Commun. Math. Phys. 66, 95–130 (1979)

Communications in
Mathematical
Physics

Micro-Analyticity of the S-Matrix and Related Functions

Masaki Kashiwara[1]*, Takahiro Kawai[1]*, and Henry P. Stapp[2]**

[1] School of Mathematics, The Institute for Advanced Study, Princeton, New Jersey 08540, USA, and Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606, Japan***

[2] Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720, USA***, and CERN, CH-1211 Geneva, Switzerland

Abstract. It is shown that the singularity spectrum of the phase space integral associated with any partially ordered Landau diagram is confined to a variety defined by a set of modified Landau equations. These equations are similar to the ordinary Landau equations but involve limiting procedures. The variety defined by the modified equations coincides with the variety defined by the ordinary equations except at points called $u=0$ points. Next the causal parts of the sets defined by the modified Landau equations are defined, in a natural way, and it is conjectured that the singularity spectrum of the S-matrix is confined to the union of the causal parts of the singularity spectra of the phase space integrals. An analogous conjecture on general bubble diagram functions asserts that the singularity spectrum of each of these functions is confined to sets defined by the modified Landau equations augmented by appropriate positive-α and negative-α conditions. Generalized Landau equations are introduced. These equations do not involve limiting procedures, but provide a useful partial characterization of the sets defined by the modified Landau equations augmented by these positive-α and negative-α conditions.

§0. Introduction

The primary purpose of this paper is to formulate a conjecture on the singularity spectrum of the S-matrix. This conjecture is designed to be compatible both with unitarity and with the macro-causality requirement that momentum-energy can be transferred over macroscopic distances only by stable particles. These requirements are severe, and our conjecture appears to provide a satisfactory point of departure for the analysis of the singularity structure of the S-matrix within the framework of the theory of holonomic functions (=functions satisfying a holonomic (=maximally overdetermined) system of (micro-)differential equations)

* Supported in part by National Science Foundation grant MCS77-18723

** Supported in part by the U.S. Energy and Development Administration, under the auspices of the Division of Physical Research and in part by the European Organization for Nuclear Research

*** Current address

0010-3616/79/0066/0095/$07.20

developed by Sato-Kawai-Kashiwara [13], hereafter referred to as S-K-K [13] for short.

The singularity spectrum of a hyperfunction is a generalization of the set of singularities of the hyperfunction. For a hyperfunction $f(p)$ of a real variable $p=(p_1, \dots, p_n)$, its singularity spectrum is a well defined set in (p, u)-space, where $u=(u_1, \dots, u_n)$ is a cotangent vector at p. The singularity spectrum of $f(p)$ specifies the set of real singular points p and, at points p where f is not analytic, gives information about the directions in Imp-space from which f is the boundary value of an analytic function or more generally characterizes possible decompositions of f into sums of boundary values of analytic functions from specified directions. This connection is described in S-K-K [13] Chapt. I, Proposition 1.5.4. The closely related ideas of essential support theory developed by Bros and Iagolnitzer are described in [22] and in the references cited there.

The S-matrix kernel $S(p)$ is, by virtue of its unitarity property, a distribution, hence a hyperfunction, and thus has well defined singularity spectra.

There are several conceivable methods of constructing the singularity spectrum of the S-matrix. A first method would be to assume that the singularity spectrum of the S-matrix is confined to the union of the singularity spectra of the Feynman integrals, and then to construct the singularity spectra of the Feynman integrals. However, most Feynman integrals diverge, and the calculation of their singularity spectra consequently presents technical difficulties that have not yet been resolved if we consider the on-shell amplitudes. Moreover, it would be preferable to obtain this basic information about the singularity spectrum of the S-matrix by methods that do not involve regularization, renormalization and divergent infinite series.

A second method would be to obtain the singularity spectrum of the S-matrix directly from the principle of macrocausality. This principle asserts that momentum-energy is transferred over macroscopic distances only by stable particles: any transfer of momentum-energy that cannot be ascribed to a network of stable particles has a probability to occur that falls off exponentially under space-time dilation. This property has been rigorously formulated away from certain exceptional points in p-space called $u=0$ points, and has been shown to imply (Iagolnitzer-Stapp [20], Iagolnitzer [4–6], and Pham [23]) that at non $u=0$ points the singularity spectrum of the S-matrix is confined to the union of the set of positive-α Landau varieties $\mathscr{L}^+(D)$.

The $u=0$ points of the S-matrix consist of the points $p=(p_1, \dots, p_n)$ where two or more of the momentum-energy vectors p_r are parallel or two or more of the final momentum-energy vectors are parallel. Thus the set of $u=0$ points of the S-matrix is a set of low dimension.

Macrocausality gives the singularity spectrum of the S-matrix at all points p not lying in the low dimensional set of $u=0$ points. However, the lack of information on this low dimensional set has very pernicious consequences. These $u=0$ points can occur in the domains of integration of unitarity-type products of the connected parts of several S-matrix elements. These products, called bubble diagram functions, inevitably arise when one analyzes the consequences of unitarity, and it is important to determine their analytic structures. However, the lack of information about the singularity spectrum of the S-matrix at $u=0$ points propagates into a lack of information on the singularity spectrum of the bubble diagram functions at their $u=0$ points (Stapp [2], Coster-Stapp [3], Iagolnitzer [6], Kawai-Stapp [1]). This is disastrous because the $u=0$ points of the bubble diagram functions are not confined to the low dimensional set of points where pairs of external (i.e., non-integrated) momentum-energy vectors are parallel. The $u=0$ points of a bubble diagram function $F_B(p)$ include a certain subset of the points p such that one or more of the component S-matrix functions has a $u=0$ point at some point in the domain of integration. Already for the simplest applications one encounters bubble diagram functions where these $u=0$ points cover the entire physical region in p-space. Thus the $u=0$ problem becomes magnified from a tiny problem at the level of the S-matrix to a major problem at the level of the bubble diagram functions.

In applications ([7]) this problem has been partially circumvented by the introduction of an ad hoc mixed-α cancellation assumption about how the singularities of different terms cancel among themselves. However, this assumption has no fundamental basis, and only limited applicability. Clearly a more satisfactory procedure would be to supply the missing information about the singularity spectrum of the S-matrix at $u=0$ points, and then deduce the singularity spectrum of bubble diagram functions at their $u=0$ points. As discussed in more detail later, the knowledge of the singularity spectrum of the S matrix at $u=0$ points will not be sufficient by itself to determine the singularity spectrum of bubble diagram functions at their $u=0$ points, but it will clearly be an important ingredient in this program.

The extension of the macrocausality condition to $u=0$ points is not altogether unambiguous. In contrast to the situation at $u\neq 0$ points the physical idea of macrocausality leads, at $u=0$ points, to the need to consider limiting procedures, and the precise meaning of macrocausality depends critically on the fine details of these limiting procedures.

In order to specify an appropriate limiting procedure we shall here combine the physical idea of macrocausality with the requirement that the S-matrix be unitary.

The unitary equations involve the mass-shell constraint and momentum-energy conservation-law delta functions. These delta function factors lead to necessary singularities of the S-matrix at certain points where the phase space integrals defined by these delta functions have thresholds. This is because certain terms in the iterated unitarity equations drop out when the argument p goes below these thresholds, and hence some of the remaining terms must have singularities.

Each phase space integral is associated with a Feynman-like diagram. To construct the phase space integral associated with a certain diagram D one replaces each vertex of the diagram by a momentum-energy conservation-law delta function, replaces each line by a mass-shell constraint delta function, and then integrates over the momentum-energy of each internal line. Phase space integrals exhibit the kinematic singularities arising from the simultaneous momentum-energy and mass-shell constraint delta functions.

The diagrams D corresponding to phase space integrals are required to satisfy the partial ordering requirement that they can be drawn so that positive energy flows uniformly from left to right: all diagrams that arise from the unitarity and cluster decomposition requirements on the S-matrix satisfy this partial ordering requirement.

The iterated unitarity equations, combined with cluster decomposition, lead to an infinite set of different expressions for each connected part of the S-matrix. These equations require the singularity spectrum of the S-matrix to contain all the points in the singularity spectra of all the phase space integrals, except those points where there are exact cancellations of contributions from different terms.

Exact cancellations of this kind must occur at points that are excluded from the singularity spectrum of the S-matrix by the macrocausality condition. The studies in [7] show in some detail how these systematic cancellations come about. However, no such cancellations are demanded at points of the singularity spectra of the phase space integrals that are compatible with macrocausality, and there is no reason to expect such cancellations. Thus these points should be allowed in the singularity spectrum of the S-matrix if one is to avoid the likelihood of a conflict with unitarity.

On the other hand, there is no apparent need for the singularity spectrum of the S-matrix to contain points that do not lie in the singularity spectrum of any phase space integral. We thus conjecture that the singularity spectrum of the S-matrix is confined to the union of the causal parts of the singularity spectra of the phase space integrals, where the causal parts are the parts compatible with macrocausality conditions. To convert this formulation of our conjecture into an explicit form we need explicit conditions on the singularity spectra of the phase space integrals.

Phase space integrals, unlike Feynman integrals, are convergent. The mathematical core of the present paper is a study of the singularity spectra of phase space integrals. It is shown, in Sect. 1, that the singularity spectrum of an arbitrary phase space integral is confined to the variety defined by a set of modified Landau equations. These modified Landau equations are similar to the ordinary Landau equations, but allow limit points of solutions to equations that differ from the ordinary equations by quantities that tend to zero in the limit. Except at $u=0$ points the modified equations yield the same conditions as the ordinary Landau equations.

The causal parts of the sets $\tilde{\mathscr{L}}(D)$ defined by the modified Landau equations are identified in Sect. 2 as the sets defined by the positive-α modified Landau equations. These equations are the modified Landau equations augmented by the conditions that the Landau parameters α be positive and that the closely related parameters β be real.

Our principal conjecture is then that the singularity spectrum of the S-matrix is confined to the set

$$\tilde{\mathscr{L}}^+ \equiv \bigcup_D \tilde{\mathscr{L}}^+(D),$$

where the union is over all partially ordered Landau diagrams D, and $\tilde{\mathscr{L}}^+(D)$ is the set defined by the positive-α modified Landau equations associated with D.

This conjecture gives strong but reasonable conditions on the singularity spectrum of the S-matrix at $u=0$ points. Starting from this information one would like to deduce the singularity spectrum of the bubble diagram functions at $u=0$ points. But the problems at these points do not stem exclusively from the $u=0$ points of the component S-matrix functions. There are $u=0$ points of bubble diagram functions even in cases where there are no $u=0$ points of the component S-matrix functions in the integration domain. And the standard general theorems provide no information also in these cases.

We hope, on this basis of examples we have studied, that a specification of the holonomic structure of the S-matrix will provide the needed information, and allow us to extend the previously derived $u \neq 0$ structure theorem to $u=0$ points. We conjecture that this extended theorem should be the same as the earlier $u \neq 0$ theorem, but with the ordinary Landau equations replaced by the modified Landau equations, with all α and β taken to be real, and signs of these quantities specified as in the $u \neq 0$ structure theorem. In particular, the α_l associated with a line that is an internal line of a bubble should be positive if that bubble corresponds to the connected part of the S-matrix and minus if the bubble corresponds to the connected part of $S^\dagger$. This result would confine the physical region singularities of all bubble diagram functions to codimension-one subsets of the physical region, thus eliminating the open sets of allowed singularities permitted by the ordinary Landau equations.

The modified Landau equations discussed above involve limiting procedures, and are therefore more difficult to use than the ordinary Landau equations. In order to facilitate the application of our results we define in Sect. 3 another set of equations, called the generalized Landau equations, which do not involve limiting procedures, and show that the set of points defined by the modified Landau equations is confined to the set of points defined by the generalized Landau equations, outside a set of points called generalized $u=0$ points. Some simple examples given at the end of Sect. 3 illustrate the usefulness of the generalized Landau equations.

Inspired in part by certain aspects of the present work, Iagolnitzer [27] has proposed an alternative solution to the $u=0$ problems considered here. Based on properties of the S-matrix derived from a certain extension of macrocausality to $u=0$ points he derives a $u=0$ structure theorem that gives strong conditions on the singularity spectra of bubble diagram functions. These singularity spectra are defined by limiting procedures that are similar to ours, but different. They have the advantage of involving only real quantities: each internal line is associated with a *pair* of real on-mass-shell vectors, rather than a single complex off-mass-shell vector.

The fact that Iagolnitzer derives a structure theorem makes his work an important advance over ours. On the other hand, our main effort has been to formulate a conjecture or assumption that should be compatible with unitarity. No comparable effort has yet been made on the regularity property R of S-matrix that occurs in Iagolnitzer's work. However, an examination of this question has been commenced by Iagolnitzer and one of the present authors (H.S.) with encouraging results.

Iagolnitzer has questioned the derivation of the positive-α condition given in the original version of this paper on the grounds that our extension of the macrocausality condition to $u=0$ situations did not allow for the doubling of the lines permitted by his refined macrocausality condition. We have, accordingly, in the present paper, modified our discussion in Sect. 2 so as to place the major burden of our argument on the results derived in Sect. 1, with only a minimal dependence on macrocausality. On the other hand, the doubling of lines allowed by Iagolnitzer's procedures disrupts the energy-momentum conservation law condition during the course of the limiting procedure and weakens his results compared to ours in certain situations. In particular, if Iagolnitzer's condition R could be proved for the energy-momentum conservation law factor of the S-matrix, so that his results would apply to phase space integrals, then his results would not generally yield the conservation of center-of-mass trajectory condition (sometimes called the conservation of angular momentum condition) that is entailed by our results of Sect. 1, and that is described in Sect. 3.

A detailed account of the historical development of this subject, with discussions of applications, can be found in the paper of Iagolnitzer [27]. The articles of Iagolnitzer [4–6, 21] and Kawai-Stapp [1] especially emphasize the microlocal aspects of the singularity structure of the S-matrix and the bubble diagram functions: these works reformulate the notion of the normal analytic structure of the S-matrix ([20]) in a neat and precise mathematical language that employs the notion of the cotangent bundle (see also Sato [11] and Pham [23]). In obtaining their results on bubble diagram functions Iagolnitzer [6] (resp. Kawai-Stapp [1]) uses the general results on products, integration etc. of distributions (resp. hyperfunctions). The general results Iagolnitzer [6] used were obtained in the framework of essential support theory developed by Bros-Iagolnitzer (see Iagolnitzer [22] and references cited there) and those of Kawai-Stapp [1] were obtained in the framework of microfunction theory (S-K-K [13]).

In deriving our result on the singularity spectra of the phase space integrals essential use is made of a result concerning the characteristic variety of holonomic systems of linear differential equations that hyperfunctions of the form

$$\prod_{j=1}^{d} \delta(\varphi_j(x)) \prod_{l=1}^{N} (f_l(x) + \sqrt{-1}0)^{\lambda_l}$$

or $\prod_{j=1}^{d} \delta(\varphi_j(x)) \prod_{l=1}^{N} f_l(x)_+^{\lambda_l}$ must satisfy (Kashiwara-Kawai [9, 10, 26]). In [9] a property (Lemma 2) stronger than the result used here was announced. The original form of the present work was based on that stronger property. However, after the work was essentially complete, and the results announced ([14]), a gap in the proof of Lemma 2 of [9] was discovered. Consequently, the result on phase space integrals given in this paper are slightly weaker than the ones previously announced ([14]). They are obtained by replacing the stronger property used originally by a slightly weaker one ([9], Lemma 1. See [26] for the proof). Although a correct proof of Lemma 2 of [9] has not yet been constructed, the authors believe, on the basis of considerations described in [26], that the originally announced conclusions are nevertheless true. The originally announced result of [14] is thus presented at the end of Sect. 1 as a mathematical conjecture, which, however, is not used in the analysis of the singularity spectrum of the S-matrix in Sect. 2.

We expect our modified Landau equations to play an essential role in clarifying the relationship between the analyticity and the unitarity of the S-matrix along the line proposed by Sato (Sato [11], Kawai-Stapp [12]).

In particular, a recent result of the first two named authors (M. K. and T. K.) ([28]) indicates that we might generalize the results obtained for phase space integrals to general bubble diagram functions if we could rigorously formulate and accept Sato's conjecture ([11]) on the holonomic character of the S-matrix.

For the definition of Landau diagrams, bubble diagram functions and related functions, we refer the reader to Kawai-Stapp [12] and the references cited there. For the theory of micro-differential (=pseudo-differential) equations and micro-functions, we refer to S-K-K [13]. We use in this paper the same notations as in these two articles, unless otherwise stated.

In this article all the relevant particles are supposed to be massive. It is also assumed that the set of their mass-values have no accumulation point (except possibly at infinity).

§1. Singularity Spectra of Phase Space Integrals

In this section we discuss the singularity structure of the phase space integral $I_D(p) = I_D(p_1, \ldots, p_n)$ associated with a partially-ordered, stable-particle Landau diagram D having n external lines, N internal lines and n' vertices. The external lines are indexed by r ($r = 1, \ldots, n$), the internal lines are indexed by l ($l = 1, \ldots, N$), and the vertices are indexed by j ($j = 1, \ldots, n'$). The j-th vertex shall be denoted by V_j.

A formal definition of phase space integral $I_D(p)$ is given as follows:

$$\prod_{r=1}^{n} \delta(p_r^2 - \mu_r^2) Y(p_{r,0}) \int \prod_{j=1}^{n'} \delta^4\left(\sum_{r=1}^{n} [j:r] p_r + \sum_{l=1}^{N} [j:l] k_l\right) \cdot \prod_{l=1}^{N} \delta(k_l^2 - m_l^2) Y(k_{l,0}) \prod_{l=1}^{N} d^4 k_l. \tag{1.1}$$

Because this integral involves delta functions of arguments that can be functionally dependent in the domain of integration, the general rule for defining products of hyperfunctions (S-K-K [13], Chapt. I, Corollary 2.4.2) does not provide a general rigorous definition of (1.1). In fact, the $u=0$ points (Kawai-Stapp [12], §0) are precisely the points where the immediate application of the general rule fails. Nevertheless, the integral $I_D(p)$ has, as we shall see, because of the tameness of the integrand, a natural well-defined meaning.

In order to investigate the singularity structure of $I_D(p)$, we introduce, following Riesz [15], an auxilliary integral

$$I_D(p; \lambda, \lambda') \equiv I_D(p_1, \ldots, p_n; \lambda_1, \ldots, \lambda_{n'}, \lambda'_1, \ldots, \lambda'_N)$$

defined by

$$\int \prod_{r=1}^{n} \frac{(p_r^2 - \mu_r^2)_+^{\lambda_r} Y(p_{r,0})}{\Gamma(\lambda_r + 1)} \prod_{j=1}^{n'} \delta^4\left(\sum_{r=1}^{n} [j:r] p_r + \sum_{l=1}^{N} [j:l] k_l\right) \cdot \prod_{l=1}^{N} \frac{(k_l^2 - m_l^2)_+^{\lambda'_l}}{\Gamma(\lambda'_l + 1)} Y(k_{l,0}) \prod_{l=1}^{N} d^4 k_l. \tag{1.2}$$

Here Y is the Heaviside function, and $f_+^\lambda = f^\lambda Y(f)$ if $\mathrm{Re}\,\lambda > 0$ (see Gel'fand-Shilov [16]). We call $I_D(p;\lambda,\lambda')$ the generalized phase space integral associated with D.

For $\mathrm{Re}\,\lambda_r$, $\mathrm{Re}\,\lambda'_l \gg 0$, the integrand of (1.2) is well-defined as a product of continuous functions. Furthermore, the integral (1.2) is a proper integral in the sense that the support of its integrand is confined to a compact set as long as p is confined to a compact set. This property follows from the fact that the diagram D is partially-ordered. That means that the vertices of D can be indexed in such a way that $j' < j$ whenever there is an l such that $[j':l] = +1$ and $[j:l] = -1$. Geometrically, this means that D can be drawn with all lines directed from left to right. One then sees that the energy $k_{r,0}$ of any line is less than the total energy of the incoming or outgoing lines. Consequently the momentum $\mathbf{k}_r$ is also bounded, by virtue of the Heaviside function $Y(k_r^2 - \mu_r^2)$.

The singularity structure of $I_D(p;\lambda,\lambda')$ is, as will be seen below, sufficiently manageable to allow us to obtain information on the singularity structure of $I_D(p)$ by making use of an analytic continuation procedure with respect to parameters λ and λ'. This analytic-continuation procedure will be explained later (Theorem 1.4).

In order to simplify the writing, we change our notation until the end of the proof of Theorem 1.7, by introducing the following definitions:

$$\begin{aligned} &p_r \equiv k_{r-n} \qquad (r = n+1, \ldots, N+n = M)\\ &\mu_r \equiv m_{r-n} \qquad (r = n+1, \ldots, N+n = M)\\ &\lambda_r \equiv \lambda'_{r-n} \qquad (r = n+1, \ldots, N+n = M). \end{aligned} \tag{1.3}$$

Then we consider the following distribution

$$\Phi_D(p;\lambda) \equiv \prod_{r=1}^{M} \frac{(p_r^2 - \mu_r^2)_+^{\lambda_r}}{\Gamma(\lambda_r + 1)} Y(p_{r,0}) \prod_{j=1}^{n'} \delta^4\left(\sum_{r=1}^{M} [j:r] p_r\right). \tag{1.1'}$$

First we recall the following result due to Bernstein-Gel'fand [17] and Atiyah [18], see also Kashiwara-Kawai [26].

Lemma 1.1. *$\Phi_D(p;\lambda)$ is a well-defined distribution in p which depends meromorphically on $\lambda = (\lambda_1, \ldots, \lambda_M)$. Furthermore, $\Phi_D(p;\lambda)$ is holomorphic in λ when* $\mathrm{Re}\,\lambda_r > 0$ *($r = 1, \ldots, M$).*

Proof. First note that $p_{r,0} \neq 0$ under the assumptions that $p_r^2 \geqq \mu_r^2$ ($r = 1, \ldots, M$) and that $\sum_{r=1}^{M} [j;r] p_r = 0$ ($j = 1, \ldots, n'$). Therefore it is enough to investigate

$$\tilde{\Phi}_D(p;\lambda) = \prod_{r=1}^{M} \frac{(p_r^2 - \mu_r^2)_+^{\lambda_r}}{\Gamma(\lambda_r + 1)} \prod_{j=1}^{n'} \delta^4\left(\sum_{r=1}^{N} [j:r] p_r\right) \tag{1.4}$$

on the open set $\{p \in \mathbb{R}^{4M};\ p_{r,0} > 0,\ r = 1, \ldots, M\}$. It is clear that

$$Y = \left\{p \in \mathbb{R}^{4M};\ \sum_{r=1}^{M} [j:r] p_r = 0, j = 1, \ldots, n'\right\}$$

is $4(M-n)$-dimensional affine space, hence it suffices to show that $\prod_{r=1}^{M} \frac{(p_r^2 - \mu_r^2)_+^{\lambda_r}}{\Gamma(\lambda_r + 1)}$ restricted to Y enjoys the properties claimed in Lemma 1.1. This assertion immediately follows from (the proof of) Theorem 2 of Bernstein-Gel'fand [17].

Next we establish a bound on the maximum possible singularity spectrum of $\Phi_D(p;\lambda)$ considered on $M_+=\{(p,\lambda)\in\mathbb{R}^{4M}\times\mathbb{C}^M;\ p_{r,0}$ and $\mathrm{Re}\lambda_r>0\ (r=1,\dots,M)\}$. Clearly $\Phi_D(p;\lambda)$ is holomorphic with respect to λ on M_+. That is,

$$\frac{\partial}{\partial\bar{\lambda}_r}\Phi_D(p;\lambda)=0,\qquad r=1,\dots,M \tag{1.5}$$

holds on M_+.

The following Lemma 1.2 immediately follows from Theorem 18 of Kashiwara-Kawai [26].

Lemma 1.2. S.S. $\Phi_D(p;\lambda)\subset\{(p,\lambda;\ \sqrt{-1}(u,\sigma))^1\in\sqrt{-1}S^*M_+;\ \sigma=0$ *and* $(p;u)$ *satisfies the following condition* (1.6)$\}$.

There exists a sequence of complex four vectors $p_r^{(m)}$ *and* $v_j^{(m)}$ *and complex scalars* $\alpha_r^{(m)}$ *which satisfies the following:*

$$\begin{cases}\sum_{r=1}^{M}[j:r]p_r^{((m)}=0, & j=1,\dots,n', & (1.6a)\\ p_r^{(m)}\to p_r\quad \textit{with}\quad p_{r,0}>0\quad \textit{and}\quad p_r^2\geqq\mu_r^2, & r=1,\dots,M, & (1.6b)\\ \alpha_r^{(m)}p_r^{(m)}+\sum_{j=1}^{n'}[j:r]v_j^{(m)}\to u_r, & r=1,\dots,M, & (1.6c)\\ \alpha_r^{(m)}(p_r^{(m)2}-\mu_r^2)\to 0, & r=1,\dots,M. & (1.6d)\end{cases}$$

Proof. Since $(p_r^2-\mu_r^2)_+^{\lambda_r}$ can be expressed as a linear combination of $(p_r^2-\mu_r^2+\sqrt{-1}0)^{\lambda_r}$ [or $(p_r^2-\mu_r^2)^{\lambda_r}\log(p_r^2-\mu_r^2+\sqrt{-1}0)$, if λ_r is a non-negative integer] and $(p_r^2-\mu_r^2-\sqrt{-1}0)^{\lambda_r}$ [or $(p_r^2-\mu_r^2)^{\lambda_r}\log(p_r^2-\mu_r^2-\sqrt{-1}0)$ if λ_r is a non-negative integer], Theorem 18 of Kashiwara-Kawai [26] entails that S.S. $\Phi_D(p;\lambda)$ is contained in $\{(p,\lambda;\ \sqrt{-1}(u,\sigma))\in\sqrt{-1}S^*M_+;\ \sigma=0$ and $(p;u)$ satisfies the following condition (1.7)$\}$.

There exists a sequence of complex four-vectors $p_r^{(m)}$ and $v_j^{(m)}$ and complex scalars $\alpha_r^{(m)}$ which satisfy the following:

$$\begin{cases}\sum_{r=1}^{M}[j:r]p_r^{(m)}=0, & j=1,\dots,n', & (1.7a)\\ p_r^{(m)}\to p_r\quad \text{with}\quad p_{r,0}>0, & r=1,\dots,M, & (1.7b)\\ \alpha_r^{(m)}p_r^{(m)}+\sum_{j=1}^{n'}[j:r]v_j^{(m)}\to u_r, & r=1,\dots,M, & (1.7c)\\ \alpha_r^{(m)}(p_r^{(m)2}-\mu_r^2)\to 0, & r=1,\dots,M. & (1.7d)\end{cases}$$

Furthermore, since $\Phi_D(p;\lambda)$ is zero unless $p_r^2\geqq\mu_r^2$ holds for all r, the limiting point must also satisfy $p_r^2\geqq\mu_r^2$ in order that $(p,\lambda;\sqrt{-1}(u,0))$ be contained in S.S. $\Phi_D(p;\lambda)$. This completes the proof of the lemma.

---

1 Here $\sigma\in\mathbb{C}^M\cong\mathbb{R}^{2M}$. More intrinsically, we had better use the notation $(p,\lambda;\ \sqrt{-1}(\langle u,dp\rangle+2\mathrm{Re}\langle\sigma,d\lambda\rangle)\infty)$ to denote a point in $\sqrt{-1}S^*M_+$. Since there is no fear of confusion, here we use a simpler notation

We shall now analytically continue $\Phi_D(p;\lambda)$ with respect to λ so that the point $\lambda_r = -1$ $(r=1,\dots,M)$ can be reached along a path γ of continuation and that $\Phi_D(p;-1,\dots,-1)$ is a well-defined distribution. Furthermore, the resulting distribution will be shown to coincide with $\Phi_D(p)$, whose meaning will be made precise by representing it in an unambiguous form after suitable coordinate transformations. In view of this equality, Theorem 2.2.8 of S-K-K [13], Chapt. III, proves that S.S. $\Phi_D(p)$ is confined to the set given in Lemma 1.2, as we shall discuss later on (Lemma 1.6).

In order to make this analytic continuation work, we choose a specific coordinate system so that $\Phi_D(p;\lambda)$ can be explicitly calculated. For this purpose, we first choose, for each j, some $r=r(j)$ such that $[j:r(j)]=1$ holds. That is, we choose a preferred incoming line at each vertex. We denote by R_1 the set of all preferred $r(j)$'s and by R_0 the index set for non-preferred lines. Then it is obvious that the set $\left\{p\in\mathbb{R}^{4M};\ p_r^2=\mu_r^2\ (r\in R_0) \text{ and } \sum_{r=1}^{M}[j:r]p_r=0\ (j=1,\dots,n')\right\}$ is a non-singular manifold. That is, hypersurfaces $\{p;p_r^2=\mu_r^2\}$ $(r\in R_0)$ intersect transversally on $\left\{p;\ \sum_{r=1}^{M}[j:r]p_r=0\ (j=1,\dots,n')\right\}$. Therefore we can define

$$F_D(p;\lambda)=\prod_{r\in R_0}\delta(p_r^2-\mu_r^2)Y(p_{r,0})\prod_{r\in R_1}\frac{(p_r^2-\mu_r^2)_+^{\lambda_r}}{\Gamma(\lambda_r+1)}Y(p_{r,0})\prod_{j=1}^{n'}\delta^4\left(\sum_{r=1}^{M}[j:r]p_r\right) \quad (1.8)$$

as a product of continuous functions on the manifold $\left\{p\in\mathbb{R}^{4M};\ p_r^2=\mu_r^2\ (r\in R_0),\ \sum_{r=1}^{M}[j:r]p_r=0\ (j=1,\dots,n')\right\}$, if $\operatorname{Re}\lambda_r>0$ $(r\in R_1)$. Note that

$$\prod_{r\in R_0}\frac{(p_r^2-\mu_r^2)_+^{\lambda_r}}{\Gamma(\lambda_r+1)}Y(p_{r,0})\prod_{j=1}^{n'}\delta^4\left(\sum_{r=1}^{M}[j:r]p_r\right)$$

depends holomorphically on $\lambda_r\in\mathbb{C}$ $(r\in R_0)$ [i.e., it is entire in λ_r $(r\in R_0)$] and that it reduces to

$$\prod_{r\in R_0}\delta(p_r^2-\mu_r^2)Y(p_{r,0})\prod_{j=1}^{n'}\delta^4\left(\sum_{r=1}^{M}[j:r]p_r\right)$$

for $\lambda_r=-1$ $(r\in R_0)$, since the hypersurfaces $\{p;p_r^2=\mu_r^2\}$ $(r\in R_0)$ intersect transversally on the manifold

$$\left\{p;\ \sum_{r=1}^{M}[j:r]p_r=0,\quad j=1,\dots,n'\right\}.$$

It is convenient to introduce at this stage a holomorphic transformation of coordinates. Originally all the components $p_{r,\mu}$ were measured in the same coordinate frame Σ. We now introduce a set of frames Σ_j, one for each vertex j. The components of p_r as measured in Σ_j are

$$p_{r,\mu}^{(j)} \equiv \sum_{\nu=0}^{3} B_{j\mu\nu} p_{r,\nu}, \tag{1.9}$$

where B_j is a certain Lorentz transformation. It is defined to be the unique boost (pure time-like Lorentz transformation) that transforms the total momentum-energy going out from V_j, namely

$$P_j^{\text{out}} \equiv \sum_{r \in R_-(j)} p_r, \tag{1.10}$$

into a vector with null space components:

$$(B_j P_j^{\text{out}})_\mu = 0 \qquad (\mu = 1, 2, 3). \tag{1.11}$$

The set $R_-(j)$, and some other related sets, are defined by

$$\begin{aligned} R_-(j) &\equiv \{r; [j:r] = -1\} \\ R_+(j) &\equiv \{r; [j:r] = +1\} \\ R_g(j) &\equiv \{r \in R_0; [j:r] = +1\} \\ R_f &\equiv \{r; [j:r] \neq +1 \quad \text{for all } j\} \\ R_j &\equiv \{r; r \in R_f, \quad \text{or} \quad [j':r] = +1 \quad \text{for some} \quad j' \leqq j\}. \end{aligned} \tag{1.12}$$

The vector $p' \equiv (p'_1, \ldots, p'_{n'})$ is then defined by

$$\begin{cases} p'_{r,\mu} \equiv p^{(j)}_{r,\mu} & (r \in R_+(j)) \\ \text{and} & \\ p'_{r,\mu} \equiv p_{r,\mu} & (r \in R_f). \end{cases} \tag{1.13}$$

That is, the vectors p'_r that are incoming at vertex V_j are measured in the frame Σ_j in which the sum of the $p_r^{(j)}$ going out from vertex j has null space-components. In terms of these new variables p'_r the momentum-conservation law at vertex j takes the form

$$\sum_{r \in R_+(j)} \mathbf{p}'_r = 0, \tag{1.14}$$

whereas energy conservation at vertex V_j takes the form

$$\sum_{r \in R_+(j)} p'_{r,0} - \sum_{r \in R_-(j)} p^{(j)}_{r,0} = 0. \tag{1.15}$$

The components $B_{j\mu\nu}$ of the boost B_j are real analytic functions of the components of P_j^{out}. This is readily seen from an examination of the formulas of Ref. [19]. Thus the vectors p' are real analytic functions of the vectors p.

A more detailed result can be stated if one uses the partial-ordering condition on D. This condition asserts, as was mentioned before, that the vertices V_j can be indexed so that $j' < j$ if, for any l, $[j':l] = +1$ and $[j:l] = -1$. Adopting such a convention for the indices j one finds that the vector p'_r associated with a line incoming at any vertex V_j is a function only of those $p_{r'}$ that are associated with lines that are either incoming at some $V_{j'}$ with $j' \leqq j$ or are incoming on no vertex. Thus we have

$$p'_r = p'_r(p_{r'}) \quad (r \in R_+(j), \quad r' \in R_j). \tag{1.16}$$

This result follows from the fact that B_j in (1.9) depends on the vectors $p_{r'}$ associated with the lines $r' \in R_-(j)$ that are outgoing at V_j, and for each of these either $r' \in R_f$ or $r' \in R_+(j')$ for some $j' < j$. That is, $R_-(j) \subset R_j$. This form (1.16), together with the inclusion relations $R_f \subset R_1 \subset R_2 \subset \ldots \subset R_{n'}$ and the invertibility of the Lorentz transformations B_j, ensures that the mapping $p'(p)$ has a holomorphic inverse $p(p')$ such that, for each $r \in R_+(j)$,

$$p_r = p_r(p'_{r'}) \quad (r \in R_+(j), \quad r' \in R_j) \tag{1.17}$$

with real analytic right-hand side. One can prove this result by ordering the indices r so that the $r \in R_f$ come first, then the $r \in R_+(1)$, then the $r \in R_+(2)$, etc. Then the terms in the Jacobian matrix $(\partial p'_r / \partial p_{r'})$ involving $\partial B_j / \partial p_{r'}$ will all lie on one side of the diagonal, and will not contribute to the determinant $|\partial p' / \partial p|$. This determinant will then be nonzero due to the nonsingular character of the B_j. Then for each integer h $(1 \leqq h \leqq n')$ the set of Eqs. (1.16) with $j \leqq h$ can be holomorphically inverted to give the corresponding set of inverse Eqs. (1.17) for $j \leqq h$.

Since the variables p' are holomorphically equivalent to the variables p we can use p' instead of the p, in proving results about the existence of integrals when continuation is made in λ. In the following theorem we use the variables p' almost exclusively. Thus in that theorem we shall drop the prime on p'_r and represent p'_r by p_r. The original variables p_r will be denoted by p_r^{orig}. Similarly, functions such as $\Phi'_D(p'; \lambda) \equiv \Phi_D(p(p'); \lambda)$, etc. will be represented in Theorem 1.3 by simply $\Phi_D(p; \lambda)$.

We now prove the following

Theorem 1.3. *There exists a path γ in λ-space such that $\Phi_D(p; \lambda)$ continues into a well-defined distribution $\Phi_D(p; \lambda^0)$ at $\lambda = \lambda^0 \equiv (-1, \ldots, -1)$ provided λ^0 is reached along the path γ. Moreover, $\Phi_D(p; \lambda^0)$ can be identified with the corresponding distribution $\Phi_D(p)$ associated with the phase space integral.*

Proof. Clearly it suffices to consider the distribution $F_D(p; \lambda)$ defined in (1.8) instead of $\Phi_D(p; \lambda)$. Let $f(p)$ be a C^∞-function with compact support and consider the following integral.

$$I(f; \lambda) = \int F_D(p; \lambda) f(p) dp. \tag{1.18}$$

We shall analytically continue this integral with respect to λ. Define ω_r by $\sqrt{\mathbf{p}_r^2 + \mu_r^2}$. Then $I(f; \lambda)$ can be rewritten as follows:

$$\int f(p) \prod_{r \in R_1} \frac{(p_r^2 - \mu_r^2)_+^{\lambda_r}}{\Gamma(\lambda_r + 1)} Y(p_{r,0}) \prod_{j=1}^{n'} \delta^3\Big(\sum_{r \in R_+(j)} \mathbf{p}_r\Big)$$
$$\cdot \prod_{j=1}^{n'} \delta\Big(\sum_{r \in R_+(j)} p_{r,0} - \sum_{r \in R_-(j)} p_{r,0}^{(j)}\Big) \prod_{r \in R_0} \frac{d^3\mathbf{p}_r}{2\omega_r} \prod_{r \in R_1} d^4 p_r. \tag{1.19}$$

Since the diagram D is partially-ordered, we may, as mentioned above, assume that the vertices D are indexed so that if for some l $[j:l] = +1$ and $[i:l] = -1$ then $j < i$ (see Fig. 1.1).

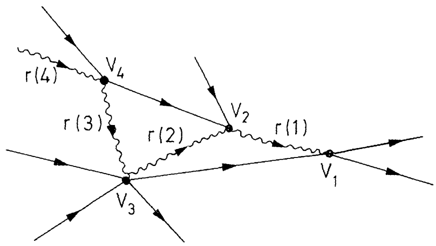

Fig. 1.1. Wiggly lines are preferred lines

We also assume now, for brevity of notation, that the lines are indexed so that $p_{r(j)}=p_j$. Then energy conservation at V_j implies

$$p_{j,0}+\sum_{r\in R_g(j)}\omega_r=\sum_{r\in R_-(j)}p^{(j)}_{r,0}\,. \tag{1.20}$$

Hence we have

$$\begin{aligned}p_{j,0}-\omega_j&=\sum_{r\in R_-(j)}p^{(j)}_{r,0}-\sum_{r\in R_+(j)}\omega_r\\&=\Big(\sum_{r\in R_-(j)}p^{(j)}_{r,0}-\sum_{r\in R_+(j)}\mu_r\Big)-\Big(\sum_{r\in R_+(j)}(\omega_r-\mu_r)\Big).\end{aligned} \tag{1.21}$$

Define a_j by $\Big(\sum_{r\in R_-(j)}p^{(j)}_{r,0}-\sum_{r\in R_+(j)}\mu_r\Big)^{1/2}$ and ϱ_j by $\Big(\sum_{r\in R_+(j)}(\omega_r-\mu_r)\Big)^{1/2}$. Since

$$(p_{j,0}-\omega_j)(p_{j,0}+\omega_j)=p_j^2-\mu_j^2\geqq 0$$

holds in the domain of integration of (1.19), $a_j^2-\varrho_j^2\geqq 0$ holds there. Since $\varrho_j\geqq 0$ holds, a_j is well-defined and non-negative.

We now invoke the following lemma, in order to make the calculation of (1.19) explicit.

Lemma 1.4. *We can introduce a coordinate system*

$$p'\equiv(\varrho_j,\Omega_{ij},\mathbf{p}_{r'})\quad(j=1,\ldots,n',i=1,\ldots,n(j)=3\#\{r;[j:r]=1\}-4,\quad r'\in R_f)^2 \tag{1.22}$$

on $\Big\{p\in\mathbb{R}^{4M};\,p_{r,0}>0\ (r=1,\ldots,M),\ p_r^2=\mu_r^2\ (r\in R_0)$ *and* $\sum_{r=1}^{M}[j:r]p_{r,j}=0\ (j=1,\ldots,n')\Big\}$ *so that the following hold:*

$$\prod_{r\in R_g(j)}\frac{d^3\mathbf{p}_r}{2\omega_r}=J_j\varrho_j^{n(j)}d\varrho_j\prod_{i=1}^{n(j)}d\Omega_{ij}\,; \tag{1.23}$$

the $p_{r,\mu}(p')$ *and* $J_j(p')$ *are real-valued analytic functions of* $p'\equiv(\varrho_j,\Omega_{ij},\mathbf{p}_{r'})$; *and*

$$a_j^2=\Big(\sum_{r\in R_-(j)}p^{(j)}_{r,0}-\sum_{r\in R_+(j)}\mu_r\Big) \tag{1.24}$$

is real analytic in $(\varrho_{j'},\Omega_{ij'},\mathbf{p}_{r'})$ $(j'<j,\ r'\in R_f)$.

2 Note that $n(j)\geqq 2$ by the assumption that D is a stable-particle diagram, i.e., $\#\{r;[j:r]=1\}\geqq 2$, where $\# A$ denotes the number of elements in set A

Proof of Lemma 1.4. The set $R_+(j)$ defined in (1.12) is the set of indices r that label the lines that are incoming at vertex V_j. For each j let r_j and Ω_{ij} $(i=1, \ldots, n(j))$ be the radial coordinates for the set of vectors $\mathbf{p}_r$ $(r \in R_+(j))$, restricted by the momentum conservation law condition (1.14). Thus

$$r_j^2 = \sum_{r \in R_+(j)} (\mathbf{p}_r)^2 \tag{1.25}$$

and

$$\mathbf{p}_r = r_j \mathbf{f}_{rj}(\Omega_{ij}) \qquad (r \in R_+(j)). \tag{1.26}$$

The variables Ω_{ij} are the angular coordinates for the $n(j)$-dimensional spherical surface defined by the intersection of the $(n(j)+3)$-dimensional spherical surface $\{r_j^2 = 1\}$ with the codimension 3 subspace

$$\left\{ \sum_{r \in R_+(j)} \mathbf{p}_r = 0 \right\}.$$

The functions $\mathbf{f}_{rj}(\Omega_{ij})$ are then real linear combinations of products of sines and cosines of the angles Ω_{ij}.

Let J_j be the Jacobian of the transformation shown in (1.23). Let J'_j be the similar Jacobian when r_j is used in place of ϱ_j. It is, apart from a constant factor and factors $2\omega_r$, just the Jacobian of the transformation from a set of $n(j)+1$ rectangular coordinates $\mathbf{p}_r$ $(r \in R_+(j))$ to the set of radial coordinates (r_j, Ω_{ij}). Thus it is, apart from the factors $2\omega_r$, a real analytic function of the angles Ω_{ij}.

Equation (1.26) shows that

$$\varrho_j(r_j, \Omega_{ij}) \equiv \left(\sum_{r \in R_+(j)} \left(\sqrt{\mu_r^2 + (\mathbf{p}_r)^2} - \mu_r \right) \right)^{1/2} \tag{1.27}$$

is, for fixed Ω_{ij}, a monotonically increasing real analytic function of r_j that is of the form $c_j r_j$ with $c_j \neq 0$ near $r_j = 0$. Thus the inverse mapping

$$r_j = r_j(\varrho_j, \Omega_{ij}) \tag{1.28}$$

is also real analytic, and the set of real variables consisting of all r_j, Ω_{ij}, and $\mathbf{p}_{r'}$ for $r' \in R_f$ is holomorphically equivalent to the set p' obtained by replacing each r_j by ϱ_j. Consequently, the functions $\mathbf{p}(p')$ and $J_j(p')$ are real and analytic over the domain of integration.

It will now be shown that each $p_{j,0}$ is a real analytic function of the set of variables

$$S_j \equiv \{\varrho_{j'}, \Omega_{ij'}, \mathbf{p}_{r'}; j' \leqq j, r' \in R_f\}. \tag{1.29}$$

This will be shown by proving by induction that the energy conservation law (1.20) can be written in the form

$$p_{j,0}(S_j) = \sum_{r \in R_-(j)} p_{r,0}^{(j)}(S_{j-1}) - \sum_{r \in R_g(j)} \omega_r(S_j), \tag{1.30}$$

where the functions on the right-hand side are real analytic functions of the indicated variables over the domain of integration.

Equations (1.26) and (1.28) give

$$\omega_r \equiv (\mu_r^2 + \mathbf{p}_r^2)^{1/2} = \omega_r(\varrho_j, \Omega_{ij}) \qquad (r \in R_g(j)) \tag{1.31}$$

with a real analytic right-hand side. Thus the second term on the right-hand side of (1.30) has the indicated form. For $r \in R_g(j)$ the mass-shell constraint gives $p_{r,0} = \omega_r$. Thus (1.30) will be true if the first term on the right-hand side has the indicated form.

Suppose h is some integer $1 \leqq h \leqq n'$, and that (1.30) holds for all $j \leqq h-1$. Then

$$p_{j,0} = p_{j,0}(S_j) \qquad (j \leqq h-1). \tag{1.32}$$

This equation is trivially true for $h=1$. Assuming (1.32) we shall show that the first term of the right-hand side of (1.30) has the required form for $j=h$. Since $S_{j-1} \subset S_j$ this Eq. (1.30) with $j=h$ then defines $p_{r,0}(S_j)$ for $j=h$, and (1.32) becomes true with h replaced by $h+1$.

Equations (1.26), (1.28), and (1.29) give, for any j and $r \in R_+(j)$

$$\mathbf{p}_r = \mathbf{p}_r(\varrho_j, \Omega_{ij}) = \mathbf{p}_r(S_j) \qquad (r \in R_+(j)) \tag{1.33}$$

with real analytic right-hand side. Then Eqs. (1.31), (1.32), and (1.33) give

$$p_r = p_r(S_j) \qquad (r \in R_j) \qquad (j \leqq h-1) \tag{1.34}$$

with real analytic right-hand side. This result and the mapping (1.17) give

$$p_r^{\mathrm{orig}} = p_r^{\mathrm{orig}}(S_j) \qquad (r \in R_j) \qquad (j \leqq h-1) \tag{1.35}$$

with real analytic right-hand side. The Lorentz boost B_j is a real analytic function of

$$P_j^{\mathrm{out}} = \sum_{r \in R_-(j) \subset R_{j-1}} p_r^{\mathrm{orig}}. \tag{1.36}$$

Hence (1.35), with j replaced throughout by $j-1$, gives

$$B_j = B_j(S_{j-1}) \qquad (j \leqq h) \tag{1.37}$$

with real analytic right-hand side. This result and (1.35) give

$$p_r^{(j)} \equiv B_j p_r^{\mathrm{orig}} = p_r^{(j)}(S_{j-1}) \qquad (r \in R_{j-1}) \qquad (j \leqq h) \tag{1.38}$$

with real analytic right-hand side. This result gives

$$\sum_{r \in R_-(j) \subset R_{j-1}} p_{r,0}^{(j)} = \sum_{r \in R_-(j) \subset R_{j-1}} p_{r,0}^{(j)}(S_{j-1}) \qquad (j \leqq h) \tag{1.39}$$

with real analytic right-hand side. Thus the first term on the right-hand side of (1.30) has the required form. Since $S_{j-1} \subset S_j$, the right-hand side of (1.30) can be used to define left-hand side. This gives (1.32) for $j=h$. Thus by finite induction the limitations involving h can be removed and (1.34) gives

$$p_r = p_r(S_j) \qquad (r \in R_j) \tag{1.40}$$

with real analytic right-hand side. Similarly, (1.39) and the definition (1.24) of a_j give

$$a_j^2 \equiv \sum_{r\in R_-(j)} p_{r,0}^{(j)}(S_{j-1}) - \sum_{r\in R_+(j)} \mu_r = a_j^2(S_{j-1}) \tag{1.41}$$

with real analytic right-hand side. This completes the proof of Lemma 1.4. Q.E.D.

Continuation of the Proof of Theorem 1.3. Using Lemma 1.4 we can rewrite integral (1.18) in the form

$$\begin{aligned} I &\equiv I(f;\lambda) \\ &= \int f(p') \prod_{j=1}^{n'} \left[\frac{1}{\Gamma(\lambda_j+1)} (a_j-\varrho_j)_+^{\lambda_j}(a_j+\varrho_j)_+^{\lambda_j} \right. \\ &\quad \left. \cdot (p_{j,0}+\omega_j)^{\lambda_j} Y(p_{j,0}) J_j \varrho_j^{n(j)} d\varrho_j \prod_{i=1}^{n(j)} d\Omega_{ij} \right] \prod_{r\in R_f} \frac{d^3\mathbf{p}_r}{2\omega_r}. \end{aligned} \tag{1.42}$$

By setting $\varrho_j = a_j \varrho_j'$ we further rewrite (1.42):

$$\begin{aligned} I &= \int \left(\int_0^1 \dots \int_0^1 \tilde{\varphi}(p') \prod_{j=1}^{n'} \left[\frac{1}{\Gamma(\lambda_j+1)} (a_j^2)_+^{(2\lambda_j+n(j)+1)/2} \right. \right. \\ &\quad \left. \left. \cdot (1-\varrho_j')^{\lambda_j}(1+\varrho_j')^{\lambda_j} d\varrho_j' \prod_{i=1}^{n(j)} d\Omega_{ij} \right] \prod_{r\in R_f} \frac{d^3\mathbf{p}_r}{2\omega_r} \right). \end{aligned} \tag{1.43}$$

Here $\tilde{\varphi}(p') \equiv f'(p') \prod_{j=1}^{n'} (p_{j,0}+\omega_j)^{\lambda_j} Y(p_{j,0}) J_j(\varrho_j')^{n(j)}$, with $f'(p') \equiv f(p)$. Recall that $a_j^2 = \sum_{r\in R-(j)} p_{r,0} - \sum_{r\in R+(j)} \mu_r$ is an analytic function of $(\varrho_{j'}, \Omega_{ij'}, \mathbf{p}_r)$ $(j'<j, r\in R_f)$. The transformation of I into the form (1.43) is legitimate, because $n(j) \geqq 2$. We shall apply the following Lemma 1.5 to (1.43).

Lemma 1.5. *Let* $\varphi(x) \equiv \varphi(x_1, x')$ *be a continuous function defined on* $L \equiv \{x_1; 0 \leqq x_1 \leqq 1\} \times K$ *for a compact set* K. *Assume that* $\left|\frac{\partial\varphi}{\partial x_1}\right|$ *is bounded and measurable on* L. *Then, for* λ *with* $\mathrm{Re}\,\lambda > -1$, *we have*

$$\begin{aligned} &\int_0^1 (\lambda+1)(1-x_1)^{\lambda} \varphi(x_1, x') dx_1 \\ &\quad = \varphi(1, x') - \int_0^1 (1-(1-x_1)^{\lambda+1}) \frac{\partial\varphi}{\partial x_1} dx_1. \end{aligned} \tag{1.44}$$

Proof of Lemma 1.5. Since

$$\frac{\partial}{\partial x_1}((1-(1-x_1)^{\lambda+1})\varphi) = (1-(1-x_1)^{\lambda+1}) \frac{\partial\varphi}{\partial x_1} + (\lambda+1)(1-x_1)^{\lambda}\varphi$$

holds, we have

$$\varphi(1, x') = \int_0^1 (1-(1-x_1)^{\lambda+1}) \frac{\partial\varphi}{\partial x_1} dx_1 + (\lambda+1) \int_0^1 (1-x_1)^{\lambda} \varphi dx_1. \quad \text{Q.E.D.}$$

The right-hand side of (1.44) is well-defined for λ with $\mathrm{Re}\lambda > -2$. Hence we can define the analytic continuation of the integral given by the left-hand side of (1.44) by the right-hand side of (1.44). Moreover, the second term on the right-hand side disappears for $\lambda = -1$.

To apply Lemma 1.5 to (1.43), it suffices to show that

$$\left|\frac{\partial}{\partial \varrho_1'}\left(\tilde{\varphi} \prod_{j=1}^{n'} (a_j^2)_+^{(2\lambda_j+n(j)+1)/2}\right)\right|$$

is bounded on the interval $0 \leqq \varrho_1' \leqq 1$. Recall that $a_j^2(\varrho_{j'})$ is an analytic function of the $\varrho_{j'}$ for $j' < j$, that $\varrho_j = \varrho_j' a_j$, and that $\tilde{\varphi}$ is C^∞ in the ϱ_j. It follows from these conditions that the above function is indeed bounded provided $\mathrm{Re}\lambda_j > 0$ for $j \geqq 2$, and $\mathrm{Re}\lambda_1 > -3/2$. To show this for the terms obtained by differentiating $\tilde{\varphi}$ one uses the general formulas

$$\frac{\partial \tilde{\varphi}}{\partial \varrho_j'} = \sum_{i=1}^{n'} \frac{\partial \tilde{\varphi}}{\partial \varrho_i} \frac{\partial \varrho_i}{\partial \varrho_j'}$$

and

$$\begin{aligned} \frac{\partial \varrho_i}{\partial \varrho_j'} &= \frac{\partial(\varrho_i' a_i(\varrho_k))}{\partial \varrho_j'} \qquad (k < i) \\ &= 0 \qquad\qquad \text{if} \quad i < j \\ &= a_i \qquad\qquad \text{if} \quad i = j \\ &= a_i + \sum_{k=j}^{i-1} \frac{1}{2a_i} \frac{\partial a_i^2}{\partial \varrho_k} \frac{\partial \varrho_k}{\partial \varrho_j'} \qquad \text{if} \quad i > j. \end{aligned}$$

Repeated use of the second formula can produce in some terms a factor as singular as $\prod_{i=2}^{n'} (2a_i)^{-1}$. But this factor is more than compensated for by the factor $\prod_{j=2}^{n'} (a_j^2)_+^{(2\lambda_j+n(j)+1)/2}$, provided $\mathrm{Re}\lambda_j > 0$ for $j \geqq 2$. If $\mathrm{Re}\lambda_j > 0$ for $j \geqq 2$ then

$$\begin{aligned} \frac{\partial}{\partial \varrho_i'} (a_j^2)_+^{(2\lambda_j+n(j)+1)/2} &= \frac{(2\lambda_j + n(j) + 1)}{2} (a_j^2)_+^{(2\lambda_j+n(j)-1)/2} \\ &\quad \cdot \left(\sum_{k=i}^{j-1} \frac{\partial(a_j^2)}{\partial \varrho_k} \frac{\partial \varrho_k}{\partial \varrho_i'} \right) \end{aligned} \tag{1.45}$$

is likewise bounded, for similar reasons. The factor $(a_1^2)_+^{(2\lambda_1+n(1)+1)/2}$ does not depend on the ϱ_j's. It remains bounded provided $\mathrm{Re}\lambda_1 > -3/2$. Therefore, if we define a segment $\{\gamma_1(t)\}_{0 \leqq t \leqq 1}$ in $\mathbb{C}^{n'}$ by

$$\gamma_1(t) = (-2t+1, \lambda_2, \ldots, \lambda_{n'}),$$

where $\mathrm{Re}\lambda_j > 0, j = 2, \ldots, n'$, then $I(f; \lambda)$ can be analytically continued from a point in its original domain of definition to a point whose first coordinate is equal to -1.

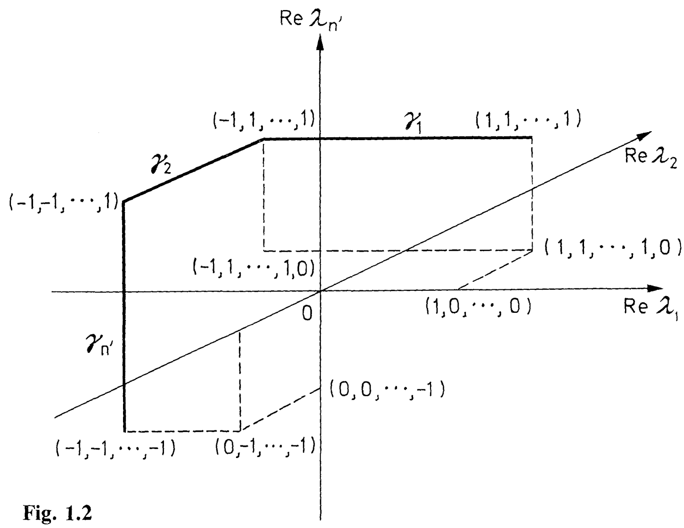

Fig. 1.2

Then we can use (1.44) to determine the value of the integral $I(f;\lambda)$ at $\lambda_1=-1$. It gives

$$I(f;\gamma_1(t))|_{t=1}=\tfrac{1}{2}\int\Bigg[\int_0^1\dots\int_0^1(\tilde{\varphi}(p')|_{\varrho_1'=1})a_1^{n(j)-1}$$
$$\cdot\prod_{j=2}^{n'}\frac{1}{\Gamma(\lambda_j+1)}(a_j^2)_+^{(2\lambda_j+n(j)+1)/2}$$
$$\cdot(1-\varrho_j')^{\lambda_j}(1+\varrho_j')^{\lambda_j}d\varrho_j'\Bigg]\prod_{j=1}^{n'}\prod_{i=1}^{n(j)}d\Omega_{ij}\prod_{r\in R_f}\frac{d^3\mathbf{p}_r}{2\omega_r}. \tag{1.46}$$

Since the same argument applies to (1.46), we can find successively a sequence of segments $\gamma_1(t),\dots,\gamma_{n'}(t)$ (see Fig. 1.2 above) along which $I(f;\lambda)$ can be analytically continued, and we obtain, finally,

$$I(f;\lambda^0)=(\tfrac{1}{2})^{n'}\int(\tilde{\varphi}(p')|_{\varrho_1'=\dots=\varrho_{n'}'=1})$$
$$\cdot\prod_{j=1}^{n'}(a_j^2)_+^{(n(j)-1)/2}\prod_{i=1}^{n(j)}d\Omega_{ij}\prod_{r\in R_f}\frac{d^3\mathbf{p}_r}{2\omega_r}. \tag{1.47}$$

On the other hand, by making use of the coordinate system introduced by Lemma 1.4, we can write

$$\tilde{\Phi}_D(p)\equiv\Phi_D(p)\prod_{r=1}^{M}d^4p_r\equiv\prod_{r=1}^{M}\delta(p_r^2-\mu_r^2)Y(p_{r,0})$$
$$\cdot\prod_{j=1}^{n'}\delta^4\left(\sum_{r=1}^{M}[j:r]p_r\right)\prod_{r=1}^{M}d^4p_r$$

in the form

$$\tilde{\Phi}_D(p)=\prod_{j=1}^{n'}\left[\delta(a_j^2-\varrho_j^2)Y(a_j)Y(\varrho_j)(p_{j,0}+\omega_j)^{-1}J_j\varrho_j^{n(j)}d\varrho_j\prod_{i=1}^{n(j)}d\Omega_{ij}\right]$$

$$\cdot\prod_{r\in R_f}d^3p_r/2\omega_r\,. \tag{1.48}$$

To evaluate this expression one can apply the standard formula

$$\prod_{i=1}^{n}[\delta(f_i(x))dx_i]=\|\partial f_i/\partial x_j\|^{-1},$$

which holds wherever the determinant on the right-hand side is nonzero. Since a_j^2 is an analytic function of the set of variables S_{j-1}, which includes none of the variables ϱ_i for $i\geqq j$, this formula gives

$$\prod_{j=1}^{n'}[\delta(a_j^2-\varrho_j^2)d\varrho_j]=\left\|\left[\prod_{j=1}^{n'}(2\varrho_j)\right]^{-1}\right\|\equiv J^{-1}\,. \tag{1.49}$$

The determinant J on the right-hand side vanishes if any ϱ_j is zero. Hence the contribution to this distribution from points where any ϱ_j vanishes is initially not well defined. However, the ill-defined functions $|\prod(2\varrho_j)|^{-1}=J^{-1}$ occurs multiplied by $\prod\varrho_j^{n(j)}$. Since each exponent $n(j)$ is at least 2 the zeros of this latter factor more than compensate for the poles of J^{-1}. Hence $\tilde{\Phi}_D(p)$ is naturally defined by inserting (1.49) into (1.48) and then using continuity to define the value of

$$\left(\frac{1}{2^{n'}}\right)\prod_{j=1}^{n'}Y(a_j)Y(\varrho_j)(p_{j,0}+\omega_j)^{-1}J_j\varrho_j^{n_j-1}$$

to be zero when any $\varrho_j=0$. This definition assigns a null contribution to the distribution $\Phi_D(p)$ from this set of zero measure where $J=0$. Then a comparison of (1.48) to (1.47) shows that $\Phi_D(p)=\Phi_D(p;\lambda^0)$. This completes the proof of Theorem 1.3. Q.E.D.

With the analytic continuation now completed, what remains to be shown is the following.

Lemma 1.6. *Let $f(x,\lambda)$ be a hyperfunction defined on $M\times\Omega$, where M is an analytic manifold and Ω is a connected open set in $\mathbb{C}$. Assume that f depends holomorphically on λ. Assume that $(x^0,\lambda^0;\ \sqrt{-1}(\xi^0,0))$ $(\in\sqrt{-1}S^*(M\times\Omega))$ does not belong to* S.S. $f(x,\lambda)$. *Then $(x^0,\lambda';\ \sqrt{-1}(\xi^0,0))$ does not belong to* S.S. $f(x,\lambda)$ *for any $\lambda'\in\Omega$.*

See Theorem 2.2.8 of S-K-K [13], Chapt. III for the proof of this lemma.

Applying Lemma 1.6 and Theorem 2.2.6 of S-K-K [6], Chapt. I successively to $\Phi_D(p;\lambda)$ considered in a neighborhood of γ_j, we see that the singularity spectrum of $\Phi_D(p;\lambda^0)$ is confined to the set of points which satisfy condition (1.6).

Furthermore, the preceding arguments show us that $\Phi_D(p;\lambda^0)$ is annihilated by any of the multiplication operator $(p_r^2-\mu_r^2)$ $(r=1,\dots,M)$. Hence the additional condition "$p_r^2\geqq\mu_r^2$" in (1.6b) can be replaced by "$p_r^2=\mu_r^2$". Hence we have the following.

Theorem 1.7. *The singularity spectrum of $\Phi_D(p;\lambda^0)$ is confined to the set of points $(p;\sqrt{-1}u)$ which satisfy condition* (1.6) *with the replacement of the condition in* (1.6b) *by a stronger condition $p_r^2=\mu_r^2$.*

Here we return to the original notations, i.e., we abandon the convention (1.3). Then, applying Theorem 2.3.1 of S-K-K [6], Chapt. I to $\int \Phi_D(p,k;\lambda^0)dk$ we obtain the following final result. Note that this integral is a proper integral.

Theorem 1.8. *The singularity spectrum of phase space integral $I_D(p)$ is confined to the following set $\tilde{\mathscr{L}}(D)$:*

$\tilde{\mathscr{L}}(D)=\{(p;\sqrt{-1}u)\in\sqrt{-1}S^*\mathbb{R}^{4n}$; *there exist real four-vectors k_l $(l=1,\dots,N)$ and sequences of complex scalars $\alpha_l^{(m)}$ $(l=1,\dots,N)$ and $\beta_r^{(m)}$ $(r=1,\dots,n)$ and complex four-vectors $p_r^{(m)}$, $u_r^{(m)}$ $(r=1,\dots,n)$, $k_l^{(m)}$ $(l=1,\dots,N)$ and $v_j^{(m)}$ $(j=1,\dots,n')$ which satisfy the following relations* (1.50)$\}$

$$\left\{\begin{array}{ll}
p_r^{(m)}\to p_r \quad \text{with} \quad p_{r,0}>0, \quad p_r^2=\mu_r^2 \quad (r=1,\dots,n), & \text{(1.50a)}\\
u_r^{(m)}\to u_r \quad (r=1,\dots,n), & \text{(1.50b)}\\
k_l^{(m)}\to k_l \quad \text{with} \quad k_{l,0}>0, \quad k_l^2=m_l^2 \quad (l=1,\dots,N), & \text{(1.50c)}\\
\alpha_l^{(m)}((k_l^{(m)})^2-m_l^2)\to 0 \quad (l=1,\dots,N), & \text{(1.50d)}\\
\beta_r^{(m)}((p_r^{(m)})^2-\mu_r^2)\to 0 \quad (r=1,\dots,n), & \text{(1.50d')}\\
\sum_{r=1}^{n}[j:r]p_r^{(m)}+\sum_{l=1}^{N}[j:l]k_l^{(m)}=0 \quad (j=1,\dots,n'), & \text{(1.50e)}\\
u_r^{(m)}=-\sum_{j=1}^{n'}[j:r]v_j^{(m)}-\beta_r^{(m)}p_r^{(m)} \quad (r=1,\dots,n), & \text{(1.50f)}\\
\sum_{j=1}^{n'}[j:l]v_j^{(m)}-\alpha_l^{(m)}k_l^{(m)}\to 0 \quad (l=1,\dots,N). & \text{(1.50g)}
\end{array}\right.$$

The arguments given above show that $I_D(p;\lambda^0)=I_D(p)$ is annihilated by the multiplication operators $(p_r^2-\mu_r^2)$ $(r=1,\dots,n)$ and $\sum_{j,r}[j:r]p_r$. Therefore it has the form

$$\delta^4\Big(\sum_{j,r}[j:r]p_r\Big)\prod_{r=1}^{n}\delta^+(p_r^2-\mu_r^2)\varphi_D(p), \tag{1.51}$$

if

$$\{p_r^2-\mu_r^2=0\} \quad (r=1,\dots,n) \quad \text{and} \quad \Big\{\sum_{j,r}[j:r]p_r=0\Big\} \tag{1.52}$$

cross normally.

Condition (1.52) is satisfied if p is not contained in $\mathscr{M}_{\text{exc}}=\{p\in\mathscr{M}_r$; all p_r's are parallel$\}$. Here $\mathscr{M}_r$ denotes the reduced mass-shell manifold, i.e.

$$\Big\{p\in\mathbb{R}^{4n};\ \sum_{j,r}[j:r]p_r=0,\ \ p_r^2=\mu_r^2\ \ (r=1,\dots,n)\Big\}.$$

In this case we may discuss the singularity spectrum of $\varphi_D(p)$ defined on $\mathcal{M}' \equiv \mathcal{M}_r - \mathcal{M}_{\text{exc}}$.

First note that the inclusion map from $\mathcal{M}'$ to $N \equiv \mathbb{R}^{4n} - \mathcal{M}_{\text{exc}}$ induces the exact sequence

$$0 \to T^*_{\mathcal{M}'}N \to T^*N \underset{N}{\times} \mathcal{M}' \to T^*\mathcal{M}' \to 0. \tag{1.53}$$

In other words, each point in $T^*\mathcal{M}'$ can be represented by $(p;u) \in \mathbb{R}^{8n}$ with the following equivalence relations:

$(p;u)$ and $(p';u')$ represent the same point in $T^*\mathcal{M}'$ if and only if both, (1.54)

$$\begin{cases} p = p' & (1.55a) \\ \text{and} \\ u_r - u'_r = -[j(r):r]a - \beta_r p_r \quad (r = 1, \ldots, n) & (1.55b) \end{cases}$$

hold for some real four-vector a and real scalars β_r $(r = 1, \ldots, n)$.

Making use of this convention, we conclude from (the proof of) Lemma 0.2 of Kawai-Stapp [12] that the following two statements are equivalent.

$$\begin{cases} (p; \sqrt{-1}u) \in \text{S.S.}\, \varphi_D(p) \subset \sqrt{-1}S^*\mathcal{M}', & (1.56a) \\ (p; \sqrt{-1}u) \in \text{S.S.}\, I_D(p) \cap \sqrt{-1}(S^*N - S^*_{\mathcal{M}'}N). & (1.56b) \end{cases}$$

Here $(p; \sqrt{-1}u)$ in (1.56a) should be understood according to the convention stated above. In other words, the variety $\tilde{\mathscr{L}}(D)$ defined in Theorem 1.8 defines a variety in $\sqrt{-1}S^*\mathcal{M}'$ by just the same relations (1.50) on the understanding of the above equivalence relations. Thus we obtain

Theorem 1.9. *The singularity spectrum of $\varphi_D(p)$, which is well-defined on $\mathcal{M}'$, is confined to $\tilde{\mathscr{L}}(D)$ considered in $\sqrt{-1}S^*\mathcal{M}'$.*

Remark 1. Since the set of points $(p;u)$ which satisfy conditions (1.6) is contained in the real locus of a Lagrangian variety (Kashiwara-Kawai [26], Proposition 15), the set defined by (1.50) is also contained in the real locus of a Lagrangian variety (Kashiwara-Kawai [10], Lemma 5). Hence the projection of the set defined by (1.50) to the base manifold $\mathcal{M}'$ has at least codimension 1: it contains no open set.

Remark 2. Condition (1.50g) is equivalent to the existence of a set of complex vectors $(\hat{u}_l^{(m)}, \delta_{l,+}^{(m)}, \delta_{l,-}^{(m)})$ and complex numbers $\beta_{l,+}^{(m)}$ and $\beta_{l,-}^{(m)}$ such that

$$\hat{u}_l^{(m)} = v_{j_\pm(l)}^{(m)} - \beta_{l,\pm}^{(m)} k_l^{(m)} + \delta_{l,\pm}^{(m)} \quad (l = 1, \ldots, N) \tag{1.50g1}$$

and

$$\delta_{l,\pm}^{(m)} \to 0 \quad (l = 1, \ldots, N), \tag{1.50g2}$$

where $[j_\pm(l):l] = \pm 1$.

Without loss of generality we may also impose the conditions

$$u_{r,0}^{(m)} = 0 \quad (r = 1, \ldots, n) \tag{1.50h1}$$

and

$$\hat{u}_{l,0}^{(m)}=0 \quad (l=1,\dots,N). \tag{1.50h2}$$

The introduction of $\hat{u}_l^{(m)}$ and of the normalizations (1.50h1) and (1.50h2) are convenient in the discussion in §3 (cf. Definition 3.1).

As mentioned in §0, a gap in the proof of Lemma 2 of [9] was discovered and this obliged us to change the statement of our results from that given in our announcement [14]. At the same time, we mention that in [14] we omitted conditions (1.50d) and (1.50d′) in defining the extended Landau variety $\tilde{\mathscr{L}}(D)$ in order to be able to state a result involving only real quantities. However, omitting conditions (1.50d) and (1.50d′) results in the failure of the claim of the Lagrangian character of $\tilde{\mathscr{L}}(D)$. Hence the first two lines of p. 143 of [14] are incorrect, as they are stated there. Theoretically this defect is too serious to be compensated by the other advantages, even though it is not so serious when we apply our results to concrete problems (see §3 and Iagolnitzer-Stapp [8]). Hence we have decided to restore the conditions (1.50d) and (1.50d′) in this paper in defining the extended Landau variety.

Although a correct proof of Lemma 2 of [9] has not yet been constructed, the claim itself is still believed to be true (Kashiwara-Kawai [26], §7, Conjecture). This conjecture yields the following conjecture:

Conjecture. *In the definition* (1.50) *of* $\tilde{\mathscr{L}}(D)$, *we can assume in addition that* $\alpha_l^{(m)}$ $(l=1,\dots,N)$ *and* $\beta_r^{(m)}$ $(r=1,\dots,n)$ *are real.*

§2. The Causal Part of $\tilde{\mathscr{L}}(D)$

The principle of macrocausality states that momentum-energy is transferred over macroscopic distances only by stable particles. More specifically, it states that any transfer of momentum-energy that cannot be attributed to a space-time network of stable particles occurs with a probability that falls off exponentially under space-time dilation. This principle can be formulated in a precise and natural way at non $u=0$ points, and entails, at these points, that the essential support of the S-matrix, and hence also its singularity spectrum ([21, 22]), is confined to the union of the positive-α Landau varieties $\mathscr{L}^+(D)$. These varieties $\mathscr{L}^+(D)$ are defined in the same way as $\tilde{\mathscr{L}}(D)$, but with all quantities real, with the parameters α restricted to positive values, and with all quantities independent of m (i.e., with no limiting procedure).

At $u=0$ points two or more of the momentum-energy vectors p_r are parallel. This leads to complications in the physical arguments, and to the need to consider limiting procedures. This was discussed in detail in the preprint version of this paper. [IAS (Princeton) Preprint, April 1978. See also the report of T. Kawai and H. P. Stapp at the symposium "Hyperfunctions and linear differential equations V." held at RIMS, Kyoto Univ. (Japan) from Oct. 13, 1976 through Oct. 16, 1976, RIMS Kôkyûroku, No. 287, pp. 170–182.] But the arguments are too lengthy to be presented here. Rather, to be more brief, we shall define the causal part of $\tilde{\mathscr{L}}(D)$

by adding to the Eqs. (1.50) that define $\tilde{\mathscr{L}}(D)$ the two extra conditions

$$\operatorname{Im} v_j^{(m)} \to 0 \quad (\text{all } j) \tag{2.1}$$

and

$$\operatorname{Re} \alpha_l^{(m)} > 0 \quad (\text{all } l, m). \tag{2.2}$$

These two conditions ensure that each point in the causal part of $\tilde{\mathscr{L}}(D)$ is associated with a sequence of diagrams that approach a sequence of real space-time networks of stable particles. The existence of such a sequence of real space-time networks leads to a breakdown of the arguments whereby one concludes from the macrocausality principle that the limit point (p, u) is absent from the singularity spectrum. Thus macrocausality cannot exclude the causal parts of the sets $\tilde{\mathscr{L}}(D)$ from the singularity spectrum of the S-matrix. On the other hand, the sequences that generate the non-causal parts of $\tilde{\mathscr{L}}(D)$ do not correspond, directly at least, to sequences of real space-time networks. Hence these sequences should not, according to macrocausality principle, be associated with singularities of the S-matrix.

This argument assigns special status to the sequences defined by the modified Landau equations. On the other hand, there might be other ways to characterize the singularity spectra of the phase space integrals, that would lead, via the same reasoning, to different conclusions. Our conjecture therefore rest ultimately on the naturalness of the modified Landau equations, within the framework of holonomic functions. Within this framework our conjectures appear likely to be compatible with unitarity, but this must eventually be shown to be the case.

Condition (2.1) entails

$$\operatorname{Im}(\alpha_l^{(m)} k_l^{(m)}) \to 0 \quad (\text{all } l) \tag{2.3}$$

or equivalently

$$\operatorname{Re} \alpha_l^{(m)} \operatorname{Im} k_l^{(m)} + \operatorname{Im} \alpha_l^{(m)} \operatorname{Re} k_l^{(m)} \to 0. \tag{2.4}$$

The imaginary part of (1.50d) gives

$$\operatorname{Im} \alpha_l^{(m)} [(\operatorname{Re} k_l^{(m)})^2 - (\operatorname{Im} k_l^{(m)})^2 - m_l^2] + 2 \operatorname{Re} \alpha_l^{(m)} \operatorname{Im} k_l^{(m)} \operatorname{Re} k_l^{(m)} \to 0 \quad (\text{all } l). \tag{2.5}$$

Equations (2.4), (2.5) and (1.50c) imply

$$\operatorname{Im} \alpha_l^{(m)} \to 0 \quad (\text{all } l). \tag{2.6}$$

But for each sequence that satisfies (1.50) and (2.6) one can construct, by setting $\operatorname{Im} \alpha_l^{(m)} = 0$, another sequence that satisfies these conditions and has the same limiting values (p, u, k). Similarly $\operatorname{Im} \beta_r^{(m)}$ can be set to zero. Thus the causal part of $\tilde{\mathscr{L}}(D)$ is contained in $\tilde{\mathscr{L}}^+(D)$, which is defined by Eqs. (1.50) augmented by the condition that $\alpha_l^{(m)}$ be real and positive, and the $\beta_r^{(m)}$ be real.

Conversely, $\tilde{\mathscr{L}}^+(D)$ is contained in the causal part of $\tilde{\mathscr{L}}(D)$, since the conditions that the $\alpha_l^{(m)}$ be real and positive and that the $\beta_r^{(m)}$ be real, together with the equations (1.50), entail (2.1) and (2.2). Thus the causal part of $\tilde{\mathscr{L}}(D)$ can be identified with $\tilde{\mathscr{L}}^+(D)$.

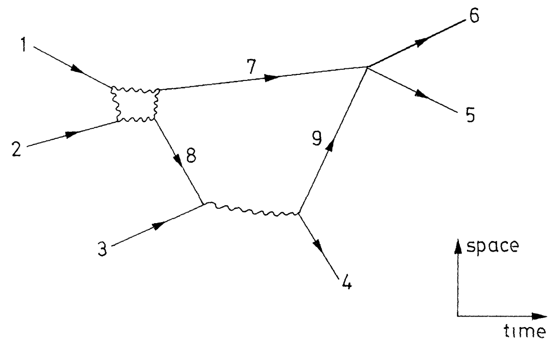

Fig. 2.1. A space-time diagram showing a typical network of mechanisms. This network transfers the energy-momentum of the initial particles 1, 2, and 3 to the final particles 4, 5, and 6. Momentum-energy is conserved at each vertex

§3. Generalized Landau Equation

On the basis of the considerations given in §1 and §2, we introduce the following conjecture on the micro-analyticity of the (on-shell) S-matrix (cf. Pham [23], Sato [11], Kawi-Stapp [12] and references cited there).

Conjecture. *The singularity spectrum of the S-matrix is confined to the set*

$$\tilde{\mathscr{L}}^+ \equiv \bigcup_D \tilde{\mathscr{L}}^+(D),$$

where D runs over all partially ordered Landau diagrams, and $\tilde{\mathscr{L}}^+(D)$ is defined in the same way as $\tilde{\mathscr{L}}(D)$ of § 1, but with the additional condition that all $\alpha_l^{(m)}$ be positive and all $\beta_r^{(m)}$ be real.

This conjecture on the singularity spectrum of the S-matrix is a special case of the following conjecture on the singularity spectrum of general bubble diagram functions:

Conjecture. *The singularity spectrum of $F_B(p)$ is confined to the set*

$$\tilde{\mathscr{L}}^B \equiv \bigcup_{D \in \bar{B}} \tilde{\mathscr{L}}^\sigma(D)$$

where $\bar{B}$ is the set of Landau diagrams that fit in B (see [7]) and $\tilde{\mathscr{L}}^\sigma(D)$ is defined in the same way as $\tilde{\mathscr{L}}^+(D)$ except that for every line l of D that lies inside a plus bubble of B one imposes the condition $\alpha_l^{(m)} > 0$, for every line l of D that lies inside a minus bubble of B one imposes the condition $\alpha_l^{(m)} < 0$, and one allows all other $\alpha_l^{(m)}$ to be either positive or negative.

This conjecture is a natural extension to $u=0$ points of the Structure Theorem proved in [1–7] for $u \neq 0$ points.

The extended Landau variety $\tilde{\mathscr{L}}^\sigma(D)$ is a natural object, both from the mathematical and physical viewpoints. But the need to consider limiting procedures and complex quantities makes the set $\tilde{\mathscr{L}}^\sigma(D)$ difficult to use in practical calculations. However, there is a partial characterization of $\tilde{\mathscr{L}}^\sigma(D)$ that involves only real quantities and no limiting procedures, and that is often adequate for practical purposes. This characterization is in terms of a variety $\mathscr{L}_g^\sigma(D)$ called the generalized Landau variety.

The generalized Landau variety $\mathscr{L}_g^\sigma(D)$ is defined by generalized Landau equations. Before defining these equations we first recall the definition of ordinary Landau equations. The expression of these equations given below is a little different from the usual one (e.g., the equations used in Kawai-Stapp [1]), but its content is the same.

Definition 3.1 (Landau equations). A set $(p_1, \dots, p_n; u_1, \dots, u_n) \equiv (p; u)$ consisting of n real four-vectors p_r and n real four-vectors u_r is said to be a solution of Landau equations associated with the Landau diagram D if and only if there are sets of real four-vectors k_l and $\hat{u}_l$ $(l=1, \dots, N)$ and v_j $(j=1, \dots, n')$ and real scalars $\beta_{l,+}$ and $\beta_{l,-}$ $(l=1, \dots, N)$ and β_r $(r=1, \dots, n)$ such that the following equations are satisfied:

$$\begin{cases} p_r^2 = \mu_r^2, \quad p_{r,0} > 0 & (r=1, \dots, n), & (3.1a) \\ k_l^2 = m_l^2, \quad k_{l,0} > 0 & (l=1, \dots, N), & (3.1b) \\ \sum_{r=1}^{n} [j:r] p_r + \sum_{l=1}^{N} [j:l] k_l = 0 & (j=1, \dots, n'), & (3.1c) \\ u_r = -[j(r):r](v_{j(r)} - \beta_r p_r) & (r=1, \dots, n), & (3.1d) \\ \hat{u}_l = v_{j_\pm(l)} - \beta_{l,\pm} k_l & (l=1, \dots, N), & (3.1e) \\ \sigma_l(\beta_{l,+} - \beta_{l,-}) \equiv \sigma_l \alpha_l \geqq 0 & \text{for all signed lines } l. & (3.1f) \end{cases}$$

The Landau variety $\mathscr{L}^\sigma(D)$ is the set of points $(p; \sqrt{-1}u)$ such that a solution $(p, u, k, \hat{u}, v, \beta)$ of these Landau equations exists. The positive-α Landau variety $\mathscr{L}^+(D)$ is defined in the same way except that (3.1f) is replaced by the conditions $\alpha_l \geqq 0$ for all l.

Definition 3.2. A solution of the Landau equations is called *star-shaped* if and only if all vertices and lines of the diagram D that represent this solution lie on a finite set of rays that originate from a single point P (see Fig. 3.0).

For any star-shaped solution s let J_s be the set of indices j such that V_j does not lie at P.

Definition 3.3 (Generalized Landau equations). The generalized Landau equations associated with a Landau diagram D consist of a set of alternative sets of equations. The first alternative set consists of the ordinary Landau equations

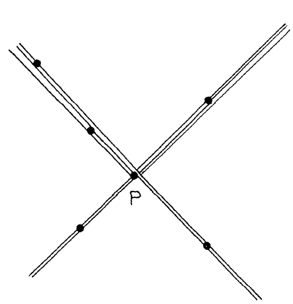

Fig. 3.0

associated with D. There is another alternative set for each star-shaped solution s of the ordinary Landau equations associated with D. This alternative set fixes the momentum-energy vectors p and k to be the same as in s, but allows the vertices to be shifted. In particular, a set of real displacement vectors $\eta_{j,r}$ and $\hat{\eta}_{j,l}$ is introduced, and the alternative set of equations consists of

(i) the condition that the p_r and k_l have the values that they have in s,

(ii) the ordinary Landau Eqs. (3.1a), (3.1b), (3.1c), and (3.1f), and

(iii) the equations

$$\begin{cases} u_r = -[j(r):r](v_{j(r)}+\eta_{j,r})-\beta_r p_r & (r=1,\dots,n), & (3.1\text{d}') \\ \hat{u}_l = v_{j+(l)}+\hat{\eta}_{j+(l),l}-\beta_{l,+}k_l & (l=1,\dots,N), & (3.1\text{e}'1) \\ \hat{u}_l = v_{j-(l)}+\hat{\eta}_{j-(l),l}-\beta_{l,-}k_l & (l=1,\dots,N), & (3.1\text{e}'2) \\ \eta_{j,r}=0 \quad \text{and} \quad \hat{\eta}_{j,l}=0 & (j\notin J_s) & (3.1\text{g}') \\ \text{and} & & \\ \sum_{r=1}^{n}[j:r](p_{r,0}(\eta_{j,r})_\nu - p_{r,\nu}(\eta_{j,r})_0) + \sum_{l=1}^{N}[j:l](k_{l,0}(\hat{\eta}_{j,l})_\nu - k_{l,\nu}(\hat{\eta}_{j,l})_0) & & \\ \quad =0 \quad (j\in J_s, \quad \nu=1,2,3). & & (3.1\text{h}') \end{cases}$$

The set of points $(p;\sqrt{-1}u)$ such that $(p;u)$ satisfies these generalized Landau equations is called $\mathscr{L}_g^\sigma(D)$. The positive-α generalized equations and the associated positive-α variety $\mathscr{L}_g^+(D)$ are defined in the same way with the sign condition (3.1f) replaced by the requirement that all α_l be nonnegative.

Definition 3.4. A point p is called a generalized $u=0$ point associated with D if and only if the generalized Landau Eqs. (3.1) associated with D have a $u=0$ solution $(p;u)=(p;0)$ in which some vertex not at the origin has incident upon it a pair of lines with nonparallel momentum-energy vectors. The set $\mathscr{L}_g^{\sigma,u=0}(D)$ is defined by $\mathscr{L}_g^{\sigma,u=0}(D)\equiv\{(p;\sqrt{-1}u);\, p$ is a generalized $u=0$ point associated with D and the set of signs $\sigma\equiv\{\sigma_l\}\}$.

Theorem 3.1. *Let D be any connected partially ordered Landau diagram. Then $\tilde{\mathscr{L}}^\sigma(D)$ is confined to $\mathscr{L}_g^\sigma(D)\cup\mathscr{L}_g^{\sigma,u=0}(D)$.*

Proof. Suppose $(p;\sqrt{-1}u)$ lies on the extended Landau variety $\tilde{\mathscr{L}}^\sigma(D)$. Then there is a sequence $(p_r^{(m)}, u_r^{(m)}, k_l^{(m)}, v_j^{(m)}, \alpha_l^{(m)}, \beta_r^{(m)})$ that satisfies (1.50), where the limits of $p^{(m)}$ and $u^{(m)}$ are the given vectors p and u. It will be shown that the existence of this sequence allows one either to construct a solution $(p_r, u_r, k_l, \hat{u}_l, v_j, \beta_r, \beta_{l,+}, \beta_{l,-}, \eta_{j,r}, \hat{\eta}_{j,l})$ of the associated generalized Landau equations, or to show that p is a generalized $u=0$ point associated with D.

The conditions (1.50d) and (1.50d′) are not used in the proof. Therefore, we can ignore the imaginary parts of the vectors, and consider all the quantities $p^{(m)}$, $u^{(m)}$, etc., to be real. [Note that the conditions in (1.50) other than (1.50d) are linear in $p^{(m)}$, $u^{(m)}$, etc.]

The limiting values of the sequences $p_r^{(m)}$, $k_l^{(m)}$, and $u_r^{(m)}$ are denoted by p_r, k_l, and u_r, respectively. Let a vertex V_j of D be called a *parallel vertex* if and only if all the vectors p_r and k_l associated with lines of D incident upon V_j are parallel. Let the set of indices j of the parallel vertices be denoted by J_p, and for each $j \in J_p$ let e_j be the unit vector in common direction of the vectors p_r and k_l of the lines incident upon V_j:

$$e_j = p_r/|p_r| \quad \text{if} \quad [j:r] \neq 0 \quad \text{and} \quad j \in J_p$$

$$e_j = k_l/|k_l| \quad \text{if} \quad [j:l] \neq 0 \quad \text{and} \quad j \in J_p .$$

We consider now two cases:

Case (i). The vectors $v_j^{(m)}$ $(j \notin J_p)$ are bounded.

In this case one can, in a suitable Lorentz frame, construct a subsequence such that

(1) for each $j \notin J_p$,

$$v_j^{(m)} \to v_j ;$$

(2) for each $j \in J_p$, either

(i) $v_j^{(m)} \to v_j$,

or

(ii) $v_{j,0}^{(m)}$ goes monotonically to $+\infty$,

or

(iii) $v_{j,0}^{(m)}$ goes monotonically to $-\infty$;

(3) the ordering of the $v_{j,0}^{(m)}$, from smallest to largest, is independent of m.

We shall henceforth consider, in this case (i), only this subsequence. [To construct a subsequence of the kind just described one first takes a subsequence in which the bounded sequence of $v_j^{(m)}$ for the first $j \notin J_p$ tends a limit. Then one takes a sub-subsequence in which $v_j^{(m)}$ for the second $j \notin J_p$ tends to a limit, etc., until one has a sequence in which all $v_j^{(m)}$ $(j \notin J_p)$ tend to limits. Then one takes a subsequence in which for the first $j \in J_p$ either $v_j^{(m)}$ tends to a limit or $|v_j^{(m)}|$ increases monotonically without bound. Next one takes a sub-subsequence in which for the second $j \in J_p$ either $v_j^{(m)}$ tends to a limit or $|v_j^{(m)}|$ increases monotonically without bound, etc., until one has a sequence in which for each $j \notin J_p$ the vectors $v_j^{(m)}$ tend to a limit, and for each $j \in J_p$ either $v_j^{(m)}$ tends to a limit or $|v_j^{(m)}|$ increases monotonically without bound. For the first $j \in J_p$ such that $|v_j^{(m)}|$ increases monotonically without bound one can choose a subsequence so that $v_j^{(m)}/|v_j^{(m)}|$ approaches a limit. Then one picks a sub-subsequence such that the same is true for the second such $j \in J_p$, etc., until one has a sequence in which for every $j \in J_p$ either $v_j^{(m)}$ tends to a limit or $|v_j^{(m)}|$ increases monotonically without bound and $v_j^{(m)}/|v_j^{(m)}|$ tends to a limit. A Lorentz transformation mixes space and time components. Thus in almost any frame those j such that $|v_j^{(m)}|$ increases monotonically without bound will have unbounded $v_{j,0}$. Again taking subsequences one arrives at a subsequence in which properties (1) and (2) hold. Finally one can enumerate the possible time orderings of the vertices V_j, with $t_i = t_j$ considered as one possible ordering of t_i and t_j. If the infinite subsequence constructed above is separated into parts having different time orderings of the n' vertices V_j, then at least one of these parts will be an infinite subsequence. For this subsequence, properties (1), (2) and (3) will hold.]

Consider now for each j the quantity

$$X_j^{(m)} \equiv -\sum_{r=1}^{n} [j:r]^2 [p_{r,0}^{(m)} u_{r,v}^{(m)} - p_{r,v}^{(m)} u_{r,0}^{(m)}] + \sum_{l=1}^{N} [j:l][k_{l,0}^{(m)} \hat{u}_{l,v}^{(m)} - k_{l,v}^{(m)} \hat{u}_{l,0}^{(m)}] \quad (j=1,\dots,n', \quad v=1,2,3). \tag{3.3}$$

Replacing each nonzero $[j:r]u_r^{(m)}$ in (3.3) by its expression in terms of v_j given by (1.50f) and replacing each nonzero $[j:l]\hat{u}_l^{(m)}$ in (3.3) by its expression in terms of v_j given in (1.50g1), one obtains, after some cancellations,

$$\begin{aligned} X_j^{(m)} = & \sum_{r=1}^{n} [j:r][p_{r,0}^{(m)} v_{j,v}^{(m)} - p_{r,v}^{(m)} v_{j,0}^{(m)}] \\ & + \sum_{l=1}^{N} [j:l][k_{l,0}^{(m)} v_{j,v}^{(m)} - k_{l,v}^{(m)} v_{j,0}^{(m)}] \\ & + \sum_{l\in L_j^+} [k_{l,0}^{(m)} (\delta_{l,+}^{(m)})_v - k_{l,v}^{(m)} (\delta_{l,+}^{(m)})_0] \\ & - \sum_{l\in L_j^-} [k_{l,0}^{(m)} (\delta_{l,-}^{(m)})_v - k_{l,v}^{(m)} [\delta_{l,-}^{(m)}]_0] \end{aligned} \tag{3.4}$$

where $L_j^{\pm} \equiv \{l; [j:l] = \pm 1\}$.

The first two terms sum to zero because of (1.50e), which demands conservation of momentum-energy at vertex V_j. The last two terms approach zero because of (1.50g2). Thus

$$X_j^{(m)} \to 0 \quad (j=1,\dots,n'). \tag{3.5}$$

The set $\{1,\dots,n'\}$ can be divided into a set of disjoint sets $\tilde{J}_0, \tilde{J}_1, \tilde{J}_2, \dots, \tilde{J}_h$ such that V_j remains bounded if and only if $j \in \tilde{J}_0$, and for each g satisfying $1 \leqq g \leqq h$ there is an $\tilde{e}_g$ such that for $j \notin \tilde{J}_0$, $e_j = \tilde{e}_g$ if and only if $j \in \tilde{J}_g$. Then (3.5) implies that

$$X_g^{(m)} \equiv \sum_{j\in \tilde{J}_g} X_j^{(m)} \tag{3.7}$$

satisfies

$$X_g^{(m)} \to 0 \quad (0 \leqq g \leqq h). \tag{3.8}$$

The quantity

$$X_g^{(m)} = \sum_{j\in\tilde{J}_g} \left\{ \sum_{r=1}^{n} (-[j:r]^2 [p_{r,0}^{(m)} u_{r,v}^{(m)} - p_{r,v}^{(m)} u_{r,0}^{(m)}]) + \sum_{l=1}^{N} [j:l][k_{l,0}^{(m)} \hat{u}_{l,v}^{(m)} - k_{l,v}^{(m)} \hat{u}_{l,0}^{(m)}] \right\} \quad (1 \leqq g \leqq h) \tag{3.9}$$

can be written in the form

$$X_g^{(m)} = \sum_{r\in R_g} (-[p_{r,0}^{(m)} u_{r,v}^{(m)} - p_{r,v}^{(m)} u_{r,0}^{(m)}]) + \sum_{\substack{l\in L_g\\ j\in \tilde{J}_g}} [j:l][k_{l,0}^{(m)} \hat{u}_{l,v}^{(m)} - k_{l,v}^{(m)} \hat{u}_{l,0}^{(m)}] \quad (1\leqq g\leqq h),$$

where

$$R_g \equiv \{r; [j:r]\neq 0 \quad \text{for some} \quad j\in \tilde{J}_g\}, \tag{3.10}$$

$$L_g \equiv \{l; [j:l]\neq 0 \quad \text{for exactly one} \quad j\in \tilde{J}_g\}. \tag{3.11}$$

For each $l\in L_g$ with $1\leqq g\leqq h$ the line L_l has one end in the set of vertices $\{V_j; j\in \tilde{J}_g\}$ and the other end in the set of vertices $\{V_j; j\in \tilde{J}_0\}$, since $\tilde{e}_g \neq \tilde{e}_{g'}$ $(1\leqq g, g'\leqq h)$ if $g\neq g'$. That is, for each $l\in L_g$ $(1\leqq g\leqq h)$ either $j_+(l)\in \tilde{J}_0$ or $j_-(l)\in \tilde{J}_0$. But then the existence of the limits p_r and k_l, and v_j for $j\in \tilde{J}_0$, together with (1.50g1), (1.50g2), and (1.50h2), entail that all the $\hat{u}_l^{(m)}$ with $l\in L_g$ $(1\leqq g\leqq h)$ approach limits:

$$\hat{u}_l^{(m)} \to \hat{u}_l \qquad l\in L_g (1\leqq g\leqq h).$$

[Here we have used the fact that the existence of the limit of the sequence of vectors $v_{j+(l)}^{(m)}$ (or $v_{j-(l)}^{(m)}$), $k_l^{(m)}$ and $\delta_{l,+}^{(m)}$ (or $\delta_{l,-}^{(m)}$, resp.) entails, by virtue of the normalization (1.50h2), the existence of limit of the sequence of scalars $\beta_{l,+}^{(m)}$ (or $\beta_{l,-}^{(m)}$, resp.). Actually, the limit is equal to $\lim_{m\to\infty} [v_{j+(l),0}^{(m)}/k_{l,0}^{(m)}]$ (or $\lim_{m\to\infty} [v_{l-(l),0}^{(m)}/k_{l,0}^{(m)}]$, resp.). Note that $k_{l,0}^{(m)} \geqq m_l > 0$.]

Taking this limit in (3.9) one obtains

$$X_g = -\sum_{r\in R_g} p_{r,0} u_{r,v} + \sum_{\substack{l\in L_g\\ j\in \tilde{J}_g}} [j:l] k_{l,0} \hat{u}_{l,v}$$

$$= 0 \quad (1\leqq g\leqq h, \quad v=1,2,3). \tag{3.12}$$

The definitions

$$\begin{cases} \tilde{u}_r \equiv -[j(r):r] p_{r,0} u_r \\ \text{and} \\ \tilde{u}_l \equiv k_{l,0} \hat{u}_l \end{cases} \tag{3.13}$$

allow (3.12) to be written in the form

$$\sum_{j\in \tilde{J}_g} \Big(\sum_{r\in R_g} [j:r] \tilde{u}_r + \sum_{l\in L_g} [j:l] \tilde{u}_l \Big) = 0 \quad (1\leqq g\leqq h). \tag{3.14}$$

Consider now the diagram D_g consisting of the vertices V_j with $j\in \tilde{J}_g$ and the lines of D that are incident upon them. Let $D_{g,b}$ $(b=1,2,\ldots)$ be the connected components of D_g. Let $R_{g,b}$ and $L_{g,b}$ be the subsets of R_g and L_g that label lines of $D_{g,b}$. Then the arguments that give (3.14) give also, for each (g,b),

$$\sum_{j\in \tilde{J}_g} \Big(\sum_{R_{g,b}} [j:r] \tilde{u}_r + \sum_{L_{g,b}} [j:l] \tilde{u}_l \Big) = 0. \tag{3.15}$$

In each $D_{g,b}$ one may introduce a spanning tree diagram ([24]). This is a connected diagram without closed loops that contains every vertex of $D_{g,b}$ and every external line of $D_{g,b}$ Then a unique $\tilde{u}_l$ can be assigned to each internal line L_l of D by:

(a) setting $\tilde{u}_l = \lim \tilde{u}_l^{(m)}$ for each line L_l of D incident upon a vertex V_j of D with $j \in \tilde{J}_0$,

(b) setting $\tilde{u}_l = 0$ for each line L_l of D that belongs to some $D_{g,b}$ but not to its spanning tree diagram, and

(c) requiring for each vertex V_j of each $D_{g,b}$ the conservation-law equation

$$\sum_{r=1}^{n} [j:r]\tilde{u}_r + \sum_{l=1}^{N} [j:l]\tilde{u}_l = 0. \tag{3.16}$$

It will now be shown that these three conditions are consistent and allow us to determine vectors $\tilde{u}_r$ and $\tilde{u}_l$ uniquely.

Condition (a) ensures that each external line L_l or L_r of each $D_{g,b}$ has a well-defined $\tilde{u}_r$ or $\tilde{u}_l$. Condition (b) ensures that the $\tilde{u}_l$ of the internal lines of each $D_{g,b}$ are determined by the conservation-law conditions (c) to be unique linear combinations of the $\tilde{u}_l$ and $\tilde{u}_r$ of the external lines of that $D_{g,b}$. To see this, simply pick a preferred external line of $D_{g,b}$ and let the $\tilde{u}_l$ or $\tilde{u}_r$ associated with each other external line flow along the unique path in the spanning tree diagram that leads directly to the preferred external line. Then conditions (b) and (c) are satisfied. Equation (3.15) ensures that the total $\tilde{u}_l$ or $\tilde{u}_r$ flowing out at the preferred vertex of $D_{g,b}$ equals that flowing in along the other lines. The tree structure of the spanning tree diagram ensures uniqueness.

The vectors $\tilde{u}_r$ and $\tilde{u}_l$ constructed in this way can be converted back, by means of (3.13), to vectors u_r and $\hat{u}_l$ that satisfy for each $j \notin \tilde{J}_0$,

$$\sum_{r=1}^{n} (-[j:r]^2 p_{r,0} u_{r,\nu}) + \sum_{l=1}^{N} [j:l] k_{l,0} \hat{u}_{l,\nu} = 0, \quad j \notin \tilde{J}_0, \quad \nu = 1,2,3. \tag{3.17}$$

Since all the $p_r^{(m)}$ and $k_l^{(m)}$ tend to well-defined limits, one may construct a star-shaped solution s of the ordinary Landau equations by placing at the origin all the vertices V_j with $j \in \tilde{J}_0$, and placing on lines through the origin having direction $\tilde{e}_g$ all the vertices V_j with $j \in \tilde{J}_g$. The time ordering of the vertices on these lines can be taken to be the same as the time ordering in the subsequence defined at the beginning of the proof, except that all the vertices V_j for $j \in \tilde{J}_0$ lie at the origin, and hence at a single point. However, all the vertices V_j with $j \notin \tilde{J}_0$ can be placed away from the origin. Thus the set J_s associated with this star-shaped solution s of the ordinary Landau equations is the complement [in the set $(1, \dots, n')$] of $\tilde{J}_0$, and each $j \in \tilde{J}_g$ $(1 \leqq g \leqq h)$ belongs to J_s.

The generalized Landau equations associated with D and s have several trivial degrees of freedom. These can be removed by requiring

$$\begin{cases} u_{r,0} = 0 & (r = 1, \dots, n) \\ \hat{u}_{l,0} = 0 & (l = 1, \dots, N) \\ v_j = 0, \quad (\eta_{j,r})_0 = (\hat{\eta}_{j,l})_0 = 0 & (j \in J_s) \\ \beta_r = 0 & (j(r) \in J_s) \\ \beta_{l,+} = 0 & (j_+(l) \in J_s) \\ \text{and} & \\ \beta_{l,-} = 0. & (j_-(l) \in J_s) \end{cases} \tag{3.18}$$

[The simultaneous addition of v_j to all $\eta_{j,r}$ and $\hat{\eta}_{j,l}$ having index $j \in J_s$ changes (3.1h′) by a term that vanishes by virtue of momentum-energy conservation at the vertex V_j. And the addition of $\beta_r p_r$ to $\eta_{j,r}$, or $\beta_{l,+} k_l$ to $\hat{\eta}_{j_+(l),l}$, or $\beta_{l,-} k_l$ to $\hat{\eta}_{j_-(l),l}$, assuming $[j:r] \neq 0$ $(j \in J_s)$, or $[j:l] = +1$ $(j \in J_s)$, or $[j:l] = -1$ $(j \in J_s)$, respectively, also leaves (3.1h′) intact.] Then the generalized Landau Eqs. (3.1d′) for $j(r) \in J_s$, (3.1e′1) for $j_+(l) \in J_s$, and (3.1e′2) for $j_-(l) \in J_s$ are automatically satisfied, whereas (3.1h′) takes the form (3.17). Thus if we define the vectors p_r, k_l, u_r, and v_j for $j \notin J_s$ to be the limits of the corresponding sequences, and define the $\hat{u}_l$ in the manner described above, then we will have a solution of the generalized Landau equations, since all the generalized Landau equations other than those just mentioned are limits of equations that occur in the definition of $\tilde{\mathscr{L}}^\sigma(D)$. This completes the proof for Case (i).

Case (ii). Some vector $v_j^{(m)}$ $(j \notin J_p)$ is not bounded.

In this case one can construct from the original sequence a new rescaled sequence in which:

(a) there is some $j \notin J_p$ such that $|v_j^{(m)}| = 1$ for all m,

(b) for all other $j \notin J_p$ the inequality $|v_j^{(m)}| \leqq 1$ holds for all m, and

(c) $u_r^{(m)} \to 0$ for all r.

This rescaled solution satisfies the conditions of Case (i). Hence the arguments for Case (i) show that a $u=0$ solution of the generalized Landau equations can be constructed. This solution has $|v_j| = 1$ for some $j \notin J_p$. But then p is a generalized $u=0$ point, since our scaling procedure has nothing to do with momentum-energy vectors p and k, i.e., the momentum-energy vectors incident upon $j \notin J_p$ remain nonparallel after the rescaling. This completes the proof of Theorem 3.1.

Remark 1. The definitions and results concerning the generalized Landau equations that are given in this paper are not identical to those given in our earlier papers (Kawai-Stapp [1], and Kashiwara-Kawai-Stapp [14]). The definitions given previously are, in fact, more complicated and difficult to use than the present ones, and the results stated earlier are not completely correct as stated. Accordingly, the earlier definitions and results concerning generalized Landau equations should be replaced by the versions given above.

Remark 2. Since the reduced mass-shell variety $\mathscr{M}_r$ is singular along $\mathscr{M}_{\text{exc}}$, we cannot talk about the singularity spectrum there. One natural alternative approach is to discuss the singularity spectrum of

$$\prod_{r=1}^{n} \delta(p_r^2 - \mu_r^2) Y(p_{r,0}) \delta^4\Big(\sum_{j,r} [j:r] p_r\Big) f(p)$$

on $\sqrt{-1} S^* \mathbb{R}^{4n}$, where $f(p)$ is a function defined in a neighborhood of $p_0 \in \mathscr{M}_{\text{exc}}$ in $\mathbb{R}^{4n}$. The case where $f(p) \equiv 1$ is the simplest one of the phase space integrals, and this case is thus covered by Theorem 1.8. Furthermore, the conjectures stated at the beginning of this section apply to

$$\prod_{r=1}^{n} \delta(p_r^2 - \mu_r^2) Y(p_{r,0}) \delta^4\Big(\sum_{j,r} [j:r] p_r\Big) s(p)$$

and

$$\prod_{r=1}^{n} \delta(p_r^2 - \mu_r^2) Y(p_{r,0}) \delta^4\Big(\sum_{j,r} [j:r] p_r\Big) f_B(p),$$

respectively, if we do not employ the convention (1.54); namely, if we regard $\tilde{\mathscr{L}}^\sigma(D)$ as a subvariety of $\sqrt{-1}S^*\mathbb{R}^{4n}$. Here $s(p)$ [resp. $f_B(p)$] denotes the off-shell scattering amplitude (resp. off-shell bubble diagram function) divided by $\delta^4\left(\sum_{j,r}[j:r]p_r\right)$. The convention also applies to the generalized Landau variety $\mathscr{L}^\sigma_g(D)$. With this understanding. Theorem 3.1 also holds for varieties considered in $\sqrt{-1}S^*\mathbb{R}^{4n}$. As a matter of fact, the prescription given by $\tilde{\mathscr{L}}^\sigma(D)$ at $\mathscr{M}_{\text{exc}}$ simply reads as follows: If $p\in\mathscr{M}_{\text{exc}}$ and $(p;\sqrt{-1}u)\in\tilde{\mathscr{L}}^\sigma(D)$, then

$$\sum_{r=1}^{n}(p_{r,0}u_{r,\nu}-p_{r,\nu}u_{r,0})=0\,,\qquad \nu=1,2,3\,.$$

Note that these conditions immediately follow from the Lorentz-invariance property of the functions in question. In fact, the Lorentz-invariance property of a function f implies that f satisfies the following differential equations:

$$\sum_{r=1}^{n}\left(p_{r,\nu}\frac{\partial}{\partial p_{r,0}}+p_{r,0}\frac{\partial}{\partial p_{r,\nu}}\right)f=0\,,\qquad \nu=1,2,3\,.$$

Then denoting by η_r^ν the cotangent vector corresponding to $p_{r,\nu}$ ($\nu=0,1,2,3$), we find that

$$\text{S.S.}\,f\subset\{(p;\sqrt{-1}\eta)\in\sqrt{-1}S^*\mathbb{R}^{4n};\quad \sum_{r=1}^{n}(p_{r,0}\eta_r^\nu+p_{r,\nu}\eta_r^0)=0\,,\quad \nu=1,2,3\}\,. \tag{3.18'}$$

(S-K-K [13] Chapt. III, Theorem 2.1.1.) On the other hand, according to the usual convention that we have been using, we identify $u_r=(u_{r,0},\,u_{r,1},\,u_{r,2},\,u_{r,3})$ with $\eta_r=(\eta_r^0,\,\eta_r^1,\,\eta_r^2,\,\eta_r^3)$ by the aid of the Minkowsky metric in order to regard u-vectors as cotangent vectors, namely, u_r and η_r are related by the following relations:

$$u_{r,0}=\eta_r^0$$
$$u_{r,\nu}=-\eta_r^\nu\,,\qquad \nu=1,2,3\,.$$

Hence (3.18') reads as follows:

$$\text{S.S.}\,f\subset\{(p;\sqrt{-1}u)\in\sqrt{-1}S^*\mathbb{R}^{4n};\quad \sum_{r=1}^{n}(p_{r,0}u_{r,\nu}-p_{r,\nu}u_{r,0})=0\}\,.$$

Thus the validity of our conjecture at $\mathscr{M}_{\text{exc}}$ is guaranteed by the Lorentz-invariance property of the S-matrix and the bubble diagram functions.

It may be useful to illustrate the results obtained in this paper by some concrete examples:

Example 3.1. Let D be the diagram of Fig. 3.1. Then all physical points p are $u=0$ points of D.

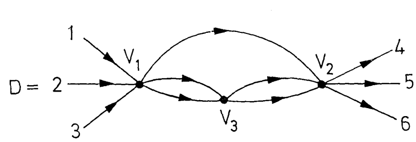

Fig. 3.1. A Landau diagram D for which all physical points p are $u=0$ points. All masses are assumed to be equal

The Landau equations corresponding to this D can be satisfied for all on-mass-shell values of the external momenta that satisfy the conservation law constraint $p_1+p_2+p_3=p_4+p_5+p_6\equiv P$. The solution is represented by a space-time diagram D in which all external trajectories pass through the same point P. Placing this point at the origin gives a $u=0$ solution. Consequently, the standard results ([6], [12]) allow every point $p\in\mathbb{R}^{24}$ such that the mass-shell and conservation-law constraints are satisfied to be a singularity of I_D.

However, Theorem 3.1 shows that $\tilde{\mathscr{L}}^{\mathbb{R}}(D)\equiv\bigcup_{\sigma}\tilde{\mathscr{L}}^{\sigma}(D)$, where σ runs over all possible signs, is limited by the conditions $P^2=9m^2$ and

$$\sum_{r=1}^{6}(p_{r,0}u_{r,\nu}-p_{r,\nu}u_{r,0})=0, \qquad \nu=1,2,3. \tag{3.19}$$

These two conditions correspond to allowing only those space-time diagrams D such that all the external lines are parallel and the center-of-mass trajectory of the initial coincides with that of the final particles. Actually, Kawai-Stapp [1] shows that the singularity spectrum of I_D for this diagram D is described in this manner. The arguments in that article were based on a specially detailed analysis in this case and were dependent on Lorentz invariance.

Note that for this diagram D the $u=0$ points cover the entire set of points that satisfy the mass-shell and conservation-law constraints, whereas the generalized $u=0$ points are an empty set. This is because the only $u=0$ solutions of generalized Landau equations associated with this D are those in which all the vertices not at the origin are parallel vertices; i.e., for any $u=0$ solution all trajectories incident upon any vertex that does not lie at the origin are parallel.

Example 3.2. Let D be the diagram shown in Fig. 3.2.

D =

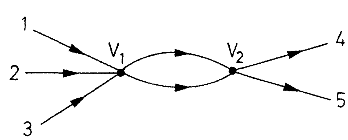

Fig. 3.2. Diagram for Example 3.2. The masses associated with the external lines 1, 2, 3 are μ and the masses associated with external lines and the internal lines 4, 5 and are m with $m\geqq\frac{3}{2}\mu$

Theorem 3.1 shows that $\tilde{\mathscr{L}}^{\mathbb{R}}(D)$ is confined to the set of points $(p;\sqrt{-1}u)\in\sqrt{-1}S^*\mathbb{R}^{20}$ that satisfy the mass-shell constraints $p_r^2=m^2$, the conservation-law constraints $p_1+p_2+p_3=p_4+p_5$, the threshold condition $(p_4+p_5)^2=4m^2$, the conservation of center-of-mass trajectory condition

$$\sum_{r=1}^{5}(p_{r,0}u_{r,\nu}-p_{r,\nu}u_{r,0})=0, \qquad \nu=1,2,3, \tag{3.20}$$

and the condition that for some four-vector v_1 and some set of scalars β'_1, β'_2, β'_3

$$u'_r\equiv\varepsilon_r u_r=v_1-\beta'_r p_r \qquad (r=1,2,3) \tag{3.21}$$

where $\varepsilon_r\equiv-[j(r):r]$.

The condition (3.21) corresponds to the requirement on the space-time diagram D that the three initial trajectories pass through the common vertex V_1. Condition (3.20) demands that the trajectory of the center-of-mass of the initial particles coincides with that of the final particles. This condition arises from the generalized Landau equations. It is not correctly represented by the ordinary Landau equations, which would require the stronger conditions analogous to (3.21), namely,

$$u'_r = v_2 - \beta'_r p_r \qquad (r=4,5). \tag{3.22}$$

The center-of-mass conditions (3.19) and (3.20) arise from substituting into (3.1h′) the expressions in (3.1) that give the $\eta_{j,r}$ and $\hat{\eta}_{j_\pm,l}$ in terms of the u_r, $\hat{u}_l$ and v_j, and then using the conservation-law condition (3.1c) to eliminate v_j. This gives, for each $j \in J_s$,

$$\sum_{r=1}^{n} [j:r](p_{r,0}u'_{r,v} - p_{r,v}u'_{r,0}) + \sum_{l=1}^{N} [j:l](k_{l,0}\hat{u}_{l,v} - k_{l,v}\hat{u}_{l,0}) = 0. \tag{3.23}$$

The same equation is true also for $j \notin J_s$, since (3.1h′) is trivially true for $j \notin J_s$ by virtue of (3.1g′). Equation (3.23) expresses the requirement that the trajectory of the center-of-mass of the particles coming into V_j coincides with that of the particles going out from V_j. Since this center-of-mass property holds at each vertex, it also holds for the diagram as a whole, because the contributions from the intermediate particles cancel out when (3.23) is summed over all j ($j=1, \ldots, n'$).

Examples 3.3. Suppose D is the diagram of Fig. 3.3(a).

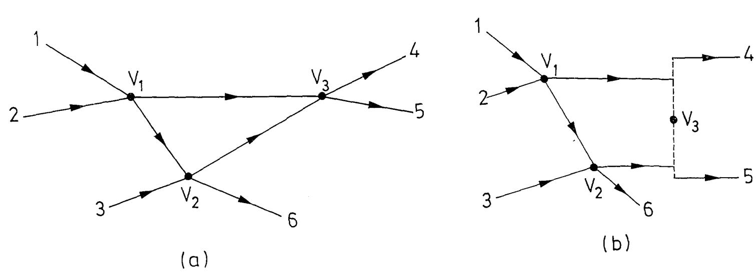

Fig. 3.3. Diagrams for Example 3.3. All masses are equal

Then Theorem 3.1 shows that $\tilde{\mathscr{L}}^{\mathrm{IR}}(D)$ is confined to a union of the following two parts. The first is described by the ordinary Landau equations associated with triangle diagram. The second is confined to the points p where the triangle diagram Landau surface in p space meets the normal threshold surface $(p_4+p_5)^2 = 4m^2$. The singularity spectrum of this second part is described by the space-time diagram of Fig. 3.3(b), where the parallel trajectories incident upon the dotted line through V_3 are constrained by a center-of-mass condition of the form (3.23) (see also [25], §4, for related topics).

Example 3.4. Suppose D is the diagram of Fig. 3.4.

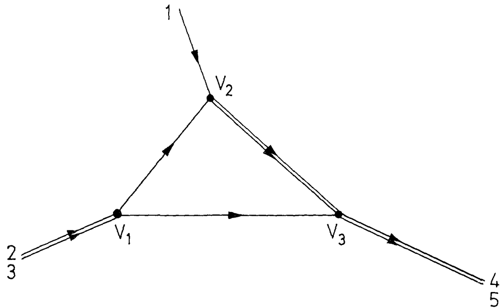

Fig. 3.4. Diagram for Example 3.4

If the masses are such that there is a solution of the ordinary Landau equations associated with D, with p_2 parallel to p_3 and p_4 parallel to p_5, but p_1 not parallel to p_2, then the points $p=(p_1, \ldots, p_5)$ corresponding to this solution are generalized $u=0$ points associated with D.

[Note that the three vertices define a plane in which all vectors lie. Conservation of center-of-mass-motion (i.e., conservation of the trajectory of the center-of-mass) implies that the three sets of external trajectories must pass through a common point; otherwise there would be a net contribution $\sum_{r=1}^{5} (p_{r,0}u_{r,v} - p_{r,v}u_{r,0})$, as one can see by placing the origin at the intersection of, say, trajectory 1 with trajectory 2 (or 3).]

The generalized $u=0$ points associated with this diagram D occur only if the masses satisfy a certain algebraic relationship. To obtain an example of a generalized $u=0$ point that can occur without a strong condition on the masses, one can remove lines 4 and 5 of Fig. 3.4 and then connect together two copies of the resulting diagram at vertex V_3. The space-time diagram formed by laying the two parts of this diagram on top of each other corresponds to a generalized $u=0$ point.

The signs of the α's are, in this case, not all positive. We know of no generalized $u=0$ point corresponding to a positive-α solution with physical particle masses.

Acknowledgment. The authors would like to acknowledge very helpful correspondence and discussions with Professor Daniel Iagolnitzer.

References

1. Kawai, T., Stapp, H.P.: Micro-local study of the S-matrix singularity structure. In: Lecture notes in physics, Vol. 39, pp. 36–48. Berlin, Heidelberg, New York: Springer 1975
2. Stapp, H.P.: J. Math. Phys. **9**, 1548–1592 (1968)
3. Coster, J., Stapp, H.P.: J. Math. Phys. **11**, 2743–2763 (1970)
4. Iagolnitzer, D.: Macro-causality, physical region analyticity and independence property in S-matrix theory. In: Lecture notes in mathematics, Vol. 449, pp. 102–132. Berlin, Heidelberg, New York: Springer 1975
5. Iagolnitzer, D.: The S-matrix. Amsterdam, Oxford, New York: North-Holland 1978
6. Iagolnitzer, D.: Commun. Math. Phys. **41**, 39–53 (1975)

7. Stapp, H.P.: Discontinuity formula for multiparticle amplitudes. In: Structural analysis of collision amplitude, pp. 191–273. Amsterdam, Oxford, New York: North-Holland 1976
8. Iagolnitzer, D., Stapp, H.P.: Commun. Math. Phys. **57**, 1–30 (1977)
9. Kashiwara, M., Kawai, T.: Proc. Japan Acad. **52**, 161–164 (1976)
(See also the full paper of Kashiwara-Kawai [26] for the detailed argument and a correction.)
10. Kashiwara, M., Kawai, T.: Proc. Japan Acad. **51**, 270–272 (1975)
11. Sato, M.: Recent development in hyperfunction theory and its application to physics. In: Lecture notes in physics, Vol. 39, pp. 13–29. Berlin, Heidelberg, New York: Springer 1975
12. Kawai, T., Stapp, H.P.: Publ. RIMS, Kyoto Univ. **12**, Suppl. 155–232 (1977)
13. Sato, M., Kawai, T., Kashiwara, M.: Microfunctions and pseudo-differential equations. In: Lecture notes in mathematics, Vol. 287, pp. 265–529. Berlin, Heidelberg, New York: Springer 1973
14. Kashiwara, M., Kawai, T., Stapp, H.P.: Publ. RIMS, Kyoto Univ. **12**, Suppl. 141–146 (1977)
15. Riesz, M.: Acta Math. **81**, 1–223 (1949)
16. Gel'fand, I.M., Shilov, G.E.: Generalized functions, I. New York: Academic Press 1964 (translated from the Russian original, 1958)
17. Bernstein, I.N., Gel'fand, S.I.: Funkts. Analiz i ego Prilozhen. **3**(1), 84–85 (1969) (in Russian) (translation: Funct. Anal. Appl. **3**, 68–69 (1969)
18. Atiyah, M.F.: Comm. Pure Appl. Math. **23**, 145–150 (1970)
19. Stapp, H.P.: Phys. Rev. **125**, 2136 (1962), App. J. 1; Phys. Rev. **160**, 1251 (1967), Eq. (2.5), line 3
20. Iagolnitzer, D., Stapp, H.P.: Commun. Math. Phys. **14**, 15–55 (1969)
21. Iagolnitzer, D.: Physical region properties of multiparticle collision amplitude. In: Structural analysis of collision amplitude, pp. 163–190. Amsterdam, Oxford, New York: North-Holland 1976
22. Iagolnitzer, D.: Analytic structure of distributions and essential support theory. In: Structural analysis of collision amplitude, pp. 295–358. Amsterdam, Oxford, New York: North-Holland 1976
23. Pham, F.: Microanalyticité de la matrice *S*. In: Lecture notes in mathematics, Vol. 449, pp. 83–102. Berlin, Heidelberg, New York: Springer 1975
24. Berge, C.: Graphs and hypergraphs, p. 25. Amsterdam, Oxford, New York: North-Holland 1973
25. Kashiwara, M., Kawai, T., Oshima, T.: Commun. Math Phys. **60**, 97–130 (1978)
26. Kashiwara, M., Kawai, T.: On holonomic systems for $\prod_{l=1}^{N}(f_l+\sqrt{-1}0)^{\lambda_l}$. Publ. RIMS, Kyoto Univ. **15**, No. 2 (1979)
27. Iagolnitzer, D.: Commun. Math. Phys. **63**, 49–96 (1978)
28. Kashiwara, M., Kawai, T.: On the characteristic variety of a holonomic system with regular singularities. RIMS, Preprint No. 276 (1979)

Communicated by H. Araki

Received May 26, 1978; in revised form October 23, 1978