for any fixed n_0, (3.36)$\to 0$ as $n \to \infty$. Putting

$$g(x) = \int_{\tilde{a}_A^m(x)} d\lambda(x_A) P_t^{n_0}(x_A|a_A^m(x)) \tilde{F}(x_A) \tag{3.40}$$

it follows

$$(3.36) = \left| \int_{\mathfrak{X}} \mu(dx)[g(S_{-t}x) - g(S_{-t}^{(n)}x)] \right| . \tag{3.41}$$

We shall prove that

$$g(S_{-t}x) = \lim_{n \to \infty} g(S_{-t}^{(n)}x) \quad \text{for} \quad \mu\text{-a.a.} \quad x \in \mathfrak{X} \tag{3.42}$$

and this will imply that (3.36)$\to 0$ as $n \to \infty$ by the use of the bound $\|g\|_{\infty} \leqq \|F\|_{\infty}$ and Lebesgue theorem.

Let us fix $x \in \mathfrak{X}_{\varphi'}$ such that $\mathscr{L}_{\varphi}(S_{-t}x) \notin \mathbb{N}$ where $\varphi'(k) = 1 \vee \log k$. Then

$$\begin{aligned} |g(S_{-t}x) - g(S_{-t}^{(n)}x)| &\leqq \|F\|_{\infty} \left| \int_{\tilde{a}_A^m(S_{-t}x)} d\lambda(x_A) P_t^{(n_0)}(x_A|a_A^m(S_{-t}x)) \right. \\ &\quad \left. - \int_{\tilde{a}_A^m(S_{-t}^{(n)}x)} d\lambda(x_A) P_t^{(n_0)}(x_A) a_A^m(S_{-t}^{(n)}x)) \right| \\ &= \left| \int_{\tilde{a}_A^m(S_{-t}x)} d\lambda(x_A)[P_t^{(n_0)}(x_A|a_A^m(S_{-t}^{(u)}x))] \right| \end{aligned} \tag{3.43}$$

where the last equality in (3.43) holds if n is sufficiently large, in virtue of Theorem 1) ii) and the definition of ξ_A^m.

Furthermore

$$P_t^{n_0}(x_A|a_A^m(S_{-t}^{(n)}x)) \xrightarrow[n \to \infty]{} P_t^{n_0}(x_A|a_A^m(S_{-t}x)) . \tag{3.44}$$

(3.44) is consequence of the following convergence

$$\exp[-h(S_{-t}^{(n_0)}(x_\Lambda \cup (S_{-t}^{(n)}x)_{\Lambda^c})_{\Lambda_{n_0+r}})] \xrightarrow[n\to\infty]{} \exp[-h(S_{-t}^{(n_0)}(x_\Lambda \cup (S_{-t}x)_{\Lambda^c})_{\Lambda_{n_0+r}})] \quad (3.45)$$

due to Theorem 1 and the continuity of h and of the Lebesgue theorem combined with the following estimate

$$\begin{aligned} h(S_{-t}^{(n)}(x_\Lambda \cup (S_{-t}^{(n)}x)_{\Lambda^c})_{\Lambda_{n_0+r}}) &\geqq cH[(x_\Lambda \cup (S_{-t}^{(n)}x)_{\Lambda^c})_{\Lambda_{n_0+r}}] \\ &\geqq c\Bigg[\sum_{(p_i,q_i)\in x_\Lambda} \{p_i^2/2 + (k-2\nu J)q_i^2 + \lambda q_i^4\} \\ &\quad + \sum_{(q_j,p_j)\in(S_{-t}^{(n)}x)_{\Lambda_{0+r}\setminus\Lambda}} \{p_j^2/2 + (k-2\nu J)q_j^2 + \lambda q_j^4\} \end{aligned} \quad (3.46)$$

In fact exp[(3.46)] is the product of two terms one of which does not depend on n and is integrable w.r.t. $d\lambda(x_\nu)$ and the other one is not depending on x_Λ but converges as $n\to\infty$ and hence is bounded.

So (3.32) and hence Theorem 4 will be proven as consequence of the following statement:

$$\mu_t(\{x|\mathscr{L}_\varphi(x)=m\})=0. \quad (3.47)$$

For every $x\in\mathfrak{X}_{\varphi'}$ then there exists a $j\in\mathbb{N}^+$ such that

$$\mathscr{L}_\varphi(S_t x)=\mathscr{L}^{(j)}(S_t x)/\varphi(j)=m.$$

So (3.47) is implied by the fact that

$$\mu_t(\{x|\mathscr{L}^{(j)}(x)=k, \quad k\in\mathbb{R}\})=0. \quad (3.48)$$

Finally, (3.48) may be obtained by considering that the set in $\mathfrak{X}_\Lambda$ where $\mathscr{L}^{(j)}(x_\Lambda)$ take a fixed value has Lebesgue measure 0, and using the locally absolute continuity of μ_t w.r.t. the Lebesgue measure. This final statement follows from the locally absolute continuity w.r.t. the Lebesgue measure of the approximating measures $\mu_t^{(h)}$.

Appendix

Proof of Theorem 1. By the use of the equation of the motion we obtain

$$\frac{d}{dt}\mathscr{L}^{(i)}(S_t^{(n)}x)=Jp_i^{(n)}(t,x)\Bigg(\sum_{j\in v_i} q_j^{(n)}(t,x)\Bigg). \quad (A.1)$$

The following estimates are obvious

$$|p_i^{(n)}(t,x)| \leqq \sqrt{2\mathscr{L}^{(i)}(S_t^{(n)}x)}\,; \qquad |q_i^{(n)}(t,x)| \leqq \sqrt{\frac{2}{k}}\sqrt{\mathscr{L}^{(i)}(S_t^{(n)}x)}. \quad (A.2)$$

(A.1) and the hypothesis on φ give the following integral inequality:

$$\mathscr{L}_\varphi(S_t^{(n)}x) \leqq \mathscr{L}_\varphi(x) + \bar{a}\int_0^t \mathscr{L}_\varphi(S_s^{(n)}x)ds \quad (A.3)$$

where $\bar{a}$ is a constant independent of n.

Hence

$$\mathscr{L}_\varphi(S_t^{(n)}x) \leqq e^{\bar{a}|t|}(\mathscr{L}_\varphi(x)). \tag{A.4}$$

Let us put:

$$u_k(t,x,n,m) = \sup_{|i|\leqq k} \{|q_i^{(n)}(t,x) - q_i^{(n+m)}(t,x)| \vee |p_i^{(n)}(t,x) - p_i^{(n+m)}(t,x)|$$

Then by using the equation of the motion and in virtue of (A.2) and (A.4), there exists a constant $\bar{c}$ for which the following estimate holds:

$$\begin{aligned} u_k(t,x,n,m) \leqq & \int_0^t \sup_{|i|\leqq k} [|p_i^{(n)}(s) - p_i^{(n+m)}(s)| \\ & \vee |K(q_i^{(n+m)}(s) - q_i^{(n)}(s)) + 4\lambda(q_i^{(n+m)}(s) - q_i^{(n)}(s)) \\ & \cdot (q_i^{(n+m)}(s)^2 + q_i^{(n+m)}(s) q_i^{(n)}(s) + q_i^{(n)}(s)^2) \\ & J \sum_{j\in v_i} |q_j^{(n)}(s) - q_j^{(n+m)}(s)|] ds \\ \leqq & \bar{c} e^{\bar{a}t} \varphi(k) \mathscr{L}_\varphi(x) \int_0^t u_{k+1}(s,x,n,m) ds. \end{aligned} \tag{A.6}$$

Interating the procedure $n-k$ times we obtain:

$$u_k(t,x,n,m) \leqq \frac{(\bar{c} e^{\bar{a}|t|} \mathscr{L}_\varphi(x) \varphi(n) t)^{n-k} \bar{c} e^{\bar{a}|t|} \mathscr{L}_\varphi(x) \varphi(n)}{(n-k)!}. \tag{A.7}$$

This is bound (2.14) with the first meaning of $(\bar{q}, \bar{p})$, that combined with (A.4) gives the assertion i). The estimates (2.12′), (2.14) with the second meaning of $(\bar{q}, \bar{p})$ and also (2.15), (2.16) may be obtained with the same arguments as above.

Now we prove ii). By i) one has:

$$\lim_{n\to\infty} \mathscr{L}^{(i)}(S_t^{(n)}x) = \mathscr{L}^{(i)}(S_t x) \qquad x \in \mathfrak{X}_\varphi. \tag{A.8}$$

Fixed now $x \in \mathfrak{X}_{\varphi'}$ it is enough to prove that there exists $b>0$ and i_n, such that $|i_n| \leqq b < +\infty$ for which

$$\mathscr{L}_\varphi(S_t^{(n)}x) = \frac{\mathscr{L}^{(i_n)}(S_t^n(x))}{\varphi(i_n)}. \tag{A.9}$$

But the estimate:

$$\frac{\mathscr{L}^{(i)}(S_t^{(n)}x)}{\varphi(i)} \leqq \mathscr{L}_{\varphi'}(x) e^{\bar{a}_3 t} \frac{\varphi'(i)}{\varphi(i)} \tag{A.10}$$

(where $\bar{a}_3$ depends only on φ') combined with the fact that $\varphi'(k)/\varphi(k) \to 0$ gives (A.9) and hence proves ii).

Proof of Theorem 2. Theorem 2 is a consequence of the estimates in [13], Corollary 2.4. In fact, denoting $P_\Lambda(dx_\Omega)$, $\Omega \subset \Lambda$, the probability distribution of x_Ω w.r.t. the measure

$$\mu_\Lambda(dx_\Lambda) = \frac{\exp[-h(x_\Lambda)]}{\text{normalization factor}}$$

one obtains the following estimates:

$$P_\Lambda(dx_\Omega) \leqq A \exp\left[-k_1 \sum_{i\in\Omega} p_i^2 - k_2 \sum_{i\in\Omega} q_i^4\right] d\lambda(x_\Omega) \tag{A.11}$$

where A, k_1, k_2 are constant independent of Λ. Compactness arguments combined with (A.11) give the existence statement [14].

Still using estimate (A.11) one obtains:

$$\int \mu(dx) e^{b\mathscr{L}^{((j)}(x)} < M \tag{A.12}$$

where M is a constant not depending on j and b is sufficiently small. Finally by the Tchebyshev inequality:

$$\mu(\{x | \mathscr{L}_{\varphi'}(x) > \vartheta\}) \leqq A e^{-B\vartheta}$$

for some constant A and B. This gives the thesis.

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